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The binary digits of $n + t$

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Abstract. The binary sum-of-digits function s counts the number of ones in the binary expansion of a nonnegative integer. For any nonnegative integer t, T. W. Cusick defined the asymptotic density c_t of integers $n \geq 0$ such that

$$s(n + t) \geq s(n).$$

In 2011, he conjectured that $c_t > 1/2$ for all t – the binary sum of digits should, more often than not, weakly increase when a constant is added. In this paper, we prove that there exists an explicit constant M_0 such that indeed $c_t > 1/2$ if the binary expansion of t contains at least M_0 maximal blocks of contiguous ones, leaving open only the "initial cases" – few maximal blocks of ones – of this conjecture. Moreover, we sharpen a result by Emme and Hubert (2019), proving that the difference $s(n + t) - s(n)$ behaves according to a Gaussian distribution, up to an error tending to 0 as the number of maximal blocks of ones in the binary expansion of t grows.

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