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Pointwise estimates of solutions to nonlinear equations for nonlocal operators

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Abstract. We study pointwise behavior of positive solutions to nonlinear integral equations, and related inequalities, of the type

$$u(x) - \int_\Omega G(x, y)\, g(u(y))d\sigma(y) = h,$$

where (Ω, σ) is a locally compact measure space, $G(x, y)\colon \Omega \times \Omega \to [0, +\infty]$ is a kernel that satisfies a weak form of the maximum principle, $h \geq 0$ is a measurable function, and $g\colon [0, \infty) \to [0, \infty)$ is a monotone increasing function.

In the special case where G is Green's function of the Laplacian (or fractional Laplacian) that satisfies the maximum principle and $h = 1$, a typical global pointwise bound for any supersolution $u > 0$ is given by

$$u(x) \geq F^{-1}(G\sigma(x)), \quad x \in \Omega,$$

where $F(t) := \int_1^t \frac{ds}{g(s)}$, $t \geq 1$, and necessarily

$$G\sigma(x) < F(\infty) = \int_1^{+\infty} \frac{ds}{g(s)},$$

for every $x \in \Omega$ such that $u(x) < \infty$.

This problem is motivated by the semilinear fractional Laplace equation

$$(-\Delta)^{\frac{\alpha}{2}} u - g(u)\sigma = \mu \quad \text{in } \Omega, \quad u = 0 \text{ in } \Omega^c,$$

with measure coefficients σ, μ, where $g(u) = u^q$, $q > 0$, and $0 < \alpha < n$, in domains $\Omega \subseteq \mathbb{R}^n$, or Riemannian manifolds, with positive Green's function G.

In a similar way, we treat positive solutions to the equation

$$u(x) + \int_\Omega G(x, y)\, g(u(y))d\sigma(y) = h,$$

and the corresponding fractional Laplace equation $(-\Delta)^{\frac{\alpha}{2}} u + g(u)\sigma = \mu$, with a monotone decreasing function g, in particular $g(u) = u^q$, $q < 0$.

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