

Hermitian curvature flow on complex homogeneous manifolds

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Abstract. In this paper we study a version of the Hermitian curvature flow (HCF) considered by the author in [25]. We focus on complex homogeneous manifolds equipped with *submersion metrics*. We prove that this finite-dimensional space of metrics is invariant under the HCF and write down the corresponding ODE on the space of Hermitian forms on the underlying Lie algebra. Using these computations we construct HCF-Einstein metrics on G -homogeneous manifolds, where G is a complexification of a compact simple Lie group. We conjecture that under the HCF any submersion metric on such a manifold pinches towards the HCF-Einstein metric. For a nilpotent or solvable complex Lie group equipped with a right-invariant metric we investigate the blow-up behavior of the HCF.

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1. Introduction

The striking success of the Hamilton's Ricci flow demonstrates that parabolic metric flows are a very powerful tool which helps constructing distinguished Riemannian metrics and studying geometry/topology of manifolds admitting certain special metrics. The Ricci flow has particularly nice long-time existence, convergence properties on Kähler manifolds and we have a reasonably good understanding of the singularity formation [16, 19, 21, 23]. However, on a general non-Kähler Hermitian complex manifold (M, g, J) the Ricci flow does not interact nicely with the complex geometry, since the Ricci form $\text{Ric}(g)$ is not necessary invariant under the operator of the almost complex structure J . Hence the evolved metric $g(t)$ on M might even not be Hermitian. To overcome this problem some natural modifications of the Ricci flow were suggested recently in the literature [7, 20, 22]. Any non-Kähler manifold admits a one-parameter family of Hermitian connections [6], among which the Chern and the Bismut connections are of a special interest. Given a general complex Hermitian manifold it is natural to consider curvatures of these

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Hermitian connections instead of the Ricci form of the Levi-Civita connection to define a Hermitian version of the Ricci flow.

In 2011 Streets and Tian [20] introduced a family of Hermitian curvature flows (HCFs) on a Hermitian manifold (M, g, J) :

$$\partial_t g = -S + Q(T), \quad (1.1)$$

where $S_{i\bar{j}} = g^{m\bar{n}} \Omega_{m\bar{n}i\bar{j}}$ is the second Chern-Ricci form of the Chern connection Ω and Q is an arbitrary symmetric J -invariant quadratic term in torsion T of ∇ . For any choice of Q equation (1.1) defines a non-linear strongly parabolic equation for the Hermitian metric g .

In this paper we study a version of the Hermitian curvature flow which was first considered by the author in [25],

$$\partial_t g_{i\bar{j}} = -S_{i\bar{j}} - \frac{1}{2} g^{m\bar{n}} g^{p\bar{s}} T_{mp\bar{j}} T_{\bar{n}si}. \quad (1.2)$$

In what follows we will refer to this flow as the HCF. It is proved in [25] that the flow (1.2) preserves Griffiths positivity of the initial Hermitian metric (see Section 2 for the precise formulation). In the present paper we focus on the behavior of this flow on a (not necessary compact) complex homogeneous manifold $M = G/H$ acted on by a complex-analytic group G with the isotropy subgroup H . There are several reasons why these manifolds are of a special interest for us:

- (i) Homogeneous manifolds form a rich family for which one can explicitly compute certain metrics and analyze behavior of a metric flow. For example, in [2, 8, 15] authors study the Ricci flow on Lie groups and homogeneous manifolds; in [4, 5] authors consider the *pluriclosed flow*, on compact homogeneous surfaces and solvmanifolds, respectively. Recently, various version of HCF on homogeneous manifolds were studied in [14, 17, 18]. We expect that our computations will shed some extra light on the geometric nature of the HCF (1.2);
- (ii) Non-symmetric rational homogeneous manifolds G/P , where G is a reductive algebraic group and P is its parabolic subgroup, are projective manifolds, however, there are ‘natural’ Hermitian metrics induced by the Killing metric of the compact form of G , which are typically non-Kähler (see [27] for an explicit computation on a complete flag manifold F_3). It is interesting to analyze whether our metric flow distinguishes these special metrics;
- (iii) In [25], we, in particular, proved that the HCF preserves Griffiths semipositivity of the initial metric. Homogeneous manifolds equipped with submersion metrics (see Definition 1.2 below) are essentially the only abundant source of such examples.

For an homogeneous manifold $M = G/H$ we denote by \mathfrak{g} and \mathfrak{h} the complex Lie algebras of G and H . Let $\mathcal{M}(M)$ be the infinite-dimensional space of Hermitian metrics on M modulo isometry. There are two distinguished subspaces of $\mathcal{M}(M)$.

Definition 1.1 (Invariant metrics). Define $\mathcal{M}^{\text{inv}}(M)$ to be the space of *G-invariant Hermitian metrics* on M . These metrics are in one-to-one correspondence with Ad_H -invariant Hermitian metrics on the vector space $\mathfrak{g}/\mathfrak{h}$, see [13, X page 3]:

$$\mathcal{M}^{\text{inv}}(M) \longleftrightarrow \{\text{Ad}_H\text{-invariant metrics on } \mathfrak{g}/\mathfrak{h}\}.$$

Clearly, the set $\mathcal{M}^{\text{inv}}(M)$ is preserved by any metric flow of the form $\frac{dg}{dt} = R(g)$, where R is a tensor field, such that its value at $m \in M$ is determined by the germ of g at m , provided that the solution to the flow is unique. Hence, any such ‘natural’ metric flow defines an ODE on the finite dimensional space $\mathcal{M}^{\text{inv}}(M)$.

The study of invariant metrics on homogeneous manifolds is a classical subject of differential geometry [1]. These metrics provide a “hands-on” construction of a multitude of explicit Riemannian manifolds with certain prescribed geometric properties. One can then translate the differential-geometric problems into the questions of basic linear algebra and representation theory. However, on a general homogeneous manifold, $\mathcal{M}^{\text{inv}}(M)$ could be empty. Below we will be interested in a different class of metrics, which is always nonempty for any complex homogeneous manifold.

Definition 1.2 (Submersion metrics). Let h be a Hermitian metric on the Lie algebra \mathfrak{g} of a complex Lie group G . Metric h defines a unique right-invariant Hermitian metric on G , such that its restriction to $T_e^{1,0}G \simeq \mathfrak{g}$ coincides with h . Then there is a unique submersion metric on G/H , which turns the projection $G \rightarrow G/H$ into a Hermitian submersion. We define $\mathcal{M}^{\text{sub}}(M)$ to be the space of *submersion metrics*. We have a map

$$\{\text{Hermitian metrics on } \mathfrak{g}\} / \text{Ad}_G \longrightarrow \mathcal{M}^{\text{sub}}(M).$$

There is an alternative description of submersion metrics. The holomorphic tangent bundle of a complex homogeneous manifold M is globally generated by $\mathfrak{g} \subset H^0(M, T^{1,0}M)$ via the infinitesimal evaluation map. Hence, any Hermitian metric h on the vector space \mathfrak{g} induces a metric on its quotient $T^{1,0}M$. Note that typically the submersion Hermitian metrics are not G -homogeneous.

Remark 1.3. Often there is a compact subgroup $K \subset G$ acting transitively on G/H from the left, e.g., any rational homogeneous space admits such an action. If we choose $h \in \text{Sym}^{1,1}(\mathfrak{g}^*)$ to be K -biinvariant, then the corresponding submersion metric on G/H will be K -invariant.

In general, there is no reason for the set $\mathcal{M}^{\text{sub}}(M)$ to be invariant under a metric flow. However, it turns out that the HCF (1.2) preserves $\mathcal{M}^{\text{sub}}(M)$ (see Theorem 5.1 below). In some sense this is an expected result, since the HCF preserves Griffiths/dual-Nakano semipositivity (see [24, 25]) and the metrics in $\mathcal{M}^{\text{sub}}(M)$ are the only known dual-Nakano semipositive metrics on a general complex homogeneous manifold. What is somewhat less expected, is that there exists an ODE on $\text{Sym}^{1,1}(\mathfrak{g})$ – the set of Hermitian metrics on \mathfrak{g}^* , which induces the HCF on all G -homogeneous manifolds *independently of the isotropy subgroup H* .

The evolution term of this ODE is given by a generalization of a familiar operation $\#$ and turns out to be very similar to the ODE defined by the zero-order part of the evolution equation for the Riemannian curvature (respectively Chern curvature) under the Ricci flow (respectively HCF). Namely, in the case of the Ricci flow the curvature operator $R \in \text{Sym}^2(\mathfrak{so}(n))$ evolves (in the moving frame) according to the equation

$$\frac{dR}{dt} = \Delta R + R^2 + R^\#$$

and the relevant ODE is

$$\frac{dR}{dt} = R^2 + R^\#.$$

For the HCF on the set of submersion metrics $\mathcal{M}^{\text{sub}}(M)$, the ODE for $B \in \text{Sym}^{1,1}(\mathfrak{g})$ takes form

$$\frac{dB}{dt} = B^\#.$$

This is a Riccati-type system of equations, as the evolution term $B^\#$ is quadratic in B .

In a special case when G is the complexification of a compact simple Lie group, we use the ODE for $B \in \text{Sym}^{1,1}(\mathfrak{g})$ to construct scale-static solutions to the HCF on any homogeneous manifold of G . Such a metric corresponds to the Killing metric of the compact real form of G and can be thought of as the HCF-analogue of the Einstein metric. We explicitly solve the ODE in some simple examples (see Example 5.3 for the diagonal Hopf surface and Example 6.5 for the Iwasawa threefold) and formulate conjectural pinching behavior of this differential equation (Conjecture 5.4). It seems that at this point we need a better understanding of the algebraic properties of the operation $\#$ to resolve the conjecture and to understand the behavior of the HCF on $\mathcal{M}^{\text{sub}}(G/H)$ for a general G .

In Section 6, we make the first step towards this understanding and study the blow-up behavior of ODE $\frac{dB}{dt} = B^\#$ on an arbitrary complex Lie group. Namely, we analyze how the growth rate of a solution $B(t)$ depends on the algebraic properties of \mathfrak{g} (see Theorems 6.2 and 6.3).

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2. Hermitian curvature flow

In this section we fix some notations and review the definition of the Hermitian curvature flow.

In what follows for a complex vector space V we denote by \overline{V} the conjugate vector space. We denote by $\text{Sym}^{1,1}(V) \subset V \otimes \overline{V}$ the real vector space of Hermitian forms, *i.e.*, the set of complex sesquilinear forms h on V^* such that $h(\xi, \bar{\eta}) = h(\eta, \bar{\xi})$. For $h_1, h_2 \in \text{Sym}^{1,1}(V)$ we say that $h_1 \geq h_2$, if for any $\xi \in V^*$ the inequality $h_1(\xi, \bar{\xi}) \geq h_2(\xi, \bar{\xi})$ holds. A form $h \in \text{Sym}^{1,1}(V)$ is said to be *positive*, if $h(\xi, \bar{\xi}) > 0$ for any nonzero $\xi \in V^*$.

Let (M, g, J) be a Hermitian manifold, where J is an integrable complex structure and g is a Hermitian metric. Let $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ be the decomposition of the complexified tangent bundle into the $\pm i$ eigenspaces of J . Denote by ∇ the Hermitian connection on TM uniquely characterized by the properties:

- (i) $\nabla g = 0$;
- (ii) $\nabla J = 0$;
- (iii) $T(X, JY) = T(JX, Y)$ for any $X, Y \in TM$, where $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is the torsion tensor;
- (iv) $\nabla^{0,1} = \bar{\partial}$.

Remark 2.1. Given (i) and (ii) properties (iii) and (iv) are equivalent. It is useful to think about (iii) as the vanishing of $(1, 1)$ -type part of T , *i.e.*, $T(\xi, \bar{\eta}) = 0$ for $\xi, \eta \in T^{1,0}M$.

Definition 2.2. The *Chern curvature* of a Hermitian manifold (M, g, J) is the curvature of ∇ .

$$\Omega(X, Y)Z := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})Z,$$

where $X, Y, Z \in TM$. We also define a tensor with 4 vector arguments by lowering one index.

$$\Omega(X, Y, Z, W) := g((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})Z, W).$$

Tensor Ω satisfies a number of symmetries.

Proposition 2.3. *For any real vectors $X, Y, Z, W \in TM$ one has*

$$\begin{aligned} \Omega(X, Y, Z, W) &= -\Omega(Y, X, Z, W), \quad \Omega(X, Y, Z, W) = -\Omega(X, Y, W, Z); \\ \Omega(JX, JY, Z, W) &= \Omega(X, Y, Z, W), \quad \Omega(X, Y, JZ, JW) = \Omega(X, Y, Z, W). \end{aligned}$$

Symmetries of Ω imply that for $\xi, \eta, \zeta, \nu \in T^{1,0}M$

$$\Omega(\xi, \eta, \cdot, \cdot) = \Omega(\cdot, \cdot, \zeta, \nu) = 0, \quad \Omega(\xi, \bar{\eta}, \zeta, \bar{\nu}) = \overline{\Omega(\eta, \bar{\xi}, \nu, \bar{\zeta})},$$

in particular $\Omega(\xi, \bar{\xi}, \eta, \bar{\eta}) \in \mathbb{R}$. It is easy to check that the values $\Omega(\xi, \bar{\xi}, \eta, \bar{\eta})$ for $\xi, \eta \in T^{1,0}M$ completely determine tensor Ω . In what follows, by abuse of notation we sometimes write TM as a shorthand for the holomorphic tangent bundle $T^{1,0}M$.

The main object of our study is the following version of a general Hermitian curvature flow (1.1)

$$\begin{cases} \partial_t g_{i\bar{j}} = -\Psi_{i\bar{j}}(g(t)), \\ g(0) = g_0, \end{cases} \quad (2.1)$$

where $\Psi(g) \in \text{Sym}^{1,1}(T^*M)$ is given by

$$\Psi(g)_{i\bar{j}} := g^{m\bar{n}} \Omega_{m\bar{n}i\bar{j}} + \frac{1}{2} g^{m\bar{n}} g^{p\bar{s}} T_{pm\bar{j}} T_{\bar{s}n\bar{i}}$$

is the sum of the second Chern-Ricci curvature and a certain quadratic torsion term for $g = g(t)$. We refer to $\Psi(g)$ as the *torsion-twisted Chern-Ricci form*. By [20, Proposition 5.1] on a compact manifold there exists unique solution to equation (2.1) on some maximal time interval $[0, \tau)$. If the initial metric g_0 is Kähler, the HCF coincides with the Kähler-Ricci flow.

Important property of the flow (2.1) is preservation of the following curvature positivity condition.

Definition 2.4. A complex Hermitian manifold (M, g, J) has *Griffiths positive* (respectively non-negative) curvature, if its Chern curvature Ω for any non-zero $\xi, \eta \in T^{1,0}M$ satisfies

$$\Omega(\xi, \bar{\xi}, \eta, \bar{\eta}) > 0, \text{ (respectively } \geq 0\text{).}$$

Theorem 2.5 ([25, Theorem 0.1]). *Let $g(t), t \in [0, \tau)$ be the solution to the HCF on a compact complex Hermitian manifold (M, g_0, J) . Assume that the Chern curvature Ω^{g_0} at the initial moment $t = 0$ is Griffiths non-negative (respectively positive). Then for $t \in [0, \tau)$ the Chern curvature $\Omega(t) = \Omega^{g(t)}$ remains Griffiths non-negative (respectively positive).*

If, moreover, the Chern curvature Ω^{g_0} is Griffiths positive at least at one point $x \in M$, then for any $t \in (0; \tau)$ the Chern curvature is Griffiths positive everywhere on M .

One of the main sources of complex manifolds admitting metrics of Griffiths non-negative curvature is the set of manifolds with *globally generated* tangent bundle. To be precise, if $\mathbb{C}^m \rightarrow TM$ is a surjective map from a trivial bundle over a complex manifold M , then any constant metric on \mathbb{C}^m induces a metric with dual-Nakano/Griffiths non-negative curvature [9, Section 0.5]. In the next section we investigate the behavior of HCF on such manifolds, equipped with a submersion metric.

3. HCF on a globally generated tangent bundle

We start this section with deriving the coordinate expression for the torsion-twisted Chern-Ricci form

$$\Psi_{i\bar{j}} = g^{m\bar{n}} \Omega_{m\bar{n}i\bar{j}} + \frac{1}{2} g^{m\bar{n}} g^{p\bar{s}} T_{pm\bar{j}} T_{\bar{s}n\bar{i}}$$

on (M, g, J) . To simplify subsequent applications of this computation we provide a formula for the g -dual of $\Psi_{i\bar{j}}$

$$\Psi^{i\bar{j}} := g^{i\bar{n}} g^{m\bar{j}} \Psi_{m\bar{n}},$$

i.e., we use the metric g to identify Ψ with a section of $\text{Sym}^{1,1}(TM)$.

Proposition 3.1.

$$\Psi^{i\bar{j}} = g^{m\bar{n}} \partial_m \partial_{\bar{n}} g^{i\bar{j}} - \partial_m g^{i\bar{n}} \partial_{\bar{n}} g^{m\bar{j}}. \quad (3.1)$$

Proof. Recall that for the Chern connection on (TM, g) we have:

$$\Gamma_{ij}^k = g^{k\bar{l}} \partial_i g_{j\bar{l}}.$$

Hence for the Chern curvature we have:

$$\begin{aligned} g^{m\bar{l}} \Omega_{i\bar{j}m}^k &= -g^{m\bar{l}} \partial_{\bar{j}} (g^{k\bar{n}} \partial_i g_{m\bar{n}}) = -g^{m\bar{l}} \partial_{\bar{j}} \left(-g^{k\bar{n}} g_{m\bar{s}} g_{p\bar{n}} \partial_i g^{p\bar{s}} \right) \\ &= g^{m\bar{l}} \partial_{\bar{j}} \left(g_{m\bar{s}} \partial_i g^{k\bar{s}} \right) = g^{m\bar{l}} \partial_{\bar{j}} g_{m\bar{s}} \partial_i g^{k\bar{s}} + \partial_i \partial_{\bar{j}} g^{k\bar{l}} \\ &= -g_{p\bar{s}} \partial_{\bar{j}} g^{p\bar{l}} \partial_i g^{k\bar{s}} + \partial_i \partial_{\bar{j}} g^{k\bar{l}}. \end{aligned}$$

For the torsion tensor T_{mp}^i we compute:

$$T_{mp}^i = g^{i\bar{l}} (\partial_m g_{p\bar{l}} - \partial_p g_{m\bar{l}}) = g_{m\bar{l}} \partial_p g^{i\bar{l}} - g_{p\bar{l}} \partial_m g^{i\bar{l}}.$$

Therefore

$$\begin{aligned} \frac{1}{2} g^{m\bar{n}} g^{p\bar{s}} T_{mp}^i T_{\bar{n}\bar{s}}^{\bar{j}} &= \frac{1}{2} g^{m\bar{n}} g^{p\bar{s}} \left(g_{m\bar{l}} \partial_p g^{i\bar{l}} - g_{p\bar{l}} \partial_m g^{i\bar{l}} \right) \left(g_{k\bar{n}} \partial_{\bar{s}} g^{k\bar{j}} - g_{k\bar{s}} \partial_{\bar{n}} g^{k\bar{j}} \right) \\ &= g^{p\bar{s}} g_{k\bar{l}} \partial_p g^{i\bar{l}} \partial_{\bar{s}} g^{k\bar{j}} - \partial_p g^{i\bar{n}} \partial_{\bar{n}} g^{p\bar{j}}. \end{aligned}$$

Using the above formulas together we get the expression for $\Psi^{i\bar{j}}$.

$$\Psi^{i\bar{j}} = g^{m\bar{n}} \partial_m \partial_{\bar{n}} g^{i\bar{j}} - \partial_p g^{i\bar{s}} \partial_{\bar{s}} g^{p\bar{j}}.$$

After relabeling indices we get the stated formula. \square

Combining Proposition 3.1 and the equation of the HCF flow (2.1) we get the following corollary.

Corollary 3.2. *Let (M, J, g_0) be a Hermitian manifold. Assume that $\tilde{g}(t) \in \text{Sym}^{1,1}(TM)$ is a solution to the PDE on $M \times [0, \tau)$*

$$\begin{cases} \partial_t \tilde{g}^{i\bar{j}} = \tilde{g}^{m\bar{n}} \partial_m \partial_{\bar{n}} \tilde{g}^{i\bar{j}} - \partial_m \tilde{g}^{i\bar{n}} \partial_{\bar{n}} \tilde{g}^{m\bar{j}}, \\ \tilde{g}(0) = g_0^{-1}, \end{cases} \quad (3.2)$$

such that $\tilde{g}^{i\bar{j}}(t)$ is positive definite for $t \in [0, \tau)$. Then $g(t) := \tilde{g}^{-1}(t)$ is the solution to the HCF on M .

An interesting feature of equation (3.2) is that its right-hand side depends only on \tilde{g} and not on \tilde{g}^{-1} . In particular, there might exist a solution $\tilde{g}(t)$ starting with a degenerate or indefinite form. It would be interesting to find a geometric interpretation of such solution. However, in this case the equation is not elliptic anymore.

Now, we explicitly compute the HCF on a complex homogeneous manifold, equipped with a submersion metric $g = p_*h$, $h \in \text{Sym}^{1,1}(\mathfrak{g}^*)$ (see Definition 1.2). Denote by $\{s_\alpha\}$ a basis of holomorphic vector fields induced by one-parameter subgroups of G , *i.e.*, $s_\alpha = p(e_\alpha)$, where $\{e_\alpha\}$ is the basis of \mathfrak{g} , and

$$p: \mathfrak{g} \rightarrow T^{1,0}M$$

is the evaluation map.

Let $s_\alpha = a_\alpha^i \partial/\partial z^i$ be the local coordinate expression for the vector field s_α , and denote $a_\alpha^{\bar{i}} := \overline{a_\alpha^i}$. Functions a_α^i are holomorphic. In the coordinates, the submersion metric $g = g_{i\bar{j}}$ is given by the expression

$$g_{i\bar{j}} = \left(a_\alpha^i a_\beta^{\bar{j}} (h^{-1})^{\alpha\bar{\beta}} \right)^{-1}. \quad (3.3)$$

Indeed, $a_\alpha^i a_\beta^{\bar{j}} (h^{-1})^{\alpha\bar{\beta}}$ is the Hermitian metric on $\Lambda^{1,0}(M)$ induced by the inclusion $p^*: \Lambda^{1,0}(M) \rightarrow \mathfrak{g}^*$, and the Hermitian metric $g_{i\bar{j}}$ on $T^{1,0}M$ is its inverse.

Proposition 3.3. *Let $(G/H, g, J)$ be a complex homogeneous manifold, with a submersion metric $g = p_*h$, $h \in \text{Sym}^{1,1}(\mathfrak{g}^*)$. Let $\{e_\alpha\}_{\alpha=1}^m$ be an h -orthonormal frame of \mathfrak{g} , with $s_\alpha := p_*(e_\alpha)$. Then the evolution term for the HCF considered as a section of $\text{Sym}^{1,1}(T^{1,0}M)$ is given by*

$$\Psi(g) = \frac{1}{2} \sum_{\alpha, \beta=1}^m [s_\alpha, s_\beta] \otimes \overline{[s_\alpha, s_\beta]},$$

where $[\cdot, \cdot]$ is the commutator of vector fields on $M = G/H$.

Proof. First we note, that since vector fields $s_\alpha = a_\alpha^i \partial/\partial z^i$ are holomorphic, all the derivatives $\bar{\partial}a_\alpha^i$, $\partial a_\beta^{\bar{j}}$ vanish. Using this fact and equations (3.1) and (3.3) we find

$$\begin{aligned} \Psi(g)^{i\bar{j}} &= g^{m\bar{n}} \partial_m \partial_{\bar{n}} g^{i\bar{j}} - \partial_m g^{i\bar{n}} \partial_{\bar{n}} g^{m\bar{j}} \\ &= (h^{-1})^{\gamma\bar{\delta}} a_\gamma^m a_{\bar{\delta}}^{\bar{n}} (h^{-1})^{\alpha\bar{\beta}} \partial_m a_\alpha^i \partial_{\bar{n}} a_{\beta}^{\bar{j}} - (h^{-1})^{\gamma\bar{\delta}} \partial_m a_\gamma^i a_{\bar{\delta}}^{\bar{n}} h^{\alpha\bar{\beta}} a_\alpha^m \partial_{\bar{n}} a_{\beta}^{\bar{j}} \\ &= (h^{-1})^{\alpha\bar{\beta}} h^{\gamma\bar{\delta}} \left(a_\gamma^m \partial_m a_\alpha^i \cdot a_{\bar{\delta}}^{\bar{n}} \partial_{\bar{n}} a_{\beta}^{\bar{j}} - a_\alpha^m \partial_m a_\gamma^i \cdot a_{\bar{\delta}}^{\bar{n}} \partial_{\bar{n}} a_{\beta}^{\bar{j}} \right) \\ &= \frac{1}{2} (h^{-1})^{\alpha\bar{\beta}} (h^{-1})^{\gamma\bar{\delta}} \left(a_\gamma^m \partial_m a_\alpha^i - a_\alpha^m \partial_m a_\gamma^i \right) \cdot \left(a_{\bar{\delta}}^{\bar{n}} \partial_{\bar{n}} a_{\beta}^{\bar{j}} - a_{\beta}^{\bar{n}} \partial_{\bar{n}} a_{\bar{\delta}}^{\bar{j}} \right). \end{aligned} \quad (3.4)$$

In the last equality we used the fact that the whole expression is symmetric under the change $(\alpha\bar{\beta}) \leftrightarrow (\gamma\bar{\delta})$. The last two multiples in the last expressions

are exactly the i and \bar{j} coordinates of the Lie brackets $[a_\gamma^m \partial/\partial z_m, a_\alpha^m \partial/\partial z_m]$ and $[a_{\bar{\delta}}^{\bar{n}} \partial/\partial z_{\bar{n}}, a_{\bar{\beta}}^{\bar{n}} \partial/\partial z_{\bar{n}}]$. Since $\{e_\alpha\}_{\alpha=1}^m$ is an h -orthonormal basis of \mathfrak{g} , we get the stated formula. \square

The expression (3.4) for $\Psi(g)$ suggests that the HCF with $g_0 = p_* h \in \mathcal{M}^{\text{sub}}(M)$ is governed by the Lie algebra structure on the space of holomorphic vector fields $H^0(M, T^{1,0}M)$. In the next sections we study this relation.

4. Operation # on a Lie algebra

Before studying the HCF on complex homogeneous manifolds following Hamilton [10] we define an algebraic operation on the second tensor power of a Lie algebra and list its basic properties.

Definition 4.1 (Operation #). Let $\mathfrak{g}_{\mathbb{R}}$ be a real Lie algebra. Define a symmetric bilinear $\text{ad}_{\mathfrak{g}_{\mathbb{R}}}$ -invariant operation

$$\#: \mathfrak{g}_{\mathbb{R}}^{\otimes 2} \otimes \mathfrak{g}_{\mathbb{R}}^{\otimes 2} \rightarrow \mathfrak{g}_{\mathbb{R}}^{\otimes 2},$$

by the formula

$$(v_1 \otimes v_2) \# (w_1 \otimes w_2) = [v_1, w_1] \otimes [v_2, w_2].$$

If we choose a basis $\{e_\alpha\}$ of $\mathfrak{g}_{\mathbb{R}}$ and denote by $c_{\alpha\beta}^\gamma$ its structure constants, then for $B = \{B^{\alpha\beta}\}$, $D = \{D^{\alpha\beta}\}$, $B, D \in \mathfrak{g}_{\mathbb{R}}^{\otimes 2}$

$$(B \# D)^{\alpha\beta} = c_{\epsilon\delta}^\alpha c_{\gamma\theta}^\beta B^{\epsilon\gamma} D^{\delta\theta}.$$

Clearly, the operation $\#$ preserves the parity of the decomposition $\mathfrak{g}_{\mathbb{R}}^{\otimes 2} = \text{Sym}^2(\mathfrak{g}_{\mathbb{R}}) \oplus \Lambda^2(\mathfrak{g}_{\mathbb{R}})$, *i.e.*, defines the maps

$$\begin{aligned} \#: \text{Sym}^2(\mathfrak{g}_{\mathbb{R}}) \otimes \text{Sym}^2(\mathfrak{g}_{\mathbb{R}}) &\rightarrow \text{Sym}^2(\mathfrak{g}_{\mathbb{R}}), \\ \#: \Lambda^2(\mathfrak{g}_{\mathbb{R}}) \otimes \Lambda^2(\mathfrak{g}_{\mathbb{R}}) &\rightarrow \text{Sym}^2(\mathfrak{g}_{\mathbb{R}}), \\ \#: \Lambda^2(\mathfrak{g}_{\mathbb{R}}) \otimes \text{Sym}^2(\mathfrak{g}_{\mathbb{R}}) &\rightarrow \Lambda^2(\mathfrak{g}_{\mathbb{R}}). \end{aligned}$$

Now, let \mathfrak{g} be a complex Lie algebra. Similarly to the real case, we denote by the symbol $\#$ the map

$$\begin{aligned} \#: (\mathfrak{g} \otimes \bar{\mathfrak{g}}) \otimes (\mathfrak{g} \otimes \bar{\mathfrak{g}}) &\rightarrow (\mathfrak{g} \otimes \bar{\mathfrak{g}}), \\ (v_1 \otimes \bar{v}_2) \# (w_1 \otimes \bar{w}_2) &= [v_1, w_1] \otimes [\bar{v}_2, \bar{w}_2]. \end{aligned}$$

As in the real case, $\#$ preserves the parity of $\mathfrak{g} \otimes \bar{\mathfrak{g}}$, in particular, it induces a bilinear map on the set of Hermitian elements of $\mathfrak{g} \otimes \bar{\mathfrak{g}}$

$$\#: \text{Sym}^{1,1}(\mathfrak{g}) \otimes \text{Sym}^{1,1}(\mathfrak{g}) \rightarrow \text{Sym}^{1,1}(\mathfrak{g}).$$

We will write $B^\#$ for $\frac{1}{2}B\#B$.

Remark 4.2. The $\#$ -operation was introduced by Hamilton in the context of the Ricci flow, see [10]. Definition 4.1 differs from the one of Hamilton in several aspects:

- (i) Originally $\#$ was defined only for the real Lie algebra $\mathfrak{so}(n)$, while our definition makes sense for any Lie algebra;
- (ii) Hamilton used $\#$ only for the part of $\text{Sym}^2(\mathfrak{so}(n))$ satisfying the first Bianchi identity;
- (iii) Hamilton used the Killing metric to interpret $\#$ as a bilinear operator on the space self-adjoint operations.

This operation and its algebraic properties play the key role in the characterization of compact manifolds with 2-positive curvature operator [3]. For an arbitrary real metric Lie algebra this operation was also considered by Wilking in [26, Section 3].

Remark 4.3. Operation $\#$ is natural, *i.e.*, if $\rho: \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism of Lie algebras, then $\rho(B)\#\rho(D) = \rho(B\#D)$ for any $B, D \in \text{Sym}^{1,1}(\mathfrak{g})$.

The following lemma easily follows from the definition by considering a basis of \mathfrak{g} which diagonalizes the two forms.

Lemma 4.4. *Let \mathfrak{g} be a real (respectively complex) Lie algebra. Assume that forms $B, D \in \text{Sym}^2(\mathfrak{g})$ (respectively $B, D \in \text{Sym}^{1,1}(\mathfrak{g})$) are symmetric (respectively Hermitian) positive definite. Then $B\#D$ is positive semidefinite with $\ker(B\#D) = \text{Ann}([\mathfrak{g}, \mathfrak{g}]) \subset \mathfrak{g}^*$.*

Example 4.5. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(2)$ and denote by $\langle \cdot, \cdot \rangle$ the invariant metric on $\mathfrak{g}_{\mathbb{R}}$ normalized in such a way that $\langle [e_1, e_2], e_3 \rangle = \pm 1$ for any orthonormal triple e_1, e_2, e_3 . Take $B_{\mathbb{R}} \in \text{Sym}^2(\mathfrak{g}_{\mathbb{R}})$ and chose a $\langle \cdot, \cdot \rangle$ -orthonormal basis e_1, e_2, e_3 , which diagonalizes $B_{\mathbb{R}}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then

$$\begin{aligned} B_{\mathbb{R}}^{\#} &= 1/2(\lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \lambda_3 e_3 \otimes e_3) \# (\lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \lambda_3 e_3 \otimes e_3) \\ &= (\lambda_2 \lambda_3 e_1 \otimes e_1 + \lambda_1 \lambda_3 e_2 \otimes e_2 + \lambda_1 \lambda_2 e_3 \otimes e_3) \end{aligned}$$

is diagonalized in the same basis with the eigenvalues $\lambda_2 \lambda_3, \lambda_1 \lambda_3, \lambda_1 \lambda_2$. In particular, if $B_{\mathbb{R}}$ is proportional to the dual of the metric $\langle \cdot, \cdot \rangle$ then so is $B_{\mathbb{R}}^{\#}$.

Example 4.6. Let $\mathfrak{g}_{\mathbb{R}}$ be a compact simple Lie algebra with an invariant metric $\langle \cdot, \cdot \rangle$. Let $B_{\mathbb{R}} = \langle \cdot, \cdot \rangle^{-1}$ be the dual of $\langle \cdot, \cdot \rangle$, *i.e.*, for an orthonormal basis e_1, \dots, e_m let $B_{\mathbb{R}} = \sum e_i \otimes e_i$. Then $B_{\mathbb{R}}^{\#}$ is proportional to $B_{\mathbb{R}}$ with a positive factor.

Indeed, since $\mathfrak{g}_{\mathbb{R}}$ is simple, $[\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}] = \mathfrak{g}_{\mathbb{R}}$, and by Lemma 4.4 both $B_{\mathbb{R}}$ and $B_{\mathbb{R}}^{\#}$ are positive definite ad $\mathfrak{g}_{\mathbb{R}}$ -invariant elements in $\text{Sym}^2(\mathfrak{g}_{\mathbb{R}})$. As $\mathfrak{g}_{\mathbb{R}}$ is simple, such an element is unique up to multiplication by a positive constant, so $B_{\mathbb{R}}^{\#} = \lambda B_{\mathbb{R}}$, $\lambda > 0$.

We expect that proportionality $B^{\#} = \lambda B$ characterizes the ad $\mathfrak{g}_{\mathbb{R}}$ -invariant positive definite forms on any compact simple Lie algebra.

Question. Let $\mathfrak{g}_{\mathbb{R}}$ be a compact simple real Lie algebra with an invariant metric $\langle \cdot, \cdot \rangle$. Let $B \in \text{Sym}^2(\mathfrak{g}_{\mathbb{R}})$ be a positive definite form such that $B^\# = \lambda B$. Is B^{-1} proportional to $\langle \cdot, \cdot \rangle$?

Remark 4.7. If $\mathfrak{g}_{\mathbb{R}}$ is a real Lie algebra equipped with an invariant metric $\langle \cdot, \cdot \rangle$ extended in an obvious way to all tensor products of $\mathfrak{g}_{\mathbb{R}}$, then a trilinear form

$$P(B, C, D) := \langle B \# C, D \rangle, \quad B, C, D \in \mathfrak{g}_{\mathbb{R}}^{\otimes 2},$$

is totally symmetric. It follows from the coordinate expression for $B \# C$ through the structure constants and the fact that in any $\langle \cdot, \cdot \rangle$ -orthonormal basis the structure constants are totally skew-symmetric.

Remark 4.8. If $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ is the complexification of a real Lie algebra, then a real symmetric form $B_{\mathbb{R}} \in \text{Sym}^2(\mathfrak{g}_{\mathbb{R}})$ defines a Hermitian form $\varphi(B_{\mathbb{R}}) \in \text{Sym}^{1,1}(\mathfrak{g})$: in the basis $\{e_\alpha\}$ of $\mathfrak{g}_{\mathbb{R}}$ and the corresponding basis of \mathfrak{g} this form is given by $\varphi(B_{\mathbb{R}})_{\alpha\bar{\beta}} := (B_{\mathbb{R}})_{\alpha\beta}$. It is clear that $\varphi(B_{\mathbb{R}})^\# \varphi(B_{\mathbb{R}}) = \varphi(B_{\mathbb{R}} \# B_{\mathbb{R}})$.

5. ODE for the submersion metrics on complex homogeneous manifolds

In this section we turn back to a complex homogeneous manifold $M = G/H$, equipped with a submersion metric $g = p_*h$, $h \in \text{Sym}^{1,1}(\mathfrak{g}^*)$. Note that if a group K acts on $M = G/H$ and actions of G and K commute, then the metric g is K -invariant. We will use this observation in Example 5.3 below.

With Proposition 3.3 and Definition 4.1 we can reduce the HCF on a complex homogeneous manifold (M, g_0, J) to an ODE for $h^{-1} = B \in \text{Sym}^{1,1}(\mathfrak{g})$.

Theorem 5.1 (HCF of a submersion metric). *Let $M = G/H$ be a complex homogeneous manifold equipped with a submersion Hermitian metric $g_0 = p_*h_0 \in \mathcal{M}^{\text{sub}}(M)$, where $h_0 \in \text{Sym}^{1,1}(\mathfrak{g}^*)$. Let $B(t)$ be the solution to the ODE*

$$\begin{cases} \frac{dB}{dt} = B^\#, \\ B(0) = h_0^{-1}. \end{cases} \quad (5.1)$$

Then $g(t) = p_(B(t)^{-1})$ solves the HCF on (M, g_0, J) . In particular $g(t) \in \mathcal{M}^{\text{sub}}(M)$.*

Proof. If $B(t)$ solves (5.1), then by Proposition 3.3 the Hermitian metric $\tilde{g}(t)$ on $\Lambda^{1,0}(M)$, induced from $B(t) \in \text{Sym}^{1,1}(\mathfrak{g})$ via the map $\Lambda^{1,0}(M) \rightarrow \mathfrak{g}^*$, satisfies the partial differential equation

$$\begin{cases} \frac{d\tilde{g}}{dt} = \Psi(\tilde{g}^{-1}), \\ \tilde{g}(0) = g_0^{-1}, \end{cases}$$

where, as in Proposition 3.1, Ψ is identified with a section of $\text{Sym}^{1,1}(T^{1,0}M)$. Hence $g(t) = \tilde{g}(t)^{-1}$ is the solution to the HCF on (M, g_0, J) . \square

Surprising consequence of this theorem is that ODE (5.1) gives solutions to the HCF on all G -homogeneous manifolds $M = G/H$ equipped with a submersion metric independently of the isotropy subgroup H .

Example 5.2. Let $G = SL(2, \mathbb{C})$. The Lie algebra of G has the compact real form $\mathfrak{su}(2)$, *i.e.*, $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$. Assume that $B_0 \in \text{Sym}^{1,1}(\mathfrak{sl}(2, \mathbb{C}))$ corresponds to $B_{\mathbb{R}} \in \text{Sym}^{1,1}(\mathfrak{su}(2))$ (see Remark 4.8). Then ODE (5.1) reduces to the equation for $B_{\mathbb{R}} \in \text{Sym}^2(\mathfrak{su}(2))$

$$\frac{dB_{\mathbb{R}}}{dt} = B_{\mathbb{R}}^{\#}.$$

Let $\langle \cdot, \cdot \rangle$ be a positive definite multiple of the Killing form of $\mathfrak{su}(2)$. Assume that $\langle \cdot, \cdot \rangle$ is normalized in such way that for any orthonormal basis e_1, e_2, e_3 we have $\langle [e_1, e_2], e_3 \rangle = \pm 1$.

Let e_1, e_2, e_3 be an orthonormal basis of $\mathfrak{su}(2)$ diagonalizing $B_{\mathbb{R}}$. Denote the eigenvalues of $B_{\mathbb{R}}$ with respect to $\langle \cdot, \cdot \rangle$ by $\lambda_1, \lambda_2, \lambda_3$. In this basis the evolution equation takes the form

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_2 \lambda_3 \\ \frac{d\lambda_2}{dt} = \lambda_1 \lambda_3 \\ \frac{d\lambda_3}{dt} = \lambda_1 \lambda_2. \end{cases} \quad (5.2)$$

These equations imply that $\det B_{\mathbb{R}} = \lambda_1 \lambda_2 \lambda_3$ satisfies

$$\frac{d \det B_{\mathbb{R}}}{dt} = (\lambda_2 \lambda_3)^2 + (\lambda_1 \lambda_3)^2 + (\lambda_1 \lambda_2)^2 \geq 3(\det B_{\mathbb{R}})^{4/3}.$$

So $\det B_{\mathbb{R}}(t) \geq 1/(C - t)^3$ for some $C > 0$, and the solution of (5.2) blows up as $t \rightarrow t_{\max} < \infty$. Moreover, for any $i, j \in \{1, 2, 3\}$

$$\frac{d}{dt}(\lambda_i^2 - \lambda_j^2) = 0,$$

hence all $\lambda_i \rightarrow +\infty$ as $t \rightarrow t_{\max}$, $i \in \{1, 2, 3\}$ and $\lambda_i/\lambda_j \rightarrow 1$. It follows that $B_{\mathbb{R}}(t)$ pinches towards the (dual of the) Killing form:

$$B_{\mathbb{R}}(t)/|B_{\mathbb{R}}(t)|_{\infty} \rightarrow \langle \cdot, \cdot \rangle^{-1} = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3.$$

Example 5.3 (Diagonal Hopf surface). Diagonal Hopf surface is the quotient $M = (\mathbb{C}^2 \setminus (0, 0))/\Gamma$, where the generator of $\Gamma \simeq \mathbb{Z}$ acts as $(z^1, z^2) \mapsto (\mu z^1, \mu z^2)$ for some $\mu \in \mathbb{C}$ with $|\mu| > 1$. M is a compact complex manifold diffeomorphic to $S^3 \times S^1$. Note that the natural action of $SL(2, \mathbb{C})$ on \mathbb{C}^2 commutes with Γ , hence descends to the transitive action on M .

As in Example 5.2, any element $B_{\mathbb{R}} \in \text{Sym}^2(\mathfrak{su}(2))$ defines a metric g_0 on M . By the computation of Example 5.2 and Theorem 5.1, under the HCF this metric converges after normalization to the metric $g_{\infty} = p_*(\langle \cdot, \cdot \rangle)$.

In coordinates z^1, z^2 on $\mathbb{C}^2 \setminus \{0\} \subset \mathbb{C}^2$ this metric is given by

$$(g_{\infty})_{i\bar{j}} = \frac{1}{|z|^4} (\delta^{ij} |z|^2 + \bar{z}^i z^j).$$

Example 5.2 demonstrates the expected behavior of the ODE (5.1) for any complex Lie group G with a simple compact real form. Namely, assume that $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ is the complexification of a simple compact real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ with an invariant metric $\langle \cdot, \cdot \rangle$. Let us denote by

$$\kappa \in \text{Sym}^{1,1}(\mathfrak{g})$$

the element corresponding to $\langle \cdot, \cdot \rangle^{-1} \in \text{Sym}^2(\mathfrak{g}_{\mathbb{R}})$ (as in Remark 4.8). We refer to κ as the Killing form. For such G and κ we propose the following conjecture.

Conjecture 5.4. Let $B(t)$ be the solution to the ODE (5.1) on the maximal time interval $[0, t_{\max})$. Then there exists $\gamma \in G$, $\lambda \in \mathbb{R}$ such that $B(t)$ pinches towards $\lambda \text{Ad}_{\gamma}(\kappa)$:

$$B(t)/|B(t)|_{\infty} \rightarrow \lambda \text{Ad}_{\gamma}(\kappa), \quad t \rightarrow t_{\max}.$$

Let complex Lie group G and $\kappa \in \text{Sym}^{1,1}(\mathfrak{g})$ be as above. The following result demonstrates that the submersion metric induced by κ on a G -homogeneous manifold, is HCF-Einstein, *i.e.*, scale-static under the flow. This observation provides some evidence for Conjecture 5.4 to be true.

Theorem 5.5. Let G be the complexification of a simple compact Lie group. Let $M = G/H$ be a complex homogeneous manifold equipped with the Hermitian metric $g_0 = p_*(\kappa^{-1})$. Then g_0 is scale-static under the HCF, *i.e.*, $\Psi(g_0) = \lambda g_0$ for some positive constant λ .

Proof. In Example 4.6 we observed that for $B_{\mathbb{R}} = \langle \cdot, \cdot \rangle^{-1}$

$$B_{\mathbb{R}}^{\#} = \lambda B_{\mathbb{R}}.$$

Hence for $\kappa = \varphi(B_{\mathbb{R}})$ (see Remark 4.8) we have $\kappa^{\#} = \lambda \kappa$. This fact together with Theorem 5.1 imply that g_0 is scale static under the HCF. \square

6. Blow-up behavior of the ODE

In this section we study the HCF on a complex Lie group G , equipped with a submersion (= right-invariant) metric $g_0 \in \mathcal{M}^{\text{sub}}(G)$ (g_0 is identified with its

restriction to $\mathfrak{g} \simeq T_{\text{id}}G$). By Theorem 5.1 the HCF reduces to the ODE for $B(t) \in \text{Sym}^{1,1}(\mathfrak{g})$ with $B_0 = g_0^{-1}$

$$\begin{cases} \frac{dB}{dt} = B^\#, \\ B(0) = B_0. \end{cases} \quad (6.1)$$

It turns out, that the growth rate of a solution $B(t)$ is completely determined by the algebraic properties of the underlying Lie algebra. Namely, $B(t)$ has polynomial, exponential growth, or a finite time blow-up, depending on whether \mathfrak{g} is nilpotent, solvable, or admits a semisimple quotient (see Theorem 6.2, Theorem 6.3 and Remark 6.4).

Before we state and prove the results on the growth rate of a solution to (6.1) let us make the following elementary observation.

Proposition 6.1. *Let $B(t), D(t) \in \text{Sym}^{1,1}(\mathfrak{g})$ be the solutions to (6.1) with the initial conditions $B(0) = B_0$ and $D(0) = D_0$ such that $B_0 \geq D_0 \geq 0$. Then for $t \geq 0$*

$$B(t) \geq D(t).$$

Proof. We claim that for $B, D \in \text{Sym}^{1,1}(\mathfrak{g})$ if $B \geq D \geq 0$, then $B^\# \geq D^\#$. Indeed, let $\{e_i\}$ be a basis diagonalizing simultaneously B and D :

$$B = \sum a_i e_i \otimes \bar{e}_i, \quad k = \sum b_i e_i \otimes \bar{e}_i$$

with $a_i \geq b_i \geq 0$. Then $B^\# = \sum_{i,j} a_i a_j [e_i, e_j] \otimes \overline{[e_i, e_j]} \geq \sum_{i,j} b_i b_j [e_i, e_j] \otimes \overline{[e_i, e_j]} = D^\#$.

Now, fix a background positive-definite form $I \in \text{Sym}^{1,1}(\mathfrak{g})$, and define

$$\rho(t) = \sup_{\xi \in \mathfrak{g}^*, \ I(\xi, \bar{\xi})=1} (D(t)(\xi, \bar{\xi}) - B(t)(\xi, \bar{\xi})). \quad (6.2)$$

We have $\rho(0) \leq 0$, $D(t) \leq B(t) + \rho(t)I$. In particular, by the claim above, $D^\#(t) \leq B^\#(t) + \rho(t)I^\#B + \rho^2(t)I^\#$. As in the proof of Hamilton's maximum principle,

$$\frac{d\rho}{dt} \leq \sup(D^\#(t)(\xi, \bar{\xi}) - B^\#(t)(\xi, \bar{\xi})),$$

where the supremum is taken over all such ξ that in (6.2) the supremum is achieved. Therefore, on any fixed time interval with $|\rho(t)|$ bounded, we have

$$\frac{d\rho}{dt} \leq C_1 \rho(t) + C_2 \rho^2(t) \leq C |\rho(t)|$$

for some constant C . Hence $\rho(t) \leq 0$, provided $\rho(0) \leq 0$. \square

Theorem 6.2. *For a complex Lie algebra \mathfrak{g} and a positive definite Hermitian form $B_0 \in \text{Sym}^{1,1}(\mathfrak{g})$ let $B(t)$ be the solution to the ODE (6.1) on the maximal time interval $[0; t_{\max})$, $0 < t_{\max} \leq +\infty$. Then the following are equivalent:*

1. \mathfrak{g} is a nilpotent Lie algebra;
2. For any initial data B_0 , the solution $B(t)$ has at most polynomial growth, i.e., $t_{\max} = +\infty$, and there exists a polynomial p such that

$$B(t) < p(t)B_0;$$

3. For some initial data B_0 the solution $B(t)$ has subexponential growth, i.e., $t_{\max} = +\infty$, and for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that for $t > T_\epsilon$

$$B(t) < e^{\epsilon t} B_0.$$

Proof. We prove implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. By Ado's theorem for nilpotent Lie algebras [11] there exists a faithful representation of \mathfrak{g} into some $\mathfrak{gl}(V)$ such that \mathfrak{g} acts by nilpotent endomorphisms. With the use of the basic theory of Lie algebras [12, Section 3.3] one can assume that the image of this representation lies in $\mathfrak{n}(n)$ – the Lie algebra of strictly upper-triangular $n \times n$ matrices:

$$\rho: \mathfrak{g} \rightarrow \mathfrak{n}(n).$$

We extend ρ to a map $\rho: \text{Sym}^{1,1}(\mathfrak{g}) \rightarrow \text{Sym}^{1,1}(\mathfrak{n}(n))$ in the obvious way.

Let $\{E_{i,j} \mid 1 \leq i < j \leq n\}$ be the elementary matrices spanning $\mathfrak{n}(n)$. We fix a collection of positive real numbers $\{f_0^{(k)}\}_{k=1}^{n-1}$ such that the Hermitian form $f_0 \in \text{Sym}^{1,1}(\mathfrak{n}(n))$

$$f_0 := \sum_{1 \leq i < j \leq n} f_0^{(j-i)} E_{i,j} \otimes \overline{E_{i,j}}$$

is greater than $\rho(B_0)$. Consider the solution $f(t) \in \text{Sym}^{1,1}(\mathfrak{n}(n))$ to the ODE

$$\frac{df}{dt} = f^\#$$

with the initial condition $f(0) = f_0$. After expanding the definition of $f^\#$ we see that this ODE is equivalent to a system of $n - 1$ scalar equations

$$\frac{df^{(k)}}{dt} = \frac{1}{2} \sum_{j=1}^{k-1} f^{(j)} f^{(k-j)}, \quad k = 1, \dots, n-1.$$

Solving these equations inductively for $k = 1, \dots, n - 1$ we get $f^{(k)}(t) = p_{k-1}(t)$, where $p_{k-1}(t)$ is a polynomial of degree $(k - 1)$.

Hermitian forms $\rho(B(t))$ and $f(t)$ satisfy the same ODE with the initial conditions $\rho(B_0) < f_0$. Therefore Proposition 6.1 implies that $\rho(B(t)) \leq f(t)$. Since $f(t)$ has polynomial growth, we get

$$\rho(B(t)) < p(t) f_0$$

for some polynomial $p(t)$. Finally, using the fact that ρ is faithful and $B_0 \in \text{Sym}^{1,1}(\mathfrak{g})$ is positive definite, we find a constant C such that

$$B(t) < Cp(t)B_0.$$

As $B(t)$ is bounded on any interval $[0, t_{\max})$, the solution extends to the whole $[0, +\infty)$.

$2 \Rightarrow 3$. Is trivially true.

$3 \Rightarrow 1$. Assume that 1 does not hold, and \mathfrak{g} is not nilpotent. Then by Engel's theorem for some $x \in \mathfrak{g}$ the operator ad_x is not nilpotent. Hence the map $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ has non-zero eigenvalue λ :

$$[x, y] = \lambda y, y \neq 0.$$

Consider $f_0 := a_0x \otimes \bar{x} + b_0y \otimes \bar{y} \in \text{Sym}^{1,1}(\mathfrak{g})$. Note that $f_0^\# = |\lambda|^2 a_0 b_0 y \otimes \bar{y}$, hence for the functions $a(t), b(t)$ with $a(0) = a_0, b(0) = b_0$ satisfying

$$\frac{da}{dt} = 0, \quad \frac{db}{dt} = |\lambda|^2 ab$$

the form $f(t) = a(t)x \otimes \bar{x} + b(t)y \otimes \bar{y}$ solves the ODE (6.1). Explicitly these functions are given by $a(t) = a_0, b(t) = b_0 e^{|\lambda|^2 a_0 t}$. If positive numbers a_0, b_0 are small enough, one has $f_0 < B_0$, hence by Proposition 6.1 $f(t) < B(t)$. Therefore $B(t)$ cannot have subexponential growth. Contradiction. \square

Theorem 6.3. *For a complex Lie algebra \mathfrak{g} and a positive definite Hermitian form $B_0 \in \text{Sym}^{1,1}(\mathfrak{g})$, let $B(t)$ be the solution to the ODE (6.1) on the maximal time interval $[0; t_{\max})$, $0 < t_{\max} \leq +\infty$. Then the following are equivalent:*

1. \mathfrak{g} is a solvable Lie algebra;
2. For any initial data B_0 , the solution $B(t)$ has at most exponential growth, i.e., $t_{\max} = +\infty$, and there exist constants C, K such that

$$B(t) < C e^{Kt} B_0;$$

3. For some initial data B_0 , the solution $B(t)$ exists on $[0, +\infty)$, i.e., $t_{\max} = +\infty$.

Proof. The proof is essentially analogous to Theorem 6.2. We prove implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. By Ado's theorem there exists a faithful representation of \mathfrak{g} into some $\mathfrak{gl}(V)$, and by Lie's theorem one can assume that the image of this representation lies in $\mathfrak{b}(n)$ – the Borel subalgebra of $\mathfrak{gl}(V)$, consisting of upper-triangular $n \times n$ matrices:

$$\rho : \mathfrak{g} \rightarrow \mathfrak{b}(n).$$

Let $\{E_{i,j} | 1 \leq i \leq j \leq n\}$ be the elementary matrices spanning $\mathfrak{b}(n)$. We fix a collection of positive real numbers $\{f_0^{(k)}\}_{k=0}^{n-1}$ such that the Hermitian form $f_0 \in \text{Sym}^{1,1}(\mathfrak{b}(n))$

$$f_0 := \sum_{1 \leq i \leq j \leq n} f_0^{(j-i)} E_{i,j} \otimes \overline{E_{i,j}}$$

is greater than $\rho(B_0)$. Consider the solution $f(t) \in \text{Sym}^{1,1}(\mathfrak{b}(n))$ to the ODE (6.1) with the initial condition $f(0) = f_0$. After expanding the definition of $f^\#$ we see that this ODE is equivalent to a system of $n-1$ scalar equations

$$\frac{df^{(k)}}{dt} = \frac{1}{2} \sum_{j=0}^k f^{(j)} f^{(k-j)}, \quad k = 1, \dots, n-1$$

with $f^{(0)}(t) \equiv f_0^{(0)}$. Solving these equations inductively for $k = 1, \dots, n-1$ we get $f^{(k)}(t) = q_k(e^{f_0^{(0)} t})$, where $q_k(t)$ is a polynomial of degree k with $q_k(0) = 0$.

Hermitian forms $\rho(B(t))$ and $f(t)$ solve the same ODE with the initial conditions $\rho(B_0) < f_0$. Therefore Proposition 6.1 implies that $\rho(B(t)) \leq f(t)$. Since $f(t)$ has exponential growth, we get

$$\rho(B(t)) < C_0 e^{Kt} f_0$$

for some constants C_0, K . Finally, using the fact that ρ is faithful and $B_0 \in \text{Sym}^{1,1}(\mathfrak{g})$ is positive definite, we find a constant C such that

$$B(t) < C e^{Kt} B_0.$$

As $B(t)$ is bounded on any interval $[0, t_{\max})$, the solution extends to the whole $[0; +\infty)$.

2 \Rightarrow 3. Is trivially true.

3 \Rightarrow 1. Assume that 1 does not hold, and \mathfrak{g} is not solvable. Denote by $\text{Rad}(\mathfrak{g})$ the maximal solvable ideal. Then the quotient $\mathfrak{g}/\text{Rad}(\mathfrak{g})$ is semisimple Lie algebra and has a simple summand \mathfrak{g}_0 . So there is a surjective homomorphism onto a simple Lie algebra.

$$\rho: \mathfrak{g} \rightarrow \mathfrak{g}_0.$$

As in the set-up for Conjecture 5.4 let $\kappa \in \text{Sym}^{1,1}(\mathfrak{g}_0)$ be a positive-definite Hermitian form corresponding to the Killing metric of the compact real form of \mathfrak{g}_0 . Then according to Example 4.6 and Theorem 5.5 $\kappa^\# = \lambda \kappa$ for some $\lambda > 0$.

Now, let $B(t) \in \text{Sym}^{1,1}(\mathfrak{g})$ be a solution to the ODE (6.1) defined on $[0, +\infty)$. Choose $\epsilon_0 > 0$ such that $\epsilon_0 \kappa < \rho(B(0))$. If $\epsilon(t)$ satisfies the equation

$$\frac{d\epsilon}{dt} = \lambda \epsilon^2, \quad \epsilon(0) = \epsilon_0,$$

then $f(t) = \epsilon(t)\kappa$ solves the ODE (6.1) for $f(t) \in \text{Sym}^{1,1}(\mathfrak{g}_0)$ with the initial data $f(0) = \epsilon_0\kappa$. Explicitly we have

$$\epsilon(t) = \frac{\epsilon_0}{1 - \epsilon_0\lambda t}.$$

On the one hand we have the solution $f(t)$ to (6.1) blowing up at the finite time $t = (\epsilon_0\lambda)^{-1}$, on the other hand $f(0) < \rho(B(0))$, hence by Proposition 6.1 for any $t \geq 0$

$$f(t) < \rho(B(t)).$$

Contradiction with the finiteness of $B(t)$ for all $t \in [0, +\infty)$. \square

Remark 6.4. Lie algebra \mathfrak{g} is *not* solvable if and only if it admits a (semi)simple quotient $\mathfrak{g} \rightarrow \mathfrak{g}_0 \rightarrow 0$. Hence, Theorem 6.3 implies that the following statements about the solution $B(t)$ to the ODE (6.1) are equivalent:

1. \mathfrak{g} admits a (semi)simple quotient;
2. There exists $t_{\max} < +\infty$ such that $\lim_{t \rightarrow t_{\max}} \|B(t)\|_{B_0} = +\infty$.

Example 6.5 (Iwasawa manifold). Let G be the 3-dimensional complex Heisenberg group

$$G := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| a, b, c \in \mathbb{C} \right\}.$$

and $\Gamma \subset G$ its discrete subgroup, consisting of matrices with $a, b, c \in \mathbb{Z}[i]$. The quotient $M = G/\Gamma$ is the *Iwasawa manifold*. M is a compact complex parallelizable manifold. M does not admit any Kähler metric; in fact, since M is not formal, there is no complex structure on M admitting a Kähler metric.

The Lie algebra of G is $\mathfrak{g} = \text{span}(\partial_a, \partial_b, \partial_c) \simeq \mathfrak{n}(3)$. Consider $g_0 = p_*(B_0^{-1})$, where $B_0 \in \text{Sym}^{1,1}(\mathfrak{g})$. Denote by $B(t)$ the solution to the ODE (6.1). Theorem 6.2 provides an explicit expression for $B(t)$ and, in particular, implies that $B(t)$ polynomially blows up as $t \rightarrow \infty$. In fact, since $B^\#$ is proportional to $\partial_b \otimes \overline{\partial_b}$ for any $B \in \text{Sym}^{1,1}(\mathfrak{g})$, we see that $\partial_b \otimes \overline{\partial_b}$ is the only coordinate of $B(t)$, which blows up. For the solution $g(t) = p_*(B(t)^{-1})$ to the HCF this means that as $t \rightarrow \infty$

$$g(t)(\partial_b, \partial_b) \rightarrow 0, \quad g(t)|_{\text{span}(\partial_a, \partial_c)} \equiv g(0)|_{\text{span}(\partial_a, \partial_c)}.$$

To get a geometric picture, consider the Gromov-Hausdorff limit of $(G/\Gamma, g(t))$. It is easy to see that the projection onto coordinates a and c defines a holomorphic fibration

$$\pi: G/\Gamma \rightarrow \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i].$$

The fibers of π are the orbits of the flow generated by $\mathbb{C} \cdot \partial_c$. The limiting behavior of $g(t)$ implies that in the Gromov-Hausdorff limit, the fibers with the submersion

metric uniformly collapse to a point as $t \rightarrow +\infty$ and G/Γ collapses to the product of elliptic curves:

$$(G/\Gamma, g(t)) \xrightarrow{GH} (\mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i], g(0)|_{\text{span}(\partial_a, \partial_c)}).$$

Using the computations of Theorem 6.2 one can show that the HCF exhibits a similar behavior on all complex nilmanifolds of the form G/Γ , where G is a complex nilpotent group, and $\Gamma \subset G$ is a cocompact lattice.

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