Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXV (2024), 611-648

Geometry of certain foliations on the complex projective plane

SAMIR BEDROUNI AND DAVID MARÍN

Abstract. Let $d \geq 2$ be an integer. The set $\mathbf{F}(d)$ of foliations of degree d on the complex projective plane can be identified with a Zariski-open set of a projective space of dimension $d^2 + 4d + 2$ on which $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ acts. We show that there are exactly two orbits, $\mathcal{O}(\mathcal{F}_1^d)$ and $\mathcal{O}(\mathcal{F}_2^d)$, of minimal dimension 6, necessarily closed in $\mathbf{F}(d)$. This generalizes known results in degrees 2 and 3. We deduce that an orbit $\mathcal{O}(\mathcal{F})$ of an element $\mathcal{F} \in \mathbf{F}(d)$ of dimension 7 is closed in $\mathbf{F}(d)$ if and only if $\mathcal{F}_i^d \not\in \overline{\mathcal{O}(\mathcal{F})}$ for $i = 1, 2$. This allows us to show that in any degree $d \geq 3$ there are closed orbits in $\mathbf{F}(d)$ other than the orbits $\mathcal{O}(\mathcal{F}_1^d)$ and $\mathcal{O}(\mathcal{F}_2^d)$, unlike the situation in degree 2. On the other hand, we introduce the notion of the basin of attraction $\mathbf{B}(\mathcal{F})$ of a foliation $\mathcal{F} \in \mathbf{F}(d)$ as the set of $\mathcal{G} \in \mathbf{F}(d)$ such that $\mathcal{F} \in \overline{\mathcal{O}(\mathcal{G})}$. We show that the basin of attraction $\mathbf{B}(\mathcal{F}_1^d)$, respectively $\mathbf{B}(\mathcal{F}_2^d)$, contains a quasi-projective subvariety of $\mathbf{F}(d)$ of dimension greater than or equal to $\dim \mathbf{F}(d) - (d-1)$, respectively $\dim \mathbf{F}(d) - (d-3)$. In particular, we obtain that the basin $\mathbf{B}(\mathcal{F}_2^3)$ contains a nonempty Zariski-open subset of $\mathbf{F}(3)$. This is an analog in degree 3 of a result on foliations of degree 2 due to Cerveau, Déserti, Garba Belko and Meziani.

Mathematics Subject Classification (2020): 37F75 (primary); 32S65, 32M25, 32M05 (secondary).