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Sharp estimates for bubbling solutions of a fourth order mean field equation

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Abstract. We consider a sequence of multi-bubble solutions u_k of the following fourth order equation

$$\Delta^2 u_k = \rho_k \frac{h(x)e^{u_k}}{\int_\Omega he^{u_k}} \text{ in } \Omega, \quad u_k = \Delta u_k = 0 \text{ on } \partial\Omega, \tag{$*$}$$

where h is a $C^{2,\beta}$ positive function, Ω is a bounded and smooth domain in $\mathbb{R}^4$, and ρ_k is a constant such that $\rho_k \le C$. We show that (after extracting a subsequence), $\lim_{k\to+\infty} \rho_k = 32\sigma_3 m$ for some positive integer $m \ge 1$, where σ_3 is the area of the unit sphere in $\mathbb{R}^4$. Furthermore, we obtain the following sharp estimates for ρ_k:

$$\rho_k - 32\sigma_3 m = c_0 \sum_{j=1}^m \epsilon_{k,j}^2 \left(\sum_{l\neq j} \Delta G_4(p_j, p_l) + \Delta R_4(p_j, p_j) + \frac{1}{32\sigma_3} \Delta \log h(p_j) \right) + o\left(\sum_{j=1}^m \epsilon_{k,j}^2 \right)$$

where $c_0 > 0$, $\log \frac{64}{\epsilon_{k,j}^4} = \max_{x\in B_\delta(p_j)} u_k(x) - \log(\int_\Omega he^{u_k})$ and $u_k \to 32\sigma_3 \sum_{j=1}^m G_4(\cdot, p_j)$ in $C^4_{\text{loc}}(\Omega\backslash\{p_1, \dots, p_m\})$.

This yields a bound of solutions as ρ_k converges to $32\sigma_3 m$ from below provided that

$$\sum_{j=1}^m \left(\sum_{l\neq j} \Delta G_4(p_j, p_l) + \Delta R_4(p_j, p_j) + \frac{1}{32\sigma_3} \Delta \log h(p_j) \right) > 0.$$

The analytic work of this paper is the first step toward computing the Leray-Schauder degree of solutions of equation $(*)$.

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