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Catching all geodesics of a manifold with moving balls and application to controllability of the wave equation

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Abstract. We address the problem of catching all speed-1 geodesics of a Riemannian manifold with a moving ball: given a compact Riemannian manifold (M, g) and small parameters $\varepsilon > 0$ and $v > 0$, is it possible to find $T > 0$ and an absolutely continuous map $x : [0, T] \to M, t \mapsto x(t)$ satisfying $\|\dot{x}\|_\infty \leqslant v$ and such that any geodesic of (M, g) traveled at speed 1 meets the open ball $B_g(x(t), \varepsilon) \subset M$ within time T? Our main motivation comes from the control of the wave equation: our results show that the controllability of the wave equation can sometimes be improved by allowing the domain of control to move adequately, even very slowly. We first prove that, in any Riemannian manifold (M, g) satisfying a geodesic recurrence condition (GRC), our problem has a positive answer for any $\varepsilon > 0$ and $v > 0$, and we give examples of Riemannian manifolds (M, g) for which (GRC) is satisfied. Then, we build an explicit example of a domain $X \subset \mathbb{R}^2$ (with flat metric) containing convex obstacles, not satisfying (GRC), for which our problem has a negative answer if ε and v are small enough, *i.e.*, no sufficiently small ball moving sufficiently slowly can catch all geodesics of X.

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