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A quantitative characterisation of functions with low Aviles Giga energy on convex domains

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Abstract. Given a connected Lipschitz domain Ω we let $\Lambda(\Omega)$ be the set of functions in $W^{2,2}(\Omega)$ with $u = 0$ on $\partial\Omega$ and whose gradient (in the sense of trace) satisfies $\nabla u(x) \cdot \eta_x = 1$, where η_x is the inward pointing unit normal to $\partial\Omega$ at x. The functional $I_\epsilon(u) = \frac{1}{2}\int_\Omega \epsilon^{-1}\left|1 - |\nabla u|^2\right|^2 + \epsilon\left|\nabla^2 u\right|^2 dz$, minimised over $\Lambda(\Omega)$, serves as a model in connection with problems in liquid crystals and thin film blisters. It is also the most natural higher order generalisation of the Modica and Mortola functional. In [16] Jabin, Otto and Perthame characterised a class of functions which includes all limits of sequences $u_n \in \Lambda(\Omega)$ with $I_{\epsilon_n}(u_n) \to 0$ as $\epsilon_n \to 0$. A corollary to their work is that if there exists such a sequence (u_n) for a bounded domain Ω, then Ω must be a ball and (up to change of sign) $u := \lim_{n\to\infty} u_n$ is equal $\mathrm{dist}(\cdot, \partial\Omega)$. We prove a quantitative generalisation of this corollary for the class of bounded convex sets. Namely we show that there exists a positive constant γ_1 such that, if Ω is a convex set of diameter 2 and $u \in \Lambda(\Omega)$ with $I_\epsilon(u) = \beta$, then $|B_1(x)\triangle\Omega| \leq c\beta^{\gamma_1}$ for some x and

$$\int_\Omega \left|\nabla u(z) + \frac{z-x}{|z-x|}\right|^2 dz \leq c\beta^{\gamma_1}.$$

A corollary of this result is that there exists a positive constant $\gamma_2 < \gamma_1$ such that if Ω is convex with diameter 2 and C^2 boundary with curvature bounded by $\epsilon^{-\frac{1}{2}}$, then for any minimiser v of I_ϵ over $\Lambda(\Omega)$ we have

$$\|v - \zeta\|_{W^{1,2}(\Omega)} \leq c(\epsilon + \inf_y |\Omega\triangle B_1(y)|)^{\gamma_2},$$

where $\zeta(z) = \mathrm{dist}(z, \partial\Omega)$. Neither of the constants γ_1 or γ_2 are optimal.

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