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Reverse approximation of gradient flows as Minimizing Movements: a conjecture by De Giorgi

FLORENTINE FLEISSNER AND GIUSEPPE SAVARÉ

Abstract. We consider the Cauchy problem for the gradient flow

$$u'(t) = -\nabla\phi(u(t)), \quad t \geq 0; \quad u(0) = u_0, \tag{$\star$}$$

generated by a continuously differentiable function $\phi : \mathbb{H} \to \mathbb{R}$ in a Hilbert space $\mathbb{H}$ and study the reverse approximation of solutions to ($\star$) by the De Giorgi Minimizing Movement approach.

We prove that if $\mathbb{H}$ has finite dimension and ϕ is quadratically bounded from below (in particular if ϕ is Lipschitz) then for *every* solution u to ($\star$) (which may have an infinite number of solutions) there exist perturbations $\phi_\tau : \mathbb{H} \to \mathbb{R}$ ($\tau > 0$) converging to ϕ in the Lipschitz norm such that u can be approximated by the Minimizing Movement scheme generated by the recursive minimization of $\Phi(\tau, U, V) := \frac{1}{2\tau}|V - U|^2 + \phi_\tau(V)$:

$$U_\tau^n \in \operatorname{argmin}_{V\in\mathbb{H}} \Phi\big(\tau, U_\tau^{n-1}, V\big) \quad n \in \mathbb{N}, \quad U_\tau^0 := u_0. \tag{$\star\star$}$$

We show that the piecewise constant interpolations with time step $\tau > 0$ of *all* possible selections of solutions $(U_\tau^n)_{n\in\mathbb{N}}$ to ($\star\star$) will converge to u as $\tau \downarrow 0$. This result solves a question raised by Ennio De Giorgi in [9].

We also show that even if $\mathbb{H}$ has infinite dimension the above approximation holds for the distinguished class of minimal solutions to ($\star$), that generate all the other solutions to ($\star$) by time reparametrization.

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