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On an analogue of the Gauss circle problem for the Heisenberg groups

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Abstract. We consider the problem of estimating the error term $\mathcal{E}_q(x) = \big|\mathbb{Z}^{2q+1} \cap \delta_x\mathcal{B}\big| - vol\big(\mathcal{B}\big)x^{2q+2}$ which occurs in the counting of integer lattice points in Heisenberg dilates of the Cygan-Korányi ball:

$$\mathcal{B} = \big\{(v, w) \in \mathbb{R}^{2q} \times \mathbb{R} : \mathcal{N}(v, w) \le 1\big\}\ ; \quad \delta_x\big(v, w\big) = \big(xv, x^2w\big)$$

where $\mathcal{N}(v, w) = \big(|v|_2^4 + w^2\big)^{1/4}$ is the Cygan-Korányi norm, and $|\cdot|_2$ denotes the Euclidean norm. This lattice point counting problem arises naturally in the context of the Heisenberg groups, and may be viewed as a non-commutative analogue of the classical lattice point counting problem for Euclidean balls. In a previous paper, we have shown that the exponent in the upper bound $|\mathcal{E}_1(x)| \ll x^2 \log x$ obtained by Garg, Nevo and Taylor is best possible, thereby solving the problem for $q = 1$. In the higher dimensional case, the behavior of the error term is of an entirely different nature, and is closely related both in shape and form to the error term in the Gauss circle problem as soon as $q \ge 3$, while $q = 2$ marks somewhat of an intermediate point. In the present paper, we shall prove three type of results regarding the order of magnitude of $\mathcal{E}_q(x)$, which are valid for any $q \ge 3$. An upper bound estimate of the form $|\mathcal{E}_q(x)| \ll x^{2q-2/3}$. A sharp second moment estimate, which shows that $\mathcal{E}_q(x)$ has order of magnitude x^{2q-1} in mean-square. An Ω-estimate of the form $\mathcal{E}_q(x) = \Omega\big(x^{2q-1}\big(\log x\big)^{1/4}\big(\log\log x\big)^{1/8}\big)$. Consequently, we obtain the lower bound $\kappa_q = \sup\big\{\alpha > 0 : \big|\mathcal{E}_q(x)\big| \ll x^{2q+2-\alpha}\big\} \ge \frac{8}{3}$ for $q \ge 3$, and conjecture that $\kappa_q = 3$.

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