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On the Luo-Yin results concerning Gevrey regularity and analyticity for Camassa-Holm-type systems

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Abstract. Luo and Yin established a generalized Ovsyannikov theorem in [1]. However, Lemma 3.7 on which Theorem 3.1 is crucially based is incorrect. In this note, we present a sufficient condition to Lemma 3.7 to ensure that Theorem 3.1 remains true. The cost is that the life-span of solutions will be smaller.

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In [1], Luo and Yin established a generalized Ovsyannikov theorem (see [1, Theorem 3.1]). However, Lemma 3.7 on which Theorem 3.1 is crucially based is incorrect. More precisely, the integral $\int_0^t \frac{\|u(\tau)\|_{\delta(\tau)}}{(\delta(\tau)-\delta)^\sigma} d\tau$ on the right hand side of (3.6) in [1] is not well-defined, and hence the proof is incorrect. Fortunately, we find a new condition (see (1.1) below) which ensures that $u(\tau) \in X_{\delta(\tau)}$, and so the proof of Lemma 3.7 remains true.

[1, Lemma 3.7] can be revised as follows.

Lemma 3.7 (revised). *Let $\sigma \geq 1$. For every $a > 0$, $u \in E_a$, $0 < \delta < 1$, and*

$$0 \leq t < D_\sigma \frac{a(1-\delta)^\sigma}{2^{\sigma+1}}, \tag{1.1}$$

with $D_\sigma \triangleq \frac{1}{2^\sigma - 2 + \frac{1}{2^{\sigma+1}}}$, we have

$$\int_0^t \frac{\|u(\tau)\|_{\delta(\tau)}}{(\delta(\tau)-\delta)^\sigma} d\tau \leq \frac{a 2^{2\sigma+3} \|u\|_{E_a}}{(1-\delta)^\sigma} \sqrt{\frac{a(1-\delta)^\sigma}{a(1-\delta)^\sigma - t}}, \tag{1.2}$$

where

$$\delta(\tau) = \frac{1}{2}(1+\delta) + \left(\frac{1}{2}\right)^{2+\frac{1}{\sigma}} \left\{ \left[(1-\delta)^\sigma - \frac{\tau}{a}\right]^{\frac{1}{\sigma}} - \left[(1-\delta)^\sigma + (2^{\sigma+1}-1)\frac{\tau}{a}\right]^{\frac{1}{\sigma}} \right\} \in (\delta, 1).$$

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Proof. We first verify that the norm $\|u(\tau)\|_{\delta(\tau)}$ is well-defined, for all $0 \le \tau \le t < D_\sigma \frac{a(1-\delta)^\sigma}{2^{\sigma+1}}$. Indeed, it follows from (1.1) that

$$0 \le t < \frac{1}{2^{\sigma+1}} \frac{2^\sigma - 1}{2^\sigma - 2 + \frac{1}{2^{\sigma+1}}} \frac{a(1-\delta)^\sigma}{2^\sigma - 1} \le \frac{a(1-\delta)^\sigma}{2^\sigma - 1}.$$

So we can use [1, Lemma 3.6] and obtain

$$(1-\delta(\tau))^\sigma \ge \frac{1}{2^{\sigma+1}}\left((1-\delta)^\sigma - \frac{\tau}{a}\right) + \frac{\tau}{a} = \frac{(1-\delta)^\sigma}{2^{\sigma+1}} + \left(1 - \frac{1}{2^{\sigma+1}}\right)\frac{\tau}{a}. \quad (1.3)$$

By using condition (1.1), we have

$$\frac{(1-\delta)^\sigma}{2^{\sigma+1}} > \frac{\tau}{a}\left(2^\sigma - 2 + \frac{1}{2^{\sigma+1}}\right). \quad (1.4)$$

Combing (1.3) and (1.4) leads to

$$(1-\delta(\tau))^\sigma > \frac{\tau}{a}\big(2^\sigma - 1\big).$$

From the definition of the norm $\|\cdot\|_{\delta(\tau)}$, we find that $u(t) \in X_{\delta(\tau)}$. The rest of the proof is the same as [1, Lemma 3.7], and we omit the details here. ☐

Even though the condition (1.1) keeps Lemma 3.7 to be valid, the life-span of the solutions will be smaller. For completeness, we give the sketch of proof for Theorem 3.1 by modifying the arguments in [1].

Theorem 3.1 (revised). *Let $\{X_\delta\}_{0<\delta<1}$ be a scale of decreasing Banach spaces, namely, for any $\delta' < \delta$ we have $\|\cdot\|_{\delta'} \le \|\cdot\|_\delta$. Consider the Cauchy problem*

$$\begin{cases} \dfrac{du}{dt} = F(t, u(t)) \\ u|_{t=0} = u_0. \end{cases} \quad (1.5)$$

Let $T,\ R > 0, \ge 1$. For given $u_0 \in X_1$, assume that F satisfies the following conditions:

(1) *If for $0 < \delta' < \delta < 1$ the function $tu(t)$ is holomorphic in $|t| < T$ and continuous on $|t| < T$ with values in X_s and*

$$\sup_{|t|<T} \|u(t)\|_\delta < R$$

then $t \to F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in X_δ;

(2) *For any* $0 < \delta' < \delta < 1$ *and any* $u, v \in \overline{B(u_0, R)} \subset X_\delta$, *there exists a positive constant* L *depending on* u_0 *and* R *such that*

$$\sup_{|t|<T} \|F(t,u) - F(t,v)\|_\delta \le \frac{L}{(\delta - \delta')^\sigma}\|u - v\|_\delta;$$

(3) *For any* $0 < \delta < 1$, *there exists a positive constant* M *depending on* u_0 *and* R *such that*

$$\sup_{|t|<T} \|F(t,u_0)\|_\delta \le \frac{M}{(1-\delta)^\sigma}.$$

Then there exist a $T_0 \in (0, T)$ *and a unique solution* $u(t)$ *to the Cauchy problem* (1.5), *which for every* $\delta \in (0, 1)$ *is holomorphic in* $|t| < D_\sigma \frac{T_0(1-\delta)^\sigma}{2^{\sigma+1}}$ *with values in* X_δ. *Indeed,*

$$T_0 = \min\left\{\frac{1}{2^{2\sigma+4}L}, \frac{2^{\sigma+1}R}{2^{\sigma+1}2^{2\sigma+3}LR + MD_\sigma}\right\}, \quad D_\sigma = \frac{1}{2^\sigma - 2 + \frac{1}{2^{\sigma+1}}}.$$

Proof of Theorem 3.1. By following the same method in [1], we just give the sketch of proof. For any $a > 0$ and $u(t) \in \overline{B(u_0, R)} \in E_a$, we define

$$G(u(t)) \triangleq u_0 + \int_0^t F(\tau, u(\tau))d\tau, \quad 0 \le t < D_\sigma \frac{a(1-\delta)^\sigma}{2^{\sigma+1}},$$

where D_σ is defined in the revised Lemma 3.7.

Step 1: Similarly to [1], we have

$$\|G(u(t)) - u_0\|_\delta \le \frac{a2^{2\sigma+3}LR}{(1-\delta)^\sigma}\sqrt{\frac{a(1-\delta)^\sigma}{a(1-\delta)^\sigma - t}} + \frac{tM}{(1-\delta)^\sigma},$$

which implies that

$$\|G(u(t)) - u_0\|_{E_a} \le a2^{2\sigma+3}LR + \frac{aM}{2^{\sigma+1}}D_\sigma.$$

By choosing $a \le \frac{2^{\sigma+1}R}{2^{\sigma+1}2^{2\sigma+3}LR+MD_\sigma}$, the G maps E_a into itself.

Step 2: Similarly to [1], we have $\|G(u(t)) - G(v(t))\|_{E_a} \le \frac{1}{2}\|u - v\|_{E_a}$ for any $a \le \frac{1}{2^{2\sigma+4}L}$. So G is a contraction map on $\overline{B(u_0, R)} \subseteq E_a$.

Step 3: By choosing

$$a \le T_0 = \min\left\{\frac{1}{2^{2\sigma+4}L}, \frac{2^{\sigma+1}R}{2^{\sigma+1}2^{2\sigma+3}LR + MD_\sigma}\right\},$$

the map G has a unique fixed point in $\overline{B(u_0, R)} \in E_a$. Moreover, the unique solution is holomorphic in $|t| < D_\sigma \frac{T_0(1-\delta)^\sigma}{2^{\sigma+1}}$. □

Remark. Notice that the lower bound of life-span $D_\sigma \frac{T_0(1-\delta)^\sigma}{2^{\sigma+1}}$ is smaller than that obtained in [1, Theorem 3.1], which is caused by the additional condition (1.1).

References

[1] W. LUO and Z. YIN, *Gevrey regularity and analyticity for Camassa-Holm type systems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **18** (2018), 1061–1079.

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