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Non-negative curvature obstructions in cohomogeneity one and the Kervaire spheres

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Dedicated to Eugenio Calabi on his 80th birthday

Abstract. In contrast to the homogeneous case, we show that there are compact cohomogeneity one manifolds that do not support invariant metrics of non-negative sectional curvature. In fact we exhibit infinite families of such manifolds including the exotic Kervaire spheres. Such examples exist for any codimension of the singular orbits except for the case when both are equal to two, where existence of non-negatively curved metrics is known.

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Most examples of manifolds with non-negative sectional curvature are obtained via product and quotient constructions, starting from compact Lie groups with bi-invariant metrics. In [GZ1] a new large class of non-negatively curved compact manifolds was constructed by using Lie group actions whose quotients are one-dimensional, so called cohomogeneity one actions. It was shown that if the action has two singular orbits of codimension two, then the manifold carries an invariant metric with non-negative sectional curvature. This in particular gives rise to non-negatively curved metrics on all sphere bundles over $\mathbb{S}^4$, which include the Milnor exotic spheres. In [GZ1] it was also conjectured that any cohomogeneity one manifold should carry an invariant metric with non-negative sectional curvature. The most interesting application of this conjecture would be the Kervaire spheres, which can be presented as particular $2n-1$ dimensional Brieskorn varieties $M_d^{2n-1} \subset \mathbb{C}^{n+1}$ defined by the equations

$$z_0^d + z_1^2 + \cdots z_n^2 = 0\,, \qquad |z_0|^2 + \cdots |z_n|^2 = 1.$$

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For n odd and d odd, they are homeomorphic to spheres, and are exotic spheres if $2n-1 \equiv 1 \bmod 8$. As was discovered by E.Calabi in dimension 5 and later generalized in [HH], the Brieskorn variety M_d^{2n-1} carries a cohomogeneity one action by SO(2)SO(n) defined by $(e^{i\theta}, A)(z_0, \cdots, z_n) = (e^{2i\theta} z_0, e^{id\theta} A(z_1, \cdots, z_n)^t)$. It was observed in [BH] (see [Se] for $n = 3$) that M_d^{2n-1} does not admit an $\mathrm{SO}(2)\,\mathrm{SO}(n)$ invariant metric with positive curvature, if $n \geq 3$, $d \geq 2$. On the other hand, the non-principal orbits of this action have codimension 2 and $n-1$. Hence, if $n = 3$, they carry an invariant metric with non-negative curvature. If $n = 2$ the action is a linear action on the lens space $L(1, d)$. If $d = 1$ the action is the linear tensor product action on $\mathbb{S}^{2n-1}$ and if $d = 2$ it is a linear action on $\mathrm{T}_1\,\mathbb{S}^n$. Hence in all these cases one has an invariant metric with non-negative curvature. In all remaining cases we prove:

Theorem A. *For $n \geq 4$ and $d \geq 3$, the Brieskorn variety M_d^{2n-1} does not support an* $\mathrm{SO}(2)\,\mathrm{SO}(n)$ *invariant metric with non-negative sectional curvature.*

If $n = 4$, the normal SU(2) subgroup in SO(4) acts freely on M_d^7. Its quotient is $\mathbb{S}^4$, since the induced cohomogeneity one action on the base is the sum action of $\mathrm{SO}(2)\,\mathrm{SO}(3)$ on $\mathbb{S}^4$. In [GZ1] it was shown that such principal bundles admit another cohomogeneity one action by SO(4) with non-principal orbits of codimension two and hence an invariant metric with non-negative sectional curvature. We do not know if any of the remaining manifolds M_d^{2n-1} admit non-negative curvature.

In addition to the cohomogeneity one Brieskorn varieties, we construct several other infinite families of cohomogeneity one manifolds that cannot support invariant metrics of non-negative curvature (see Theorem 3.1). In particular, we get such an infinite family for any codimensions bigger than two (see Corollary 3.2). Together with the examples of the Brieskorn varieties we obtain the following counterpart to the positive result of [GZ1]:

Theorem B. *For any pair of integers $(\ell_-, \ell_+) \neq (2, 2)$ with $\ell_\pm \geq 2$, there is an infinite family of cohomogeneity one manifolds M with singular orbits of codimensions $\ell_\pm$ and no invariant metric of non-negative sectional curvature.*

Our results are in contrast to the homogeneous case, where all compact manifolds admit an invariant metric of non-negative curvature, and they leave the classification of non-negatively curved cohomogeneity one manifolds wide open. We point out though that all cohomogeneity one manifolds admit invariant metrics with almost non-negative sectional curvature (see [ST]). We also point out that our obstructions are global in nature since each half, as homogeneous disc bundles over the singular orbits, naturally admit invariant metrics with non-negative curvature.

In Section 1 we describe the cohomogeneity one action on M_d^{2n-1} and the invariant metrics. In Section 2 we derive the obstructions to non-negative curvature and prove Theorem A. In Section 3 we exhibit other infinite families of cohomogeneity one manifolds that do not admit invariant metrics with non-negative curvature.

1. Cohomogeneity one Brieskorn varieties and invariant metrics

For any integer $d \geq 1$ the Brieskorn variety M_d^{2n-1} is the smooth $(2n-1)$-dimensional submanifold of $\mathbb{C}^{n+1}$ defined by the equations

$$\begin{cases} z_0^d + z_1^2 + z_2^2 + \ldots + z_n^2 = 0, \\ |z_0|^2 + |z_1|^2 + \ldots + |z_n|^2 = 1. \end{cases}$$

When $d = 1$, M_1^{2n-1} is diffeomorphic to the standard sphere via $(z_0, Z) \to Z/|Z|$. When $d = 2$, M_2^{2n-1} is diffeomorphic to the unit tangent bundle of S^n since $X + iY$ lies in M_2^{2n-1} iff $\langle X, X\rangle = \langle Y, Y\rangle = \frac{1}{2}$, $\langle X, Y\rangle = 0$. When $n = 2$, M_d^3 is the Lens space $L(1, d)$ while, for $n \geq 3$ the Brieskorn varieties are simply connected.

The group $\mathrm{G} = \mathrm{SO}(2)\,\mathrm{SO}(n)$ acts on M_d^{2n-1} by

$$(e^{i\theta}, A)(z_0, Z) = (e^{2i\theta} z_0, e^{id\theta} AZ)\,,\ (z_0, Z) \in \mathbb{C} \oplus \mathbb{C}^n.$$

$|z_0|$ is invariant under this action and one easily sees that two points belong to the same G-orbit if and only if they have the same value of $|z_0|$. Furthermore, using the Schwarz inequality, we have that $0 \leq |z_0| \leq t_0$, where t_0 is the unique positive solution of $t_0^d + t_0^2 = 1$. In particular, the manifold is cohomogeneity one and $M/\mathrm{G} = [0, t_0]$.

In order to describe the possible orbit types for this action, we consider the Lens space $M_d^3 \subset M_d^{2n-1}$, defined by

$$M_d^3 = \{(z_0, z_1, z_2, 0, \ldots, 0)\} \cap M_d^{2n-1}.$$

The points of M_d^3 such that $0 < |z_0| < t_0$ have a common isotropy subgroup, namely

$$\mathrm{H} = \mathbb{Z}_2 \times \mathrm{SO}(n-2) = \begin{cases} (\epsilon, \mathrm{diag}(\epsilon, \epsilon, A)) & \text{if } d \text{ odd} \\ (\epsilon, \mathrm{diag}(1, 1, A)) & \text{if } d \text{ even} \end{cases}$$

where $\epsilon = \pm 1$ and $A \in \mathrm{SO}(n-2)$. Conversely, M_d^3 coincides with the fixed point set of H. There are three orbit types, the principal ones, corresponding to the interior of the interval $[0, t_0]$, and two singular ones corresponding to the boundary points.

One singular orbit, denoted by B_-, is given by $z_0 = 0$ and has codimension 2. Choosing $p_- = (0, 1, i, 0, \ldots, 0) \in M_d^3$, the isotropy subgroup of p_- is

$$\mathrm{K}^- = \mathrm{SO}(2)\,\mathrm{SO}(n-2) = (e^{-i\theta}, \mathrm{diag}(R(d\theta), A)).$$

where $R(\theta)$ is a counterclockwise rotation with angle θ.

The second singular orbit, B_+, is given by $|z_0| = t_0$ and has codimension $n-1$. If we choose $p_+ = (t_0, i\sqrt{t_0^d}, 0, \dots, 0)$, the corresponding isotropy subgroup is

$$\mathrm{K}^+ = \begin{cases} \mathrm{O}(n-1) = (\det B, \operatorname{diag}(\det B, B)) & \text{if } d \text{ odd} \\ \mathbb{Z}_2 \times \mathrm{SO}(n-1) = (\epsilon, \operatorname{diag}(1, B')) & \text{if } d \text{ even} \end{cases}$$

where $\epsilon = \pm 1$, $B \in \mathrm{O}(n-1)$ and $B' \in \mathrm{SO}(n-1)$.

A tubular neighborhood of the singular orbit $\mathrm{G}/\mathrm{K}^{\pm}$ is equivariantly diffeomorphic to the homogeneous vector bundle $\mathrm{G} \times_{\mathrm{K}^{\pm}} \mathbb{D}_{\pm}$, where $\mathbb{D}_{\pm}$ can be identified with the normal disc to the orbit $B_{\pm} = \mathrm{G}\, p_{\pm} = \mathrm{G}/\mathrm{K}^{\pm}$ at $p_{\pm}$. The group $\mathrm{K}^{\pm}$ acts linearly on $\mathbb{D}_{\pm}$, and transitively on the sphere $\partial\mathbb{D}_{\pm} = \mathbb{S}_{\pm} = \mathrm{K}^{\pm}/\mathrm{H}$. Furthermore, M is the union of these tubular neighborhoods $M = \mathrm{G} \times_{\mathrm{K}^-} \mathbb{D}_- \cup \mathrm{G} \times_{\mathrm{K}^+} \mathbb{D}_+$ glued along a principal orbit G/H. The isotropy groups $\mathrm{K}^{\pm}$ are isotropy groups of the end points of a minimal geodesic joining the singular orbits for an invariant metric on M.

The largest normal subgroup common to G and K^-, which is the ineffective kernel of the G action on B_-, is the cyclic group generated by $(e^{-i\pi/d}, -\mathrm{Id})$ for n even and $(e^{-2i\pi/d}, \mathrm{Id})$ for n odd. On the other hand, the action of $e^{i\theta} \in \mathrm{SO}(2) \subset \mathrm{K}^-$ on the slice $\mathbb{D}_-$ is given by $R(2\theta)$ since $e^{i\pi} \in \mathrm{H}$. Hence the singular orbit B_- is the fixed point set of a group of isometries and thus is totally geodesic if $d \geq 3$. This implies, as was observed in [BH], that M cannot carry a G invariant metric with positive curvature if $n \geq 4$ and $d \geq 3$ since B_- does not admit a homogeneous metric with positive curvature.

We now describe the Riemannian metrics g on M_d^{2n-1} invariant under the action of $\mathrm{G} = \mathrm{SO}(2)\,\mathrm{SO}(n)$. Let γ be a geodesic orthogonal to one and hence all G orbits, which we can assume lies in M^{H} and ends at p_+ chosen as above. Its beginning point $\gamma(0)$ will lie in $B_- \cap M^{\mathrm{H}}$, which consists of the two circles $\{(0, z, iz) \mid |z| = 1\}$ and $\{(0, z, -iz) \mid |z| = 1\}$. By changing the sign of the third basis vector if necessary, we can assume that $\gamma(0)$ lies in the first circle where the isotropy group is constant equal to K^-. We can hence assume, changing p_- if necessary, that γ is parameterized by arc length with $\gamma(0) = p_-$ and $\gamma(L) = p_+$ and all isotropy groups are described as above.

For $0 < t < L$, $\gamma(t)$ is a regular point with constant isotropy group H and the metric on the principal orbits $\mathrm{G}\,\gamma(t) = \mathrm{G}/\mathrm{H}$ is a family of homogeneous metrics g_t. Thus on the regular part the metric is determined by

$$g_{\gamma(t)} = d\,t^2 + g_t$$

and since the regular points are dense it also describes the metric on M.

By means of Killing vector fields, we identify the tangent space to G/H at $\gamma(t)$, $t \in (0, L)$ with an $\mathrm{Ad}(\mathrm{H})$-invariant complement $\mathfrak{n}$ of the isotropy subalgebra $\mathfrak{h}$ of H in $\mathfrak{g}$ and the metric g_t is identified with an $\mathrm{Ad}(\mathrm{H})$ invariant inner product on $\mathfrak{n}$. The complement $\mathfrak{n}$ can be chosen as the orthogonal complement to $\mathfrak{h}$ with the respect to a fixed $\mathrm{Ad}(\mathrm{H})$-invariant scalar product Q in $\mathfrak{g}$. In our case a convenient choice of Q on $\mathfrak{so}(2)+\mathfrak{so}(n)$ is

$$Q(a+A,b+B)=d^2ab-\frac{1}{2}\operatorname{tr}(A\cdot B)$$

where $a,b\in\mathfrak{so}(2)$ and $A,B\in\mathfrak{so}(n)$. We obtain a Q-orthogonal decomposition:

$$T_{\gamma(t)}\,\mathrm{G}(\gamma(t))\simeq\mathfrak{n}\simeq\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{m}_1+\mathfrak{m}_2$$

for $t\in(0,L)$, where $\mathfrak{p}_i$ are one-dimensional trivial Ad(H)-modules and $\mathfrak{m}_i$ are $(n-2)$-dimensional irreducible Ad(H)-modules. A Q-orthonormal basis of $\mathfrak{n}$ is given by

$$X=\left(\frac{1}{d}I+E_{12}\right)/\sqrt{2}\in\mathfrak{p}_1\,,\ Y=\left(\frac{1}{d}I-E_{12}\right)/\sqrt{2}\in\mathfrak{p}_2\,,$$
$$E_i=E_{1(i+2)}\in\mathfrak{m}_1\,,\ F_i=E_{2(i+2)}\in\mathfrak{m}_2$$

for $i=1\ldots n-2$, where E_{ij} is the standard basis for $\mathfrak{so}(n)$ and $I=E_{12}\in\mathfrak{so}(2)$.

The same identification at the singular points takes the form

$$\begin{aligned}\mathfrak{k}_-&=\mathfrak{h}+\mathfrak{p}_1, &\qquad T_{\gamma(0)}B_-&\simeq\mathfrak{p}_2+\mathfrak{m}_1+\mathfrak{m}_2,\\ \mathfrak{k}_+&=\mathfrak{h}+\mathfrak{m}_2, &\qquad T_{\gamma(L)}B_+&\simeq\mathfrak{p}_1+\mathfrak{p}_2+\mathfrak{m}_1.\end{aligned}$$

Any Ad(H)-invariant scalar product on $\mathfrak{n}$ must be a multiple of Q on the irreducible H-modules $\mathfrak{m}_i$ and the inner products between $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are described in terms of an Ad(H)-equivariant isomorphism. Up to a factor, the only such isomorphism $\mathfrak{m}_1\to\mathfrak{m}_2$ is given by $E_i\to F_i$. Furthermore, $\mathfrak{p}_1+\mathfrak{p}_2$ must be orthogonal to $\mathfrak{m}_1+\mathfrak{m}_2$. In terms of this basis, g_t can therefore be described as follows:

$$\begin{aligned}&g_t(X,X)=f_1^2(t), &\quad &g_t(Y,Y)=f_2^2(t), &\quad &g_t(X,Y)=f_{12}(t)\\ &g_t(E_i,E_i)=h_1^2(t), &\quad &g_t(F_i,F_i)=h_2^2(t), &\quad &g_t(E_i,F_i)=h_{12}(t)\end{aligned}$$

with all other inner products being 0.

If these 6 functions are smooth, and g_t is positive definite, they define a smooth metric on the regular part of M. Some further conditions are required to ensure that the metric $g_t+d\,t^2$ can be smoothly extended to the singular orbits. The precise necessary and sufficient conditions were determined in [BH]. For our purposes it is sufficient to consider only a few of these smoothness conditions. They are, assuming that $d\geq 3$:

$$\begin{aligned}&f_1(0)=0\,, &&f_1'(0)=\sqrt{2}/d\,, &&f_{12}(0)=f_{12}'(0)=0,\\ &h_1(0)=h_2(0)\,, &&h_1'(0)=h_2'(0)=0\,, &&h_{12}(0)=h_{12}'(0)=0\,,\\ &h_1'(L)=0\,, &&h_2(L)=0\,.\end{aligned}\tag{1.1}$$

The first two follow from the fact that $\mathfrak{p}_1$ collapses at B_- and that the circle K^-/H has length $2\pi d/\sqrt{2}$ in the metric Q. The inner product $f_{12}(0) = 0$ since the slice $\mathbb{D}_-$ and B_- are orthogonal. Since K^- fixes Y, it follows that $\langle X, Y\rangle$, restricted to the slice, is a function of the square of the distance to the origin. Hence $f_{12}(t)$ is an even function of t which implies that $f'_{12}(0) = 0$. We have $h_{12}(0) = 0$ and $h_1(0) = h_2(0)$ since the isotropy representation of K^- on $\mathfrak{m}_1 + \mathfrak{m}_2$ is irreducible. The remaining derivatives vanish at $t = 0$ since B_- is totally geodesic. Now $h'_1(L) = 0$ means that the second fundamental form of B_+ restricted to $\mathfrak{m}_1$ is 0. Indeed, this follows from the fact that the second fundamental form is equivariant under the action of K^+, and that this action on the slice $\mathbb{D}_+$ is equivalent to its action on the subspace of $T_{p_+}(B_+)$ spanned by $X + Y = E_{12}$ and $\mathfrak{m}_1$. Finally, $h_2(L) = 0$ since $\mathfrak{m}_2$ collapses at p_+.

2. Obstructions and Proof of Theorem A

From now on we assume that $M = M_d^{2n-1}$ with $d \geq 3$ and $n \geq 4$ and that M is equipped with a $\mathrm{G} = \mathrm{SO}(2)\,\mathrm{SO}(n)$-invariant C^2-metric g. Our starting point is the following global fact:

Lemma 2.1. *If (M, g) is non-negatively curved, then $h_{12}(t) = 0$ and $h_1(t)$ is constant for $t \in [0, L]$.*

Proof. Let $E_i \in \mathfrak{m}_1$ as before and consider the vector field $\tilde{E}_i$ obtained by parallel transport of $E_i(\gamma(L))$ along $\gamma(t)$ for $t \in [0, L]$. By Ad(H)-equivariance, the parallel transport along γ preserves $\mathfrak{m}_1 + \mathfrak{m}_2$ and hence $\tilde{E}_i(\gamma(0)) \in \mathfrak{m}_1 + \mathfrak{m}_2$. Recall that B_- is totally geodesic and that the second fundamental form of B_+ restricted to $\mathfrak{m}_1$ is 0. Since the normal geodesic connecting p_- to p_+ is a minimizing connection between B_- and B_+, the second variation formula implies that $\langle R(\tilde{E}_i, \gamma')\gamma', \tilde{E}_i\rangle = 0$. Since g has non-negative curvature, $R(\tilde{E}_i, \gamma')\gamma' = 0$ and hence $\tilde{E}_i$ is a Jacobi field along γ, which has the same initial conditions as E_i at $\gamma(L)$. Thus both must agree, which means that E_i is parallel along γ and hence h_1 is constant for $t \in [0, L]$.

The orthogonal complement $\mathfrak{m}_1^\perp$ of $\mathfrak{m}_1$ in $\mathfrak{m}_1 + \mathfrak{m}_2$ is both invariant under parallel translation and under the symmetric endomorphism $R(\cdot, \gamma')\gamma'$. Therefore any Jacobi field with initial conditions in $\mathfrak{m}_1^\perp$ stays in $\mathfrak{m}_1^\perp$. By (1.1) we have $\mathfrak{m}_1^\perp = \mathfrak{m}_2$ for $t = 0$ and hence $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonal everywhere. ☐

The remainder will be a proof by contradiction. By computing the sectional curvature of two particular 2-planes, the non-negativity condition gives upper and lower bounds for $h'_2(t)$ for t small, which when combined yields $d \leq 2$.

We can replace all functions, e.g. $h_1(t)$, by $ah_1(t/a)$ corresponding to multiplying the metric by a. From now on we will normalize the metric so that $h_1(0) = 1$. It follows from (1.1) that $h_2(0) = 1$ as well.

Let $A, B \in \mathfrak{n}$ be linearly independent. The corresponding Killing vector fields span a two plane tangent to the principal orbits at the regular points of $\gamma(t)$. The sectional curvature of such a two plane can be computed using the Gauss equations

$$R(A,B,B,A)=\overline{R}(A,B,B,A)-\tfrac{1}{4}\,g_t(A,B)'^2+\tfrac{1}{4}\,g_t(A,A)'\,g_t(B,B)'$$

where $\overline{R}(A,B,A,B)$ is the intrinsic curvature of the principal orbit G/H. The latter can be computed by any of the well known formulas for the curvature tensor of a homogeneous space (cf. [Be, 7.30] or [GZ2]).

We choose the vectors $A=E_1+F_2$ and $B=E_2+F_1$, which is possible since $n\geq 4$. Using the fact that $[A,B]=0$, a curvature computation shows:

$$R(A,B,B,A)=\tfrac{1}{2}\,(1-h_2^2)^2\frac{f_1^2+f_2^2-2\,f_{12}^2}{f_1^2\,f_2^2-f_{12}^2}-h_2^2\,h_2'^2\,.$$

In the following we set

$$\delta(t)=1-h_2(t)^2$$

and hence the non-negative curvature condition can be formulated as:

$$(\delta')^2\leq 2\frac{f_1^2+f_2^2-2\,f_{12}^2}{(f_1^2\,f_2^2-f_{12}^2)}\delta^2 \tag{2.1}$$

Since F_1/h_2 is a parallel vectorfield, it follows that $\sec(\gamma'(t),F_1)=-h_2''/h_2$, and hence the function $h_2(t)$ is concave. Using the smoothness condition $h_2'(0)=0$ and $h_2(0)=1$ it follows that $\delta(t)\geq 0$. We first claim that $\delta(t)>0$ for $t>0$. If not, there would be a $t_0>0$ with $\delta(t_0)=0$ but $\delta(t)>0$ for $t>t_0$ since $h_2(L)=0$ (cf. (1.1)). But then (2.1) implies that $\delta'/\delta=\log(\delta)'$ is bounded from above, which by integrating from $t_0+\epsilon$ to $t_0+\eta$ leads to a contradiction when $\epsilon\to 0$.

The smoothness conditions (1.1) imply that $f_1(0)=0$ and $\lim_{t\to 0}\frac{f_{12}}{f_1}=0$ and since $f_2(0)\neq 0$, (2.1) implies that:

$$\log(\delta)'\leq(1+\eta)\frac{\sqrt{2}}{f_1}\ \text{for } t\in(0,L) \tag{2.2}$$

where $\eta(t)$ is a positive function with $\lim_{t\to 0}\eta(t)=0$.

To obtain a lower bound, we consider the family of 2-planes spanned by $A_r=X+r\,Y$ and $B=F_1$. A necessary condition for $R(A,B,B,A)$ to be non-negative for all r is

$$R(X,F_1,X,F_1)\,R(Y,F_1,Y,F_1)\geq R(X,F_1,Y,F_1)^2\,.$$

A computation shows that

$$\begin{aligned}
R(X,F_1,X,F_1)&=-\tfrac{1}{8}\delta\big(4-2f_1^2-2f_{12}-\delta\big)+\tfrac{1}{8}(f_1^2+f_{12})^2+\tfrac{1}{2}f_1f_1'\delta'\\
R(Y,F_1,Y,F_1)&=-\tfrac{1}{8}\delta\big(4-2f_2^2-2f_{12}-\delta\big)+\tfrac{1}{8}(f_2^2+f_{12})^2+\tfrac{1}{2}f_2f_2'\delta'\\
R(X,F_1,Y,F_1)&=-\tfrac{1}{8}\delta\big(4-f_1^2-f_2^2-2f_{12}-\delta\big)+\tfrac{1}{8}(f_1^2+f_{12})(f_2^2+f_{12})\\
&\quad+\tfrac{1}{4}f_{12}'\delta',
\end{aligned}$$

If we set $a = f_2(0)^2$, and using $\delta(0) = \delta'(0) = 0$, it follows that

$$R(X, F_1, X, F_1)\, R(Y, F_1, Y, F_1) - R(X, F_1, Y, F_1)^2 = -\tfrac{a^2}{16}(1+\eta_1)\delta \\ +\tfrac{a^2}{16}(1+\eta_2) f_1 f_1' \delta' \geq 0$$

where η_i are functions with $\lim_{t\to 0} \eta_i(t) = 0$, $i = 1, 2$. Combining with inequality (2.2) gives

$$(1-\eta_3(t))\tfrac{1}{f_1(t) f_1'(t)} \leq \log(\delta)' \leq (1+\eta_3(t))\frac{\sqrt{2}}{f_1(t)}$$

for a suitable positive function η_3 with $\lim_{t\to 0} \eta_3(t) = 0$. In particular $f_1'(0) \geq \frac{1}{\sqrt{2}}$ and since $f_1'(0) = \frac{\sqrt{2}}{d}$ (cf.(1.1)), we obtain $d \leq 2$. This finishes the proof of our main Theorem.

Remark. The first inequality is already enough to rule out the existence of invariant non-negatively curved metrics of class C^{d+1} for $d = 2m + 1 > 2$. Indeed, since $f_1'(0) = \frac{\sqrt{2}}{d}$, (2.2) implies that

$$\log(\delta)' \leq (1 + \tilde{\eta}(t))\frac{d}{t}.$$

Integrating gives that for each $\alpha > d$ we have $\delta(t) \geq t^\alpha$ provided that t lies in a sufficiently small interval $(0, \varepsilon(\alpha))$. On the other hand, using the action of the cohomogeneity one Weyl group at B_- (see e.g. [AA]), one shows that if d is odd $\delta(t) = 0$ for $t \in [-L, 0]$. Thus the metric can not be of class C^k for any integer $k > d$.

The choice of the given 2-planes can be motivated as follows. For the first set of 2-planes, the contribution given by the second fundamental form is non-positive. This forces the intrinsic sectional curvature to be non-negative. Since $[A, B] = 0$, the 2 plane has intrinsic curvature 0 in a bi-invariant metric, and non-negative curvature of M gives a strong upper bound for h_2'.

For the second family, we choose 2-planes whose intrinsic sectional curvature should be negative, since it forces the contribution from the second fundamental form to be positive, giving a lower bound for h_2'. This can be done by considering two planes containing $E_{12} \in \mathfrak{so}(2)$ since non-negative curvatures for such 2-planes would imply that SO(2) splits off isometrically, contradicting the fact that $X = \frac{1}{d}A + E_{12}$ must vanish at the singular orbit B_-.

3. Other examples and proof of Theorem B

It turns out that there are many more examples of compact cohomogeneity one manifolds which do not admit invariant non-negatively curved metrics. Recall that any choice of groups $\mathrm{H} \subset \{\mathrm{K}^-, \mathrm{K}^+\} \subset \mathrm{G}$ with $\mathrm{K}^\pm/\mathrm{H} = \mathbb{S}^{l_\pm}$ define a cohomogeneity one manifold $M = \mathrm{G} \times_{\mathrm{K}^-} \mathbb{D}^{l_-+1} \cup \mathrm{G} \times_{\mathrm{K}^+} \mathbb{D}^{l_++1}$.

Suppose $\mathrm{K}'/\mathrm{H}' = \mathbb{S}^{\ell-1}$ is a sphere, and let $sl\colon \mathrm{K}' \to \mathrm{O}(\ell)$ be the corresponding representation. Suppose furthermore that $\mu\colon \mathrm{K}' \to \mathrm{SO}(k)$ is an irreducible faithful representation with $\mu(\mathrm{H}') \subset \mathrm{SO}(k-1)$. We can then define a cohomogeneity one manifold M with $\mathrm{G} := \mathrm{SO}(n)$. View $\mathrm{SO}(k)$ as the upper $k \times k$-block of $\mathrm{SO}(n)$ and set

$$
\begin{aligned}
\mathrm{K}^- &= \mu(\mathrm{K}') \cdot \mathrm{SO}(n-k) \subset \mathrm{SO}(k)\,\mathrm{SO}(n-k) \subset \mathrm{SO}(n) \\
\mathrm{K}^+ &= \mu(\mathrm{H}') \cdot \mathrm{SO}(n-k+1) \subset \mathrm{SO}(k-1)\,\mathrm{SO}(n-k+1) \subset \mathrm{SO}(n) \\
\mathrm{H} &= \mu(\mathrm{H}') \cdot \mathrm{SO}(n-k) \subset \mathrm{K}^\pm .
\end{aligned}
\tag{3.1}
$$

Theorem 3.1. *Let* $\mathrm{K}'/\mathrm{H}' = \mathbb{S}^{\ell-1}$ *with* $\ell \geq 3$ *and* $\mu\colon \mathrm{K}' \to \mathrm{SO}(k)$ *an irreducible faithful representation such that*

a) $\mu(\mathrm{H}') \subset \mathrm{SO}(k-1)$
b) $\mu(\mathrm{K}')$ *does not act transitively on* $\mathbb{S}^{k-1}$
c) sl *is not a subrepresentation of* $S^2\mu$
d) $n \geq k+2$.

Then M *as defined by* (3.1) *does not admit a* G *invariant* C^2 *metric of non-negative sectional curvature.*

Proof. We assume that there is an invariant non-negatively curved metric on M. Let $c\colon [0, L] \to M$ denote a shortest geodesic from the singular orbit G/K_- to G/K_+ and extend c to a geodesic $c\colon \mathbb{R} \to M$. We can assume that H leaves c pointwise fixed. Let $W \subset \mathfrak{n} = \mathfrak{h}^\perp \subset \mathfrak{so}(n)$ be the subspace spanned by $E_{i,j}$ with $1 \leq i \leq k$, $k+1 \leq j \leq n$ and $W(t)$ its image in $T_{c(t)}M$. Then $W(t)$ is invariant under parallel translation since it is complementary and hence orthogonal to the fixed point set of $\mathrm{SO}(n-k) \subset \mathrm{H}$ in $\mathfrak{n}$.

Notice that $W(0)$ and $W(2L)$ are tangent to $B_- = \mathrm{G}/\mathrm{K}^-$ and we first claim that the second fundamental form of B_- vanishes on them. Indeed, the action of K^- on W is given by $\mu\hat{\otimes}\rho_{n-k}$, where as usual we denote by ρ_{n-k} the representation of $\mathrm{SO}(n-k)$ on $\mathbb{R}^{n-k}$. Hence $S^2(W) = S^2(\mu)\hat{\otimes}S^2(\rho_{n-k}) \oplus \Lambda^2(\mu)\hat{\otimes}\Lambda^2(\rho_{n-k})$. Furthermore, $\Lambda^2(\rho_{n-k}) \simeq \mathfrak{so}(n-k)$ is irreducible and $S^2(\rho_{n-k})$ is a trivial representation plus an irreducible one. The claim follows from equivariance of the second fundamental form under the action of K^- since the slice representation of K^- is given by $sl\hat{\oplus}\mathrm{Id}$ and by assumption $sl \notin S^2(\mu)$.

We now consider the geodesic $c_{[0,2L]}$ as a point in the space of all path in M which start and end in G/K_-. Clearly c is a critical point of the energy functional and we claim that it has index $n-k$ with respect to any G-invariant metric. To see this, observe that since geodesics from G/K_- to G/K_+ are minimal, this path space can be approximated by broken geodesics starting and ending orthogonal to

G/K_- and with break points in G/K_+ without changing the index and nullity of the critical point c. On this subspace of dimension $\dim(K_-/H) + \dim(K_+/H)$ the energy clearly has nullity $\dim(K_-/H)$ and index $\dim(K_+/H) = n - k$.

On the other hand, we can extend each vector in $W(0)$ to a parallel vectorfield along c and by the previous observation the boundary terms in the variational formula for the energy vanish. Hence the Hessian of the energy is negative semidefinite on this $k(n-k)$-dimensional vectorspace. Since the index is only $(n-k)$ it follows that there exists a subspace $W_1 \subset W$ of dimension at least $(k-1)(n-k)$ which is contained in the nullspace of the Hessian and hence consists of parallel Jacobifields. As in Lemma 2.1, it follows that these vectorfields are actually given by Killing fields. Since W_1 is also invariant under the action of $SO(n-k) \subset H$ we can assume, via a change of basis, that W_1 is spanned by the first $k-1$ rows in W. Let W_2 be last row in W. Since μ is irreducible, it follows that $W_1(0)$ is orthogonal to $W_2(0)$ and hence, as in the proof of Lemma 2.1, $W_1(t)$ is orthogonal to $W_2(t)$ for all t.

Using again that μ is irreducible, we can normalize the metric so that $E_{i,j} \in W$ is an orthonormal basis at $t = 0$. By the above it follows that $E_{i,j} \in W_1$ remains orthonormal for all t. Since $SO(n-k) \subset H$ acts irreducibly on W_2, we can set $h(t) := \|E_{i,j}\|$ for $E_{i,j} \in W_2$. non-negative curvature implies that the function h is concave, and since $h(0) = 1$, $h'(0) = 0$, it follows that $h(t) \leq 1$. As before we set $\delta = 1 - h^2 \geq 0$.

Let $Y_1 = E_{k,k+1}$ and $Y_2 = E_{k,k+2}$ and set $X_1 = \mathrm{Ad}_a E_{k-1,k+1}$ and $X_2 = \mathrm{Ad}_a E_{k-1,k+2}$, where $a \in SO(k-1)$ is an element in the upper $(k-1) \times (k-1)$–block. We now consider the 2-plane spanned by $A = X_1 + Y_2$ and $B = X_2 + Y_1$. Clearly A and B commute. A computation shows that:

$$\begin{aligned} R(A, B, B, A) = {} & \tfrac{1}{4}\delta^2 Q([Y_2, X_2] - [X_1, Y_1], P_t^{-1}([Y_2, X_2] - [X_1, Y_1])) \\ & - \tfrac{1}{4}\delta'^2 \geq 0 \end{aligned} \tag{3.2}$$

where $Q(X, Y) = -\frac{1}{2}\operatorname{tr}(XY)$ is a bi-invariant metric of $SO(n)$ and $g_t(X, Y) = Q(X, P_t(Y))$ for $X, Y \in \mathfrak{n}$. Since $h(L) = 0$, it follows as in the previous section that $\delta(t) > 0$ when $t > 0$. Notice that $[Y_2, X_2] - [X_1, Y_1] = 2\sum_{s=1}^{s=k-1} a_{s,k-1} E_{s,k}$ if $a = (a_{i,j}) \in SO(k-1)$ and is hence an arbitrary vector in the last column of the upper $k \times k$-block. Since this column can be regarded as the tangent space of $SO(k)/SO(k-1) = \mathbb{S}^{k-1}$ and since by assumption K' does not act transitively on $\mathbb{S}^{k-1}$, we can choose $a \in SO(k-1)$ such that $[Y_2, X_2] - [X_1, Y_1]$ is perpendicular to the Lie algebra of K_- with respect to Q. This implies that there exists $C > 0$ and $\epsilon > 0$ such that $Q([Y_2, X_2] - [X_1, Y_1], P_t^{-1}([Y_2, X_2] - [X_1, Y_1])) \leq C$ for $t \in (0, \epsilon)$. Hence (3.2) implies that $\log(\delta)' \leq C$ which again contradicts $\delta(0) = 0$ and $\delta(t) > 0$ for $0 < t < \epsilon$. □

By the classification of transitive actions on spheres, condition (b) in Theorem 3.1 only excludes $\mu = sl$ and the spin representations of Spin(7) and Spin(9). But notice that in the case of $\mu = sl$, the above cohomogeneity one manifold M

does admit an invariant non-negatively curved metric, since it is diffeomorphic to the homogeneous space $\mathrm{SO}(n+1)/\mu(\mathrm{K}) \cdot \mathrm{SO}(n-k+1)$ endowed with the cohomogeneity one action of $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$.

On the other hand, condition (a) is very restrictive. For $\mathrm{K}'/\mathrm{H}' = \mathrm{SO}(\ell)/\mathrm{SO}(\ell-1)$, such representations are called *class one representations* of the symmetric pair $(\mathrm{SO}(\ell), \mathrm{SO}(\ell-1))$ and it is well known (see e.g. [Wa]) that they consist of the irreducible representations μ_m of $\mathrm{SO}(\ell)$ on the homogeneous harmonic polynomials of degree m in ℓ variables. It is now easy to see that for these special representations, condition (c) is satisfied as well. In particular we obtain:

Corollary 3.2. *For each $\ell \geq 3$ and $m \geq 2$, the pair* $(\mathrm{K}', \mathrm{H}') = (\mathrm{SO}(\ell), \mathrm{SO}(\ell-1))$ *and the representation μ_m defines a cohomogeneity one manifold as in* (3.1) *that does not admit an invariant metric with non-negative curvature.*

In these examples the codimension of the singular orbits are $\ell \geq 3$ and $n-k+1 \geq 3$. In particular, for any pair of integers (ℓ_-, ℓ_+) with $\ell_\pm \geq 3$, there exists an infinite family of cohomogeneity one manifold with singular orbits of codimensions $\ell_\pm$ and no invariant metric of non-negative sectional curvature. This, together with Theorem A, implies Theorem B.

For a general $\mathrm{K}'/\mathrm{H}' = \mathbb{S}^{\ell-1}$ it is clear that $S^m(sl)$ restricted to H' has a fixed vector. This implies that at least one of the irreducible subrepresentations μ of $S^m(sl)$ also satisfies condition (a) and one again shows that condition (c) is satisfied. This gives rise to at least one representation satisfying the conditions of Theorem 3.1 for any K'/H'.

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