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Erratum to "Heights of points with bounded ramification"

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Abstract. This note is to inform that Lemma 5.8 in [1] is incorrect. We give a counterexample, locate the error in the proof and discuss the consequences to Theorem 5.9 in [1]. All other results in [1] are not affected by this error.

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1. [1, Lemma 5.8] is false

All citations refer to the original paper [1] and we also use the notation therein. The following counterexample to Lemma 5.8 was found by Francesco Amoroso and Lea Terracini.

Example 1.1. The prime 3 is totally ramified in $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$. Denote by w_0 the unique place of $\mathbb{Q}(\sqrt{3})$ with $w_0 \mid 3$. Then $\sqrt{1+2\sqrt{3}} \in \mathbb{Q}(\sqrt{3})^{nr,w_0}$.

Assume for the sake of contradiction that $\sqrt{1+2\sqrt{3}} \in \mathbb{Q}^{nr,3}(\sqrt{3})$. Then there must exist $a, b \in \mathbb{Q}^{nr,3}$ such that $\sqrt{1+2\sqrt{3}} = a + b\sqrt{3}$. Squaring this equation yields $1+2\sqrt{3} = (a^2+3b^2)+2ab\sqrt{3}$. Since $\sqrt{3} \notin \mathbb{Q}^{nr,3}$, it follows that $b = a^{-1}$ and $a^2+3a^{-2} = 1$. The latter implies that a is a root of x^4-x^2-3. After a short calculation we know that the discriminant of $\mathbb{Q}(a)$ is -8112 and therefore 3 ramifies in $\mathbb{Q}(a)$. In particular, $a \notin \mathbb{Q}^{nr,3}$ contradicting our assumption. Hence, $\sqrt{1+2\sqrt{3}} \notin \mathbb{Q}^{nr,3}(\sqrt{3})$ and

$$\mathbb{Q}\left(\sqrt{3}\right)^{nr,w_0} \neq \mathbb{Q}^{nr,3}\left(\sqrt{3}\right).$$

Setting $K = \mathbb{Q}$, $v = 3$ and $\alpha = \sqrt{3}$, this is a counterexample to the statement of Lemma 5.8.

Remark 1.2. The mistake in the proof of Lemma 5.8 is the following. It is correct that $K^{nr,v}(\alpha) \subseteq \cap_{i=1}^{n} K(\alpha)^{nr,w_i}$. Moreover, it is true that $\beta \in L^w(\alpha)$ for all

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$w \in M_{K(\alpha,\beta)}$, $w \mid v$. However, it **not** true that this implies the existence of a number field $L \subseteq \cap_w L^w$ such that $\beta \in L(\alpha)$, where w runs through all extensions of v to $K(\alpha, \beta)$.

Due to this failure, the statement of Theorem 5.9 has to be replaced by the following result.

Theorem 1.3. *Let E be an elliptic curve defined over a number field K. The field $K^{nr,v}(\alpha)$ has the Bogomolov property relative to $\widehat{h_E}$, if there is a $w \in M_{K(\alpha)}$, $w \mid v$, such that $E/K(\alpha)$ has bad reduction at w.*

This follows immediately from [1, Theorem 4.1, Corollary 5.1, Proposition 5.4] and the correct inclusion $K^{nr,v}(\alpha) \subseteq \cap_{i=1}^n K(\alpha)^{nr,w_i}$.

Remark 1.4. The *and only if*-part of Theorem 5.9 is not justified anymore. Hence, the only known cases where the Bogomolov property relative to $\widehat{h_E}$ is not preserved under finite field extensions are those described in Example 5.7. It remains open whether there are further examples.

References

[1] L. POTTMEYER, *Heights of points with bounded ramification*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), 965–981.

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