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Nori's fundamental group over a non algebraically closed field

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Abstract. Let X be a connected reduced scheme over a field k, and $x \in X(k)$ be a k-rational point. M. V. Nori constructed in his Ph.D thesis a fundamental group scheme $\pi^N(X, x)$ which generalizes A. Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(X, x)$ by including infinitesimal covers. However, Nori's fundamental group scheme carries little arithmetic information, and it behaves like the étale fundamental group only when k is algebraically closed. For example, if $X = \text{Spec}\,(k)$, then Nori's fundamental group scheme is always trivial while the étale fundamental group $\pi_1^{\text{ét}}(X, x) = \text{Gal}(\bar{k}/k)$. In this paper, we study a slightly modified version of Nori's fundamental group scheme: we take x to be a geometric point instead of a rational point. It is very surprising to the author that this tiny little modification of Nori's original definition brings a lot of arithmetic information and makes the fundamental group scheme more like $\pi_1^{\text{ét}}(X, x)$. For example, now if we take $X = \text{Spec}\,(k)$ again, with $\bar{x} \in X(\bar{k})$, then we get a profinite group scheme $\pi^N(k/k, \bar{x})$ over k which admits $\text{Gal}(\bar{k}/k)$ as a (pro-constant) quotient of its. Thus not only the Galois extensions, but also the purely inseparable extensions of k are encoded into $\pi^N(k/k, \bar{x})$. We call $\pi^N(k/k, \bar{x})$ the *Nori-Galois group* of k. We also studied the fundamental sequence which relates the Nori-Galois group to the geometric fundamental group. It turns out that the expected fundamental exact sequence is always a complex and exact on the right, but fails to be exact in the middle and on the left. Then we give conditions to determine when the exactness holds.

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