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Range-decreasing group homomorphisms and holomorphic maps between generalized loop spaces

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Abstract. Let $\mathcal{G}$ (respectively M) be a positive-dimensional Lie group (respectively a connected complex manifold without boundary) and let V be a finite-dimensional C^∞ compact connected manifold, possibly with boundary. Fix a smoothness class $\mathcal{F} \in \{C^\infty$, Hölder $C^{k,\alpha}$, Sobolev $W^{k,p}\}$. The space $\mathcal{F}(V, \mathcal{G})$ (respectively $\mathcal{F}(V, M)$) of all $\mathcal{F}$ maps $V \to \mathcal{G}$ (respectively $V \to M$) is a Banach/Fréchet Lie group (respectively a complex manifold). Let $\mathcal{F}^0(V, \mathcal{G})$ (respectively $\mathcal{F}^0(V, M)$) be the component of $\mathcal{F}(V, \mathcal{G})$ (respectively $\mathcal{F}(V, M)$) containing the identity (respectively constants). A map f from a domain $\Omega \subset \mathcal{F}_1(V, M)$ to $\mathcal{F}_2(W, M)$ is called range-decreasing if $f(x)(W) \subset x(V)$, for every $x \in \Omega$. We prove that if $\dim_{\mathbb{R}} \mathcal{G} \geq 2$ then any range-decreasing group homomorphism $f : \mathcal{F}_1^0(V, \mathcal{G}) \to \mathcal{F}_2(W, \mathcal{G})$ is the pullback by a map $\phi : W \to V$. We also provide several sufficient conditions for a range-decreasing holomorphic map $\Omega \to \mathcal{F}_2(W, M)$ to be a pullback operator. Then we apply these results to study certain decompositions of holomorphic maps $\mathcal{F}_1(V, N) \supset \Omega \to \mathcal{F}_2(W, M)$. In particular, we identify some classes of holomorphic maps $\mathcal{F}_1^0(V, \mathbb{P}^n) \to \mathcal{F}_2(W, \mathbb{P}^m)$, including all automorphisms of $\mathcal{F}^0(V, \mathbb{P}^n)$.

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