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Thin actions on CAT(0)-spaces

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Abstract. We study packed, geodesically complete, CAT(0)-spaces X admitting a *thin action*, that is, a discrete isometry group Γ of X whose systole at every point is sufficiently small (smaller than a universal constant only depending on the packing constants of X —the *Margulis constant*). We show that, if X admits an almost-cocompact thin action, then it can be decomposed as a union $\bigcup_i \mathrm{Min}(g_i)$ of minimal sets of isometries with small displacement, with the property that, if $\mathrm{Min}(g_i)\cap\mathrm{Min}(g_j) \neq \emptyset$, then g_i, g_j generate a virtually nilpotent group. This generalizes some well-known structure theorems for manifolds, due to Buyalo and Cao-Cheeger-Rong, to CAT(0)-spaces. We then deduce some splitting and rigidity results for thin actions by slim groups (*i.e.*, isometry groups whose elliptic elements have fixed-point set with empty interior): for instance, every packed, geodesically complete, CAT(0)-space admitting a thin, almost cocompact action by a slim group Γ always splits a non-trivial Euclidean factor, provided that Γ does not contain a subgroup isomorphic to $\mathbb{Z}\times\mathbb{Z}$; also, if $\dim(X) \leq 2$, then X is isometric to the product of $\mathbb{R}$ with a (possibly trivial) tree, while in the cocompact case we always deduce that $X = \mathbb{R}$ necessarily. Moreover, we show that if a space X as above has some strictly negative curvature behaviour (*e.g.*, if X is a visibility space, or it has an open subset which is CAT($-\varepsilon$)) then it cannot support a thin action; this generalizes the classical Margulis lemma to a broader class of singular spaces.

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