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Trudinger-Moser inequalities on a closed Riemannian surface with the action of a finite isometric group

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Abstract. Let (Σ, g) be a closed Riemannian surface, $W^{1,2}(\Sigma, g)$ be the usual Sobolev space, $\mathbf{G}$ be a finite isometric group acting on (Σ, g), and $\mathscr{H}_{\mathbf{G}}$ be the function space including all functions $u \in W^{1,2}(\Sigma, g)$ with $\int_\Sigma u dv_g = 0$ and $u(\sigma(x)) = u(x)$ for all $\sigma \in \mathbf{G}$ and all $x \in \Sigma$. Denote the number of distinct points of the set $\{\sigma(x) : \sigma \in \mathbf{G}\}$ by $I(x)$ and $\ell = \min_{x\in\Sigma} I(x)$. Let $\lambda_1^{\mathbf{G}}$ be the first eigenvalue of the Laplace-Beltrami operator on the space $\mathscr{H}_{\mathbf{G}}$. Using blow-up analysis, we prove that if $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta \leq 4\pi\ell$, then there holds

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\, \int_\Sigma |\nabla_g u|^2 dv_g - \alpha \int_\Sigma u^2 dv_g \leq 1} \int_\Sigma e^{\beta u^2} dv_g < \infty;$$

if $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta > 4\pi\ell$, or $\alpha \geq \lambda_1^{\mathbf{G}}$ and $\beta > 0$, then the above supremum is infinity; if $\alpha < \lambda_1^{\mathbf{G}}$ and $\beta \leq 4\pi\ell$, then the above supremum can be attained. Moreover, similar inequalities involving higher order eigenvalues are obtained. Our results partially improve original inequalities of J. Moser [17], L. Fontana [9] and W. Chen [4].

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