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Squeezing functions and Cantor sets

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Abstract. We construct "large" Cantor sets whose complements resemble the unit disk arbitrarily well from the point of view of the squeezing function, and we construct "large" Cantor sets whose complements do not resemble the unit disk from the point of view of the squeezing function. Finally we show that complements of Cantor sets arising as Julia sets of quadratic polynomials have degenerate squeezing functions, despite of having Hausdorff dimension arbitrarily close to two.

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1. Introduction

Recently there have been many studies of the boundary behaviour of the squeezing function (see Section 2 for the definition and references) in one and several complex variables. In one complex variable there are two extremes:

(1) if $\gamma \subset b\Omega$ is an isolated boundary component of a domain Ω which is not a point, then
$$\lim_{\Omega \ni z \to \gamma} S_\Omega(z) = 1; \tag{1.1}$$

(2) if $\gamma \subset b\Omega$ is an isolated boundary component of a domain Ω which is a point, then
$$\lim_{\Omega \ni z \to \gamma} S_\Omega(z) = 0. \tag{1.2}$$

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This suggests studying the boundary behaviour of $S_\Omega(z)$ where $\Omega = \mathbb{P}^1 \setminus K$, and K is a Cantor set. In [1] Ahlfors and Beurling showed that there exist Cantor sets in $\mathbb{P}^1$ whose complements admit bounded injective holomorphic functions. In particular, such complements admit a non-degenerate squeezing function, and so this class of domains is nontrivial from the point of view of the squeezing function.

Our first result is the following:

Theorem 1.1. *For any $\epsilon > 0$ there exists a Cantor set $Q \subset I^2$ with 2-dimensional Lebesgue measure greater than $1-\epsilon$, such that*

$$\lim_{\Omega \ni z \to Q} S_\Omega(z) = 1, \tag{1.3}$$

and, moreover, $S_\Omega(z) \geq 1 - \epsilon$ for all $z \in \Omega$, where $\Omega = \mathbb{P}^1 \setminus Q$.

We also show that there exist Cantor sets with completely different behaviour.

Theorem 1.2. *There exists a Cantor set $Q \subset \mathbb{P}^1$ such that the following hold:*

(1) *for any point $x \in Q$ and any neighbourhood U of x we have that $U \cap Q$ has positive 2-dimensional Lebesgue measure;*
(2) *S_Ω achieves any value between zero and one on $U \cap \Omega$, where $\Omega = \mathbb{P}^1 \setminus Q$.*

Finally we show that certain Julia sets arising in one dimensional complex dynamics are Cantor sets which are degenerate from the point of view of the squeezing function, although they can have Hausdorff dimension arbitrarily close to two and thus their complements admit bounded holomorphic functions. Recall that a compact set of Hausdorff dimension strictly larger than one has strictly positive analytic capacity, hence its complement admits bounded holomorphic functions (see, *e.g.* [15, part (B) of Theorem 64, page 74]).

Theorem 1.3. *Let $f_c(z) = z^2 + c$ with $c \notin \mathcal{M}$. Then $\mathbb{P}^1 \setminus \mathcal{J}_c$ does not admit a bounded injective holomorphic function.*

Here, $\mathcal{J}_c$ denotes the Julia set for the function f_c, and $\mathcal{M}$ denotes the Mandelbrot set, so that J_c is a Cantor set if and only if $c \notin \mathcal{M}$.

Other Cantor sets of this type were constructed by Ahlfors and Beurling [1].

2. A "large" Cantor Set whose complement resembles the unit disk Proof of Theorem 1.1

We give some definitions. Let $\triangle \subset \mathbb{C}$ denote the unit disc, and let $B_r(p) \subset \mathbb{C}$ denote the disk of radius r centered at p. Let $\Omega \subset \mathbb{P}^1$ be a domain and let $x \in \Omega$. If $\varphi : \Omega \to \triangle$ is an injective holomorphic function such that $\varphi(x) = 0$, we set

$$S_{\Omega,\varphi}(x) := \sup\{r > 0 : B_r(0) \subset \varphi(\Omega)\}, \tag{2.1}$$

and we set

$$S_\Omega(x) := \sup_{\varphi}\{S_{\Omega,\varphi}(x)\}, \tag{2.2}$$

where the supremum is taken over all injective holomorphic functions $\varphi : \Omega \to \triangle$ such that $\varphi(x) = 0$. The function S_Ω is called the squeezing function. If the domain Ω does not admit any bounded injective holomorphic functions, then the squeezing function is called degenerate. The concept of squeezing function goes back to work by Liu-Sun-Yau, see [12,13] and S.-K-Yeung [17]. More recently, Deng-Guan-Zhang, see [2] initiated a basic study of the squeezing function. After that the squeezing function has been investigated by several authors, among them, Fornæss-Wold [7], Nikolov-Trybula-Andreev [14], Deng-Guan-Zhang [3], Joo-Kim [10], Kim-Zhang [11], Zimmer [18], Fornæss-Rong [5], Fornæss-Shcherbina [6], Diederich-Fornæss [4] and Fornæss-Wold [8]. We will introduce an auxiliary function R that will enable us to bound the squeezing function from below on the limit of a certain increasing sequence of domains. Let $\Omega \subset \mathbb{P}^1$ be a domain which admits an injective holomorphic map $\psi : \Omega \hookrightarrow \triangle$. Then for any point $x \in \Omega$ it is known (see, for example, Theorem 1 in [16]) that Ω also admits a circular slit map, that is an injective holomorphic map $\varphi : \Omega \hookrightarrow \triangle$ onto a circular slit domain S, such that $\varphi(x) = 0$. By definition, S is a circular slit domain if $\triangle \setminus S$ consists of arcs lying on concentric circles centred at the origin (the arcs may degenerate to points). If $x \in \Omega$, we let $\mathsf{Slit}_x(\Omega)$ denote the set of all circular slit maps that sends x to the origin. For a domain $\Omega \subset \mathbb{P}^1$ we define

$$R_\Omega(x) := \sup_{\varphi \in \mathsf{Slit}_x(\Omega)} \{S_{\Omega,\varphi}(x)\}. \tag{2.3}$$

Notice that by definition $R_\Omega \leq S_\Omega$.

Definition 2.1. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of domains in $\mathbb{P}^1$, and set $K_j := \mathbb{P}^1 \setminus \Omega_j$. We say that Ω_j converges *strongly* to a domain $\Omega \subset \mathbb{P}^1$ with $K := \mathbb{P}^1 \setminus \Omega$, if the compact sets K_j converge to K in the Hausdorff distance, and we write $\Omega_j \overset{s}{\to} \Omega$. If $x_j \in \Omega_j$ and if $x_j \to x \in \Omega$ we write $(\Omega_j, x_j) \overset{s}{\to} (\Omega, x)$.

Proposition 2.2. *Let $\Omega \subset \mathbb{P}^1$ be a finitely connected domain such that no boundary component of Ω is a point. Let $N \in \mathbb{N}$, and let $\{\Omega_j\}$ be a sequence of domains, where each Ω_j is m_j-connected with $m_j \leq N$. Assume that $(\Omega_j, x_j) \overset{s}{\to} (\Omega, x)$. Then $R_{\Omega_j}(x_j) \to R_\Omega(x)$.*

Proof. Let $K_1, ..., K_m$ denote the complementary components of Ω. Then for each $1 \leq k \leq m$ there is a *unique* slit map $\varphi_k : \Omega \to \triangle$ such that φ_k identifies K_k with $b\triangle$, $\varphi_k(x) = 0$ and $\varphi_k'(x) > 0$ (see, for example, Theorem 7 in [16]). In particular, $R_\Omega(x)$ is realised by one (or more) of these maps.

Similarly, each Ω_j has complementary components K_k^j for $1 \leq k \leq m_j$, and for the K_k^j's that are not points, there are unique slit maps φ_k^j identifying K_k^j with $b\triangle$, $\varphi_k^j(x_j) = 0$ and $(\varphi_k^j)'(x_j) > 0$.

After re-grouping to simplify notation, we may assume that there is a sequence $s_j \leq m_j$ such that the compact set $K_1^j \cup \cdots \cup K_{s_j}^j$ converges in the Hausdorff distance to a complementary component of Ω, say K_1, and groups of the other K_i^j's converge to the other complementary components of Ω. Since the diameter of K_1 is strictly positive, we may assume that there is a lower bound for the diameters of the sets $\{K_1^j\}$.

We first claim that there exists a constant $c > 0$ such that $(\varphi_1^j)'(x_j) > c$ for all j. Notice that, in view of Koebe's $\frac{1}{4}$-theorem, our claim implies that all slits are bounded away from zero. Assume by contradiction that there is such a sequence $(\varphi_1^j)'(x_j) \to 0$. For any convergent subsequence of the sequence (φ_1^j), the limit map is constantly equal to 0. Choose such a convergent subsequence and denote it still by (φ_1^j). After possibly having to pass to a subsequence, we may now choose a nontrivial loop $\tilde{\gamma} := bB_r(0)$, where $0 < r < 1$, which is contained in $\varphi_1^j(\Omega_j)$ for all j, such that the Kobayashi length of $\tilde{\gamma}$ in $\varphi_1^j(\Omega_j)$ is uniformly bounded from above.

Let U be any (small) open neighbourhood of K_1. Then for j sufficiently large, we have that $\varphi_1^j(\Omega_j \setminus U)$ is contained in the disk bounded by $\tilde{\gamma}$. Set $\tilde{\gamma}^j := (\varphi_1^j)^{-1}(\tilde{\gamma})$. Then since φ_1^j identifies K_1^j with $b\triangle$, we have that $\Omega_j \setminus U$ is on one side of $\tilde{\gamma}_j$ and K_1^j is on the other. Then the spherical lengths of the $\tilde{\gamma}_j$'s are bounded uniformly from below, since the diameters of the K_1^j's are bounded uniformly from below. But then the Kobayashi length of $\tilde{\gamma}^j$ in Ω_j goes to infinity, a contradiction. So we may extract a subsequence from φ_1^j converging uniformly on compact subsets of Ω to an injective map $\tilde{\varphi} : \Omega \to \triangle$.

We claim that $\tilde{\varphi}$ maps Ω onto a slit domain. First we show that the slits cannot close up to a circle of radius strictly less than one. Indeed, fix a compact set L in Ω. Then we can assume that the sequence converges uniformly on L. This implies that if there is a slit S of minimal radius $r < 1$ which closes up to a circle in the limit, then eventually all the images of L must be contained in the disc of radius r. Then arguing as above we can pick a circle of radius $r < s < 1$ so that on the preimages of the circle, the Kobayashi length is arbitrarily large. This is impossible.

We can assume that the slits converge. The complement of the limiting slits is connected. Pick any compact subset F of the complement of the limiting slits. Consider the inverse maps of the φ_k^j. This is a normal family. Indeed, we can remove a small disc around a point where the sequence φ_k^j is uniformly convergent. After this removal the family of inverses is a normal family. The limit map of the inverses is then the inverse of a slit map on Ω, which proves that the limit map $\tilde{\varphi}$ is a slit map.

So φ is, up to rotation, the unique slit map which identifies K_1 with $b\triangle$, and by choosing other complementary components than K_1 in the above construction, all the possible slit maps $\varphi_k : \Omega \to \triangle$ may be obtained as such limits. So we would arrive at a contradiction if we did not have $R_{\Omega_j}(x_j) \to R_\Omega(x)$. □

Lemma 2.3. *Let* $Q_j = [a_j, b_j] \times [c_j, d_j] \subset \mathbb{C}$ *be pairwise disjoint cubes for* $j = 1, ..., m$*, and set*

$$\Omega := \mathbb{P}^1 \setminus (Q_1 \cup \cdots \cup Q_m). \tag{2.4}$$

For each j*, set*

$$\Gamma_j := \{a_j + (1/2)(b_j - a_j)\} \times [c_j, d_j], \tag{2.5}$$

and for $k \in \mathbb{N}$ *denote by* $\Gamma_j(1/k)$ *the open* $\frac{1}{k}$*-neighborhood of* Γ_j*, and by* $Q^l_{j,k}$ *and* $Q^r_{j,k}$ *the left and right connected components of* $Q_j \setminus \Gamma_j(1/k)$ *respectively. Set*

$$\Omega_k := \mathbb{P}^1 \setminus (Q^l_{1,k} \cup Q^r_{1,k} \cup \cdots \cup Q^l_{m,k} \cup Q^r_{m,k}). \tag{2.6}$$

Then for any $\epsilon > 0$ *there exist* $\delta > 0$ *and* $N \in \mathbb{N}$ *such that, for all* $k \geq N$*,*

$$R_{\Omega_k}(z) \geq 1 - \epsilon \text{ if } z \in \Omega_k \cap Q_j(\delta) \text{ for some } j, \tag{2.7}$$

and

$$|R_{\Omega_k}(z) - R_\Omega(z)| < \epsilon \text{ if } z \notin Q_j(\delta) \text{ for all } j, \tag{2.8}$$

where $Q_j(\delta)$ *denotes the* δ*-neighborhood of* Q_j*.*

Proof. Since all cases are similar, to avoid notation, we prove (2.7) for $j = 1$. We may also assume that $Q_1 = [-1, 1] \cup [-1, 1]$.

For each $k \in \mathbb{N}$ there is a unique conformal map $\phi_k : \mathbb{P}^1 \setminus Q^l_{1,k} \cup Q^r_{1,k} \to \mathbb{P}^1$ such that the image is the complement of two closed disks B^1_k and B^2_k, normalized by the condition

$$\phi_k(z) = z + \sum_{j=1}^{\infty} a^k_j (1/z)^j \tag{2.9}$$

near infinity (see, e.g. [9, Theorem 2, page 237]). Then by uniqueness, $\phi_k(z) = -\phi_k(-z)$, so the two disks have the same size. Moreover, since each ϕ_k is normalized to have derivative one at infinity, the radii of the disks have to be bounded from above and from below: we can assume that the centers and the radii (in the spherical metric) converge. Indeed, by the Koebe 1/4-theorem the discs must all be in a bounded region of $\mathbb{C}$. Hence the radii are bounded above. Next we assume that the radii converge to 0. Let p, q denote the limits of the centers. Then the inverses are a normal family in the complement of the two points, hence the limit must be constant. This is only possible if $p, q = \infty$ contradicting the uniform boundedness of the discs.

So by scaling and rotation, we may then assume that

$$B^1_k = B_1(-1 - \delta_k) \text{ and } B^2_k = B_1(1 + \delta_k), \tag{2.10}$$

for some $\delta_k > 0$, where in general $B_r(p)$ denotes the disk of radius r centered at p (however, we have now possibly destroyed the normalization condition).

We now show that $\delta_k \to 0$. Otherwise, consider the circles $|z - (1 + \delta_k)| = 1 + \delta_k$. These have uniformly bounded Kobayashi length. However their preimage goes around one of the rectangles and passes between the rectangles, where the Kobayashi metric is arbitrarily large. Hence their Kobayashi length is unbounded, contradiction.

Since we may assume that the sequence $\{\phi_k\}$ converges to a conformal map, we get (2.7) from Lemma 2.4 below. And since all cases are similar, we conclude that (2.7) holds for any $j = 1, ..., m$. Finally (2.8) follows from Proposition 2.2. □

Lemma 2.4. *Set* $\Omega := \mathbb{P}^1 \setminus (\overline{B}_1(-1) \cup \overline{B}_1(1) \cup K)$ *be a domain, with* K *a compact set with finitely many connected components, disjoint from* $\overline{B}_1(-1) \cup \overline{B}_1(1)$. *Let* $\delta_j \searrow 0$, *and suppose that*

$$\Omega_j := \mathbb{P}^1 \setminus (\overline{B}_1(-1-\delta_j) \cup \overline{B}_1(1+\delta_j) \cup K_j) \tag{2.11}$$

is a sequence of domains such that $K_j \to K$ *with respect to Hausdorff distance, and such that the number of connected components of* K_j *is uniformly bounded. Then for any* $\epsilon > 0$ *there exists* $\eta > 1$ *such that* $R_{\Omega_j}(z) \geq 1 - \epsilon$ *for all* j *large enough such that* $\overline{B}_1(-1-\delta_j) \subset B_\eta(-1)$ *and* $\overline{B}_1(1+\delta_j) \subset B_\eta(1)$, *and for all* $z \in (B_\eta(-1) \cup B_\eta(1)) \cap \Omega_j$.

Proof. Assume to get a contradiction that there exist $\epsilon > 0$ and sequences $\eta_k \searrow 1$, $j_k \to \infty$, such that

$$\overline{B}_1(-1-\delta_{j_k}) \subset B_{\eta_k}(-1), \quad \overline{B}_1(1+\delta_{j_k}) \subset B_{\eta_k}(1)$$

and a sequence $z_k \in (B_{\eta_k}(-1) \cup B_{\eta_k}(1)) \cap \Omega_{j_k}$ such that $R_{\Omega_{j_k}}(z_k) < 1 - \epsilon$. We may assume that $\mathrm{Re}(z_k) \geq 0$ for all k.

Set $f_k(z) := z - (1+\delta_{j_k})$, $\Omega'_{j_k} := f_k(\Omega_{j_k})$, and $z'_k := f_k(z_k)$. Note that $1 < |z'_k| \leq 2\eta_k - 1$ and that $f_k(-2-\delta_{j_k}) = -3(1+(2/3)\delta_{j_k})$. Next set $g_k(z) := 1/z$, $\Omega''_{j_k} := g_k(\Omega'_{j_k})$, and $z''_k := g_k(z'_k)$. Then $|z''_k| \geq \frac{1}{2\eta_k - 1}$ and $|g_k(f_k(-2-\delta_{j_k}))| < 1/3$.

To sum up: Ω''_{j_k} is a domain obtained by removing a disk D_k and the compact set $g_k(f_k(K_{j_k}))$ from the unit disk, the point q_k on the boundary of D_k closest to the origin is of modulus less than one third, and there is a point $z''_k \in \Omega''_{j_k}$ with $|z''_k| \geq \frac{1}{2\eta_k - 1}$ for which $R_{\Omega''_{j_k}}(z''_k) < 1 - \epsilon$.

Clearly, the Poincaré distances between z''_k and q_k, and z''_k and $g_k(f_k(K_{j_k}))$, goes to infinity as $k \to \infty$, so if we set $\psi_k(z) := \frac{z - z''_k}{1 - \bar{z}''_k z}$, after possibly having to pass to a subsequence and in view of the following below sublemma, the domains $\psi_k(\Omega''_{j_k})$ converge to a simply connected domain with respect to strong convergence. Applying Proposition 2.2 and using one more time the following sublemma, this implies that $R_{\psi_k(\Omega''_{j_k})}(0) \to 1$ as $k \to \infty$ - a contradiction. □

Sublemma 2.5. We have that $\liminf_{k\to\infty} d_P(z''_k, D_k) > 0$, where d_P denotes the Poincaré distance.

Proof. Note that $\mathrm{Re}(z_k') \geq -1-\delta_{j_k}$, so that if we set $\gamma_k := \{z \in \mathbb{C} : \mathrm{Re}(z) = -1-\delta_{j_k}\}$, then any curve connecting z_k'' and D_k will have to pass through $\tilde{\gamma}_k = g_k(\gamma_k)$. So it is enough to find a lower bound for the Poincaré distance between D_k and $\tilde{\gamma}_k$ for large k. Now the real points on $\tilde{\gamma}_k$ are 0 and $\frac{1}{-1-\delta_{j_k}}$, and the real points on bD_k are $\frac{1}{-1-2\delta_{j_k}}$ and $\frac{1}{-3(1+(2/3)\delta_{j_k})}$, and using the fact that Poincaré disks are Euclidean disks, it suffices to control the distance between $\frac{1}{-1-\delta_{j_k}}$ and $\frac{1}{-1-2\delta_{j_k}}$, and between 0 and $\frac{1}{-3(1+(2/3)\delta_{j_k})}$. The last distance is clearly bounded away from zero, so we compute the first. We have that

$$\lim_{k\to\infty} \log \frac{1+\frac{1}{1+\delta_{j_k}}}{1-\frac{1}{1+\delta_{j_k}}} - \log \frac{1+\frac{1}{1+2\delta_{j_k}}}{1-\frac{1}{1+2\delta_{j_k}}} = \lim_{k\to\infty} \log \frac{1-\frac{1}{1+2\delta_{j_k}}}{1-\frac{1}{1+\delta_{j_k}}} = \log 2. \qquad \square$$

Lemma 2.6. *Let $\Omega \subset \mathbb{P}^1$, $x \in \Omega$, and suppose that $\mathbb{P}^1 \setminus \Omega$ contains at least three points. Suppose that $\Omega_j \overset{s}{\to} \Omega$. Then*

$$r := \limsup_{j\to\infty} S_{\Omega_j}(x) \leq S_\Omega(x).$$

Proof. If $r = 0$ this is clear, so we assume that $r > 0$. Then, after possibly having to pass to a subsequence, there exists a sequence $\varphi_j : \Omega_j \to \triangle$ of embeddings, $\varphi_j(x) = 0$, $B_{r_j}(0) \subset \varphi_j(\Omega_j)$, $r_j \to r$. Let $a_j \in \mathbb{P}^1$ be distinct points such that $a_i \notin \Omega$ for $i = 1, 2, 3$. For any $\delta > 0$ the ball $B_\delta(a_i)$ is not contained in Ω_j for all j large enough. So we may fix $0 < \delta << 1$, and assume that there exist points $a_i^j \in B_\delta(a_i)$ such that $a_i^j \notin \Omega_j$ for all j and for $i = 1, 2, 3$. Since there is a compact family of Möbius transformations mapping the triples $\{a_1^j, a_2^j, a_3^j\}$ to the triple $\{a_1, a_2, a_2\}$, and since the complement of three points is Kobayashi hyperbolic, we may assume that for all $0 < r' < r$ the sequence $\varphi_j^{-1}|_{B_{r'}(0)}$ is convergent. Hence the derivatives of $\varphi_j(x)$ are uniformly bounded below and above. Therefore we can assume that the φ_j converge to an injective holomorphic map from Ω to $\triangle$. Moreover the image contains the disc of radius r. $\square$

Proof of Theorem 1.1. Set $Q^1 = I^2$. By alternating Lemma 2.3 and its horizontal analogue we obtain a decreasing sequence Q^j of disjoint unions of cubes, such that

(1) $Q := \cap_{j\geq 1} Q^j$ is a Cantor set with 2-dimensional Lebesgue measure arbitrarily close to one,
(2) $R_{\mathbb{P}^1\setminus Q^j} \geq 1-\epsilon$, and
(3) For any sequence (z_j) in $\mathbb{P}^1 \setminus Q$ converging to Q and any $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $R_{\mathbb{P}^1\setminus Q^i}(z_j) > 1-\delta$ whenever $j \geq N$ and i is large enough (depending on j).

By (1) the two-dimensional Lebesgue measure of Q can be arbitrarily close to one.
By (2) and by Lemma 2.6 it follows that $S_{\mathbb{P}^1\setminus Q}(z) \geq 1-\epsilon$.
By (3) and by Lemma 2.6 it follows that $S_{\mathbb{P}^1\setminus Q}(z_j) \geq 1-\delta$ for all $j \geq N$, and hence $\lim_{\mathbb{P}^1\setminus Q \ni z\to Q} S_{\mathbb{P}^1\setminus Q}(z) = 1$. $\square$

3. A "large" Cantor Set whose complement does not resemble the unit disk Proof of Theorem 1.2

We modify the construction in the previous section. For an inductive construction, assume that we have constructed a family $\mathcal{Q}^j := \{Q^j_1, \dots, Q^j_{m(j)}\}$ of $m(j)$ disjoint cubes. We may choose $m(j)$ closed loops Γ^j_i, each surrounding and being so close to one of the cubes, that $S_{\Omega_j}(z) \geq 1 - 1/j$ if $z \in \Gamma^j_i$ for some $1 \leq i \leq m(j)$, where $\Omega_j = \mathbb{P}^1 \setminus \cup_i Q^j_i$. Further we may choose a finite number of points p^j_1, ..., $p^j_{k(j)}$ in Ω_j such that we find a point p^j_ℓ in any $1/j$-neighbourhood of any point in $b\Omega_j$, and such that $S_{\Omega'_j}(z) \geq 1 - 2/j$ if $z \in \cup_{1 \leq i \leq m(j)} \Gamma^j_i$, where we denote $\Omega'_j := \Omega_j \setminus \cup_{1 \leq \ell \leq k(j)} \{p^j_\ell\}$. The reason is that the removal of a set sufficiently close to the boundary of a domain, will essentially not disturb a lower bound for neither S nor R.

Then by Lemma 3.2 below and Proposition 2.2 we may choose an arbitrarily small $\delta_j > 0$ and arbitrarily small cubes $\tilde{Q}^j_\ell \subset \{|z - p^j_\ell| < \delta_j\}$ such that

(i) $S_{\Omega''_j}(z) \leq 1/j$ if $|z - p^j_\ell| = \delta_j$ for some $1 \leq \ell \leq k(j)$;
(ii) $S_{\Omega''_j}(z) \geq 1 - 3/j$ if $z \in \Gamma^j_i$ for some $1 \leq i \leq m(j)$,

where we denote $\Omega''_j := \Omega_j \setminus \cup_\ell \tilde{Q}^j_\ell$. By applying Lemma 2.3 twice we may divide each cube in the collection

$$\{Q^j_1, \dots, Q^j_{m(j)}, \tilde{Q}^j_1, \dots, \tilde{Q}^j_{k(j)}\}$$

into four, creating a new collection of cubes $\mathcal{Q}^{j+1}$ such that

(i) $S_{\Omega_{j+1}}(z) \leq 2/j$ if $|z - p^j_\ell| = \delta_j$ for some $1 \leq \ell \leq k(j)$;
(ii) $S_{\Omega_{j+1}}(z) \geq 1 - 4/j$ for $z \in \Gamma^j_i$ for some $1 \leq i \leq m(j)$,

where Ω_{j+1} denotes the complement of the cubes in $\mathcal{Q}^{j+1}$. The inductive step may be repeated indefinitely so as to ensure that for all $k > 1$ we still have that

(i′) $S_{\Omega_{j+k}}(z) < 3/j$ for $|z - p^j_\ell| = \delta_j$ for some ℓ;
(ii′) $S_{\Omega_{j+k}}(z) > 1 - 5/j$ for $z \in \Gamma^j_i$ for some i.

We now define

$$Q := \limsup_{j \to \infty} \bigcup_{Q^j_l \in \mathcal{Q}^j} Q^j_l.$$

If in each step of the construction the points p^j_i were chosen close enough to each of the previously constructed cubes, it follows that any connected component of Q must be a point (since the diameters of the cubes go to zero), and no point will be isolated. Hence Q is a Cantor set. It follows from Lemma 3.2 that we may arrange that statement corresponding to (i') holds in the limit. The statement corresponding to (ii') holds in the limit by Lemma 2.6 since Ω_j converges strongly to $\mathbb{P}^1 \setminus Q$.

Lemma 3.1. *Let $K \subset \mathbb{P}^1$ be a compact set such that $\Omega = \mathbb{P}^1 \setminus K$ admits a bounded injective holomorphic function. Let $p_1, ..., p_m \in \Omega$ be distinct points, and set $\Omega' = \Omega \setminus \{p_1, ..., p_m\}$. Then*

$$\lim_{\Omega' \ni z_j \to p_j} S_{\Omega'}(z) = 0, \tag{3.1}$$

for $j = 1, ..., m$.

Proof. We consider p_1. Assume to get a contradiction that there exists a sequence $\Omega' \ni z_j \to p_1$ and injective holomorphic maps $\varphi_j : \Omega' \to \triangle, \varphi_j(z_j) = 0$, and $B_r(0) \subset \varphi_j(\Omega')$ for some $r > 0$. All maps extend holomorphically across $p_1, ..., p_m$, and we may extract a subsequence converging to a limit map φ, with $\varphi(p_1) = 0$. Since $|\varphi(p_1) - \varphi(z_j)| \to 0$ as $j \to \infty$, this leads to a contradiction. □

Lemma 3.2. *Let $K \subset \mathbb{P}^1$ be a compact set such that $\Omega = \mathbb{P}^1 \setminus K$ admits a bounded injective holomorphic function. Let $p_1, ..., p_m \in \Omega$ be distinct points, and let $\epsilon > 0$. Then there exist $\delta_1 > 0$ (arbitrarily small) and $0 < \delta_2 << \delta_1$, such that for any domain $\Lambda \subset \mathbb{P}^1$ with $\mathbb{P}^1 \setminus \Lambda \subset K(\delta_2) \cup (\cup_{j=1}^m B_{\delta_2}(p_j))$ (with at least one complementary component in each $B_{\delta_2}(p_j)$), we have that $S_\Lambda(z) < \epsilon$ for all $z \in \Lambda$ with $|z - p_j| = \delta_1$ for some j. Here $K(\delta_2)$ denotes the δ_2-neighbourhood of K.*

Proof. Let $0 < \mu << 1$ (to be determined). Fix $\delta_1 > 0$ such that the Kobayashi length in $\Omega' = \Omega \setminus \{p_1, ..., p_m\}$ of each loop $|z - p_j| = \delta_1$ is strictly less than μ. Let $f_\theta : \triangle \to \Omega'$ be a continuous family of universal covering maps with $f_\theta(0) = p_1 + \delta_1 e^{i\theta}$. Then the Kobayashi metric $g_K^{\Omega'}(p_1 + \delta_1 e^{i\theta})$ is equal to $1/|f_\theta'(0)|$. Fix any $0 < r < 1$. Then for any domain $\Omega'' \subset \mathbb{P}^1$ that covers the union $\cup_\theta f_\theta(\overline{\triangle_r})$, which is a compact subset of Ω', we have that $g_K^{\Omega''}(p_1 + \delta_1 e^{i\theta})$ is bounded from above by $1/|r \cdot f_\theta'(0)|$. So for r sufficiently close to 1 the Kobayashi length of the loop $|z - p_1| = \delta_1$ in Ω'' is less than μ for any such domain. The same argument may be applied to all points p_j.

Now for any such domain Ω'' we estimate the squeezing function with respect to μ. Write $S_{\Omega''}(p_j + \delta_1 e^{i\theta}) = s$, let $g : \Omega'' \to \triangle$ be a map that realises the squeezing function at $p_j + \delta_1 e^{i\theta}$, and let Γ_j denote the loop $|z - p_j| = \delta_1$. Then, since $g(\Gamma_j)$ is a nontrivial loop in $g(\Omega'')$ we have that $l_K(\Gamma_j) \geq \log(\frac{1+s}{1-s})$. Then $s \leq \frac{e^\mu - 1}{e^\mu + 1} \to 0$ as $\mu \to 0$, and so the lemma follows. □

4. Julia sets for quadratic polynomials – Proof of Theorem 1.3

Fix a quadratic polynomial $f_c(z) = z^2 + c$ and assume that $c \notin \mathcal{M}$, where $\mathcal{M}$ denotes the Mandelbrot set. Then the critical point 0 is in the basin of attraction of infinity Ω_∞, and the Julia set $\mathcal{J}_c = \mathbb{P}^1 \setminus \Omega_\infty$ is a Cantor set. We let $G_c(z)$ denote the negative Green's function associated to f_c. It satisfies the following properties:

(1) G_c is continuous on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus \mathcal{J}_c$;
(2) $G_c(z) = -\log|z| + O(1)$ near ∞;
(3) $G_c(f^n(z)) = 2^n G_c(z)$ for all $z \in \mathbb{C}$.

We regard G_c as an exhaustion function of Ω_∞. Let $t_0 = G_c(0)$. The exhaustion may be described as follows. For $t < t_0$ the level sets $\Gamma_t = \{G_c = t\}$ are smooth connected embeddings of S^1, shrinking around infinity as t decreases to $-\infty$. Considering the picture in $\mathbb{C}$, as t increases to t_0, the family Γ_t is a decreasing family of embedded S^1's, decreasing to Γ_{t_0}, which is a figure eight, the origin being the figure eight crossing point. In general, the level sets $\Gamma_{2^{-n}t_0}$ consists of 2^n pairwise disjoint figure eights, and for $2^{-n}t_0 < t < 2^{-n+1}t_0$ the level set Γ_t consists of 2^{n+1} disjoint smoothly embedded copies of S^1, one contained in each hole of a figure eight in $\Gamma_{2^{-n}t_0}$.

We now assume to get a contradiction that there exists a bounded holomorphic injection $\varphi : \Omega_\infty \to \triangle$, and we may assume that $\varphi(\infty) = 0$. We will first use the exhaustion just described to get a description of $\varphi(\Omega_\infty)$ that will allow us to modify φ in a useful way. Set $H = G_c \circ \varphi^{-1}$, defined on $\varphi(\Omega_\infty)$.

Start by choosing $s_0 << 0$ and let D_0 be the disk bounded by $\gamma_{s_0} = \{H = s_0\}$, a single closed loop. Increasing s between s_0 and t_0 we get an increasing family of single loops γ_s, but when s crosses the critical value t_0 it breaks into two loops, say $\gamma^1_{s_1}, \gamma^2_{s_1}$, for s close to t_0. One of these loops is going to enclose the other, and we relabel it γ_{s_1}. Next, increasing s between s_1 and $2t_0$ we follow a path of loops starting from γ_{s_1}, until s crosses $2t_0$, and it again breaks into two loops, say $\gamma^1_{s_2}$ and $\gamma^2_{s_2}$ for s_2 close to $2t_0$. Again, single out the one enclosing the other, and relabel it γ_{s_2}. Continuing in this fashion, we obtain a family of loops γ_{s_j} such that γ_{s_j} encloses $\gamma_{s_{j-1}}$, and such that the disk D_j bounded by γ_{s_j} contains the whole sublevel set $\{H < s_j\}$. We have that $\{D_j\}$ is an increasing family of disk, we denote by D its increasing union, and we let $\psi : D \to \triangle$ be the Riemann map satisfying $\psi(0) = 0$, $\psi'(0) > 0$. Our modified map will be $\tilde{\varphi} := \psi \circ \varphi$.

Next we will use the map f_c to find some other loops $\tilde{\gamma}_j$ in Ω_∞, each one in the same free homotopy class as $\varphi^{-1}(\gamma_{s_j})$. Start by defining $\tilde{\gamma}_0$ as the level set $G_c = t$ for some $t < t_0$ close to t_0. Then $f_c^{-1}(\tilde{\gamma}_0)$ consists of two disjoint loops, one of them free homotopic to $\varphi^{-1}(\gamma_{s_1})$. Single this out, and label it $\tilde{\gamma}_1$. Next $f_c^{-1}(\tilde{\gamma}_1)$ consists of two disjoint loops, and one of them is free homotopic to $\varphi^{-1}(\gamma_{s_2})$. Single it out, and denote it by $\tilde{\gamma}_2$. Continue in this fashion indefinitely.

We are now ready to reach the contradiction. On the one hand, since the family $\tilde{\varphi}(\tilde{\gamma}_j)$ will increase towards $b\triangle$, it follows that the Kobayashi lengths of $\tilde{\gamma}_j$ in Ω_∞ will increase towards infinity. On the other hand, let $C \subset \Omega_\infty$ denote the forward and backward orbit of the critical point 0. Then the Kobayashi length of each $\tilde{\gamma}_j$ in $\Omega_\infty \setminus C$ is longer than the Kobayashi length in Ω_∞. But $f_c : \Omega_\infty \setminus C \to \Omega_\infty \setminus C$ is a covering map, and so the Kobayashi lengths of all the $\tilde{\gamma}_j$'s in $\Omega_\infty \setminus C$ are the same. A contradiction.

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