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Young measure approach to the weak convergence theory in the calculus of variations and strong materials

MIKHAIL A. SYCHEV AND NINA NIKOLAEVNA SYCHEVA

Abstract. Let $L(x, u, v) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a Carathéodory integrand with superlinear growth in $v \in \mathbb{R}^{m\times n}$.

Under these assumptions we clarify conditions on L-gradient Young measures which imply validity of the weak convergence theory for the associated integral functional $J(u) := \int_\Omega L(x, u(x), Du(x))dx$. Weak convergence theory includes lower semicontinuity with respect to the weak convergence of Sobolev functions, the convergence in energy property (weak convergence of Sobolev functions and convergence in energy imply the strong convergence of the functions), the integral representation for the relaxed energy and related questions. The results of the weak convergence theory follow from a characterization of gradient Young measures associated with the functional. Then we apply the general theory to establish validity of the weak convergence theory for two new classes of integral functionals. Note that the approach of gradient Young measures to the weak convergence theory has certain advantages comparing with Γ-convergence approach, since homogeneous L-gradient Young measures can be characterized for completely arbitrary extended-valued integrands L with infinite growth.

We also disscus the weak convergence theory for so-called strong materials introduced by the first author. We claim that this theory has a better form for such energies. In particular strong materials with $p(x)$-growth admit this theory without any further assumptions on the function $p(x) : \Omega \to \mathbb{R}$ when previously all experts studied the case when $p(\cdot)$ satisfies Zhikov's condition.

Finally we also suggest five conjectures which aim to attract the attention of specialists to further possible developments in the weak convergence theory and to the role of strong materials in Mathematical Theory of Elasticity.

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