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On Shafarevich–Tate groups and analytic ranks in families of modular forms, I. Hida families

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Abstract. This is the first article in a two-part project whose aim is to study algebraic and analytic ranks in p-adic families of modular forms. Let f be a newform of weight 2, square-free level and trivial character, let A_f be the Abelian variety attached to f and for every good ordinary prime p for f let $\boldsymbol{f}^{(p)}$ be the p-adic Hida family through f. We prove that, for all but finitely many primes p as above, if A_f is an elliptic curve such that the Mordell–Weil group $A_f(\mathbb{Q})$ has rank 1 and the p-primary part of the Shafarevich–Tate group of A_f over $\mathbb{Q}$ is finite, then all specializations of $\boldsymbol{f}^{(p)}$ of weight congruent to 2 modulo $2(p-1)$ and trivial character have finite p-primary Shafarevich–Tate group and 1-dimensional image of the relevant p-adic étale Abel–Jacobi map. An analogous result is obtained also in the rank 0 case. As a second contribution, with no restriction on the dimension of A_f but assuming the nondegeneracy of certain height pairings *à la* Gillet–Soulé between Heegner cycles, we show that, for all but finitely many p's, if f has analytic rank 1, then all specializations of $\boldsymbol{f}^{(p)}$ of weight congruent to 2 modulo $2(p-1)$ and trivial character have analytic rank 1. This result provides some evidence in rank 1 and weight larger than 2 for a conjecture of Greenberg predicting that the analytic ranks of even-weight modular forms in a Hida family should be as small as allowed by the functional equation, with at most finitely many exceptions.

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