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Uniqueness of positive solutions to elliptic equations with critical exponential growth on the unit disc and applications

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Abstract. In the past few decades, the uniqueness of positive solutions to elliptic equations with polynomial growth has been extensively studied. However, the corresponding problems associated with elliptic equations with critical exponential growth given by the Trudinger-Moser inequalities still remain open. For this kind of equations, the classic non-degeneracy method based on the Pohozaev identity and the study of the linearized equation do not seem to work. In this paper, we will solve this uniqueness problem. More precisely, we obtain uniqueness of positive solutions to equations of the form

$$\begin{cases} -\Delta u = \lambda u e^{u^2} & x \in B_1 \subset \mathbb{R}^2 \\ u > 0 & x \in B_1 \\ u = 0 & x \in \partial B_1, \end{cases}$$

where $0 < \lambda < \lambda_1(B_1)$ and $\lambda_1(B_1)$ denotes the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition. Our method relies on a delicate and difficult analysis of radial solutions to the above equation and a careful asymptotic expansion of solutions near the boundary. Building on our uniqueness result, we develop a new strategy to establish a quantization property for elliptic equations with critical exponential growth in balls of hyperbolic spaces, and we obtain the multiplicity and non-existence of positive critical points for the supercritical Trudinger-Moser functional. Our method for the quantization property and non-existence of the critical points avoids using the complicated blow-up analysis employed in the literature. This method can also be applied to study similar problems in balls of high-dimensional Euclidean spaces $\mathbb{R}^n$ or hyperbolic spaces, provided the uniqueness for the corresponding quasilinear elliptic equations with the critical exponential growth holds true.

Mathematics Subject Classification (2020): 35A02 (primary); 35B33, 35J60 (secondary).

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