Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. V (2006), 137-157

Multivariate moment problems: Geometry and indeterminateness

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Abstract. The most accurate determinateness criteria for the multivariate moment problem require the density of polynomials in a weighted Lebesgue space of a generic representing measure. We propose a relaxation of such a criterion to the approximation of a single function, and based on this condition we analyze the impact of the geometry of the support on the uniqueness of the representing measure. In particular we show that a multivariate moment sequence is determinate if its support has dimension one and is virtually compact; a generalization to higher dimensions is also given. Among the one-dimensional sets which are not virtually compact, we show that at least a large subclass supports indeterminate moment sequences. Moreover, we prove that the determinateness of a moment sequence is implied by the same condition (in general easier to verify) of the push-forward sequence via finite morphisms.

Mathematics Subject Classification (2000): 44A60 (primary); 14P05 (secondary).

1. Introduction

Let μ be a positive measure on the real line, rapidly decreasing at infinity. The asymptotic expansion of the associated Markov function

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t-z} \approx -\frac{a_0}{z} - \frac{a_1}{z^2} - \cdots, \quad \mathrm{Im(z)} > 0,$$

and the uniquely determined continued fraction development

$$-\frac{a_0}{z} - \frac{a_1}{z^2} - \ldots = -\cfrac{a_0}{z - \alpha_0 - \cfrac{\beta_0^2}{z - \alpha_1 - \cfrac{\beta_1^2}{z - \alpha_2 - \cfrac{\beta_2^2}{\ddots}}}}, \quad \alpha_k, \beta_k \in \mathbb{R},$$

The first author was partially supported by the National Science Foundation Grant DMS-0350911. The second author was partially supported by the European RTNetwork RAAG, HPRN-CT-2001-00271.
Received November 28, 2005; accepted May 26, 2006.

depend solely on the sequence of moments

$$a_k = \int_{\mathbb{R}} t^k \mathrm{d}\mu(t), \quad k \geq 0.$$

The moment sequence (a_k) is called *determinate* if there exists a unique representing positive measure. This corresponds precisely to the convergence of the associated continued fraction at non-real points z. Originally, all determinateness criteria were obtained via very laborious, and often ad hoc, computations involving continued fractions; for an excellent presentation of these aspects see the monograph by Perron [15]. The progress in function theory and early functional analysis has provided a better grasp of this uniqueness problem (*cf.* the works of Nevanlinna and Carleman), but it was only when von Neumann's spectral theory of unbounded symmetric operators was completed that determinateness was understood in simple and efficient terms, see Akhiezer's monograph [1]. For instance, it is well known today that the determinateness of the moment sequence (a_k) cannot be characterized by a growth condition of the a_k's, but rather via indirect approximation properties, such as M. Riesz' density criterion: $(1 + x^2)\mathbb{R}[x]$ is dense in the Hilbert space completion of the polynomials $\mathbb{R}[x]$ with respect to the integration functional defined by the moments.

Much less is known in the case of moments of positive measures acting on polynomials of several variables. Riesz' density criterion has several multivariate counterparts, well exposed in Fuglede's article [7]; however, they provide only sufficient uniqueness conditions. The best numerical uniqueness conditions are nowadays obtained via rather evolved functional analytic methods (*cf.* [6, 12, 14]), but again they remain far from being also necessary.

Our note starts with the simple and old observation that, in order to have uniqueness in the multivariate moment problem (with prescribed supports), one has to enlarge the polynomial ring to an algebra having sufficiently many elements, so that they separate all measures rapidly decreasing at infinity; and secondly one has to keep track in this process of the possible positive extensions of the integration functional. In practical terms, we show that if a well-chosen, single non-polynomial function (whose inverse serves as a general denominator in the extended ring of functions) is approximable (in the canonically attached L^2-space) by polynomials, then the uniqueness of the representing measure follows. This criterion has a well known parallel in M. Riesz' work on one-dimensional moment problems [18].

The main theme of our note is derived from the above observation and can roughly be stated as: *The geometry of the support of a measure in $\mathbb{R}^d$ affects, and can be used to test, its determinateness.* Apparently this phenomenon was not identified before.

The contents is the following. In Section 2 we prove a general uniqueness criterion, based on the approximation by polynomials of a single external function. This will guide the rest of the article. Section 3 contains the main results, showing in particular that a moment problem supported by a one-dimensional set K is necessarily determinate, provided that K is virtually compact. (See Remark 3.4 for the notion of virtual compactness.) We also prove that determinateness on affine curves can be checked on the push-forward measure via a finite covering. In Section 4, using the direct image morphism (of integration along the fibers of a finite map) we show on the other hand that large classes of non-virtually compact sets of dimension one carry indeterminate moment sequences. In Section 5 we go back to the founders of the theory of moments (Stieltjes and Markov), to one of the first growth conditions ensuring the determinateness of a moment sequence. We adapt their ideas, via elementary computations, to the multivariate setting. Section 6 contains a variety of examples, illustrating on one hand the limits and on the other hand the flexibility and universality of our geometric study.

ACKNOWLEDGEMENTS. The present work was started while the first author was Visiting Professor at the University of Konstanz. He warmly thanks his hosts for hospitality and excellent working conditions.

2. Preliminaries

Throughout this note we adopt the following notation. If X is an affine $\mathbb{R}$-variety, then $\mathbb{R}[X]$ is its $\mathbb{R}$-algebra of regular (that is, polynomial) functions, and $X(\mathbb{R})$ is the set of $\mathbb{R}$-points of X. If $f\colon X \to Y$ is a morphism of affine $\mathbb{R}$-varieties, we often write $f^*\colon \mathbb{R}[Y] \to \mathbb{R}[X]$ for the associated homomorphism of $\mathbb{R}$-algebras.

The real coordinates in affine space are denoted by $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, and $\mathbb{R}[x]$ stands for the polynomial algebra.

We write in short $a(\mu) = (a_\alpha)_\alpha$ for the multi-sequence of moments of a positive measure μ on $\mathbb{R}^d$:

$$a_\alpha = \int x^\alpha \mathrm{d}\mu(x), \quad \alpha \in \mathbb{Z}_+^d.$$

The linear functional associated to μ is written

$$L_\mu\colon \mathbb{R}[x] \to \mathbb{R}, \quad L_\mu(p) = \int p(x)\mathrm{d}\mu(x).$$

(By using this notation we tacitly imply that all moments exist and are finite.) If μ is a positive measure on $X(\mathbb{R})$, where X is an affine $\mathbb{R}$-variety, we may consider L_μ as a linear functional $\mathbb{R}[X] \to \mathbb{R}$.

Let K be a closed subset of $X(\mathbb{R})$, and let $L\colon \mathbb{R}[X] \to \mathbb{R}$ be a linear functional which is representable by some positive Borel measure μ on K:

$$L(p) = \int_K p(x)\mu(\mathrm{d}x)$$

for all $p \in \mathbb{R}[X]$. (Recall that this holds for the given L if and only if $L(p) \geq 0$ for every $p \in \mathbb{R}[X]$ which is non-negative on K, by Haviland's theorem [10].) We will say that the moment problem defined by L on K is *determinate* (or that the K-moment problem defined by L is determinate) if μ is uniquely determined by L. Otherwise, the moment problem defined by L on K is said to be *indeterminate*.

We stress that the question of determinateness depends not only on L, but also on the set K, in general. It may happen that L has a unique representing measure on K, but has more than one representing measures on a larger closed set, that is, on $X(\mathbb{R})$. Therefore, it will be necessary to be clear about the underlying set K when we speak about determinateness questions. Note also that the Zariski closure of the support of any representing measure does not depend on the measure, but only on the moment functional L.

The following technical criterion will be useful:

Proposition 2.1. *Let K be a closed subset of $\mathbb{R}^n$, and let μ and ν be positive Borel measures on K satisfying $L_\mu = L_\nu$. Let $f \in \mathcal{C}(K, \mathbb{R})$ be a function satisfying $f \geq 1$, a.e. on K.*[1]

Assume that there exists a sequence of polynomials p_n in $\mathbb{R}[x]$ such that $p_n \to \frac{1}{f}$ under the norm $||\cdot|| = ||\cdot||_{2,\mu} + ||\cdot||_{2,\nu}$, and let

$$A_0 := A_0(K, f) := \left\{ \frac{p}{f^k} : \ p \in \mathbb{R}[x], \ k \geq 0, \ \frac{p(x)}{f(x)^k} \to 0 \ \textit{for} \ |x| \to \infty, \ x \in K \right\}.$$

If A_0 separates the points of K, then $\mu = \nu$.

Proof. All fractions $\frac{p}{f^k}$ (with $p \in \mathbb{R}[x]$ and $k \geq 0$) are integrable with respect to both μ and ν. We prove by induction on k that

$$\int \frac{p}{f^k} \mathrm{d}\mu = \int \frac{p}{f^k} \mathrm{d}\nu.$$

For $k = 0$ this holds by assumption. For the induction step $k \to k+1$ note that

$$\frac{p}{f^{k+1}} = \lim_{n \to \infty} \left(p_n \cdot \frac{p}{f^k} \right),$$

both in $L^1(\mu)$ and $L^1(\nu)$ (since $||fg||_1^2 = \langle |f|, |g| \rangle^2 \leq ||f||_2^2 \cdot ||g||_2^2$ by the Cauchy-Schwarz inequality), and hence

$$\int \frac{p}{f^{k+1}} \mathrm{d}\mu = \int \frac{p}{f^{k+1}} \mathrm{d}\nu.$$

Let B denote the subalgebra of $\mathcal{C}(K, \mathbb{R})$ consisting of the functions ϕ for which the limit of $\phi(x)$, for $x \in K$ and $|x| \to \infty$, exists in $\mathbb{R}$. So $B = \mathcal{C}(K^+, \mathbb{R})$ where

[1] Almost everywhere with respect to both μ and ν. In practice we will have $f \geq 1$ throughout.

$K^+ = K \cup \{\infty\}$ is the one-point compactification of K. (If K is already compact, put $K^+ = K$.) By assumption, the subalgebra $A := \mathbb{R}1 \oplus A_0$ of B separates the points of K^+. Therefore, by the Stone-Weierstraß theorem, A is dense in B under uniform convergence. Therefore the measures μ and ν coincide as linear functionals on B, because if $b \in B$ and a_n in A with $a_n \to b$ under $||\cdot||_\infty$, then $\int_K a_n \to \int_K b$ for both μ and ν since $\mu(K) = \nu(K) < \infty$. It follows that $\mu = \nu$. (Recall that on a σ-compact, locally compact, metrizable space X, a Borel measure is determined by the integrals of continuous functions with compact support, see for instance [2]). □

Remarks 2.2. Here are some particular situations where this proposition will be useful:

1. Assume we have $f \in \mathbb{R}[x]$ with $f \geq 1$ on K for which there is a sequence p_n in $\mathbb{R}[x]$ with $||1 - fp_n||_{L,2} \to 0$. If $A_0(K, f)$ separates the points of K, then the moment problem on K given by L is determinate. (Indeed, $||\frac{1}{f} - p_n||_2 \leq ||1 - fp_n||_2$ since $f \geq 1$.) Note that, contrary to the preceding proposition, the condition $||1 - fp_n||_{L,2} \to 0$ is intrinsic in L and its values on polynomials.

2. Assume that an entire function is given in form of its Taylor series

$$\frac{1}{f} = \sum_\alpha c_\alpha x^\alpha.$$

Assume that the algebra $A_0(K, f)$ fulfills the separation condition in the statement of the Proposition, and let μ be a positive measure on K with moments (a_α). The normal convergence condition

$$\limsup_\alpha \big(|c_\alpha| \cdot a_{2\alpha}\big)^{1/|\alpha|} < 1$$

will assure that the partial sums converge to $1/f$ in $L^2(\mu)$, and hence the above result is applicable. Section 5 contains an illustration in this respect.

3. There are natural choices of continuous functions $f \geq 1$ for which $A_0(K, f)$ separates the points of K, like $f = 1 + \sum_i x_i^2$ or $f = e^{x^2}$.

4. On the real line ($d = 1$) one has a strong converse, going back to the work of M. Riesz [18] : The moment problem given by L is known to be determinate if and only if there exists a sequence of polynomials p_n for which $||1 - (1 + x^2)p_n||_{L,2} \to 0$. See also [1] Section 2.5.

3. Main results

We start by recording an obvious way of producing new indeterminate moment problems from given ones:

Lemma 3.1. *Let $\varphi\colon X \to X'$ be a morphism of affine $\mathbb{R}$-varieties, let K be a closed subset of $X(\mathbb{R})$ and K' a closed subset of $X'(\mathbb{R})$ with $\varphi(K) \subset K'$. If $\varphi|_K$ is injective, and if there exists an indeterminate K-moment problem on X, then there exists an indeterminate K'-moment problem on X'.*

Proof. If μ and ν are positive Borel measures on $X(\mathbb{R})$ which have the same moments, then the direct image measures $\varphi_*\mu$ and $\varphi_*\nu$ on $X'(\mathbb{R})$ have the same moments as well, since

$$\int_{X'(\mathbb{R})} q(x')(\varphi_*\mu)(\mathrm{d}x') = \int_{X(\mathbb{R})} (\varphi^* q)(x)\mu(\mathrm{d}x), \quad q \in \mathbb{R}[X'],$$

where $\varphi^*\colon \mathbb{R}[X'] \to \mathbb{R}[X]$ is the homomorphism associated to φ. Since $\varphi|_K$ is injective, $\mu \neq \nu$ implies $\varphi_*\mu \neq \varphi_*\nu$. Moreover it is clear that $\operatorname{supp}(\mu) \subset K$ implies $\operatorname{supp}\varphi_*(\mu) \subset \overline{\varphi(K)} \subset K'$. □

If $K \subset \mathbb{R}^n$ is compact, it is well-known that every K-moment problem is determinate, as a consequence of the Weierstraß approximation theorem. The following proposition generalizes this fact. It is a variation of Proposition 2.1.

Proposition 3.2. *Let X be an affine $\mathbb{R}$-variety, and let K be a closed subset of $X(\mathbb{R})$. If the algebra $H(K) = \{p \in \mathbb{R}[X]\colon p$ is bounded on $K\}$ separates the points of K, then every K-moment problem is determinate.*

Proof. First assume that $H = H(K)$ is generated by finitely many elements $h_1, \dots, h_m$ as an $\mathbb{R}$-algebra. The map $h := (h_1, \dots, h_m)\colon K \to \mathbb{R}^m$ is injective, and the subset $\overline{h(K)}$ of $\mathbb{R}^m$ is compact. From Lemma 3.1 we infer that every K-moment problem is determinate.

If H fails to be finitely generated, we are saved by the following lemma:

Lemma 3.3. *Let X be an affine $\mathbb{R}$-variety, let K be a subset of $X(\mathbb{R})$, and let B be an $\mathbb{R}$-subalgebra of $\mathbb{R}[X]$ which separates the points of K. Then there exists a finitely generated subalgebra of B with the same property.*

Given any $\mathbb{R}$-subalgebra C of $\mathbb{R}[X]$, let

$$J_C = \ker\left(\mathbb{R}[X] \otimes_{\mathbb{R}} \mathbb{R}[X] \twoheadrightarrow \mathbb{R}[X] \otimes_C \mathbb{R}[X]\right),$$

and let $W_C = V(J_C)$ be the closed subvariety of $X \times X$ associated with the ideal J_C. Then C separates the points of K if and only if $W_C(\mathbb{R}) \cap (K \times K) \subset \Delta(\mathbb{R})$, where $\Delta \subset X \times X$ is the diagonal. Indeed, the ideal J_C is generated by the elements $c \otimes 1 - 1 \otimes c$ ($c \in C$), as follows directly from the definition of the tensor product.

Now write $B = \bigcup_\alpha B_\alpha$ as the ascending union of its finitely generated subalgebras. Then J_B is the ascending union of the J_{B_α}. Since the ideal J_B is finitely generated, there is an index α with $J_B = J_{B_\alpha}$. By the previous remark, it follows that B_α separates the points of K. □

We are now discussing cases to which Proposition 3.2 applies. First let us look at one-dimensional examples.

Remark 3.4. Let X be an affine curve over $\mathbb{R}$, and let $\overline{X}$ be its (good) completion. That is, the unique (up to unique isomorphism) projective curve which contains X as a Zariski dense open subset and whose points in the complement of X are nonsingular. Let $S = \overline{X} - X$ (a finite set), and let K be a closed subset of $X(\mathbb{R})$. For simplicity, assume that X is irreducible. Following [19] we will say that K is *virtually compact* if S contains at least one point which is either non-real or does not lie in the closure $\overline{K}$, the closure being taken in $\overline{X}(\mathbb{R})$.

Let $H = H(K)$ be the subring of $\mathbb{R}[X]$ consisting of all regular functions which are bounded on K. Regarding elements $p \in \mathbb{R}[X]$ as rational functions on $\overline{X}$, p is bounded on K if and only if none of the points of $\overline{K} \cap S$ is a pole of p. So $H = \mathcal{O}(\overline{X} - T)$, where T is the set of points in S which do not lie in $\overline{K}$. Therefore we see (*cf.* [19, Lemma 5.3]) that K is virtually compact if and only if $H \neq \mathbb{R}$, and that in this case H separates the points of $X(\mathbb{R})$. Hence:

Theorem 3.5. *Let X be an irreducible affine curve over $\mathbb{R}$, and let K be a closed subset of $X(\mathbb{R})$. If K is virtually compact then every moment problem on K is determinate.* □

The condition that X is irreducible can be removed (see [19, Definition 5.1 and Lemma 5.3]). See Example 6.3 below for examples of sets K which are virtually compact but not compact.

Remark 3.6. For the case of one-dimensional sets K, and for the determinateness question, this leaves us with the case where K is not virtually compact. In other words, the case where every polynomial which is bounded on K is constant on K. With an eye on Proposition 2.1, the following observation is of interest:

Lemma 3.7. *Assume that the affine curve X is irreducible, and let $K \subset X(\mathbb{R})$ be a closed subset. Then $A_0(K, f)$ separates the points of $X(\mathbb{R})$ for every non-constant f in $\mathbb{R}[X]$ with $f \geq 1$ on $X(\mathbb{R})$.*

(Instead of $f \geq 1$ it is only needed here that f vanishes nowhere on $X(\mathbb{R})$.)

Proof. Let $Y \subset \overline{X}$ be the open set where f is regular. So Y is affine, contains X, and the points in $\overline{X} - Y$ are nonsingular on $\overline{X}$. Since the rational function $\frac{1}{f}$ vanishes in the points of $\overline{X} - Y$, there exists for every $q \in \mathbb{R}[Y]$ an integer $k \geq 1$ such that $\frac{q}{f^k}$ vanishes at all points of $\overline{X} - Y$. Let

$$I = \big\{q \in \mathbb{R}[Y] \colon \forall\, y \in Y - X : q(y) = 0\big\},$$

an ideal of $\mathbb{R}[Y]$. If $q \in I$, and if $k \geq 1$ is chosen for q as before, the rational function $\frac{q}{f^k}$ lies in $A_0(K, f)$. Since the elements of I separate the points of $X(\mathbb{R})$, the lemma follows. □

Applying Remark 2.2.1 we conclude:

Corollary 3.8. *Let X be an irreducible affine curve and $K \subset X(\mathbb{R})$ a closed set. If there are a non-constant $f \in \mathbb{R}[X]$ with $f \geq 1$ on K and a sequence p_n in $\mathbb{R}[X]$ with $fp_n \to 1$ under $||\cdot||_{L,2}$, the K-moment problem L is determinate.* □

There are higher-dimensional cases as well which are non-compact and to which Proposition 3.2 applies. Here is a class of examples:

Example 3.9. Let K_1 be a compact subset of $\mathbb{R}^n$, let $f: K_1 \to \mathbb{R}$ be a continuous function, and let $K = \{(x,t) \in K_1 \times \mathbb{R} : tf(x) = 1\}$, a closed subset of $\mathbb{R}^{n+1}$. Then $H(K)$ separates the points of K. By Proposition 3.2, therefore, any K-moment problem is determinate.

Remark 3.10. Let K be a closed semi-algebraic subset of $\mathbb{R}^n$, and let X be the Zariski closure of K (a closed $\mathbb{R}$-subvariety of $\mathbb{A}^n$). For convenience, assume that X is irreducible. If the subring $H(K)$ of $\mathbb{R}[X]$ separates the points of K, then $H(K)$ can be shown to be "large" in the sense that it every polynomial on X can be written as the quotient of two elements of $H(K)$ (D. Plaumann, unpublished). The converse is true if $\dim(K) = 1$, as remarked in Remark 3.4, but it usually fails in higher dimensions. As an example, take the subset

$$K = \{(x,y) : 0 \leq x \leq 1,\ y \geq 0,\ xy \leq 1\}$$

of the plane $\mathbb{R}^2$. Here $H(K) = \mathbb{R}[x, xy]$ has the same quotient field as the polynomial ring $\mathbb{R}[x,y]$. But $H(K)$ does not separate the points of the positive y-axis from each other, and so Proposition 3.2 does not apply. In fact, there *do exist* indeterminate K-moment problems, since K contains a half-line.

Given a morphism $\pi : X \to Y$ of affine $\mathbb{R}$-varieties, we associate to any linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ on X its push-forward $\pi_* L = L \circ \pi^* : \mathbb{R}[Y] \to \mathbb{R}$ on Y.

The second main result states very roughly that a moment problem is determinate, provided that its push-forward under a suitable finite morphism is determinate and a suitable technical condition is fulfilled. Here is the precise formulation:

Proposition 3.11. *Let $\pi : X \to Y$ be a finite morphism of affine $\mathbb{R}$-varieties, and let K be a closed subset of $X(\mathbb{R})$. Let μ, ν be positive Borel measures on K such that the moment functionals $L_\mu, L_\nu : \mathbb{R}[X] \to \mathbb{R}$ exist and are equal.*

Assume there is a continuous function $f : \pi(K) \to \mathbb{R}$ which satisfies the conditions of Proposition 2.1 *with respect to the measures $\pi_*(\mu)$ and $\pi_*(\nu)$ and the (closed) set $\pi(K)$. Assume moreover that for every $q \in \mathbb{R}[Y]$ there exists $k \geq 1$ with $\frac{q}{f^k} \in A_0(\pi(K), f)$.*

Then $\mu = \nu$.

Proof. Consider the pull-back $\pi^* f = f \circ \pi$ of f via π, a continuous real function on K. By assumption there is a sequence q_n in $\mathbb{R}[Y]$ with $q_n \to \frac{1}{f}$ both in $L^2(\pi_* \mu)$

and $L^2(\pi_*\nu)$. It follows that the sequence $p_n = \pi^* q_n \in \mathbb{R}[X]$ converges to $\frac{1}{\pi^* f}$ both in $L^2(\mu)$ and $L^2(\nu)$. By assumption,

$$A_0(\pi K, f) = \left\{ \frac{q}{f^k} : q \in \mathbb{R}[Y],\ k \geq 0,\ \frac{q(y)}{f(y)^k} \to 0 \text{ for } |y| \to \infty,\ y \in \pi(K) \right\}$$

separates the points of $\pi(K)$.

It is obvious that $A_0(K,\pi^* f)$ contains all pull-backs of functions in $A_0(\pi K, f)$. To see that $A_0(K,\pi^* f)$ separates the points of K, let $x \neq x'$ in K. Choose $a \in \mathbb{R}[X]$ with $a(x) \neq a(x')$. Since π is finite, there is an identity $a^n = \sum_{j=1}^{n} \pi^*(b_j) a^{n-j}$ in $\mathbb{R}[X]$, with $n \geq 1$ and $b_j \in \mathbb{R}[Y]$. By the hypothesis there is $k \geq 1$ such that $\frac{b_j}{f^l} \in A_0(\pi K, f)$ for $j = 1, \dots, n$ and $l \geq k$. From

$$\left(\frac{a}{\pi^* f^k} \right)^n = \sum_{j=1}^{n} \pi^* \left(\frac{b_j}{f^{jk}} \right) \cdot \left(\frac{a}{\pi^* f^k} \right)^{n-j}$$

it follows that $\frac{a}{\pi^* f^k}$ is bounded on K, and hence $\frac{a}{\pi^* f^{k+1}}$ lies in $A_0(K, \pi^* f)$. Clearly, this function separates x and x'. □

If the target variety is the affine line, we can eliminate the technical condition from Proposition 3.11 and get the following unconditional statement:

Theorem 3.12. *Let X be an affine real curve, and let $\pi : X \to \mathbb{A}^1$ be a finite morphism. If L is a moment functional on X, and if $\pi_* L$ is determinate (on $\mathbb{R}$), then L is determinate (on $X(\mathbb{R})$).*

Proof. By M. Riesz's theorem, there is an approximation $(1 + x^2) p_n(x) \to 1$ in $\mathbb{R}[x]$ under the L^2-norm given by $\pi_* L$ (see Remark 2.2.4). Pull it back to X and use Proposition 3.11. □

The converse is not true. For an example see Example 6.1 below.

4. New indeterminate moment problems from old ones

Assume we are given an indeterminate moment functional on the affine $\mathbb{R}$-variety X. We will discuss constructions by which we can produce indeterminate moment functionals on other $\mathbb{R}$-varieties Y which are related to X in some suitable way. In particular, this will give us new insights on one-dimensional sets which are not virtually compact.

First we propose to use actions of finite, or more generally compact, groups.

Let G be a finite group acting on the affine $\mathbb{R}$-variety X (by morphisms), and let $\pi : X \to X/G = Y$ be the quotient. So Y is affine with coordinate ring $\mathbb{R}[Y] = \mathbb{R}[X]^G$, the ring of G-invariants, and π corresponds to the inclusion $\mathbb{R}[Y] \subset \mathbb{R}[X]$.

Let K be a closed subset of $X(\mathbb{R})$ which is G-invariant. The image set $\pi(K)$ is a closed subset of $Y(\mathbb{R})$, and is topologically the quotient space of $X(\mathbb{R})$ by the action of G. In particular, the G-invariants in $\mathbb{R}[X]$ separate the G-orbits.

Given a Borel measure ν on the set $\pi(K)$, there exists a unique G-invariant Borel measure μ on K for which $\pi_*(\mu) = \nu$. We will denote it by $\mu =: \pi^*(\nu)$. Namely, for any measurable non-negative function $f\colon K \to \mathbb{R}$, we have

$$\int_K f(x)\mu(\mathrm{d}x) = \int_{\pi(K)} \phi_f(y)\nu(\mathrm{d}y),$$

where $\phi_f : \pi(K) \to \mathbb{R}$ is given by averaging f over the orbits:

$$\phi_f(y) = \frac{1}{|G|}\sum_{g\in G} f(gx)$$

for $x \in K$ with $y = \pi(x)$. Hence the operations π_* and π^* set up a bijective correspondence between G-invariant positive Borel measures on K and all positive Borel measures on $\pi(K)$.

All of this carries over, essentially unchanged, to the case where G is a linear algebraic group over $\mathbb{R}$ which acts morphically on X, as long as $G(\mathbb{R})$, the group of its real points, is compact. See [17] for the non-trivial proofs. The quotient variety is then denoted $Y = X/\!/G$. The correspondence between G-invariant measures on $X(\mathbb{R})$ and all measures on $\pi(X(\mathbb{R}))$ is certainly folklore, but it seems hard to locate a proper reference. More details will be contained in [5].

Proposition 4.1. *Assume that we have an algebraic group action of G on the affine $\mathbb{R}$-variety X, and that $G(\mathbb{R})$ is compact. Let K be a closed G-invariant subset of $X(\mathbb{R})$. Assume moreover that $\nu_1 \neq \nu_2$ are two Borel measures on $\pi(K)$ which induce the same moment functional on $\mathbb{R}[Y] = \mathbb{R}[X]^G$ (all moments are supposed to exist). Then $\pi^*\nu_1 \neq \pi^*\nu_2$ are two Borel measures on K which induce the same moment functional on $\mathbb{R}[X]$.* □

In particular, from any indeterminate moment problem on the closed subset $\pi(K)$ of $Y(\mathbb{R})$ we get an indeterminate moment problem on K itself.

The following construction generalizes the previous one (case where the group G is finite). It has the advantage that it applies to much more general situations.

Theorem 4.2. *Let $\pi : X \to Y$ be a finite and locally free morphism of affine $\mathbb{R}$-varieties, of (constant) degree n. Let M be a closed subset of $Y(\mathbb{R})$, and let Z be the set of all $y \in Y(\mathbb{R})$ which have less than n preimages in $X(\mathbb{R})$. Assume that there exist positive Borel measures $\mu \neq \nu$ on $Y(\mathbb{R})$ with support in M for which all moments exist and are equal and for which Z is a null set. Then there exists an indeterminate $\pi^{-1}(M)$-moment problem on X.*

Before going into the proof, we need to recall an algebraic construction. Given an extension $A \subset B$ of (commutative) rings which makes B into a projective A-module of finite type, there is a canonical trace map

$$\mathrm{tr} = \mathrm{tr}_{B/A}\colon B \to A,$$

which is A-linear (*i.e.*, $\mathrm{tr}(ab) = a\,\mathrm{tr}(b)$ for $a \in A$, $b \in B$), satisfies $\mathrm{tr}(1) = n$ if B has constant rank n over A, and commutes with base change. The latter means that for every A-algebra C one has

$$\mathrm{tr}_{B\otimes_A C/C}(b \otimes c) = \mathrm{tr}_{B/A}(b) \cdot c,$$

for $b \in B$ and $c \in C$. In particular, if $B \cong A^n = A \times \cdots \times A$ as an A-algebra, then the trace is simply the sum of the entries: $\mathrm{tr}_{B/A}(a_1, \ldots, a_n) = \sum_i a_i$.

Concretely, the trace generalizes the well-known construction from field theory: If B is free as an A-module, then $\mathrm{tr}_{B/A}(b)$ is the trace of the multiplication map $B \to B$ by b, considered as an A-linear endomorphism of B. The same is true in general, *i.e.*, when B is only assumed to be a projective (*i.e.*, locally free) A-module, with the proper definition of trace in this case. (See [4] ch. II § 4.3, for example.)

Proof. Define the measure $\tilde{\mu}$ on $X(\mathbb{R})$ by

$$\tilde{\mu}(A) := \frac{1}{n} \int_{Y(\mathbb{R})} \phi_A(y)\mu(\mathrm{d}y)$$

(for $A \subset X(\mathbb{R})$ any Borel set), where we have put

$$\phi_A(y) := \left| A \cap \pi^{-1}(y) \right|$$

for $y \in Y(\mathbb{R})$. Define $\tilde{\nu}$ on $X(\mathbb{R})$ similarly, using ν instead of μ. Since $\mu(Z) = \nu(Z) = 0$, we have $\pi_*\tilde{\mu} = \mu$ and $\pi_*\tilde{\nu} = \nu$. Hence, in particular, $\tilde{\mu} \neq \tilde{\nu}$. On the other hand, again using $\mu(Z) = 0$, we have

$$\int_{X(\mathbb{R})} p(x)\tilde{\mu}(\mathrm{d}x) = \frac{1}{n} \int_{Y(\mathbb{R})} (\mathrm{tr}\, p)(y)\mu(\mathrm{d}y)$$

for every $p \in \mathbb{R}[X]$. Here $\mathrm{tr} = \mathrm{tr}_{\mathbb{R}[X]/\mathbb{R}[Y]} \colon \mathbb{R}[X] \to \mathbb{R}[Y]$ is the trace map of X over Y, see above. The key point is that

$$(\mathrm{tr}\, p)(y) = \sum_{x \in X(\mathbb{R}) \colon \pi(x) = y} p(x)$$

holds for every $y \in Y(\mathbb{R})$, $y \notin Z$, since the trace commutes with base change.

Similarly for $\tilde{\nu}$. Since μ and ν have the same moments on Y, it follows that $\tilde{\mu}$ and $\tilde{\nu}$ have the same moments on X. Moreover it is clear that both have support in $\pi^{-1}(M)$. □

Remarks 4.3. 1. As soon as the (common) moment functional of μ and ν on $\mathbb{R}[Y]$ is known explicitly, the proof shows that one also gets an *explicit* indeterminate moment functional on $X(\mathbb{R})$. See Example 6.6 below for an illustration.

2. Note that if X and Y are curves, and if Y is non-singular, then any finite morphism $X \to Y$ is automatically locally free. This is so since every torsion-free module of finite type over a discrete valuation ring is free.

Corollary 4.4. *Let X be an affine curve over $\mathbb{R}$, and suppose that there exists a finite morphism $\pi: X \to \mathbb{A}^1$ of degree n such that the set*

$$\{t \in \mathbb{R} : t \text{ has } n \text{ preimages in } X(\mathbb{R})\}$$

is unbounded. Then there exists an indeterminate moment problem on $X(\mathbb{R})$.

Proof. The set of $t \in \mathbb{R}$ with n real preimages is semi-algebraic, hence it contains a closed half-line. Since there exist indeterminate Stieltjes moment problems (see Example 6.4), the assertion follows from Theorem 4.2. □

Examples 4.5. 1. Let X be the plane affine curve given by an equation $f(x, y) = 0$ of total degree d. If the leading form $f_d(x, y)$ of f has at least $d - 1$ different real linear factors, then Corollary 4.4 applies, showing that there exist indeterminate moment problems on $X(\mathbb{R})$.

Indeed, any general linear projection from X to a line in the plane is a finite morphism of degree d, whose fibre at infinity is entirely real and consists of at least $d - 1$ different points. Therefore, all real fibres in a neighborhood must contain d different real points.

2. Let X be $\mathbb{A}^1$ minus finitely many real points. Concretely, we may embed X into the plane via the equation

$$y(x - a_1) \cdots (x - a_m) = 1,$$

where $a_1, \dots, a_m$ are pairwise different real numbers. Any linear projection from X to a line whose fibres are neither horizontal nor vertical is a finite morphism of degree $m+1$. If the direction of the projection is chosen properly, the fibres (parallel lines) meet X in $m + 1$ real points when one moves to infinity. We conclude that there exist indeterminate moment problems on $X(\mathbb{R})$.

3. From Theorem 3.5 it is clear that $X(\mathbb{R})$ can never be virtually compact, if a finite morphism $X \to \mathbb{A}^1$ as in Corollary 4.4 exists. Of course, it is easy to verify this directly.

4. Unfortunately, the criterion of Corollary 4.4 does not always apply when $X(\mathbb{R})$ is not virtually compact. An example where it fails to apply is the curve

$$X : \quad x^4 + y^3 + y = 0.$$

It has genus 3, and has a single point at infinity, which is real. Hence in any finite morphism $\pi: X \to \mathbb{A}^1$, the fibres $\pi^{-1}(t)$ cannot contain more than two real points for $|t| \gg 0$. On the other hand, the degree of any such morphism is at least 3.

We close this section by stating an open problem: *Given an irreducible affine curve X over $\mathbb{R}$ and a closed subset K of $X(\mathbb{R})$ which is not virtually compact, does there always exist an indeterminate K-moment problem on X?*

5. A classical criterion for determinateness

We have already mentioned in Remark 2.2.2 a general Taylor series method of proving that a non-polynomial function f is approximable in $L^2(\mu)$, as a first step towards fulfilling the conditions in the uniqueness Theorem 2.1. This idea, in an even more general format, goes back to M. Riesz [18] and, contrary to the finer uniqueness criteria, such as Carleman's, is easily adaptable to the multivariate setting.

Before Riesz, a simple numerical criterion for the convergence of continued fraction associated to a Cauchy transform of a positive measure has appeared in the pioneering works of Stieltjes and Markov. A couple of decades later, the same criterion was refined by Perron and Hamburger (see [15, 8, 9]). We adapt below the Stieltjes-Markov-Perron-Hamburger criterion to the multi-variable framework proposed in this note. The second proof below is adapted after Perron [15] Satz 14 § 72 (note that the later editions of his monograph do not contain this material). This particular proof has the advantage of being self-contained and elementary; possibly different integral transforms, than the Fourier transform, can lead in a very similar way to determinateness criteria.

The reader should be aware that the investigation of the strong commutativity of symmetric operators (going back to the works of Devinatz, Nelson and Nussbaum [6, 7, 13, 14]) provides today stronger sufficient conditions for the uniqueness of the representing measure.

Theorem 5.1. *Let $(a_\alpha)_{\alpha\in\mathbb{Z}_+^d}$ be the sequence of moments of a positive measure on $\mathbb{R}^d$. If*

$$\liminf_{k\to\infty}\left(\sum_{|\alpha|=k}\frac{a_{2\alpha}}{\alpha!}\right)^{1/k}<\infty,$$

then the representing measure is unique.

First proof. This proof works under the slightly stronger assumption

$$\sup_k\left(\sum_{|\alpha|=k}\frac{a_{2\alpha}}{\alpha!}\right)^{1/k}<\infty.$$

To show that this condition implies uniqueness, consider the entire function

$$f_\lambda=f_\lambda(x)=\exp(-\lambda|x|^2)=\sum_\alpha(-\lambda)^{|\alpha|}\frac{x^{2\alpha}}{\alpha!},$$

where λ is a (real) parameter.

Let μ be a representing measure, and let $\mathcal{H}$ be the Hilbert space completion of $\mathbb{R}[x]$ in $L^2(\mu)$. We treat f_λ as a power series in λ, with coefficients in $\mathcal{H}$. The condition in the statement means that the radius of convergence of this power series is positive. Thus, there exists $\lambda_0 > 0$ with the property that the series for f_{λ_0} converges in $\mathcal{H}$. Therefore, Proposition 2.1 applies to the function f_{λ_0} and its partial sums approximations. □

Second proof. The condition in Theorem 5.1 means that there exists a positive number $R > 0$ with the property that

$$\frac{1}{k(j)!}\int |x|^{2k(j)}\mathrm{d}\mu(x) \leq \sum_{|\alpha|=k(j)} \frac{a_{2\alpha}}{\alpha!} < R^{k(j)},$$

along a sequence $k(j) \to \infty$ of integers.

Let ν be another representing measure, and denote by $u \cdot v$ the scalar product in $\mathbb{R}^d$. We will prove the equality of the Fourier transforms:

$$\int e^{iu\cdot x}\mathrm{d}\mu(x) = \int e^{iu\cdot x}\mathrm{d}\nu(x) \tag{5.1}$$

for all $u \in \mathbb{R}^d$. In particular, by general distribution theory, this will imply that the two measures are equal.

Let $n \geq 1$ be a fixed integer. Then

$$e^t - \sum_{j=0}^{n-1} \frac{t^j}{j!} = e^t \int_0^t \frac{e^{-s}s^{n-1}\mathrm{d}s}{(n-1)!}.$$

Fix a vector $u \in \mathbb{R}^d$, and apply the above formula to $t = iu \cdot x$. The result is

$$e^{iu\cdot x} - \sum_{j=0}^{n-1} \frac{(iu\cdot x)^j}{j!} = \frac{i^n e^{iu\cdot x}}{(n-1)!}\int_0^1 (u\cdot x)^n e^{isu\cdot x}s^{n-1}\mathrm{d}s.$$

By integrating against the two representing measures one finds

$$\left|\int e^{iu\cdot x}\mathrm{d}\mu(x) - \int e^{iu\cdot x}\mathrm{d}\nu(x)\right| \leq \frac{1}{n!}\int |u\cdot x|^n d\big(\mu(x)+\nu(x)\big).$$

(The extra factor n in the denominator appears from the integral $\int_0^1 s^{n-1}\mathrm{d}s$.) By the Cauchy-Schwarz inequality,

$$|u\cdot x|^n \leq |u|^n\,|x|^n,$$

and from the choice $n = n(j) = 2k(j)$ we infer, for j large,

$$\left|\int e^{iu\cdot x}\mathrm{d}\mu(x) - \int e^{iu\cdot x}\mathrm{d}\nu(x)\right| \leq 2\frac{k(j)!}{(2k(j))!}(|u|^2R)^{k(j)}.$$

For $j \to \infty$ (and arbitrary u), the right hand side goes to zero, implying

$$\int e^{iu\cdot x}\mathrm{d}\mu(x) = \int e^{iu\cdot x}\mathrm{d}\nu(x)$$

for all u, which suffices to conclude $\mu = \nu$. □

Third proof. We assume the lim-inf condition in the statement holds true and consider the marginal sequences

$$s_k(n) = a_{(0,\dots,0,n,0,\dots,0)}, \quad n \geq 0,$$

where the non-zero entry is on the k-th position. Then for a fixed k,

$$\liminf_{n\to\infty} \left(\frac{s_k(2n)}{n!}\right)^{1/n} < \infty,$$

and according to Perron's criterion, the univariate moment sequence $(s_k(n))_n$ is determinate. In view of Petersen's Theorem [15] (see also [14, 12]) the whole multi-sequence $(a_\alpha)_\alpha$ is determinate. □

The first proof, where we have assumed a slightly stronger condition, is easily adaptable to other power series, such as $\frac{|x|}{\sinh|x|}$, $\frac{1}{\cosh|x|}$ or Bessel's functions $J_n(x)$, or even entire functions of infinite order. For instance, the following general result is within reach with the same methods.

Proposition 5.2. *Let $F(z) = \sum c_\alpha z^\alpha$ be an entire function on $\mathbb{C}^d$ without zeros on $\mathbb{R}^d$ and satisfying $\lim_{|x|\to\infty} |x|^N |F(x)| = 0$ for $x \in \mathbb{R}^d$ and all $N > 1$. Let (a_α) be the moment sequence of a positive measure on $\mathbb{R}^d$. If*

$$\sup_\alpha \left(|c_\alpha|^2 a_{2\alpha}\right)^{1/|2\alpha|} < \infty,$$

then the sequence (a_α) is determinate.

Proof. Let

$$F_\lambda(x) = \sum c_\alpha x^\alpha \lambda^{|\alpha|}.$$

The first proof above applies, *mutatis mutandis*, and yields the uniqueness of the representing measure. □

Note that in Proposition 2.1, the function $\frac{1}{F}$ was denoted by f and its modulus was assumed to be bounded from below by a positive constant.

A theoretical illustration of the above proposition is contained in the next section.

6. Examples

Example 6.1. The converse of Theorem 3.12 is not true, by an example due to Schmüdgen [20]. To be more precise, let σ be an indeterminate and Nevanlinna extremal measure on the line. So that $\mathbb{C}[x]$ is dense in $L^2(\sigma)$, but $(x+i)\mathbb{C}[x]$ is not dense in $L^2(\sigma)$. In particular, the function 1 cannot be approximated by elements of $(x+i)\mathbb{C}[x]$, see [1] Theorem 2.5.1 or Section 4.1. We consider the measure

$d\nu(x) = d\sigma(x)/(1+x^2)$ on the line, and its direct image $\mu = i_*\nu$ in $\mathbb{R}^2$ via the embedding $i(x) = (x, x^2)$.

Then ν as well as μ are determinate measures, on $\mathbb{R}$ resp. $\mathbb{R}^2$. For ν simply check that $(x+i)\mathbb{C}[x]$ is dense in $L^2(\nu)$, and for μ use the fact that the projection π_1 on the first coordinate is an isomorphism between the parabola and the axis.

Claim 6.2. The measure $\tau = (\pi_2)_*\mu$ on $\mathbb{R}$ is indeterminate. Note that this measure is carried by $[0, \infty)$. Assume by contradiction that τ is determinate, that is, $(1+y)$ $\mathbb{C}[y]$ is dense in $L^2(\tau)$. In particular, there exists a sequence of polynomials p_n such that

$$\lim_n \int_0^\infty \left|(1+y)p_n(y) - \sqrt{1+y}\right|^2 d\tau(y) = 0,$$

or equivalently,

$$\begin{aligned} 0 &= \lim_n \int_{-\infty}^{\infty} \left|(1+x^2)p_n(x^2) - \sqrt{1+x^2}\right|^2 \frac{d\sigma(x)}{1+x^2} \\ &= \lim_n \int_{-\infty}^{\infty} \left|\sqrt{1+x^2}\,p_n(x^2) - 1\right|^2 d\sigma(x), \end{aligned}$$

a contradiction. Indeed, Proposition 2.1 applies to the function $f(x) = \sqrt{1+x^2}$ and yields the uniqueness of the representing measure.

Similarly we prove that the skew projection π along the parallel lines $y - 2cx =$ const., with $c \in \mathbb{R}$ fixed, gives an indeterminate measure. For instance, denote the new variable on the projected line by $t = y - 2cx$, and assume as before that

$$\int \left|(t + c^2 + 1)\, p_n(t) - 1\right|^2 d\pi_*\mu(t) \to 0$$

$(n \to \infty)$. This means

$$\int \left|(x^2 - 2cx + c^2 + 1)\, p_n(x^2 - 2cx) - 1\right|^2 \frac{d\sigma(x)}{(1+x^2)} \to 0,$$

or

$$\int \left|\frac{x-c+i}{x+i}(x-c-i)\, p_n(x^2-2cx) - \frac{1}{x+i}\right|^2 d\sigma(x) \to 0.$$

The function $\left|\frac{x-c+i}{x+i}\right|$ is uniformly bounded from below and above by positive real numbers, so that

$$\int \left|(x-c-i)\, p_n(x^2-2cx) - \frac{1}{x-c+i}\right|^2 d\sigma(x) \to 0,$$

and hence

$$\int \left|p_n(x^2-2cx) - \frac{1}{(x-c)^2+1}\right|^2 d\sigma(x) \to 0.$$

In view of Proposition 2.1 we deduce that σ is an indeterminate measure, contradiction.

Example 6.3. To illustrate Theorem 3.5, we give a couple of examples of one-dimensional sets which are virtually compact but not compact. For simplicity we stick to subsets of the plane. Let $p \in \mathbb{R}[x, y]$ be an irreducible polynomial, let X denote the plane affine curve $p = 0$.

If the leading form (i.e., highest degree form) of $p(x, y)$ is not a product of linear real factors, then every closed subset K of $X(\mathbb{R})$ is virtually compact. Thus, every K-moment problem is determinate.

But also if the leading form of p is a product of real linear forms, $X(\mathbb{R})$ may be virtually compact (let alone closed subsets K). The reason is that, although the Zariski closure of the curve X in the projective plane contains only real points at infinity, some of them may be singular and may blow up to one or more non-real points. An example is given by the curve $p = x + xy^2 + y^4 = 0$.

Finally, even if the entire curve $X(\mathbb{R})$ itself fails to be virtually compact, suitable non-compact closed subsets K may still be. For example, this is so for the hyperelliptic curves $y^2 = q(x)$, where q is monic of even degree, not a square. For example, one easily checks that if $\deg(q)$ is divisible by 4, then K is virtually compact if (and only if) the coordinate function y is bounded on K from above or from below.

Example 6.4. We reproduce here from [8] one of the simplest indeterminate Stieltjes moment sequences. This particular example was discovered by Stieltjes and refined by Hamburger.

Let ρ and δ be positive constants, and denote

$$\alpha = \frac{1}{2+\delta}, \quad \gamma = \rho^{-\alpha}.$$

Then

$$a_n = (2+\delta)\,\rho^{n+1}\,\Gamma\left((2+\delta)(n+1)\right) = \int_0^\infty x^n e^{-\gamma x^\alpha}\mathrm{d}x, \quad n \geq 0,$$

is a moment sequence on the positive semiaxis. A residue integral argument implies

$$\int_0^\infty x^n \sin\bigl(\gamma x^\alpha \tan(\pi\alpha)\bigr)\, e^{-\gamma x^\alpha}\mathrm{d}x = 0, \quad n \geq 0,$$

see [8] Lemma II. Hence

$$a_n = \int_0^\infty x^n\bigl(1 + t\sin(\gamma x^\alpha \tan(\pi\alpha))\bigr)e^{-\gamma x^\alpha}\mathrm{d}x,$$

for all $n \geq 0$ and $t \in (-1, 1)$. This shows that the moment sequence (a_n) is indeterminate with respect to the support $[0, \infty)$.

Example 6.5. Let X be the (plane affine) curve $y^2 = q(x)$, where $q(x) \in \mathbb{R}[x]$ is a polynomial. Let K be a closed semi-algebraic subset of $X(\mathbb{R})$ which is invariant under the reflection $(x, y) \mapsto (x, -y)$. If K is not compact, then there exist indeterminate moment problems on K.

Indeed, this follows from Proposition 4.1: The group G of order two acts on X as above, and the quotient variety is $X/G = \mathbb{A}^1$. By hypothesis, the image $\pi(K)$ under $\pi\colon X(\mathbb{R}) \to \mathbb{R}$ contains a closed half-line. We conclude using the existence of indeterminate Stieltjes moment problems (previous example).

Example 6.6. Let X be the plane affine curve defined by an equation $f(x, y) = 0$, where f has total degree d. Assume that the highest degree form $f_d(x, y)$ has at least $d-1$ different real linear factors. After a linear change of coordinates we can assume that f contains the monomial x^d. Thus the projection

$$\pi\colon X \to \mathbb{A}^1, \quad \pi(x, y) = y$$

to the y-axis satisfies the condition from Example 4.5.1. By that reasoning, there exist indeterminate moment problems on $X(\mathbb{R})$. We indicate how our construction makes some of them explicit. The set

$$\{y \in \mathbb{R}\colon \text{the equation } f(x, y) = 0 \text{ has } d \text{ different real roots}\}$$

is unbounded. Thus, by a linear change $x \to \pm x + c$ of coordinates we can achieve that it contains $[0, \infty)$. (If $f_d(x, y)$ has d different real factors, a change $x \to x + c$ is sufficient.) Let

$$f(x, y) = x^d + \sum_{i=1}^{d} (-1)^i\, g_i(y)\, x^{d-i}$$

with $g_i(y) \in \mathbb{R}[y]$. The trace tr of $\mathbb{R}[X] = \mathbb{R}[x, y]/(f)$ over $\mathbb{R}[y]$ satisfies $\mathrm{tr}(1) = d$ and $\mathrm{tr}(x) = g_1(y)$. The traces of the higher powers of x are the Newton sums, characterized by the recursion

$$\mathrm{tr}(x^k) + \sum_{i=1}^{k-1} (-1)^i\, g_i(y)\ \mathrm{tr}(x^{k-i}) + (-1)^k k\, g_k(y) = 0$$

($k = 1, \ldots, d$). Thus we have

$$\mathrm{tr}(1) = d,\ \mathrm{tr}(x) = g_1,\ \mathrm{tr}(x^2) = g_1^2 - 2g_2,\ \mathrm{tr}(x^3) = g_1^3 - 3g_1g_2 + 3g_3,$$

and so forth. Let $(a_n)_{n\ge 0}$ be an indeterminate Stieltjes moment sequence, e.g. the one from Example 6.4. Define the linear functional $L\colon \mathbb{R}[x, y] \to \mathbb{R}$ by setting $L(y^n) = a_n$ ($n \ge 0$), by extending this linearly to $\mathbb{R}[y]$, and by

$$L\big(h(y)x^k\big) := \frac{1}{d}\, L\Big(h(y)\, \mathrm{tr}(x^k)\Big)$$

($k = 1, \ldots, d-1$, $h \in \mathbb{R}[y]$). Finally, extend L to arbitrary $p \in \mathbb{R}[x, y]$ by setting $L(p) := L(r)$, where r is the remainder of dividing p by f; that is, $p = qf + r$ with $q, r \in \mathbb{R}[x, y]$ and $\deg_x(r) < d$.

Let $K := \{(x, y) \in \mathbb{R}^2\colon f(x, y) = 0,\ y \ge 0\}$, the part of $X(\mathbb{R})$ contained in the closed upper half plane. Then L is a K-moment sequence, and as such it is indeterminate.

Example 6.7. A simple way to produce universal denominators as in the statement of Proposition 5.2 is via the Fourier-Laplace transform, and the theory of completely monotonic functions of several variables, *cf.* for instance [3].

Let $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\phi \geq 0$, be a non-negative, rapidly decreasing (with all derivatives) function in the Schwartz space. Assume that the support of ϕ is contained in the closed positive octant $K = (\mathbb{R}_+)^d$. Then the Laplace transform

$$F(z) = \int_K e^{-z \cdot x}\, \phi(x) \mathrm{d}x$$

is an analytic function in the tube domain $\mathrm{int}(K) \times i\mathbb{R}^d$. Although it may not be an entire function, it is close enough to the hypotheses of Proposition 5.2:

$$F(x) > 0, \quad x \in K,$$

and

$$\lim_{|x| \to \infty} |x|^N F(x) = 0, \quad x \in K$$

for all $N \in \mathbb{N}$.

Suppose we want to test the determinateness of a moment sequence (a_α), with prescribed support on K. That is, given two positive measures on K with the same moments,

$$a_\alpha = \int_K x^\alpha \mathrm{d}\mu(x) = \int_K x^\alpha \mathrm{d}\nu(x), \quad \alpha \in \mathbb{Z}_+^d,$$

we want to deduce, under additional assumptions, that $\mu = \nu$. Let $\mathcal{H}$ be the closure of polynomials in $L^2(\mu)$.

In order to adapt Proposition 5.2, we start with the expansion, with t as a positive parameter:

$$F_t(y) = \int_K \left(\sum_\alpha (-1)^{|\alpha|}\, t^{|\alpha|}\, \frac{y^\alpha x^\alpha}{\alpha!} \right) \phi(x) \mathrm{d}x.$$

Remark that $F_1 = F$. Let us denote

$$a_\alpha(\phi) = \int_K y^\alpha \phi(y) \mathrm{d}y, \quad \alpha \in \mathbb{Z}_+^d.$$

As before, we want to assure the convergence of above series for some value of the parameter t, in the Hilbert space $\mathcal{H}$ (associated to the moment sequence (a_α)). That is, it is sufficient to have

$$\sum_\alpha t_0^{|\alpha|} \cdot \|y^\alpha\|_{\mathcal{H}} \cdot \|x^\alpha\|_{1,\phi} < \infty$$

for some $t_0 > 0$. In other terms, we are led to the following abstract criterion:

The moment sequence (a_α) is determinate with respect to the first octant K whenever

$$\sup_\alpha \left(\frac{a_\alpha(\phi)\sqrt{a_{2\alpha}}}{\alpha!} \right)^{1/|\alpha|} < \infty, \tag{6.1}$$

for at least one non-negative Schwartz function ϕ with support in K.

For instance one can take $\phi(x) = \prod_{j=1}^d \psi(x_j)$, where $\psi(y) = e^{-y^{-2}(M-y)^{-2}}$ for $y \in (0, M)$, and $\psi(y) = 0$ for $y \geq M$, where $M > 0$ is fixed. Denoting

$$A(k) = \int_0^M y^k e^{-y^{-2}(M-y)^{-2}} \mathrm{d}y, \quad k \geq 0,$$

one finds

$$a_\alpha(\phi) = A(\alpha_1) \cdots A(\alpha_n).$$

The crude estimate for the (non-essential) normalization $M = 1$,

$$A(k) \leq \int_0^1 y^k \mathrm{d}y = \frac{1}{k+1}$$

yields a result comparable to Theorem 5.1: *If*

$$\sup_\alpha \left(\frac{\sqrt{a_{2\alpha}}}{(\alpha+1)!} \right)^{1/|\alpha|} < \infty, \tag{6.2}$$

then the moment sequence (a_α) is determinate. (Here we have put $\alpha + 1 := (\alpha_1 + 1, \ldots, \alpha_d + 1)$.)

As a matter of fact, one can prove that the range of applicability of this particular method is not bigger, due to the fact that the test function ϕ is bounded from below on a rectangle contained in the octant K:

$$\phi(x) \geq \delta_1 > 0, \quad x_k \in [\gamma_k, \lambda_k], \quad 1 \leq k \leq d.$$

Let $m = \min_k(\lambda_k - \gamma_k)$. Assuming also that the support of ϕ is contained in the cube $[0, M] \times \cdots \times [0, M]$, and that $\phi(x) \leq \delta_2$, $x \in \mathbb{R}^d$, we find

$$\delta_1 \frac{m^{|\alpha|+d}}{(\alpha_1+1)\cdots(\alpha_d+1)} \leq \delta_1 \prod_{k=1}^d \frac{\lambda_k^{\alpha_k+1} - \gamma_k^{\alpha_k+1}}{(\alpha_k+1)} \leq a_\alpha(\phi) \leq \delta_2 \frac{M^{|\alpha|+d}}{(\alpha_1+1)\cdots(\alpha_d+1)}.$$

Thus, condition (6.1) holds, if and only if (6.2) is true.

A parallel analysis of the Fourier-Laplace transform of positive measures μ carried by the first octant in $\mathbb{R}^d$ is presented in [11]. However, there the study is focused on the integral operator $\int_K (p_0 - p \cdot x)_+ \mathrm{d}\mu(x)$ and the related Cauchy-Fantappiè transform $\int_K \frac{\mathrm{d}\mu(x)}{p_0 - p\cdot x}$.

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