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Joint normality of representations of numbers: an ergodic approach

VITALY BERGELSON AND YOUNGHWAN SON

Abstract. We introduce an ergodic approach to the study of *joint normality* of representations of numbers. For example, we show that, for any integer $b \geq 2$, almost every number $x \in [0, 1)$ is jointly normal with respect to the b-expansion and the continued-fraction expansion. This fact is a corollary of the following result which deals with *pointwise joint ergodicity*:

Let $T_b : [0, 1] \to [0, 1]$ be the times b map defined by $T_b x = bx \mod 1$ and let $T_G : [0, 1] \to [0, 1]$ be the Gauss map defined by $T_G(x) = \{\frac{1}{x}\}$ for $x \neq 0$ and $T_G(0) = 0$. (Here $\{\cdot\}$ denotes the fractional part.) For any $f, g \in L^\infty(\lambda)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T_b^n x)\, g(T_G^n x) = \int f \, d\lambda \cdot \int g \, d\mu_G \quad \text{for almost every } x \in [0, 1],$$

where λ is the Lebesgue measure on $[0, 1]$ and μ_G is the Gauss measure on $[0, 1]$ given by $\mu_G(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx$ for any measurable set $A \subset [0, 1]$.

We show that the phenomenon of the pointwise joint ergodicity takes place for a wide variety of number-theoretical maps of the interval and derive the corresponding corollaries pertaining to joint normality.

We also establish the equivalence of various forms of normality and joint normality for representations of numbers, hereby providing a general framework for classical normality results.

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