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A boxing inequality for the fractional perimeter

AUGUSTO C. PONCE AND DANIEL SPECTOR

Abstract. We prove the Boxing inequality

$$\mathcal{H}_{\infty}^{d-\alpha}(U) \leq C\alpha(1-\alpha) \int_{U} \int_{\mathbb{R}^d \setminus U} \frac{\mathrm{d}y \, \mathrm{d}z}{|y-z|^{\alpha+d}}$$

for every $\alpha \in (0,1)$ and every bounded open subset $U \subset \mathbb{R}^d$, where $\mathcal{H}_{\infty}^{d-\alpha}(U)$ is the Hausdorff content of U of dimension $d-\alpha$ and the constant $C > 0$ depends only on d. We then show how this estimate implies a trace inequality in the fractional Sobolev space $W^{\alpha,1}(\mathbb{R}^d)$ that includes Sobolev's $L^{\frac{d}{d-\alpha}}$ embedding, its Lorentz-space improvement, and Hardy's inequality. All these estimates are thus obtained with the appropriate asymptotics as α tends to 0 and 1, recovering in particular the classical inequalities of first order. Their counterparts in the full range $\alpha \in (0,d)$ are also investigated.

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