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Homotopy groups of free group character varieties

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Abstract. Let G be a connected, complex reductive Lie group with maximal compact subgroup K, and let $\mathfrak{X}_r$ denote the moduli space of G– or K–valued representations of a rank-r free group. In this article we develop methods for studying the low-dimensional homotopy groups of these spaces and of their subspaces $\mathfrak{X}_r^{\mathrm{irr}}$ of irreducible representations.

Our main result is that when G is $\mathsf{GL}_n(\mathbb{C})$ or $\mathsf{SL}_n(\mathbb{C})$, the second homotopy group of $\mathfrak{X}_r$ is trivial. The proof depends on a new general position-type result in a singular setting. This result is proven in the Appendix and may be of independent interest.

We also obtain new information regarding the homotopy groups of the subspaces $\mathfrak{X}_r^{\mathrm{irr}}$. Recent work of Biswas and Lawton determined $\pi_1(\mathfrak{X}_r)$ for general G, and we describe $\pi_1(\mathfrak{X}_r^{\mathrm{irr}})$. Specializing to the case $G = \mathsf{GL}_n(\mathbb{C})$, we explicitly compute the homotopy groups of the smooth locus $\mathfrak{X}_r^{\mathrm{sm}} = \mathfrak{X}_r^{\mathrm{irr}}$ in a large range of dimensions, finding that they exhibit Bott Periodicity.

As a further application of our methods (and in particular our general position result) we obtain new results regarding centralizers of subgroups of G and K, motivated by a question of Sikora.

Additionally, we use work of Richardson to solve a conjecture of Florentino–Lawton about the singular locus of $\mathfrak{X}_r$, and we give a topological proof that for $G = \mathsf{GL}_n(\mathbb{C})$ or $G = \mathsf{SL}_n(\mathbb{C})$, the space $\mathfrak{X}_r$ is not a rational Poincaré Duality Space for $r \geqslant 4$ and $n = 2$.

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