Surface braid groups, finite Heisenberg covers and double Kodaira fibrations

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Abstract. We exhibit new examples of double Kodaira fibrations using finite Galois covers of a product $\Sigma_b \times \Sigma_b$, where Σ_b is a smooth projective curve of genus $b \geq 2$. Each cover is obtained by providing an explicit group epimorphism from the pure braid group $\mathsf{P}_2(\Sigma_b)$ to some finite Heisenberg group. In this way, we are able to show that every curve of genus b is the base of a double Kodaira fibration; moreover, the number of pairwise non-isomorphic Kodaira fibred surfaces fibering over a fixed curve Σ_b is at least $\omega(b + 1)$, where $\omega \colon \mathbb{N} \longrightarrow \mathbb{N}$ stands for the arithmetic function counting the number of distinct prime factors of a positive integer. As a particular case of our general construction, we obtain a real 4-manifold of signature 144 that can be realized as a real surface bundle over a surface of genus 2, with fibre genus 325, in two different ways. This provides (to our knowledge) the first "double solution" to a problem from Kirby's problem list in low-dimensional topology.