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Relative linear extensions of sextic del Pezzo fibrations over curves

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Abstract. In this paper, we study a sextic del Pezzo fibration over a curve comprehensively. We obtain certain formulae of several basic invariants of such a fibration. We also establish the embedding theorem of such a fibration which asserts that every such a fibration is a relative linear section of a Mori fiber space with a general fiber $(\mathbb{P}^1)^3$ and that with a general fiber $(\mathbb{P}^2)^2$. As an application of this embedding theorem, we classify singular fibers of such a fibration and answer a question of T. Fujita about the existence of non-normal fibers.

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1. Introduction

1.1. Motivations

A smooth del Pezzo surface S of degree d is defined to be a smooth projective surface whose anti-canonical divisor $-K_S$ is ample with $(-K_S)^2 = d$. It is a famous result that for any integer $d \in \{1, \dots, 9\}$, there exists a certain variety V_d such that every del Pezzo surface S of degree d is a weighted complete intersection of V_d. For example, when $d = 3$ (respectively $d = 4$), we take $V_3 = \mathbb{P}^3$ (respectively $V_4 = \mathbb{P}^4$) and every del Pezzo surface of degree 3 (respectively 4) is a cubic hypersurface on $\mathbb{P}^3$ (respectively a complete intersection of two quadrics on $\mathbb{P}^4$). When $d = 6$, we can take not only $V_6 = (\mathbb{P}^1)^3$ but also $V_6 = (\mathbb{P}^2)^2$. Then every del Pezzo surface of degree 6 is a hyperplane section of $(\mathbb{P}^1)^3$ and also a codimension 2 linear section of $(\mathbb{P}^2)^2$ with respect to the Segre embeddings. These descriptions are classic and useful to study del Pezzo surfaces.

In this paper, we mainly discuss how to relativize these descriptions for del Pezzo fibrations. A relativization of these embeddings is important for the study of

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del Pezzo fibrations; such relativizations have been used by several researchers (*e.g.* [28]). As we will see in the next subsection, relativizations of these descriptions for del Pezzo fibrations of degree d have been established when $d \neq 6$. One of the main results of this paper is to give a relativization when $d = 6$.

1.2. Known results

In this paper, we employ the following definition for del Pezzo fibrations as in the context of Mori theory. Let X be a smooth projective 3-fold whose canonical divisor K_X is not nef. By virtue of Mori theory, X has an extremal contraction $\varphi\colon X \to C$, which is a surjective morphism onto a normal projective variety C with connected fibers satisfying that $\rho(X/C) = 1$ and $-K_X$ is φ-ample. When $\dim C = 1$, we call φ a *del Pezzo fibration*, which is one of the final outputs of the minimal model program. In this case, a general φ-fiber F is a del Pezzo surface. Then the *degree* of a del Pezzo fibration $\varphi\colon X \to C$ is defined to be $(-K_F)^2$.

Let $\varphi\colon X \to C$ be a del Pezzo fibration of degree d. In the paper [19], Mori proved that $1 \leq d \leq 9$ and $d \neq 7$. Moreover, he proved that if $d = 9$ then φ is $\mathbb{P}^2$-bundle, and if $d = 8$ then there exists an embedding of X into a $\mathbb{P}^3$-bundle over C containing X as a quadric fibration [19, Theorem (3.5)]. When $d = 1$ or 2, Fujita proved that there exists a weighted projective space bundle containing X as a relative weighted hypersurface [9, (4.1),(4.2)].

Now we assume that $3 \leq d \leq 6$. Then $\varphi\colon X \to C$ has a natural embedding into the $\mathbb{P}^d$-bundle $p \to \mathbb{P}_C(\varphi_*\mathcal{O}(-K_X)) \to C$. D'Souza [7, (2.2.1) and (2.3.1)] and Fujita [9, (4.3) and (4.4)] proved that if $d = 3$ or 4, then X is a relative complete intersection in $\mathbb{P}_C(\varphi_*\mathcal{O}(-K_X))$. More precisely, when $d = 4$ for example, they proved that there is a rank 2 vector bundle $\mathcal{E}$ on C such that X is the zero scheme of a global section of $\mathcal{O}_{\mathbb{P}_C(\varphi_*\mathcal{O}(-K_X))}(2) \otimes p^*\mathcal{E}$. When $d = 5$ or 6, $\mathbb{P}_C(\varphi_*\mathcal{O}(-K_X))$ does not contain X as a relative complete intersection and hence it seems to be difficult to treat such an X as a submanifold of $\mathbb{P}_C(\varphi_*\mathcal{O}(-K_X))$. When $d = 5$ and $C = \mathbb{P}^1$, however, K. Takeuchi claimed that X is relatively defined in $\mathbb{P}_{\mathbb{P}^1}(\varphi_*\mathcal{O}(-K_X))$ by the Pfaffian of the 4×4 diagonal minors of a 5×5 skew-symmetric matrix [28, Theorem (3.3) (v)].

1.3. Main results

In this paper, we mainly treat a *sextic* del Pezzo fibration $\varphi\colon X \to C$, *i.e.*, that of degree 6.

1.3.1. *Associated coverings*

For every sextic del Pezzo fibration $\varphi\colon X \to C$, we will define smooth projective curves B, T with a double covering structure $\varphi_B\colon B \to C$ and a triple covering structure $\varphi_T\colon T \to C$ respectively associated to φ. These coverings φ_B and φ_T are deeply related to the relative Hilbert scheme of twisted cubics and conics respectively (see Lemma 3.3). In particular, when all φ-fibers are normal, the coverings B and T coincide with the coverings $\mathscr{Z}_3$ and $\mathscr{Z}_2$ over C that are defined by Kuznetsov [16] (see Lemma 3.3 and the proof of Lemma 6.12). We refer to Definition 3.4 for the precise definition.

1.3.2. *Formulae for invariants* $(-K_X)^3$ *and* $h^{1,2}(X)$

For a sextic del Pezzo fibration $X \to C$, the associated coverings are closely related to the invariants $(-K_X)^3$ and $h^{1,2}(X)$.

Theorem A. *Let* $\varphi\colon X \to C$ *be a sextic del Pezzo fibration. Let* $\varphi_B\colon B \to C$ *be the double covering and* $\varphi_T\colon T \to C$ *the triple covering associated to* φ *(see Definition* 3.4*). Then the following assertions hold:*

(1) $\mathcal{J}(X) \times \mathrm{Jac}(C)$ *is isomorphic to* $\mathrm{Jac}(B) \times \mathrm{Jac}(T)$ *as complex tori, where* $\mathcal{J}(X)$ *is the intermediate Jacobian of* X*. Moreover, if* $C = \mathbb{P}^1$*, then* $\mathcal{J}(X)$ *is isomorphic to* $\mathrm{Jac}(B) \times \mathrm{Jac}(T)$ *as principally polarized Abelian varieties, where the polarization of* $\mathcal{J}(X)$ *is defined as in* [5]*;*
(2) *It holds that* $(-K_X)^3 = 22 - (6g(B) + 4g(T) + 12g(C))$.

This theorem shows that the invariants $(-K_X)^3$ and $h^{1,2}(X)$ can be interpreted by the genera of three curves C, B, and T.

1.3.3. *Relative linear extensions*

Let us recall that a smooth sextic del Pezzo surface is a hyperplane section of $(\mathbb{P}^1)^3$ and also a codimension 2 linear section of $(\mathbb{P}^2)^2$ under the Segre embeddings. In the following two theorems, we relativize these embeddings for every sextic del Pezzo fibration.

Theorem B. *Let* $\varphi\colon X \to C$ *be a sextic del Pezzo fibration and* $\varphi_B\colon B \to C$ *the double covering associated to* φ*. Set* $\mathcal{L} := \mathrm{Cok}(\mathcal{O}_C \to \varphi_{B*}\mathcal{O}_B) \otimes \mathcal{O}(-K_C)$*. Then there exists a smooth projective* 4*-fold* Y*, an extremal contraction* $\varphi_Y\colon Y \to C$ *and a divisor* H_Y *on* Y *satisfying the following conditions:*

(1) *Every smooth fiber of* φ_Y *is isomorphic to* $(\mathbb{P}^1)^3$*;*
(2) $\mathcal{O}_Y(K_Y + 2H_Y) = \varphi_Y^*\mathcal{L}$ *holds;*
(3) Y *contains* X *as a member of* $|\mathcal{O}(H_Y) \otimes \varphi_Y^*\mathcal{L}^{-1}|$.

Theorem C. *Let* $\varphi\colon X \to C$ *be a sextic del Pezzo fibration and* $\varphi_T\colon T \to C$ *the triple covering associated to* φ*. Set* $\mathcal{G} := \mathrm{Cok}(\mathcal{O}_C \to \varphi_{T*}\mathcal{O}_T) \otimes \mathcal{O}(-K_C)$*. Then there exists a smooth projective 5-fold* Z*, an extremal contraction* $\varphi_Z\colon Z \to C$ *and a divisor* H_Z *on* Z *satisfying the following conditions:*

(1) *Every smooth fiber of* φ_Z *is isomorphic to* $(\mathbb{P}^2)^2$*;*
(2) $\mathcal{O}_Z(K_Z + 3H_Z) = \varphi_Z^* \det \mathcal{G}$ *holds;*
(3) *There exists a section* $s \in H^0(Z, \mathcal{O}_Z(H_Z) \otimes \varphi_Z^*\mathcal{G}^\vee)$ *such that* X *is isomorphic to the zero scheme of* s.

Remark 1.1. Note that the sheaf $\mathcal{L}$ (respectively $\mathcal{G}$) in Theorem B (respectively Theorem C) is invertible (respectively locally free of rank 2).

One of the most different points from the case where the degree is not 6 is that φ_Y and φ_Z in Theorems B and C may have singular fibers, and must have when $C = \mathbb{P}^1$ by the invariant cycle theorem (*cf.* [29, II, Theorem 4.18]). We will classify singular fibers of φ_Y and φ_Z in Theorem D. Moreover, as an application of Theorems B and C, we will classify singular fibers of sextic del Pezzo fibrations in Theorems D and F.

1.3.4. *Classification of singular fibers of sextic del Pezzo fibrations*

Let us recall Fujita's result about singular fibers of del Pezzo fibrations [9].

Theorem 1.2 ([9, (4,10)]). *Let $\varphi\colon X \to C$ be a del Pezzo fibration. If φ is not of degree* 6*, then every fiber of φ is normal.*

However, singular fibers of sextic del Pezzo fibrations are yet to be classified. Indeed, Fujita proposed the following question.

Question 1.3 ([9, (3,7)]). Do there exist sextic del Pezzo fibrations containing non-normal fibers?

Another main result of this paper is a classification of singular fibers of sextic del Pezzo fibrations $\varphi\colon X \to C$. For the proof, we will use the embeddings $X \hookrightarrow Y$ and $X \hookrightarrow Z$ as in Theorems B and C. In summary, we will show the following theorem.

Theorem D. *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration. Let $\varphi_B\colon B \to C$ and $\varphi_T\colon T \to C$ be the coverings associated to φ. Let $X \hookrightarrow Y$ and $X \hookrightarrow Z$ be the embeddings as in Theorems B and C respectively. For $t \in C$, we set $B_t := \varphi_B^{-1}(t)$ and $T_t := \varphi_T^{-1}(t)$.*

Then for every $t \in C$, the numbers $(\#(B_t)_{\mathrm{red}}, \#(T_t)_{\mathrm{red}})$ determine the isomorphism classes of Y_t, Z_t and the possibilities of those of X_t as in Table 1.1.

$\#(B_t)_{\mathrm{red}}$	$\#(T_t)_{\mathrm{red}}$	X_t	Y_t	Z_t
2	3	(2,3)	$(\mathbb{P}^1)^3$	$(\mathbb{P}^2)^2$
2	2	(2,2)	$\mathbb{P}^1 \times \mathbb{Q}_0^2$	$(\mathbb{P}^2)^2$
2	1	(2,1)	$\mathbb{P}^{1,1,1}$	$(\mathbb{P}^2)^2$
1	3	(1,3)	$(\mathbb{P}^1)^3$	$\mathbb{P}^{2,2}$
1	2	(1,2) or (n2)	$\mathbb{P}^1 \times \mathbb{Q}_0^2$	$\mathbb{P}^{2,2}$
1	1	(1,1) or (n4)	$\mathbb{P}^{1,1,1}$	$\mathbb{P}^{2,2}$

Table 1.1. The singular fibers of φ, φ_Y, and φ_Z.

For the definitions of (i,j)*,* (n2)*, and* (n4)*, we refer to Theorem* 6.1*. $\mathbb{Q}_0^2$ denotes the quadric cone. For the definition of $\mathbb{P}^{1,1,1}$ (respectively $\mathbb{P}^{2,2}$), we refer to Definition* 6.6 *(respectively Definition* 6.8*).*

In particular, if X_t is normal, then the isomorphism class of X_t is determined by the pair $(\#(B_t)_{\mathrm{red}}, \#(T_t)_{\mathrm{red}})$ and the number of lines in X_t is equal to $\#(B_t)_{\mathrm{red}} \times \#(T_t)_{\mathrm{red}}$.

As applications of Theorems A and D, we have the following properties of a sextic del Pezzo fibration $\varphi\colon X \to C$.

Corollary E. *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration.*

(1) *It holds that $(-K_X)^3 \leq 22$ and $(-K_X)^3 \neq 20$;*
(2) *If $(-K_X)^3 > 0$, then $C \simeq \mathbb{P}^1$. In particular, X is rational since φ admits a section (*cf. [18, Theorem 4.2]*);*
(3) *It holds that $(-K_{X/C})^3 \leq 0$ and the equality holds if and only if φ is a smooth morphism.*

1.3.5. *Existence of non-normal fibers*

We will give an answer to Question 1.3 by presenting sextic del Pezzo fibrations with non-normal fibers. More precisely, as a consequence of Examples 7.4, 7.5, and 7.6, we will show the following theorem.

Theorem F. *Let X_0 be a sextic Gorenstein del Pezzo surface, which is possibly non-normal. Suppose that X_0 is not a cone over an irreducible curve of arithmetic genus* 1*. Then there exists a sextic del Pezzo fibration $\varphi\colon X \to C$ containing X_0.*

In particular, Theorem F gives an affirmative answer for Question 1.3.

1.3.6. *Relative double projection*

Our proof of the main results are based on the *relative double projection* from a section of φ. This is a relativization of the *double projection* from a general point x of a sextic del Pezzo surface S, which is given as follows. Under the embedding $S \hookrightarrow \mathbb{P}^6$ given by the anti-canonical system, we consider the projection $\mathbb{P}^6 \dashrightarrow \mathbb{P}^3$ from the tangent plane $\mathbb{T}_x S = \mathbb{P}^2 \subset \mathbb{P}^6$ of S at x. The proper image of S under this birational map is a smooth quadric surface $\mathbb{Q}^2$ and the map $S \dashrightarrow \mathbb{Q}^2$ is birational. This birational map is what is called the *double projection* from the point x on S. In Proposition 2.1, we will establish a relativization of this birational map $S \dashrightarrow \mathbb{Q}^2$ for a sextic del Pezzo fibration.

1.4. Organization of this paper

We organize this paper as follows.

In Section 2, we will establish a relativization of the double projection from a point on a sextic del Pezzo surface (=Proposition 2.1).

In Section 3, we will collect some preliminary results for quadric fibrations to define the associated coverings and prove Theorem A. Furthermore, we will see the following two statements: a characterization of a certain nef vector bundle of rank 3 on a quadric surface (=Proposition 3.6), and a variant of the Hartshorne-Serre correspondence on a family of surfaces with a multi-section (=Theorem 3.9). These two statements will be necessary for our proving Theorem C. As Theorem 3.9 is formulated in a slightly general form, we will postpone its proof to Appendix A.

In Section 4 and 5, we will prove Theorems B and C respectively.

In Section 6, using Theorems B and C, we will classify the singular fibers of the sextic del Pezzo fibrations and prove Theorem D and Corollary E.

In Section 7, we will prove Theorem F by using results in Section 6 and give explicit examples.

1.5. Notation and definitions

Throughout this paper, we work over the complex number field $\mathbb{C}$. We basically adopt the terminology of [11,15]. Vector bundles and line bundles just mean locally free sheaves and invertible sheaves. For a locally free sheaf $\mathcal{E}$ on X, $\mathbb{P}_X(\mathcal{E})$ is defined to be $\operatorname{ProjSym}\mathcal{E}$ in this paper. Then the Hirzebruch surface $\mathbb{F}_n$ is defined to be $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$. On $\mathbb{F}_n$, h denotes a tautological divisor, f a fiber, and $C_0 \in |h - nf|$ the negative section.

Additionally, we use the following notation: $\mathbb{Q}^n$ denotes the non-singular hyperquadric in $\mathbb{P}^{n+1}$, and $\mathbb{Q}_0^2$ denotes a quadric surface in $\mathbb{P}^3$ with an ordinary double point. Note that $\mathbb{Q}_0^2$ is given by contracting the (-2)-curve on $\mathbb{F}_2$.

Definition 1.4. For an irreducible and reduced quadric surface $Q \subset \mathbb{P}^3$, $\mathcal{O}_Q(1)$ denotes the very ample line bundle with respect to the embedding. Moreover, under a fixed linear embedding $Q \hookrightarrow \mathbb{Q}^3$, we define $\mathscr{S}_Q := \mathscr{S}_{\mathbb{Q}^3}|_Q$, where $\mathscr{S}_{\mathbb{Q}^3}$ is the spinor bundle on $\mathbb{Q}^3$ in the sense of [24, Definiton 1.3].

In this paper, we employ the following definition for del Pezzo fibrations and quadric fibrations.

Definition 1.5. We say that $\varphi\colon X \to C$ is a *del Pezzo fibration* if φ is an extremal contraction from a non-singular projective 3-fold X onto a smooth projective curve C, *i.e.*, $-K_X$ is φ-ample and $\rho(X/C) = 1$, as in Section 1.2.

$q\colon Q \to C$ is called a *quadric fibration* if q is a del Pezzo fibration of degree 8. In particular, we assume that Q is a smooth projective 3-fold and $\rho(Q/C) = 1$. By [19, Theorem (3.5)], every smooth fiber of a quadric fibration $q\colon Q \to C$ is actually isomorphic to $\mathbb{Q}^2$.

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2. Relative double projections

For the proof of Theorems B and C, the relative double projection from a section of a sextic del Pezzo fibration plays an important role. We devote this section to prove Proposition 2.1 that establishes this technique.

2.1. Relativizations of double projections

The following proposition is a method of relativizing the double projection of a sextic del Pezzo surface. The proof will be done by Takeuchi's 2-ray game argument. Before the statement of the proposition, we recall the fact that φ admits a section by [18, Theorem 4.2].

Proposition 2.1. *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration. Let C_0 be a φ-section. Let $\mu\colon \widetilde{X} = \mathrm{Bl}_{C_0} X \to X$ be the blow-up of X along C_0, $E = \mathrm{Exc}(\mu)$, and $\widetilde{\varphi} := \varphi \circ \mu$. Then the following assertions hold:*

1. *$\mathcal{O}(-K_{\widetilde{X}})$ is φ-globally generated and the morphism $\widetilde{X} \to \mathbb{P}_C(\varphi_*\mathcal{O}(-K_{\widetilde{X}}))$ over C given by $\mathcal{O}(-K_{\widetilde{X}})$ is a contraction onto its image $\overline{X}$;*
2. *The morphism $\psi_X\colon X \to \overline{X}$ is an isomorphism or a flopping contraction. When ψ_X is flopping, every non-trivial fiber of ψ_X is an isolated (-2)-curve in the sense of* [25, Definition 5.1]*;*
3. *When ψ_X is an isomorphism, let $\chi\colon \widetilde{X} \to \widetilde{Q}$ denote the identity map. Then there exists a unique contraction $\sigma\colon \widetilde{Q} \to Q$ of another $K_{\widetilde{Q}}$-negative ray over C. When ψ_X is a flopping contraction, let $\chi\colon \widetilde{X} \dashrightarrow \widetilde{Q}$ denote the flop of ψ_X. Then there exists the contraction $\sigma\colon \widetilde{Q} \to Q$ of the $K_{\widetilde{Q}}$-negative ray over C. In both cases, we set the morphisms as in the following commutative diagram:*

$$\begin{tikzcd}
& E \arrow[dl] \arrow[r, phantom, "\subset"] & \widetilde{X} \arrow[rr, dashed, "\chi"] \arrow[dl, "\mu"'] \arrow[dd, "\widetilde{\varphi}"] \arrow[dr, "\psi_X"] & & \widetilde{Q} \arrow[r, phantom, "\supset"] \arrow[dl, "\psi_Q"'] \arrow[dd, "\widetilde{q}"] \arrow[dr, "\sigma"] & G \arrow[dr] & \\
C_0 \arrow[r, phantom, "\subset"] & X \arrow[dr, "\varphi"'] & & \overline{X} \arrow[dl] \arrow[dr] & & Q \arrow[dl, "q"] \arrow[r, phantom, "\supset"] & T \\
& & C \arrow[rr, equal] & & C; & &
\end{tikzcd} \tag{2.1}$$

4. *The following assertions hold:*
   (a) *σ is the blow-up along a non-singular curve $T \subset Q$;*
   (b) *$\deg(q|_T) = 3$ and q is a quadric fibration.*

Moreover, if we set $G = \mathrm{Exc}(\sigma)$*, then there is a divisor* α *on* C *such that*
$$G \sim -K_{\widetilde{Q}} - 2E_{\widetilde{Q}} + \widetilde{q}^*\alpha;$$

(5) *The following equalities hold:*

$$\begin{aligned} (-K_Q)^3 &= \frac{1}{3}(4(-K_X)^3 - 16\chi(\mathcal{O}_T) + 48\chi(\mathcal{O}_C)). \\ -K_X.C_0 + K_Q.T &= \frac{1}{6}(K_X^3 + 22\chi(\mathcal{O}_T) - 54\chi(\mathcal{O}_C)). \\ \deg\alpha + K_Q.T &= \frac{1}{3}(K_X^3 + 10\chi(\mathcal{O}_T) - 18\chi(\mathcal{O}_C)); \end{aligned}$$

(6) *The proper transform* $E_Q \subset Q$ *of* $E = \mathrm{Exc}(\mu)$ *contains* T*. Moreover, it holds that* $-K_Q = 2E_Q - q^*\alpha$.

Proof. (1) Fix a point $t \in C$. Then $\widetilde{X}_t = \widetilde{\varphi}^{-1}(t)$ is the blow-up of $X_t = \varphi^{-1}(t)$ at a smooth point. For each $t \in C$, $-K_{X_t}$ is very ample and hence $-K_{\widetilde{X}_t}$ is globally generated and big. Let $\psi_X : X \to \overline{X}$ be as in the assertion (1). To show that the morphism $\psi_X : \widetilde{X} \to \overline{X}$ is a contraction, it suffices to prove that $-K_{\widetilde{X}_t}$ is simply generated, *i.e.*, the section ring $R(\widetilde{X}_t, \mathcal{O}(-K_{\widetilde{X}_t})) = \bigoplus_{m\geq 0} H^0(\widetilde{X}_t, \mathcal{O}(-mK_{\widetilde{X}_t}))$ is generated by $H^0(\mathcal{O}(-K_{\widetilde{X}_t}))$ for each $t \in C$. Since $\widetilde{X}_t$ is an irreducible, a general member $C \in |-K_{\widetilde{X}_t}|$ is integral. It is well-known that $-K_{\widetilde{X}_t}|_C$ is simply generated. Moreover, for every $m \in \mathbb{Z}_{\geq 0}$, the relative Kawamata-Viehweg vanishing gives $R^1\widetilde{\varphi}_*\mathcal{O}(-mK_{\widetilde{X}}) = 0$, which implies $H^1(\widetilde{X}_t, \mathcal{O}(-mK_{\widetilde{X}_t})) = 0$. Thus $-K_{\widetilde{X}_t}$ is simply generated.

(2) Let l be an arbitrary non-trivial ψ-fiber. First, we prove that l is the proper transform of a line in a φ-fiber meeting C_0 and that $\mathcal{N}_{l/\widetilde{X}} \simeq \mathcal{O}(-1)^2$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. Set $P := \mathbb{P}_C(\varphi_*\mathcal{O}(-K_X))$, $\widetilde{P} := \mathrm{Bl}_{C_0} P$, and $\overline{P} := \mathbb{P}_C(\widetilde{\varphi}_*\mathcal{O}(-K_{\widetilde{X}}))$. Then we have a $\mathbb{P}^1$-bundle $\pi : \widetilde{P} \to \overline{P}$. Now P contains X and hence $\widetilde{P}$ contains $\widetilde{X}$. Then the restriction $\pi|_{\widetilde{X}}$ coincides with ψ_X by (1). Hence l is the proper transform of a line in some φ-fiber X_t which meets the point $C_0 \cap X_t$. Moreover, the normal bundle $\mathcal{N}_{l/\widetilde{X}}$ is contained in $\mathcal{N}_{l/\widetilde{P}} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 6}$. Hence we have $\mathcal{N}_{l/\widetilde{X}} \simeq \mathcal{O}(-1)^2$ or $\mathcal{O} \oplus \mathcal{O}(-2)$.

The remaining part we must show is that ψ_X is a flopping contraction if ψ_X is not isomorphic. Assume that ψ_X contracts a prime divisor G. Since $\mathrm{Pic}(\widetilde{X}) = \mathbb{Z}[-K_{\widetilde{X}}] \oplus \mathbb{Z}[E] \oplus \widetilde{\varphi}^* \mathrm{Pic}(C)$, there exist $x, y \in \mathbb{Z}$ such that $G \equiv_C x(-K_{\widetilde{X}}) + yE$. Note that $\psi|_{\widetilde{X}_t} : \widetilde{X}_t \to \tau(\widetilde{X}_t)$ is the contraction of the (-2)-curves for a general t.

As shown in the above argument, the (-2)-curves are the proper transforms of the lines passing through the point $C_{0,t}$. Hence the (-2)-curves on $\widetilde{X}_t$ are disjoint. Moreover, if n denotes the number of the (-2)-curves on $\widetilde{X}_t$, then $n \leq 2$ since X_t is a del Pezzo surface of degree 6. Then we have $0 = -K_{\widetilde{X}_t}.G|_{\widetilde{X}_t} = 5x + y$, $-2n = (G|_{\widetilde{X}_t})^2 = 5x^2 + 2xy - y^2$, and $n \in \{1, 2\}$. These equalities gives $x = \pm\sqrt{\frac{n}{15}}$ and $y = \mp\sqrt{\frac{5n}{3}}$, which is a contradiction.

(3) It follows from the construction of $\widetilde{Q}$ that $2 = \rho(\widetilde{X}/C) = \rho(\widetilde{Q}/C)$. Hence $\overline{\mathrm{NE}}(\widetilde{Q}/C) \subset N_1(\widetilde{Q}/C) = \mathbb{R}^2$ is spanned by two rays. If χ is identity, then $-K_{\widetilde{Q}}$ is relatively ample over C and the assertion holds by the relative contraction theorem. If χ is a flop, then ψ_Q is the contraction of an extremal ray, say R_1. If R_2 denotes another ray, then we have $K_{\widetilde{Q}}.R_2 < 0$ since a general $\widetilde{q}$-fiber is a del Pezzo surface. Letting σ be the contraction of R_2, we complete the proof of (3).

(4) Since $\rho(\widetilde{Q}) = 3$, it holds that $\rho(Q) = 2$ and $\dim Q \geq 2$. If $\dim Q = 2$, then Q is a $\mathbb{P}^1$-bundle over C. Let $s \subset Q$ be a section of $Q \to C$ and set $G := q^*s$. Then there exist $x, y \in \mathbb{Z}$ such that $G \equiv_C x(-K_{\widetilde{Q}}) + yE_{\widetilde{Q}}$. Then for a general $\widetilde{q}$-fiber $F_{\widetilde{Q}}$, we have $G^2F_{\widetilde{Q}} = 0$ and $-K_{\widetilde{Q}}GF_{\widetilde{Q}} = 2$, which implies $5x^2 + 2xy - y^2 = 0$ and $5x + y = 2$. Solving the above equalities, we get $x = \pm\frac{6+\sqrt{6}}{15}$ and $y = \mp\sqrt{\frac{2}{3}}$, which contradicts $x, y \in \mathbb{Z}$.

Therefore, we obtain $\dim Q = 3$ and hence σ is divisorial. Set $G = \mathrm{Exc}(\sigma)$. Since every $\widetilde{\varphi}$-fiber is integral and χ is isomorphic in codimension 1, every $\widetilde{q}$-fiber is integral. Hence $T := \sigma(G)$ is not contained in any q-fiber. By [19, Theorem (3.3)], T is a non-singular curve, Q is smooth, and σ is the blow-up of Q along T. Set $m := \deg(q|_T)$. Then there exist $x, y \in \mathbb{Z}$ and $\alpha \in \mathrm{Pic}(C)$ such that $G \sim x(-K_{\widetilde{Q}}) + yE_{\widetilde{Q}} + \widetilde{q}^*\alpha$. For a general $\widetilde{q}$-fiber $F_{\widetilde{Q}}$, $\sigma|_{F_{\widetilde{Q}}}$ is a contraction of disjoint finitely many (-1)-curves. Hence we have $G^2.F = -m$ and $-K_{\widetilde{Q}}.F_{\widetilde{Q}}.G = m$, which implies $5x^2 + 2xy - y^2 = -m$ and $5x + y = m$. Hence we obtain $x = \frac{\pm\sqrt{6m^2+30m}+6m}{30}$ and $y = \mp\sqrt{\frac{m(m+5)}{6}}$, which implies $m \in \{1, 3\}$.

When $m = 1$, we have $(x, y) = (0, 1)$ since $x, y \in \mathbb{Z}$. Then $G|_{F_{\widetilde{Q}}} \equiv E|_{F_{\widetilde{Q}}}$ holds for a general $F_{\widetilde{Q}}$ and hence we obtain $G = E_{\widetilde{Q}}$. If χ is an identity, then it is a contradiction since μ is the contraction of another ray. If χ is a flop, then E is relatively ample over $\overline{X}$ and hence $-E$ is relatively ample over $\overline{X}$. However, G is relatively ample over $\overline{X}$, which is a contradiction.

Therefore, we obtain $m = 3$ and hence q is a quadric fibration. In this case, we have $(x, y) = (1, -2)$ since $x, y \in \mathbb{Z}$.

(5) Using (4), we obtain the following equations:

$$\begin{aligned}
(-K_Q)^3 - 2(-K_Q.T) - 2\chi(\mathcal{O}_T) &= (-K_{\widetilde{Q}})^3 = (-K_{\widetilde{X}})^3 \\
&= (-K_X)^3 - 2(-K_X.C_0) - 2\chi(\mathcal{O}_C), \\
-K_Q.T + 2\chi(\mathcal{O}_T) &= (-K_{\widetilde{Q}})^2.G = (-K_{\widetilde{X}})^2.\chi_*^{-1}G \\
&= ((-K_X)^3 - 2(-K_X.C_0) - 2\chi(\mathcal{O}_C)) \\
&\quad - 2(-K_X.C_0 + 2\chi(\mathcal{O}_C)) + 5\deg\alpha, \text{ and} \\
-2\chi(\mathcal{O}_T) &= -K_{\widetilde{Q}}.G^2 = -K_{\widetilde{X}}.(\chi_*^{-1}G)^2 \\
&= ((-K_X)^3 - 2(-K_X.C_0) - 2\chi(\mathcal{O}_C)) \\
&\quad - 8\chi(\mathcal{O}_C) - 4(-K_X.C_0 + 2\chi(\mathcal{O}_C)) + 6\deg\alpha.
\end{aligned}$$

The assertion directly follows from solving the above equations.

(6) By (4), we obtain $-K_Q \sim 2E_Q - q^*\alpha$. For a general point $t \in C$, $E_Q \cap Q_t$ is a hyperplane section of Q_t. Since $E_{\widetilde{Q}} \cap \widetilde{Q}_t$ is a (-1)-curve in the $\widetilde{q}$-fiber $\widetilde{Q}_t$, $E_Q \cap Q_t$ passes through the three points $T \cap Q_t$. Thus E_Q contains T. □

Definition 2.2. For a sextic del Pezzo fibration $\varphi\colon X \to C$ with a section C_0, the pair $(q\colon Q \to C, T)$ as in Proposition 2.1 is called the *relative double projection* of the pair $(\varphi\colon X \to C, C_0)$.

3. Preliminaries and proof of Theorem A

The main purpose of this section to prepare some facts about quadric fibrations (=Lemma 3.1), nef vector bundles on quadric surfaces (=Proposition 3.6), and a certain vector bundle on a family of surfaces with a multi-section (=Theorem 3.9). In this section we will also prove Theorem A by using Lemma 3.1 and Proposition 2.1.

3.1. The double covers associated to quadric fibrations

In this section we confirm some basic properties of a quadric fibration $q\colon Q \to C$. We refer to Definition 1.5 for the definition of a quadric fibration in this paper. The following lemma should be well-known for experts.

Lemma 3.1. *Let $q\colon Q \to C$ be a quadric fibration and s a q-section (note that q admits a section by* [18, Theorem 4.2]*). Let $f\colon \widetilde{Q} = \mathrm{Bl}_s Q \to Q$ be the blow-up of Q along s.*

Then there exists a divisorial contraction $g\colon \widetilde{Q} \to P$ over C such that $p\colon P \to C$ is a $\mathbb{P}^2$-bundle and g is the blow-up along a smooth irreducible curve $B \subset P$. Moreover, the morphism $q_B := p|_B\colon B \to C$ is a finite morphism of degree 2 *with the following conditions:*

1. *The branched locus of q_B coincides with the closed set $\Sigma := \{t \in C \mid Q_t = q^{-1}(t)$ is singular $\}$ with the reduced induced closed subscheme structure;*
2. *$\mathcal{J}(Q)$ and* $\mathrm{Jac}(B)$ *are isomorphic as complex tori, where $\mathcal{J}(Q)$ denotes the intermediate Jacobian of Q. If $C = \mathbb{P}^1$, then these are isomorphic as principally polarized Abelian varieties;*
3. *It holds that $(-K_Q)^3 = 40 - (8g(B) + 32g(C))$.*

Proof. By [7, Theorem (2.7.3)] and its proof, we have the contraction $\widetilde{Q} \to P$, which is the blow-up along a bisection B of P. Thus it is easy to check (1). (2) and (3) follow from similar arguments as in Proposition 2.1.

□

Definition 3.2. Let $q\colon Q \to C$ be a quadric fibration. Take a q-section C_0. Then we obtain the double covering $q_B\colon B \to C$ as in Lemma 3.1. By Lemma 3.1 (1), this double covering is independent of the choice of C_0. We call this $q_B\colon B \to C$ *the double covering associated to* q.

Lemma 3.3. *Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration and C_0 a φ-section. Let $(q\colon Q \to C, T)$ be the relative double projection as in Definition 2.2. Let $\varphi_B\colon B \to C$ be the double covering associated to q and $\varphi_T := q|_T$.*

Let $\mathrm{Hilb}_{3t+1}(X/C) \to \overline{B} \to C$ *(respectively* $\mathrm{Hilb}_{2t+1}(X/C) \to \overline{T} \to C$*) be the Stein factorization of the relative Hilbert scheme of twisted cubics (respectively conics). Let B' (respectively T') be the normalization of $\overline{B}$ (respectively $\overline{T}$). Then B (respectively T) is isomorphic to B' (respectively T') over C. In particular, φ_B and φ_T are independent of the choice of φ-sections.*

Proof. We fix a φ-section C_0 and take the diagram (2.1) as in Proposition 2.1. Let $U \subset C$ be an open set such that X_t is smooth and $-K_{\widetilde{X}_t}$ is ample for every $t \in U$. By Proposition 2.1 (2), U is not empty and the birational map $\chi\colon \widetilde{X} \dashrightarrow \widetilde{Q}$ is isomorphic over U. Set $X_U := \varphi^{-1}(U)$, $Q_U := q^{-1}(U)$, $\widetilde{X}_U := \widetilde{\varphi}^{-1}(U) \simeq \widetilde{q}^{-1}(U)$, $G_U := \widetilde{q}^{-1}(U) \cap G$, and $T_U = \varphi_T^{-1}(U)$. We set the morphisms as in the following diagram:

$$\begin{tikzcd}
& \widetilde{X}_U \arrow[dl] \arrow[dd, "\widetilde{\varphi}_U"] \arrow[dr] & \supset & G_U \arrow[dr] & \\
X_U \arrow[dr, "\varphi_U"'] & & Q_U \arrow[dl, "q_U"] & \supset & T_U \\
& U. & & &
\end{tikzcd} \tag{3.1}$$

Note that $\widetilde{\varphi}_U$, φ_U, and q_U are isotrivial.

First, we show the assertion for T. Composing the morphisms in the diagram (3.1), we have a morphism $e_U\colon G_U \to X_U$ over U. Then we can regard the $\mathbb{P}^1$-bundle $G_U \to T_U$ as a family of conics in the fibers of $X_U \to U$. By the universal property, there is a natural morphism $T_U \to \mathrm{Hilb}_{2t+1}(X_U/U)$ over U. Since φ_U is isotrivial, the morphism $\mathrm{Hilb}_{2t+1}(X_U/U) \to U$ factors an étale triple cover T'_U over U as the Stein factorization. Hence $T_U \to \mathrm{Hilb}_{2t+1}(X_U/U)$ is a section of $\mathrm{Hilb}_{2t+1}(X_U/U) \to T'_U$, which implies that $T_U \simeq T'_U$. Hence T' is isomorphic to T.

Next, we show the assertion for B. Set $B_U = \varphi_B^{-1}(U)$. Then by Lemma 3.1 (1), $\mathrm{Hilb}_{t+1}(Q_U/U) \to U$ factors through B_U such that $\mathrm{Hilb}_{t+1}(Q_U/U) \to B_U$ is a $\mathbb{P}^1$-bundle and $B_U \to U$ is an étale double covering. Let $R_U \to \mathrm{Hilb}_{t+1}(Q_U/U)$ be the universal family of the relative Hilbert scheme of lines. Set $R'_U := \widetilde{X}_U \times_{Q_U} R_U$. The flat family $R'_U \to \mathrm{Hilb}_{t+1}(Q_U/U)$ parametrizes twisted cubics on X_U over U by the evaluation $R'_U \to X_U$. Hence the universal property gives a morphism $\mathrm{Hilb}_{t+1}(Q_U/U) \to \mathrm{Hilb}_{3t+1}(X_U/U)$ over U. Let $B'_U \to U$ be the finite part of the Stein factorization of $\mathrm{Hilb}_{3t+1}(X_U/U) \to U$. Then we get a morphism $B_U \to B'_U$. Thanks to isotriviality, we can check that B'_U is an étale double covering over U and $B_U \to B'_U$ is bijective. Hence $B_U \to B'_U$ is an isomorphism. Thus B is isomorphic to B'.

The proof is complete. □

Definition 3.4. Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration. We define $\varphi_B\colon B \to C$ and $\varphi_T\colon T \to C$ as in the settings of Lemma 3.3. We call φ_B (respectively φ_T) the *associated double* (respectively *triple*) *covering* to that sextic del Pezzo fibration φ.

3.2. Proof of Theorem A

We use the same notation as in Proposition 2.1.

(1) As in Proposition 2.1 (2) and (3), the birational map $\chi : \widetilde{X} \dashrightarrow \widetilde{Q}$ is an isomorphism or a flop of isolated (-2)-curves. Then by [25, Corollary (5.6) and (5.7)], χ is a composition of blow-ups and blowing-downs along smooth rational curves. Hence $\mathcal{J}(\widetilde{X})$ is isomorphic to $\mathcal{J}(\widetilde{Q})$ as complex tori by [5, Lemma 3.11]. When $C = \mathbb{P}^1$ moreover, the isomorphism $\mathcal{J}(\widetilde{X}) \simeq \mathcal{J}(\widetilde{Q})$ preserves the polarizations. Therefore, the assertion follows from Lemma 3.1 (2) and [5, Lemma 3.11].

(2) The assertion from a direct calculation using the formulas in Lemma 3.1 (3) and Proposition 2.1 (5).

The proof of Theorem A is complete. □

3.3. Characterizations of some nef vector bundles on quadric surfaces

A locally free sheaf $\mathcal{E}$ is called a nef vector bundle if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef. In this subsection, we will obtain numerical characterizations of some nef vector bundles on an irreducible and reduced quadric surface. We will use these results for proving Theorem C.

Until the end of this subsection, we work over the following setting:

- $\mathbb{Q}$ denotes an irreducible and reduced quadric surface in $\mathbb{P}^3$. We refer the definitions of $\mathcal{O}_{\mathbb{Q}}(1)$ and $\mathscr{S}_{\mathbb{Q}}$ to Definition 1.4;
- When $\mathbb{Q}$ is smooth, we set $\pi : \mathbb{F} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1) \oplus \mathcal{O}(1)) \to \mathbb{P}^1$ and let $\sigma : \mathbb{F} \to \mathbb{Q}$ be an isomorphism;
- When $\mathbb{Q}$ is singular, we set $\pi : \mathbb{F} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathbb{P}^1$ and let $\sigma : \mathbb{F} \to \mathbb{Q}$ be the contraction of the (-2)-curve;
- Let h (respectively f) be a tautological divisor (respectively a fiber) of the $\mathbb{P}^1$-bundle $\mathbb{F} \to \mathbb{P}^1$. Note that $\sigma^*\mathcal{O}_{\mathbb{Q}}(1) = \mathcal{O}_{\mathbb{F}}(h)$ for each case.

Remark 3.5. By [24], $\mathscr{S}^{\vee}_{\mathbb{Q}}$ is a globally generated vector bundle of rank 2 with $\det \mathscr{S}^{\vee}_{\mathbb{Q}} = \mathcal{O}_{\mathbb{Q}}(1)$ and $c_2(\mathscr{S}^{\vee}_{\mathbb{Q}}) = 1$. When $\mathbb{Q}$ is smooth, it is known that $\mathscr{S}^{\vee}_{\mathbb{Q}} \simeq \mathcal{O}_{\mathbb{Q}}(h-f) \oplus \mathcal{O}_{\mathbb{Q}}(f)$.

The aim is to give the following characterization of the bundle $\mathscr{S}^{\vee}_{\mathbb{Q}} \oplus \mathcal{O}_{\mathbb{Q}}(1)$.

Proposition 3.6. *Let $\mathcal{F}_{\mathbb{Q}}$ be a nef vector bundle on $\mathbb{Q}$ of rank 3 such that $\det \mathcal{F}_{\mathbb{Q}} = \mathcal{O}_{\mathbb{Q}}(2)$, $c_2(\mathcal{F}) = 3$, and $h^0(\mathcal{F}_{\mathbb{Q}}^{\vee}) = 0$. Then $\mathcal{F}_{\mathbb{Q}}$ is isomorphic to $\mathscr{S}_{\mathbb{Q}}^{\vee} \oplus \mathcal{O}_{\mathbb{Q}}(1)$.*

First, we confirm the following lemma.

Lemma 3.7. *Let $\mathcal{E}$ be a rank 2 nef vector bundle on $\mathbb{Q}$ with $\det \mathcal{E} \simeq \mathcal{O}_{\mathbb{Q}}(1)$.*

(1) *If $c_2(\mathcal{E}) = 0$, then $\mathcal{E} \simeq \mathcal{O}_{\mathbb{Q}}(1) \oplus \mathcal{O}_{\mathbb{Q}}$;*
(2) *If $c_2(\mathcal{E}) = 1$, then $\mathcal{E} \simeq \mathscr{S}_{\mathbb{Q}}^{\vee}$.*

Proof. The assertion was already proved in [22,27] if $\mathbb{Q}$ is non-singular. We attain a proof even if $\mathbb{Q}$ is singular.

(1) When $c_2(\mathcal{E}) = 0$, then the Hirzebruch-Riemann-Roch theorem and the Leray spectral sequence gives $\chi(\mathcal{E}(-1)) = \chi(\sigma^*\mathcal{E}(-h)) = 1$. Note that $h^2(\mathbb{Q}, \mathcal{E}(-1)) = h^0(\mathcal{E}^{\vee}(-1)) = h^0(\mathcal{E}(-2)) \leq h^0(\mathcal{E}(-1))$ by the Serre duality. Thus we have $h^0(\mathcal{E}(-1)) > 0$ and hence an injection $\mathcal{O}(1) \to \mathcal{E}$. By the same arguments as in [22, Proposition 5.2], we have $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}_{\mathbb{Q}}(1)$.

(2) Suppose that $c_2(\mathcal{E}) = 1$. Set $\widetilde{\mathcal{E}} := \sigma^*\mathcal{E}$. Then it holds that $\det \widetilde{\mathcal{E}} = \mathcal{O}(h)$ and $c_2(\widetilde{\mathcal{E}}) = 1$. We have $\chi(\widetilde{\mathcal{E}}(-h+f)) = 1$ by the Hirzebruch-Riemann-Roch theorem and $h^2(\widetilde{\mathcal{E}}(-h+f)) = h^0(\widetilde{\mathcal{E}}^{\vee}(-h-f)) = h^0(\widetilde{\mathcal{E}}(-2h-f))$ by the Serre duality. Since $\widetilde{\mathcal{E}}$ is nef and $c_1(\widetilde{\mathcal{E}}) = h$, we have $\widetilde{\mathcal{E}}|_l = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ for every $l \in |f|$ and hence $h^0(\widetilde{\mathcal{E}}(-2h-f)) = 0$, which implies $h^0(\widetilde{\mathcal{E}}(-h+f)) > 0$. Hence there is an injection $\alpha\colon \mathcal{O}(h-f) \to \widetilde{\mathcal{E}}$.

If there exists a fiber $l \in |f|$ such that $\alpha|_l = 0$, then we obtain a non-zero map $\mathcal{O}(h) \to \widetilde{\mathcal{E}}$, which means $h^0(\widetilde{\mathcal{E}}(-h)) > 0$. Then we however have $\widetilde{\mathcal{E}} \simeq \mathcal{O} \oplus \mathcal{O}(h)$ by the same argument as in the proof of Lemma 3.7 (1), which contradicts $c_2(\widetilde{\mathcal{E}}) = 1$. Therefore, for every $l \in |f|$, $\alpha|_l$ is a non-zero map from $\mathcal{O}(h-f)|_l \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ to $\widetilde{\mathcal{E}}|_l \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Then $\operatorname{Cok} \alpha|_l = \mathcal{O}_l$ and hence $\operatorname{Cok} \alpha$ is the pull-back of a line bundle on $\mathbb{P}^1$. Since $\det \widetilde{\mathcal{E}} = \mathcal{O}(h)$, we obtain an exact sequence

$$0 \to \mathcal{O}(h-f) \to \widetilde{\mathcal{E}} \to \mathcal{O}(f) \to 0. \tag{3.2}$$

If $\mathbb{Q}$ is smooth, then this sequence splits and hence we obtain $\widetilde{\mathcal{E}} \simeq \mathcal{O}(h-f) \oplus \mathcal{O}(f) \simeq \mathscr{S}_{\mathbb{Q}}^{\vee}$. If $\mathbb{Q}$ is singular, then this sequence does not split since $\widetilde{\mathcal{E}}$ is nef and $\mathcal{O}(h-f)$ is not nef. The vector bundles fitting into the exact sequence (3.2) which does not split are unique up to isomorphism since $\operatorname{Ext}^1(\mathcal{O}(f), \mathcal{O}(h-f)) = H^1(\mathbb{F}_2, \mathcal{O}(h-2f)) = \mathbb{C}$. By the exactly same argument, $\sigma^*\mathscr{S}_{\mathbb{Q}}^{\vee}$ also fits into the exact sequence (3.2). Hence we obtain $\sigma^*\mathscr{S}_{\mathbb{Q}}^{\vee} \simeq \sigma^*\mathcal{E}$ and hence $\mathscr{S}_{\mathbb{Q}}^{\vee} \simeq \mathcal{E}$. □

Lemma 3.8. *Let $\mathcal{F}$ be a rank 3 nef vector bundle on $\mathbb{F}$ with $\det \mathcal{F} = \mathcal{O}_{\mathbb{F}}(2h)$ and $c_2(\mathcal{F}) = 3$. Then the following assertions hold:*

(1) *$h^0(\mathcal{F}(-2h+af)) = 0$ for any $a \in \mathbb{Z}$;*
(2) *$\mathcal{F}|_l \simeq \mathcal{O} \oplus \mathcal{O}(1)^2$ for a general member $l \in |f|$;*

(3) $h^0(\mathcal{F}(-h-f)) = 0$;
(4) $h^0(\mathcal{F}(-h)) > 0$.

Proof. (1) We may assume that $a \geq 0$. We will prove (1) by the induction on a. Assume that $a = 0$. If $h^0(\mathcal{F}(-2h)) \neq 0$, then it follows that $\mathcal{F} = \mathcal{O}(2h) \oplus \mathcal{O}^2$ from [22, Proposition 5.2], which contradicts $c_2(\mathcal{F}) = 3$. Thus we have $h^0(\mathcal{F}(-2h)) = 0$. Assume that $a > 0$. If $h^0(\mathcal{F}(-2h+af)) \neq 0$, then there exists an injection $\iota\colon \mathcal{O}(2h-af) \to \mathcal{F}$. By the assumption of the induction $h^0(\mathcal{F}(-2h+(a-1)f)) = 0$, for every member $l \in |f|$, we have $\iota|_l \neq 0$. Therefore, we obtain an exact sequence $0 \to \mathcal{O}(2h-af)|_l \to \mathcal{F}|_l \to \operatorname{Cok}\iota|_l \to 0$. Since $\mathcal{F}|_l$ is a nef vector bundle with $c_1 = 2$, it holds that $\mathcal{F}|_l = \mathcal{O}(2) \oplus \mathcal{O}^2$ and $\operatorname{Cok}\iota|_l = \mathcal{O}_l^2$. It implies that the natural morphism $\pi^*\pi_*\operatorname{Cok}\iota \to \operatorname{Cok}\iota$ is isomorphic. Thus we obtain an exact sequence $0 \to \mathcal{O}(2h-af) \to \mathcal{F} \to \mathcal{O}(a_1 f) \oplus \mathcal{O}((a-a_1)f) \to 0$ for some $a_1 \in \mathbb{Z}$. Hence we have $c_2(\mathcal{F}) = 2a$, which contradicts $c_2(\mathcal{F}) = 3$.

(2) Assume the contrary. Then we obtain $\mathcal{F}|_l \simeq \mathcal{O}^2 \oplus \mathcal{O}(2)$ for any member $l \in |f|$ by the upper semicontinuity. Then $\pi_*\mathcal{F}(-2h)$ is a line bundle on $\mathbb{P}^1$. Hence we have a non-zero map $\pi^*\pi_*\mathcal{F}(-2h) \to \mathcal{F}(-2h)$. It implies that $h^0(\mathcal{F}(2h-af)) \neq 0$ for some $a \in \mathbb{Z}$, which contradicts (1).

(3) If $h^0(\mathcal{F}(-h-f)) \neq 0$, then we have an injection $\alpha\colon \mathcal{O}(h+f) \to \mathcal{F}$. When $\mathbb{Q}$ is smooth, we consider another ruling $h-f$. For every $l' \in |h-f|$, $\alpha|_{l'}\colon \mathcal{O}(h+f)|_{l'} \to \mathcal{F}|_{l'}$ is injective by (1). Since $\mathcal{O}(h+f)|_{l'} \simeq \mathcal{O}_{\mathbb{P}^1}(2)$ and $c_1(\mathcal{F}|_{l'}) = 2$, we have $\operatorname{Cok}\alpha|_{l'} \simeq \mathcal{O}_{l'}^2$ and hence $\operatorname{Cok}\alpha$ is a nef vector bundle. Then $\operatorname{Cok}\alpha \simeq \mathcal{O} \oplus \mathcal{O}(h-f)$ holds, which contradicts $c_2(\mathcal{F}) = 3$. When $\mathbb{Q}$ is singular, we take the restriction $\alpha|_{C_0}$ on the (-2)-curve C_0. By (1), $\alpha|_{C_0}$ is a non-zero map from $\mathcal{O}(h+f)|_{C_0} \simeq \mathcal{O}(1)$ to $\mathcal{F}|_{C_0} \simeq \mathcal{O}^2$, which is a contradiction. Therefore, we have $h^0(\mathcal{F}(-h-f)) = 0$.

(4) We have $\chi(\mathcal{F}(-h)) = 1$ by the Hirzebruch-Riemann-Roch theorem and $h^2(\mathcal{F}(-h)) = h^0(\mathcal{F}^\vee(-h))$ by the Serre duality. Since $\mathcal{F}^\vee|_l = \mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1)^2$ for any $l \in |f|$, we have $h^0(\mathcal{F}^\vee(-h)) = 0$ and hence $h^0(\mathcal{F}(-h)) > 0$. □

Proof of Proposition 3.6. Set $\mathcal{F} := \sigma^*\mathcal{F}_{\mathbb{Q}}$. Lemma 3.8 (4) attains an injection $\alpha\colon \mathcal{O}(h) \to \mathcal{F}$. Letting $\mathcal{E} = \operatorname{Cok}\alpha$, we have the following exact sequence:

$$0 \to \mathcal{O}(h) \xrightarrow{\alpha} \mathcal{F} \to \mathcal{E} \to 0. \tag{3.3}$$

By Lemma 3.8 (3), $\alpha|_l$ is non-zero for every $l \in |f|$. By Lemma 3.8 (2), the cokernel of $\alpha|_l\colon \mathcal{O}(h)|_l \to \mathcal{F}|_l$ is locally free for general members $l \in |f|$. Then $\mathcal{E}$ is locally free in codimension 1 and hence torsion free by [22, Lemma 5.4]. Hence we have the following exact sequence:

$$0 \to \mathcal{E} \xrightarrow{\iota} \mathcal{E}^{\vee\vee} \to \mathcal{T} \to 0. \tag{3.4}$$

Here $\mathcal{T}$ denotes the cokernel of the natural injection $\iota\colon \mathcal{E} \to \mathcal{E}^{\vee\vee}$. Since $\mathcal{E}$ is locally free in codimension 1, $\operatorname{Supp}\mathcal{T}$ is 0-dimensional or empty. Note that $\mathcal{E}^{\vee\vee}$ is locally free since $\mathbb{F}$ is a smooth surface. By [23, Lemma 9.1], $\mathcal{E}^{\vee\vee}$ is nef. It is clear that $\det\mathcal{E}^{\vee\vee} = \det\mathcal{E} = \mathcal{O}(h)$. The Hirzebruch-Riemann-Roch theorem implies that $\chi(\mathcal{F}) = 8$, $\chi(\mathcal{O}(h)) = 4$, and $\chi(\mathcal{E}^{\vee\vee}) = 5 - c_2(\mathcal{E}^{\vee\vee})$. By the exact sequences (3.3) and (3.4), we have $\chi(\mathcal{E}) = 4$ and $5 - c_2(\mathcal{E}^{\vee\vee}) = \chi(\mathcal{E}^{\vee\vee}) = \chi(\mathcal{E}) + h^0(\mathcal{T}) \geq 4$. In particular, $c_2(\mathcal{E}^{\vee\vee}) = 0$ or 1.

If $c_2(\mathcal{E}^{\vee\vee}) = 0$, then $\mathcal{E}^{\vee\vee} \simeq \mathcal{O}(h) \oplus \mathcal{O}$ by Lemma 3.7 (1). Hence we have a morphism $\mathcal{F} \to \mathcal{O}_{\mathbb{F}}$, which is surjective at the generic point. This contradicts our assumption $h^0(\mathcal{F}_{\mathbb{Q}}^{\vee}) = 0$. Hence we have $c_2(\mathcal{E}^{\vee\vee}) = 1$ and $\mathcal{T} = 0$, which implies $\mathcal{E} \simeq \mathcal{E}^{\vee\vee} \simeq \mathscr{S}_{\mathbb{Q}}^{\vee}$ by Lemma 3.7 (2). Since $\mathrm{Ext}^1(\mathscr{S}_{\mathbb{Q}}^{\vee}, \mathcal{O}_{\mathbb{Q}}(h)) = H^1(\mathscr{S}_{\mathbb{Q}}(h)) = H^1(\mathscr{S}_{\mathbb{Q}}^{\vee}) = 0$, the exact sequence (3.3) splits. The proof of Proposition 3.6 is complete. $\square$

3.4. Certain vector bundles on families of surfaces with multi-sections

We will use the following theorem for proving Theorem C.

Theorem 3.9. *Let X, Y be smooth varieties with* $\dim X = \dim Y + 2$. *Let $f\colon X \to Y$ be a flat projective morphism with $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $Z \subset X$ be a locally complete intersection closed subscheme of codimension* 2. *Suppose that $f|_Z\colon Z \to Y$ is finite with* $\deg(f|_Z) \geq 2$. *Additionally we assume that $R^1 f_*\mathcal{O}_X$ is locally free and* $H^2(Y, f_*\omega_f \otimes R^1 f_*\mathcal{I}_Z) = 0$.

Then there exists a locally free sheaf $\mathcal{F}$ satisfying the following conditions:

(1) *$\mathcal{F}$ fits into the following exact sequence:*

$$0 \to f^*((R^1 f_*\mathcal{I}_Z)(-K_Y)) \to \mathcal{F} \to \mathcal{I}_Z(-K_X) \to 0; \tag{3.5}$$

(2) *For every closed point $y \in Y$, if the fiber X_y is reduced, then there are no surjections $\mathcal{F}|_{X_y} \to \mathcal{O}_{X_y}$.*

This theorem resembles the Hartshorne-Serre correspondence for a family of surfaces $f\colon X \to Y$ with a multi-section Z. However, even if Y is point, the Hartshorne-Serre correspondence does not imply Theorem 3.9 since $H^2(\mathcal{O}(K_X)) = \mathbb{C} \neq 0$. To prove Theorem 3.9, we use relative Ext-sheaves. We will prove this theorem in Appendix A.

4. Extensions to moderate $(\mathbb{P}^1)^3$-fibrations

We devote this section to the proof of Theorem B.

4.1. Moderate $(\mathbb{P}^1)^3$-fibrations

In the next theorem, we construct certain Mori fiber spaces $\varphi_Y\colon Y \to C$ with smooth total spaces Y whose smooth fibers are $(\mathbb{P}^1)^3$.

Theorem 4.1. *Let $f : \mathbb{F} \to C$ be a $\mathbb{P}^3$-bundle and $T \subset \mathbb{F}$ a smooth curve. Assume that* $\deg(f|_T) = 3$ *and $T_t := (f|_T)^{-1}(t)$ is non-colinear in $\mathbb{F}_t = \mathbb{P}^3$ for each $t \in C$.*

(1) *There exists a unique sub $\mathbb{P}^2$-bundle $\mathbb{E} \subset \mathbb{F}$ containing T;*
(2) *There exists the following diagram:*

$$\begin{tikzcd}
& \widetilde{\mathbb{F}}^+ \arrow[dl, "\mu_{\mathbb{F}}"'] \arrow[dr, "\psi^+_{\mathbb{F}}"] & & \widetilde{\mathbb{F}} = \mathrm{Bl}_T \mathbb{F} \arrow[ll, dashed, "\Phi"'] \arrow[dl, "\psi_{\mathbb{F}}"'] \arrow[dd, "\widetilde{f}"] \arrow[dr, "\sigma_{\mathbb{F}}"] & \\
Y \arrow[dr, "\varphi_Y"'] & & \overline{\mathbb{F}} \arrow[dl] \arrow[dr] & & \mathbb{F} \arrow[dl, "f"] \\
& C \arrow[rr, equal] & & C, &
\end{tikzcd} \tag{4.1}$$

where

- $\sigma_{\mathbb{F}} : \widetilde{\mathbb{F}} := \mathrm{Bl}_T \mathbb{F} \to \mathbb{F}$ *is the blow-up of $\mathbb{F}$ along T with exceptional divisor $G_{\mathbb{F}} := \mathrm{Exc}(\sigma_{\mathbb{F}})$;*
- $\Phi : \widetilde{\mathbb{F}} \dashrightarrow \widetilde{\mathbb{F}}^+$ *is a family of Atiyah flops over C;*
- *Y is smooth and $\varphi_Y : Y \to C$ is a Mori fiber space;*
- *$\mu_{\mathbb{F}}$ is the blow-up along a φ_Y-section C_0 and contracts the proper transform $\widetilde{\mathbb{E}}^+$ of $\mathbb{E}$.*

(3) *If we set $G_Y \subset Y$ be the proper transform of $G_{\mathbb{F}}$, then it holds that $-K_Y \sim 2G_Y - \varphi_Y^*(K_C + \det f_*\mathcal{O}_{\mathbb{F}}(\mathbb{E}))$;*
(4) *Every smooth φ_Y-fiber is isomorphic to $(\mathbb{P}^1)^3$.*

In this paper, we call this Mori fiber space $\varphi_Y : Y \to C$ the moderate $(\mathbb{P}^1)^3$-fibration *with respect to the pair $(f : \mathbb{F} \to C, T)$.*

Remark 4.2. In the setting of Theorem 4.1, for every $t \in C$, the scheme $T_t = f^{-1}(t) \cap T$ is reduced, or a union of one reduced point and a non-reduced length 2 subscheme, or a curvilinear scheme of length 3, *i.e.*, $T_t \simeq \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^3)$.

Proof of Theorem 4.1. We proceed in 4 steps.

Step 1. First, we show (1). Let $\mathcal{O}_{\mathbb{F}}(1)$ be a tautological line bundle. Then $f_*(\mathcal{O}_{\mathbb{F}}(1) \otimes \mathcal{I}_T)$ is a line bundle $\mathcal{L}$ on C. Let $\mathbb{E} \in |\mathcal{O}_{\mathbb{F}}(1) \otimes \mathcal{I}_T \otimes f^*\mathcal{L}^{-1}|$ be a member corresponding to a nowhere vanishing section of $\mathcal{O}_C \simeq f_*(\mathcal{O}_{\mathbb{F}}(1) \otimes \mathcal{I}_T) \otimes \mathcal{L}^{-1}$. Then $\mathbb{E} \to C$ is a $\mathbb{P}^2$-bundle and $\mathbb{E}_t$ is the linear span of T_t for every $t \in C$, which proves (1). Replacing $\mathcal{O}_{\mathbb{F}}(1)$ by $\mathcal{O}_{\mathbb{F}}(\mathbb{E})$, we have $-K_{\mathbb{F}} = 4\mathbb{E} - f^*(K_C + \det f_*\mathcal{O}_{\mathbb{F}}(\mathbb{E}))$.

Step 2. Let $\sigma_{\mathbb{F}} : \widetilde{\mathbb{F}} \to \mathbb{F}$ and $\sigma_{\mathbb{E}} : \widetilde{\mathbb{E}} \to \mathbb{E}$ be the blow-ups along T and $G_{\mathbb{F}} \subset \widetilde{\mathbb{F}}$ and $G_{\mathbb{E}} \subset \widetilde{\mathbb{E}}$ the exceptional divisors of $\sigma_{\mathbb{F}}$ and $\sigma_{\mathbb{E}}$ respectively. Set

$$L_{\widetilde{\mathbb{F}}} := \sigma_{\mathbb{F}}^*\mathcal{O}_{\mathbb{F}}(2) - G_{\mathbb{F}} \text{ and } L_{\widetilde{\mathbb{E}}} := L_{\widetilde{\mathbb{F}}}|_{\widetilde{\mathbb{E}}}. \tag{4.2}$$

Note that

$$-K_{\widetilde{\mathbb{F}}} = 2L_{\widetilde{\mathbb{F}}} - (f \circ \sigma)^*(K_C + \det f_*\mathcal{O}_{\mathbb{F}}(1)) \sim_C 2L_{\widetilde{\mathbb{F}}}. \tag{4.3}$$

The following claim is the 2-ray game of $\widetilde{\mathbb{E}}$ over C.

Claim 4.3.

(1) $\mathcal{O}(L_{\widetilde{\mathbb{E}}})$ is globally generated and big over C and $(f|_{\mathbb{E}} \circ \sigma_{\mathbb{E}})_*\mathcal{O}(L_{\widetilde{\mathbb{E}}})$ is a vector bundle of rank 3;

(2) Set $\overline{\mathbb{E}} := \mathbb{P}_C((f|_{\mathbb{E}} \circ \sigma_{\mathbb{E}})_*\mathcal{O}(L_{\widetilde{\mathbb{E}}}))$ and let $\psi_{\mathbb{E}} \colon \widetilde{\mathbb{E}} \to \overline{\mathbb{E}}$ denote the morphisms over C defined by $|L_{\widetilde{\mathbb{E}}}|$. Then $\psi_{\mathbb{E}}$ is the blow-up of $\overline{\mathbb{E}}$ along a non-singular curve $\overline{T} \subset \overline{\mathbb{E}}$. Moreover, the composite morphism $\overline{T} \hookrightarrow \overline{\mathbb{E}} \to C$ is a triple covering.

Proof. (1) For every $t \in C$, there is a smooth conic $C \subset \mathbb{E}_t = \mathbb{P}^2$ containing T_t. By the exact sequence $0 \to \mathcal{I}_{C/\mathbb{E}_t} \to \mathcal{I}_{T_t/\mathbb{E}_t} \to \mathcal{I}_{T_t/C} \to 0$, we see that $\mathcal{O}_{\mathbb{E}_t}(2) \otimes \mathcal{I}_{T_t/\mathbb{E}_t}$ is globally generated and $h^0(\mathcal{O}_{\mathbb{E}_t}(2) \otimes \mathcal{I}_{T_t/\mathbb{E}_t}) = 3$ for every t, which proves (1).

(2) For a general point $t \in C$, $\overline{\mathbb{E}}_t \leftarrow \widetilde{\mathbb{E}}_t \to \mathbb{E}_t = \mathbb{P}^2$ is nothing but the Cremona involution. Thus $\psi_{\mathbb{E}}$ is a birational morphism onto the $\mathbb{P}^2$-bundle $\overline{\mathbb{E}} \to C$. Since $-K_{\widetilde{\mathbb{E}}} \sim_C L_{\mathbb{E}} + (\sigma|_{\mathbb{E}})^*\mathcal{O}_{\mathbb{E}}(1)$ is ample over C and $\rho(\widetilde{\mathbb{E}}) = 3$, $\psi_{\mathbb{E}}$ is the contraction of an extremal ray. Then $\psi_{\mathbb{E}}$ is the blow-up along a non-singular curve $\overline{T}$ by [19, Theorem (3.3)]. For a general $t \in C$, $\widetilde{\mathbb{E}}_t \to \overline{\mathbb{E}}_t = \mathbb{P}^2$ is the blow-up at three points. Hence $\overline{T} \to C$ is a triple covering. □

Step 3. Next we play the 2-ray game of $\widetilde{\mathbb{F}}$ over C by using Claim 4.3.

Claim 4.4.

(1) $\mathcal{O}(L_{\widetilde{\mathbb{F}}})$ is globally generated and big over C;

(2) Let $\psi_{\mathbb{F}} \colon \widetilde{\mathbb{F}} \to \overline{\mathbb{F}}$ denote the Stein factorization of the morphism over C defined by $|L_{\widetilde{\mathbb{F}}}|$. Then $\psi_{\mathbb{F}}|_{\widetilde{\mathbb{E}}} = \psi_{\mathbb{E}}$ and $\mathrm{Exc}(\psi_{\mathbb{F}}) = \mathrm{Exc}(\psi_{\mathbb{E}})$;

(3) $\psi_{\mathbb{F}}$ is a family of Atiyah's flopping contractions over C.

Proof. (1) For any $t \in C$, we have $(L_{\widetilde{\mathbb{F}}}|_{\widetilde{\mathbb{F}}_t})^3 = 5$ by a direct calculation. Since $\widetilde{\mathbb{E}} \sim \sigma^*\mathcal{O}_{\mathbb{F}}(1) - G_{\mathbb{F}}$, we have $\widetilde{\mathbb{E}} + \sigma^*\mathcal{O}_{\mathbb{F}}(1) \sim L_{\widetilde{\mathbb{F}}}$. Then we obtain an exact sequence $0 \to \sigma^*\mathcal{O}_{\mathbb{F}}(1) \to \mathcal{O}_{\widetilde{\mathbb{F}}}(L_{\widetilde{\mathbb{F}}}) \to \mathcal{O}_{\widetilde{\mathbb{E}}}(L_{\widetilde{\mathbb{E}}}) \to 0$. Since $R^1(f \circ \sigma)_*(\sigma^*\mathcal{O}_{\mathbb{F}}(1)) = 0$, $\mathcal{O}(L_{\widetilde{\mathbb{F}}})$ is globally generated over C by Claim 4.3 (1).

(2) Let $\gamma \subset \widetilde{\mathbb{F}}$ be an irreducible curve with $L_{\widetilde{\mathbb{F}}}.\gamma = 0$. Then $L_{\widetilde{\mathbb{F}}} \sim_C \widetilde{\mathbb{E}} + \sigma^*\mathcal{O}_{\mathbb{F}}(1)$ and $\sigma^*\mathcal{O}_{\mathbb{F}}(1).\gamma > 0$ since $L_{\widetilde{\mathbb{F}}}$ is ample over C. Thus we obtain $\widetilde{\mathbb{E}}.\gamma < 0$, which implies $\gamma \subset \widetilde{\mathbb{E}}$. Since $L_{\widetilde{\mathbb{E}}}.\gamma = 0$, γ is a fiber of $\mathrm{Exc}(\psi_{\mathbb{E}}) \to \overline{T}$. Conversely, every curve γ contracted by $\psi_{\mathbb{E}}$ is also contracted by $\psi_{\mathbb{F}}$. Then we have $\psi_{\mathbb{F}}|_{\widetilde{\mathbb{E}}} = \psi_{\mathbb{E}}$ by the rigidity lemma and hence $\mathrm{Exc}(\psi_{\mathbb{F}}) = \mathrm{Exc}(\psi_{\mathbb{E}})$.

(3) Let l be a fiber of $\mathrm{Exc}(\psi_{\mathbb{F}}) \to \overline{T}$. Then we have $l \simeq \mathbb{P}^1$ and the exact sequence $0 \to \mathcal{N}_{l/\widetilde{\mathbb{E}}} \to \mathcal{N}_{l/\widetilde{\mathbb{F}}} \to \mathcal{N}_{\widetilde{\mathbb{E}}/\widetilde{\mathbb{F}}}|_l \to 0$, which implies that $\mathcal{N}_{l/\widetilde{F}} = \mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}$. The proof is complete. $\square$

Step 4. Let $\psi_{\mathbb{F}}^+ : \widetilde{\mathbb{F}}^+ \to \overline{\mathbb{F}}$ be the flop of $\psi_{\mathbb{F}}$. Let $\widetilde{\mathbb{E}}^+$ and $G_{\mathbb{F}}^+$ be the proper transforms of $\widetilde{\mathbb{E}}$ and $G_{\mathbb{F}}$ on $\widetilde{\mathbb{F}}^+$ respectively.

Claim 4.5. There exists a birational morphism

$$\mu_{\mathbb{F}} : \widetilde{\mathbb{F}}^+ \to Y$$

over C such that Y is non-singular and $\mu_{\mathbb{F}}$ blows $\widetilde{\mathbb{E}}^+$ down to a φ_Y-section $C_{0,Y}$, where $\varphi_Y : Y \to C$ is the induced morphism.

Proof. By the construction of the flop $\widetilde{\mathbb{F}} \dashrightarrow \widetilde{\mathbb{F}}^+$, the proper transform $\widetilde{\mathbb{E}}^+ \subset \overline{\mathbb{F}}^+$ is a $\mathbb{P}^2$-bundle over C. By the equality (4.3), we have $-K_{\widetilde{\mathbb{F}}_t}|_{\widetilde{\mathbb{E}}_t} = \mathcal{O}_{\mathbb{P}^2}(2)$ for every point t. Hence it holds that $-K_{\widetilde{\mathbb{F}}_t^+}|_{\widetilde{\mathbb{E}}_t^+} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ and hence $\mathcal{N}_{\widetilde{\mathbb{E}}_t^+/\widetilde{\mathbb{F}}_t^+} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ for every $t \in C$. Thus we obtain the morphism $\mu_{\mathbb{F}}$. $\square$

Since $\mu_{\mathbb{F}}$ is the contraction of an extremal ray, we obtain $\rho(Y) = 2$. Therefore, φ_Y is the contraction of a K_Y-negative ray. The proof of Theorem 4.1 (2) is complete.

Since $-K_{\widetilde{\mathbb{F}}^+} \sim 4\widetilde{\mathbb{E}}^+ + 2G_{\mathbb{F}}^+ - (\varphi_Y \circ \mu_{\mathbb{F}})^*(K_C + \det f_*\mathcal{O}_{\mathbb{F}}(1))$, if we set $G_Y = \mu_{\mathbb{F}*}G_{\mathbb{F}}^+$, then we have $-K_Y \sim 2G_Y - \varphi_Y^*(K_C + \det f_*\mathcal{O}_{\mathbb{F}}(\mathbb{E}))$, which proves Theorem 4.1 (3).

To confirm Theorem 4.1 (4), we take a point $t \in C$ such that $Y_t = \varphi_Y^{-1}(t)$ is smooth. Then Y_t is a smooth Fano 3-fold with index 2, which is so-called a del Pezzo 3-fold. Since $(-K_{Y_t})^3 = (-K_{\widetilde{\mathbb{F}}_t^+})^3 + 8 = (-K_{\widetilde{\mathbb{F}}_t})^3 + 8 = 48$ and $\rho(Y_t) = \rho(\widetilde{\mathbb{F}}_t^+) - 1 = \rho(\widetilde{\mathbb{F}}_t) - 1 = 3$, we have $Y_t \simeq (\mathbb{P}^1)^3$ by Fujita's classification of the del Pezzo manifolds [13, Theorem 3.3.1]. The proof of Theorem 4.1 is complete. $\square$

4.2. Proof of Theorem B

In this section we prove Theorem B, which asserts that there exists a $(\mathbb{P}^1)^3$-fibration containing a given sextic del Pezzo fibration as a relative hyperplane section.

Proof of Theorem B. Let $\varphi : X \to C$ be a sextic del Pezzo fibration. Let C_0 be q φ-section. Let $(q : Q \to C, T)$ be the relative double projection of $(\varphi : X \to C, C_0)$ as in Definition 2.2. Let $E_Q \subset Q$ be the proper transform of $\mathrm{Exc}(\mu : \mathrm{Bl}_{C_0} X \to X)$. By Proposition 2.1 (6), there exists a divisor α on C such that $-K_Q \sim 2E_Q - q^*\alpha$. Then consider the following projective bundle over C:

$$f : \mathbb{F} := \mathbb{P}_C(q_*\mathcal{O}_Q(E_Q)) \to C.$$

This f is a $\mathbb{P}^3$-bundle and there is a natural closed embedding $Q \hookrightarrow \mathbb{F}$ over C. When $\mathcal{O}_{\mathbb{F}}(1)$ denotes the tautological bundle of f, we have $\mathcal{O}_{\mathbb{F}}(1)|_Q = \mathcal{O}_Q(E_Q)$. Since $-K_Q \sim (\mathcal{O}_{\mathbb{F}}(2) - f^*\alpha)|_Q$, we have the linear equivalence

$$Q \sim \mathcal{O}_{\mathbb{F}}(2) + f^*(\alpha - (K_C + \det(q_*\mathcal{O}_Q(E_Q)))) \tag{4.4}$$

by the adjunction formula.

Claim 4.6. There exists a unique member $\mathbb{E} \in |\mathcal{O}_{\mathbb{F}}(1)|$ such that

(1) $\mathbb{E} \cap Q = E_Q$ and $f|_{\mathbb{E}} \colon \mathbb{E} \to C$ is $\mathbb{P}^2$-bundle;
(2) for every $t \in C$, $T_t = f^{-1}(t) \cap T$ is non-colinear 0-dimensional subscheme of length 3 in $\mathbb{P}^3 = \mathbb{F}_t$ and spans $\mathbb{E}_t$.

In particular, the $\mathbb{P}^3$-bundle $f \colon \mathbb{F} \to C$ and T satisfy the condition of the setting in Theorem 4.1 and hence $\mathbb{E}$ is the sub $\mathbb{P}^2$-bundle as in Theorem 4.1 (1).

Proof. Consider an exact sequence $0 \to \mathcal{O}_{\mathbb{F}}(-Q) \otimes \mathcal{O}_{\mathbb{F}}(1) \to \mathcal{O}_{\mathbb{F}}(1) \to \mathcal{O}_Q(E_Q) \to 0$. Since $\mathcal{O}_{\mathbb{F}}(Q) \sim_C \mathcal{O}_{\mathbb{F}}(2)$, we obtain $R^i f_*(\mathcal{O}_{\mathbb{F}}(-Q) \otimes \mathcal{O}_{\mathbb{F}}(1)) = 0$ for all $i \geq 0$. Then the restriction morphism $H^0(\mathbb{F}, \mathcal{O}_{\mathbb{F}}(1)) \to H^0(Q, \mathcal{O}_Q(E_Q))$ is an isomorphism and hence there exists a unique member $\mathbb{E} \in |\mathcal{O}_{\mathbb{F}}(1)|$ such that $\mathbb{E} \cap Q = E_Q$. Since E_Q is a prime divisor of Q, we obtain $\dim \mathbb{E}_t = 2$ for all $t \in C$, which implies that $\mathbb{E} \to C$ is a $\mathbb{P}^2$-bundle. Thus (1) follows. To prove (2), we assume that there exists a line $l \subset \mathbb{F}_t$ such that $T_t \subset l$ for some t. Since $T_t \subset Q_t$ and Q_t is a quadric surface, we obtain $l \subset Q_t$. Let $\sigma \colon \mathrm{Bl}_T\, Q \to Q$ denote the blow-up as in Proposition 2.1. Then we have $-K_{\mathrm{Bl}_T Q}.\sigma_*^{-1} l < 0$, which is a contradiction since $-K_{\mathrm{Bl}_T Q}$ is nef over C from Proposition 2.1 (1) and (3). Hence the linear span of T_t, say $\langle T_t \rangle$, is a 2-plane in $\mathbb{F}_t$ and thus we deduce that $\mathbb{E}_t = \langle T_t \rangle$ for every $t \in C$. $\square$

By Theorem 4.1, $\mathbb{F} \to C$ can be birationally transformed into a Mori fiber space Y over C with a general fiber $(\mathbb{P}^1)^3$ as in the diagram (4.1). Note that $\widetilde{\mathbb{F}} = \mathrm{Bl}_T\, \mathbb{F}$ contains $\widetilde{Q} = \mathrm{Bl}_T\, Q$ in this setting.

We use the same notation as in Theorem 4.1 and its proof. The only remaining part is to show the following claim.

Claim 4.7.

(1) $\psi_{\mathbb{F}}|_{\widetilde{Q}}$ coincides with ψ_Q in Proposition 2.1 (3);
(2) The proper transform $\widetilde{Q}^+ \subset \widetilde{\mathbb{F}}^+$ of Q is isomorphic to $\widetilde{X}$;
(3) It holds that $\mu_{\mathbb{F}}|_{\widetilde{X}} = \mu_X$ and $C_{0,Y} = C_0$.
1. X is a member of $|G_Y + \varphi_Y^*\beta|$, where we set $\beta := \alpha - (K_C + \det(q_*\mathcal{O}_Q(E_Q)))$;
(4) If we set $H_Y := G_Y - \varphi_Y^*\alpha$ and $\delta := \alpha + \beta$, then we have $-K_Y = 2H_Y + \varphi_Y^*\delta$ and $X \in |H_Y + \varphi_Y^*\delta|$;
(5) Let $\varphi_B \colon B \to C$ be the associated double covering to φ. Then $\mathcal{O}_C(-\delta)$ is isomorphic to $\mathrm{Cok}(\mathcal{O}_C \to \varphi_{B*}\mathcal{O}_B) \otimes \mathcal{O}(-K_C)$, which is nothing but $\mathcal{L}$ in the statement of Theorem B.

Proof. (1) We have $Q \in |\mathcal{O}_{\mathbb{F}}(2) + f^*\beta|$ by the equality (4.4) and hence $\widetilde{Q} \in |\mathcal{O}_{\mathbb{F}}(2) - G_{\mathbb{F}} + (f \circ \sigma)^*\beta| = |L_{\widetilde{\mathbb{F}}} + (f \circ \sigma)^*\beta|$ by the equality (4.2). Recall that $\psi_{\mathbb{F}} \colon \widetilde{\mathbb{F}} \to \overline{\mathbb{F}}$ is the Stein factorization of the morphism given by $|L_{\widetilde{\mathbb{F}}}|$. For every $k > 0$, it follows that $R^1 \widetilde{f}_* \mathcal{O}_{\widetilde{\mathbb{F}}}((k-1)L_{\widetilde{\mathbb{F}}}) = 0$ and hence $\widetilde{f}_* \mathcal{O}_{\widetilde{\mathbb{F}}}(kL_{\widetilde{\mathbb{F}}}) \to (\widetilde{f}|_{\widetilde{Q}})_* \mathcal{O}_{\widetilde{Q}}(kL_{\widetilde{\mathbb{F}}})$ is surjective, which implies $\psi_{\mathbb{F}}|_{\widetilde{Q}} = \psi_Q$.

(2) We obtain $\widetilde{Q} \sim_{\overline{\mathbb{F}}} 0$ as divisors on $\widetilde{\mathbb{F}}^+$ and hence $\overline{Q} := \psi_{\mathbb{F}}(\widetilde{Q})$ is a Cartier divisor on $\overline{\mathbb{F}}$. Then we have $\widetilde{Q}^+ = (\psi_{\mathbb{F}}^+)^{-1}(\overline{Q}) \subset \widetilde{\mathbb{F}}^+$ and hence the dimension of the fibers of $\widetilde{Q}^+ \to \overline{Q}$ is less than or equal to 1. Since $\psi_{\mathbb{F}}|_{\widetilde{Q}}$ is the flopping contraction of $\widetilde{Q}$ over C, $\overline{Q} \cap \psi_{\mathbb{F}}(\mathrm{Exc}(\psi_{\mathbb{F}}))$ is a finite set or the empty set. (2) clearly holds when it is empty. Now we assume $\overline{Q} \cap \psi_{\mathbb{F}}(\mathrm{Exc}(\psi_{\mathbb{F}}))$ is not empty. Hence the morphism $\widetilde{Q}^+ \to \overline{Q}$ is a small contraction. Thus $\widetilde{Q}^+$ is regular in codimension 1 and hence normal since this is an effective divisor of a smooth variety $\widetilde{\mathbb{F}}^+$. Moreover, the birational map $\Psi|_{\widetilde{Q}} \colon \widetilde{Q} \dashrightarrow \widetilde{Q}^+$ is isomorphic in codimension 1, which implies that $E_{\widetilde{Q}} = \widetilde{\mathbb{E}}|_{\widetilde{Q}^+}$ coincides with the proper transform of $\widetilde{\mathbb{E}}^+|_{\widetilde{Q}}$. Since $-\widetilde{\mathbb{E}} \sim_{\overline{Y}} \sigma_{\mathbb{F}}^* \mathcal{O}_{\mathbb{F}}(1)$ is ample over $\overline{Y}$ and $\widetilde{\mathbb{E}}^+$ is ample over $\overline{Y}$, we conclude that $\widetilde{Q}^+$ is the flop of ψ_Q. Then we have $\widetilde{Q}^+ = \widetilde{X}$ by the uniqueness of the flop.

(3) It is enough to show that $\mu_{\mathbb{F}}^* G_Y|_{\widetilde{X}} \sim_C \mu_X^*(-K_X)$. By (4.4), $\widetilde{Q}$ is a member of $|\widetilde{\mathbb{E}} + G + (f \circ \sigma)^*\beta|$ and hence $\widetilde{X}$ is a member of $|\widetilde{\mathbb{E}}^+ + G^+ + (\varphi_Y \circ \mu_{\mathbb{F}})^*\beta|$. Since $-K_{\widetilde{\mathbb{F}}^+} \sim_C 2(\widetilde{\mathbb{E}}^+ + G_{\mathbb{F}}^+)$, we have $-K_{\widetilde{X}} \sim_C (\widetilde{\mathbb{E}}^+ + G_{\mathbb{F}}^+)|_{\widetilde{X}}$. Recalling that we set $G_Y = \mu_{\mathbb{F}*} G^+$, we have $\mu_{\mathbb{F}}^* G_Y|_{\widetilde{X}} \sim_C \mu_X^*(-K_X)$, which completes the proof of (3).

(4) This assertion follows from the equality $\mu_{\mathbb{F}}^* X = \widetilde{\mathbb{E}}^+ + \widetilde{X}$ as divisors.

(5) By Theorem 4.1 (3), we have $-K_Y = 2G_Y - \varphi_Y^*(K_C + \det q_* \mathcal{O}_Q(E_Q)) = 2G_Y + \varphi_Y^*(\beta - \alpha)$. Thus we obtain $-K_Y = 2H_Y + \varphi_Y^*\delta$ and $X \in |H_Y + \varphi_Y^*\delta|$.

(6) Set $\Sigma := \{t \in C \mid q^{-1}(t) \text{ is singular}\}_{\mathrm{red}}$. By (4.4), Q is a member of $|\mathcal{O}_{\mathbb{F}}(2) + f^*\beta|$. Let $u \in H^0(\mathrm{Sym}^2(q_* \mathcal{O}_Q(E_Q)) \otimes \mathcal{O}_C(\beta))$ be the section corresponding to Q. Then the Σ is the degeneracy locus associated to the symmetric form u. Hence we can deduce that Σ is the zero scheme of a global section of $\det(q_* \mathcal{O}(E_Q))^{\otimes 2} \otimes \mathcal{O}(4\beta)$. By Lemma 3.1 (1), Σ is the branched divisor of $\varphi_B \colon B \to C$. Then the Hurwitz formula gives $\omega_B = \varphi_B^*(\omega_C \otimes \det(q_* \mathcal{O}(E_Q)) \otimes \mathcal{O}(2\beta)) = \varphi_B^* \mathcal{O}(\alpha + \beta) = \varphi_B^* \mathcal{O}(\delta)$. By the duality of the finite flat morphism $\varphi_B \colon B \to C$, we have $\mathcal{O}_C(\delta) \otimes \varphi_{B*} \mathcal{O}_B = \varphi_{B*} \omega_B = (\varphi_{B*} \mathcal{O}_B)^\vee \otimes \omega_C$. Thus $\mathcal{O}_C(K_C - \delta)$ is the cokernel of the splitting injection $\mathcal{O}_C \to \varphi_{B*} \mathcal{O}_B$. □

The proof of Theorem B is complete. □

5. Extensions to moderate $(\mathbb{P}^2)^2$-fibrations

We devote this section to prove Theorem C. The main idea is similar to that of Theorem B.

5.1. Moderate $(\mathbb{P}^2)^2$-fibrations

In this section we will obtain a Mori fiber space $\varphi_Z\colon Z \to C$ with smooth total space Z whose smooth fibers are $(\mathbb{P}^2)^2$ as in the next theorem. We call this φ_Z a *moderate* $(\mathbb{P}^2)^2$*-fibration* in this paper.

Theorem 5.1. *Let C be a smooth projective curve and $q\colon Q \to C$ a quadric fibration. Let $T \subset Q$ be a smooth irreducible curve. Assume that $\deg(q|_T) = 3$ and $-K_{\mathrm{Bl}_T Q}$ is nef over C. Then the following assertions hold:*

(1) *There exists a locally free sheaf $\mathcal{F}$ and the following exact sequence*

$$0 \to q^*(R^1 q_* \mathcal{I}_T(-K_C)) \to \mathcal{F} \to \mathcal{I}_T(-K_Q) \to 0 \tag{5.1}$$

such that $\mathcal{F}|_{Q_t} \simeq \mathscr{S}_{Q_t}^{\vee} \oplus \mathcal{O}_{Q_t}(1)$ holds for every $t \in C$. We refer the definitions of $\mathcal{O}_{Q_t}(1)$ and $\mathscr{S}_{Q_t}$ to Definition 1.4;

(2) *There exists the following diagram:*

$$\begin{tikzcd}
& \widetilde{Z} \arrow[dl, "\mu_Z"'] \arrow[dd, "r^+"'] \arrow[dr, "\psi_Z"] & & \mathbb{P}_Q(\mathcal{F}) \arrow[ll, dashed, "\Psi"'] \arrow[dl, "\psi_{\mathcal{F}}"'] \arrow[dd, "r"] \arrow[dr, "\pi_{\mathcal{F}}"] & \\
Z \arrow[dr, "\varphi_Z"'] & & \overline{Z} \arrow[dl] \arrow[dr] & & Q \arrow[dl, "q"] \\
& C \arrow[rr, equal] & & C, &
\end{tikzcd} \tag{5.2}$$

where

- *$\mathbb{P}_Q(\mathcal{F}) \dashrightarrow \widetilde{Z}$ is a family of Atiyah flops;*
- *Z is smooth and $\varphi_Z\colon Z \to C$ is a Mori fiber space;*
- *$\mu_Z\colon \widetilde{Z} \to Z$ is the blow-up along a φ_Z-section C_0;*

(3) *Let $\xi_{\mathcal{F}}$ be a tautological divisor on $\mathbb{P}_Q(\mathcal{F})$ and set $\xi_Z := \mu_{Z*}\Psi_*\xi_{\mathcal{F}}$. Then $-K_Z \sim 3\xi_Z - \varphi_Z^*\beta$ and $\mu_Z^*\xi_Z = \Psi_*\xi_{\mathcal{F}} + \mathrm{Exc}(\mu_Z)$, where $\beta = \det(R^1 q_* \mathcal{I}_T(-K_C))$;*

(4) *Every smooth φ_Z-fiber is isomorphic to $(\mathbb{P}^2)^2$.*

In this paper, we call the Mori fiber space $\varphi_Z\colon Z \to C$ as in Theorem 5.1 the moderate $(\mathbb{P}^2)^2$-fibration *with respect to the pair $(q\colon Q \to C, T)$.*

Proof. We proceed in 4 steps.

Step 1. Let us prove (1). We apply Theorem 3.9 for $X = Q$, $Y = C$, $f = q$, and $Z = T$. Since $R^1 q_* \mathcal{O}_X = 0$ and $\dim C = 1$, Theorem 3.9 gives the exact sequence (5.1) and the locally free sheaf $\mathcal{F}$. Moreover, there are no surjections $\mathcal{F}|_{Q_t} \to \mathcal{O}_{Q_t}$ for every t since Q_t is an irreducible and reduced quadric surface. Note that $\det \mathcal{F}|_{Q_t} = \mathcal{O}(-K_{Q_t})$ and $c_2(\mathcal{F}|_{Q_t}) = 3$ follow from (5.1). If $\mathcal{F}$ is

q-nef, then we have $h^0(\mathcal{F}|_{Q_t}^{\vee}) = 0$ since every morphism $\mathcal{F} \to \mathcal{O}_{Q_t}$ must be surjective, which implies that $\mathcal{F}|_{Q_t} \simeq \mathscr{S}_{Q_t}^{\vee} \oplus \mathcal{O}_{Q_t}(1)$ by Proposition 3.6. Hence it suffices to show that $\mathcal{F}$ is q-nef. To show this nefness, we consider the projectivization $\mathbb{P}_Q(\mathcal{F})$. By the surjection $\mathcal{F} \twoheadrightarrow \mathcal{I}_T(-K_Q)$, $\mathrm{Bl}_T Q$ is embedded in $\mathbb{P}_Q(\mathcal{F})$ over Q as the zero scheme of the global section $s \in H^0(\mathbb{P}_Q(\mathcal{F}), \mathcal{O}_{\mathbb{P}_Q(\mathcal{F})}(1) \otimes q^*(R^1q_*\mathcal{I}_T(-K_C))^{\vee})$ corresponding to the injection $q^*(R^1q_*\mathcal{I}_T(-K_C)) \to \mathcal{F}$. Since $-K_{\mathrm{Bl}_T Q} = \mathcal{O}_{\mathbb{P}_Q(\mathcal{F})}(1)|_{\widetilde{Q}}$ is nef over C by assumption, so is $\mathcal{O}_{\mathbb{P}_Q(\mathcal{F})}(1)$. The proof of (1) is complete.

Step 2. Next we confirm the following claim.

Claim 5.2. There exists a unique effective divisor $E_Q \subset Q$ containing T such that $2E_Q + K_Q \sim_C 0$ and $q_*\mathcal{I}_T(E_Q) = \mathcal{O}_C$.

Proof. Take a $\mathbb{P}^3$-bundle $f\colon \mathbb{F} \to C$ such that $\mathbb{F}$ contains Q. Since $-K_{\mathrm{Bl}_T Q}$ is nef over C, the linear span of T_t is a 2-plane in $\mathbb{F}_t = \mathbb{P}^3$. By Theorem 4.1 (1), there exists a unique sub $\mathbb{P}^2$-bundle $\mathbb{E}$ containing T. Set $E_Q := \mathbb{E} \cap Q$. Then we have an exact sequence $0 \to \mathcal{O}_{\mathbb{F}}(-Q+\mathbb{E}) \to \mathcal{O}_{\mathbb{F}}(\mathbb{E}) \otimes \mathcal{I}_{T/\mathbb{F}} \to \mathcal{O}(E_Q) \otimes \mathcal{I}_{T/Q} \to 0$. Since $R^i f_*\mathcal{O}_{\mathbb{F}}(-Q+E) = 0$ for any i and $f_*(\mathcal{I}_{T/\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(\mathbb{E})) = \mathcal{O}_C$ as in the proof of Theorem 4.1 (1), we have $q_*\mathcal{I}_T(E_Q) = \mathcal{O}_C$, which proves that E_Q satisfies the conditions. For a general point $t \in C$, the fiber $(E_Q)_t$ is a unique smooth conic passing through the three points T_t. Hence the uniqueness of E_Q follows. $\square$

From now on, we fix a divisor α on C such that $-K_Q = 2E_Q - q^*\alpha$.

Claim 5.3. There exists an exact sequence

$$0 \to \mathcal{O}(E_Q - q^*\alpha) \to \mathcal{F} \to \mathcal{E} \to 0 \tag{5.3}$$

where $\mathcal{E}$ is a locally free sheaf with $\mathcal{E}|_{Q_t} \simeq \mathscr{S}_{Q_t}^{\vee}$ for every $t \in C$.

Proof. Tensoring the exact sequence (5.1) with $\mathcal{O}(-E_Q + q^*\alpha)$, we obtain the following exact sequence:

$$0 \to q^*((R^1q_*\mathcal{I}_T)(\alpha - K_C)) \otimes \mathcal{O}(-E_Q) \to \mathcal{F}(-E_Q + q^*\alpha) \to \mathcal{I}_T(E_Q) \to 0.$$

Claim 5.2 implies that $q_*\mathcal{I}_T(E_Q) = \mathcal{O}_C$. Since $R^iq_*\mathcal{O}_Q(-E_Q) = 0$ for any i, we have $q_*\mathcal{F}(-E_Q + q^*\alpha) \simeq q_*\mathcal{I}_T(E_Q) \simeq \mathcal{O}_C$. Hence we obtain an injection $\iota\colon \mathcal{O}(E_Q - q^*\alpha) \to \mathcal{F}$ with the locally free cokernel $\mathcal{E} := \mathrm{Cok}\,\iota$. Then we have an exact sequence $0 \to \mathcal{O}_{Q_t}(1) \to \mathcal{F}|_{Q_t} \to \mathcal{E}|_{Q_t} \to 0$ for every $t \in C$. Since $\mathcal{F}|_{Q_t} \simeq \mathscr{S}_{Q_t}^{\vee} \oplus \mathcal{O}_{Q_t}(1)$ and $\mathrm{Hom}(\mathcal{O}_{Q_t}(1), \mathscr{S}_{Q_t}^{\vee}) = 0$, $\mathcal{F}|_{Q_t}$ contains $\mathcal{O}_{Q_t}(1)$ as a direct summand. Hence we have $\mathcal{E}|_{Q_t} \simeq \mathscr{S}_{Q_t}^{\vee}$ for every $t \in C$. $\square$

Step 3. Let $\mathbb{P}_Q(\mathcal{E}) \subset \mathbb{P}_Q(\mathcal{F})$ be the natural inclusion from the exact sequence (5.3). We define morphisms as in the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}_Q(\mathcal{F}) \arrow[d, "\pi_{\mathcal{F}}"] \arrow[dd, bend right=60, "r_{\mathcal{F}}"'] & \mathbb{P}_Q(\mathcal{E}) \arrow[l, hook'] \arrow[dl, "\pi_{\mathcal{E}}"] \arrow[ddl, bend left=40, "r_{\mathcal{E}}"] \\
Q \arrow[d, "q"'] & \\
C. &
\end{tikzcd}$$

Let $\xi_{\mathcal{F}}$ be a tautological divisor of $\pi_{\mathcal{F}} \colon \mathbb{P}_Q(\mathcal{F}) \to Q$ and $\xi_{\mathcal{E}} := \xi_{\mathcal{F}}|_{\mathbb{P}_Q(\mathcal{E})}$. Then $\xi_{\mathcal{E}}$ is a tautological divisor of $\mathbb{P}_Q(\mathcal{E})$ and $\mathbb{P}_Q(\mathcal{E}) \subset \mathbb{P}_Q(\mathcal{F})$ is a member of $|\mathcal{O}(\xi_{\mathcal{F}} - \pi^*(E_Q)) \otimes r_{\mathcal{F}}^*\mathcal{O}(\alpha)|$. Note that $\mathcal{O}_{\mathbb{P}_Q(\mathcal{E})}(\xi_{\mathcal{E}})$ is $r_{\mathcal{E}}$-globally generated and $r_{\mathcal{E}*}\mathcal{O}_{\mathbb{P}_Q(\mathcal{E})}(\xi_{\mathcal{E}}) = q_*\mathcal{E}$ is a vector bundle of rank 4. Thus $\xi_{\mathcal{E}}$ gives a morphism $\psi_{\mathcal{E}} \colon \mathbb{P}_Q(\mathcal{E}) \to \mathbb{P}_C(q_*\mathcal{E})$. Set $G = \mathrm{Exc}(\psi_{\mathcal{E}})$ and $S = \psi_{\mathcal{E}}(G)$.

Claim 5.4.

(1) S is smooth and $\psi_{\mathcal{E}} \colon \mathbb{P}_Q(\mathcal{E}) \to \mathbb{P}_C(q_*\mathcal{E})$ is the blow-up of $\mathbb{P}_C(q_*\mathcal{E})$ along S;
(2) The morphism $S \to C$ factors a non-singular curve B such that $S \to B$ is a $\mathbb{P}^1$-bundle and $B \to C$ is a double covering. Moreover, $B \to C$ is the double covering associated to q.

Proof.

(1) Let $t \in C$ be a point and consider a morphism $\psi_{\mathcal{E},t} \colon \mathbb{P}_{Q_t}(\mathscr{S}_{Q_t}^{\vee}) \to \mathbb{P}^3$. If Q_t is smooth (respectively singular), then it is known that $\psi_{\mathcal{E},t}$ is the blow-up along union of two disjoint lines in $\mathbb{P}^3$ [20, Table 3, No 25.] (respectively the blow-up along a double line which is contained in a smooth quadric surface). This fact implies that every fiber l of $G \to S$ is isomorphic to $\mathbb{P}^1$ and satisfies $-K_{\mathbb{P}_Q(\mathcal{E})}.l = 1$. Then [2, Theorem 2.3] implies that S is non-singular and $\psi_{\mathcal{E}}$ is the blow-up along S;
(2) For any $t \in C$, $S_t \subset \mathbb{P}^3$ is a union of two disjoint lines if Q_t is smooth and $(S_t)_{\mathrm{red}}$ is a line if Q_t is singular. Let $S \to B \to C$ be the Stein factorization of $S \to C$. Since S is smooth, so is B. Then $S \to B$ is a $\mathbb{P}^1$-bundle and $B \to C$ is a double covering. The branched locus of this double cover $B \to C$ is $\{t \in C \mid Q_t \text{ is singular}\}$. Therefore, $B \to C$ is the double covering associated to $q \colon Q \to C$. □

Step 4. Let $\psi_{\mathcal{F}} \colon \mathbb{P}_Q(\mathcal{F}) \to \overline{Z}$ be the Stein factorization of the morphism given by $|\xi_{\mathcal{F}}|$ over C. Since $-K_{\mathbb{P}_Q(\mathcal{F})} \sim_C 3\xi_{\mathcal{F}}$, $\psi_{\mathcal{F}} \colon \mathbb{P}_Q(\mathcal{F}) \to \overline{Z}$ is a crepant contraction.

Claim 5.5.

(1) $\mathrm{Exc}(\psi_{\mathcal{E}}) = \mathrm{Exc}(\psi_{\mathcal{F}})$ holds and $\psi_{\mathcal{F}}$ is a 2-dimensional family of Atiyah's flopping contractions. In particular, if $\Psi\colon \mathbb{P}_Q(\mathcal{F}) \dashrightarrow \widetilde{Z}$ denotes the flop, then $\widetilde{Z}$ is non-singular;

(2) Let $E_{\widetilde{Z}}$ be the proper transform of $\mathbb{P}_Q(\mathcal{E}) \subset \mathbb{P}_Q(\mathcal{F})$ on $\widetilde{Z}$. Then there exists a birational morphism $\mu_Z\colon \widetilde{Z} \to Z$ over C such that Z is non-singular and μ_Z is the blow-up along a section $C_{0,Z}$ of the induced morphism $\varphi_Z\colon Z \to C$.

Proof.

(1) Let γ be a curve contracted by $\psi_{\mathcal{F}}$. Then we have $\xi_{\mathcal{F}}.\gamma = 0$ and hence $\mathbb{P}_Q(\mathcal{E}).\gamma = -\pi_{\mathcal{F}}^* E_Q.\gamma < 0$ since E_Q is ample over C. Thus $\mathbb{P}_Q(\mathcal{E})$ contains γ and γ is contracted by $\psi_{\mathcal{E}}$. Conversely, it is clear that every curve γ contracted by $\psi_{\mathcal{E}}$ is also contracted by $\psi_{\mathcal{F}}$. Therefore, we have $\psi_{\mathcal{F}}|_{\mathbb{P}_Q(\mathcal{E})} = \psi_{\mathcal{E}}$ by the rigidity lemma and $\mathrm{Exc}(\psi_{\mathcal{F}}) = \mathrm{Exc}(\psi_{\mathcal{E}})$. Let l be any fiber of $\mathrm{Exc}(\psi_{\mathcal{F}}) \to S$. Then we have $\mathcal{N}_{l/\mathbb{P}_Q(\mathcal{E})} = \mathcal{O}(-1) \oplus \mathcal{O}^2$ and $1 = -K_{\mathbb{P}_Q(\mathcal{E})}.l = (\xi_{\mathcal{F}} + \pi_{\mathcal{F}}^* E_Q).l = \pi_{\mathcal{F}*}l.E_Q$. Hence $\mathbb{P}_Q(\mathcal{E}).l = -1$ in $\mathbb{P}_Q(\mathcal{F})$. Considering the exact sequence $0 \to \mathcal{N}_{l/\mathbb{P}_Q(\mathcal{E})} \to \mathcal{N}_{l/\mathbb{P}_Q(\mathcal{F})} \to \mathcal{N}_{\mathbb{P}_Q(\mathcal{E})/\mathbb{P}_Q(\mathcal{F})}|_l \to 0$, we obtain $\mathcal{N}_{l/\mathbb{P}_Q(\mathcal{F})} \simeq \mathcal{O}(-1)^2 \oplus \mathcal{O}^2$. Thus $\psi_{\mathcal{F}}$ is a 2-dimensional family of Atiyah's flopping contractions;

(2) We have $E_{\widetilde{Z}} \simeq \mathbb{P}_C(q_*\mathcal{E})$ by the construction of this flop. Moreover, for each $t \in C$, if we take a line $l \subset E_{\widetilde{Z},t} \simeq \mathbb{P}^3$, then we have $-K_{\widetilde{Z}}.l = -K_{\widetilde{Z}_t}.l = 3$. Hence we obtain $-K_{\widetilde{Z}_t}|_{E_{\widetilde{Z}_t}} = \mathcal{O}_{\mathbb{P}^3}(3)$ and $\mathcal{N}_{E_{\widetilde{Z}_t}/\widetilde{Z}_t} \simeq \mathcal{O}_{\mathbb{P}^3}(-1)$ for every $t \in C$. Therefore, there exists a morphism $\mu_Z\colon \widetilde{Z} \to Z$ over C such that μ_Z blows $E_{\widetilde{Z}}$ down to a section $C_{0,Z}$ of $\varphi_Z\colon Z \to C$ and Z is smooth. □

Note that μ_Z is an extremal contraction and hence $\rho(Z) = 2$.

We show (3). Set $\beta := \det(R^1 q_* \mathcal{I}_T(-K_C))$ as in Theorem 5.1 (3). Since $-K_{\mathbb{P}_Q(\mathcal{F})} = 3\xi_{\mathcal{F}} - r_{\mathcal{F}}^*\beta$ by the exact sequence (5.1), we have $-K_Z = 3\xi_Z - \varphi_Z^*\beta$. Moreover, since μ_Z is the blow-up along a φ_Z-section, we have $\mu_Z^* K_Z = K_{\widetilde{Z}} + 3E_{\widetilde{Z}}$. Thus we have $\mu_Z^*\xi_Z = \Psi_*\xi_{\mathcal{F}} + E_{\widetilde{Z}}$ since $\xi_Z = \mu_{Z*}\Psi_*\xi_{\mathcal{F}}$ by the definition. The proof of (3) is complete.

To prove (4), let $t \in C$ be a point such that $Z_t = \varphi_Z^{-t}(t)$ is smooth. Then Z_t is a so-called del Pezzo 4-fold. From the diagram (5.2), we have $\xi_Z^4 \cdot \varphi_Z^{-1}(t) = \xi_{\mathcal{F}}^4 \cdot r_{\mathcal{F}}^{-1}(t) + 1 = 6$ by a direct calculation. Then we have $Z_t \simeq (\mathbb{P}^2)^2$ by Fujita's classification of del Pezzo manifolds [13, Theorem 3.3.1].

The proof of Theorem 5.1 is complete. □

5.2. Proof of Theorem C

Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration. Our goal is to construct a $(\mathbb{P}^2)^2$-fibration containing X as a relative linear section.

Let C_0 be a φ-section. Let $(q\colon Q \to C, T)$ be the relative double projection of $(\varphi\colon X \to C, C_0)$ as in Definition 2.2. Let $E_Q \subset Q$ be the proper transform of $\mathrm{Exc}(\mathrm{Bl}_{C_0} X \to X)$. Then E_Q is nothing but the divisor that we obtain in Claim 5.2. By Proposition 2.1 (3), we see that $\mathcal{O}(-K_{\mathrm{Bl}_T Q})$ is nef over C. Then Theorem 5.1 gives the moderate $(\mathbb{P}^2)^2$-fibration $\varphi_Z\colon Z \to C$. In order to find an embedding from X into Z over C, it suffices to show that the proper transform of Q on Z coincides with X.

Now let us use the same notation as in Theorem 5.1 and its proof. Let $\mathcal{G}$ be as in the statement of Theorem C. Considering the exact sequence $0 \to \mathcal{I}_{T/Q} \to \mathcal{O}_Q \to \mathcal{O}_T \to 0$ and taking the cohomology of q_*, we obtain

$$\mathcal{G} = R^1 q_* \mathcal{I}_T \otimes \mathcal{O}(-K_C). \tag{5.4}$$

Note that $\widetilde{Q}=\mathrm{Bl}_T Q$ is the zero scheme of the global section of $H^0(\mathbb{P}_Q(\mathcal{F}), \mathcal{O}(\xi_{\mathcal{F}})\otimes r_{\mathcal{F}}^*\mathcal{G}^\vee)$ corresponding to the injection $q^*\mathcal{G} \to \mathcal{F}$ in the sequence (5.1) under the natural isomorphism $H^0(\mathbb{P}_Q(\mathcal{F}), \mathcal{O}(\xi_{\mathcal{F}}) \otimes r_{\mathcal{F}}^*\mathcal{G}^\vee) \simeq \mathrm{Hom}_Q(q^*\mathcal{G}, \mathcal{F})$.

Now Theorem C is a consequence of the following claim.

Claim 5.6.

(1) $\psi_{\mathcal{F}}|_{\widetilde{Q}}$ coincides with ψ_Q in Proposition 2.1 (3);
(2) If $\widetilde{Q}^+ \subset \widetilde{Z}$ denotes the proper transform of $\widetilde{Q} \subset \mathbb{P}_Q(\mathcal{F})$, then the birational map $\widetilde{Q} \dashrightarrow \widetilde{Q}^+$ is the flop over C. In particular, $\widetilde{Q}^+ \simeq \widetilde{X}$ holds;
(3) It holds that $\mu_Z|_{\widetilde{X}} = \mu_X$. In particular, there exists a closed embedding $i\colon X \hookrightarrow Z$ such that $i(C_0) = C_{0,Z}$;
(4) X is the zero scheme of a global section of $\mathcal{O}_Z(\xi_Z) \otimes \varphi_Z^*\mathcal{G}^\vee$;
(5) It holds that $\mathcal{O}(K_Z + 3\xi_Z) \simeq \varphi_Z^* \det \mathcal{G}$.

Proof.

(1) It suffices to show the restriction morphism $r_{\mathcal{F}*}\mathcal{O}_{\mathbb{P}_Q(\mathcal{F})}(k\xi_{\mathcal{F}}) \to (r_{\mathcal{F}}|_{\widetilde{Q}})_*\mathcal{O}_{\widetilde{Q}}\cdot(k\xi_{\mathcal{F}})$ is surjective for every $k > 0$. Since $\widetilde{Q}$ is the zero scheme of a global section of the rank 2 vector bundle $\mathcal{O}_{\mathbb{P}_Q(\mathcal{F})}(1) \otimes r_{\mathcal{F}}^*\mathcal{G}^\vee$, we have the exact sequence $0 \to \mathcal{O}_{\mathbb{P}_Q(\mathcal{F})}((k-2)\xi_{\mathcal{F}})\otimes r_{\mathcal{F}}^* \det \mathcal{G} \to \mathcal{O}_{\mathbb{P}_Q(\mathcal{F})}((k-1)\xi_{\mathcal{F}})\otimes r_{\mathcal{F}}^*\mathcal{G} \to \mathcal{I}_{\widetilde{Q}/\mathbb{P}_Q(\mathcal{F})}(k\xi_{\mathcal{F}}) \to 0$. Thus we have $R^1 r_{\mathcal{F}*}\mathcal{I}_{\widetilde{Q}/\mathbb{P}_Q(\mathcal{F})}(k\xi_{\mathcal{F}}) = 0$ for every $k > 0$. Hence we are done;
(2) By (1), we have $\psi_{\mathcal{F}}(\widetilde{Q}) = \overline{X}$.
Since $\psi_{\mathcal{F}}$ is defined over C by $\xi_{\mathcal{F}}$, there exists a Cartier divisor $\xi_{\overline{Z}}$ on $\overline{Z}$ such that $\xi_{\overline{Z}}$ is ample over C and $\psi_{\mathcal{F}}^*\xi_{\overline{Z}} = \xi_{\mathcal{F}}$. Since $\widetilde{Q}$ is the zero scheme of a global section of $\psi_{\mathcal{F}}^*(\mathcal{O}(\xi_{\overline{Z}}) \otimes \overline{r}^*\mathcal{G}^\vee)$, $\overline{X}$ is that of $\mathcal{O}(\xi_{\overline{Z}}) \otimes \overline{r}^*\mathcal{G}^\vee$ in $\overline{Z}$ and hence $\widetilde{Q}^+$ is that of $\psi_Z^*\mathcal{O}(\xi_{\overline{Z}}) \otimes r^{+*}\mathcal{G}^\vee$ in $\widetilde{Z}$.
Then (2) follows from similar arguments as in the proof of Claim 4.7 (2);
(3) By Theorem 5.1 (3), we have $\mu_Z^*\xi_Z \sim \xi_{\widetilde{Z}} + E_{\widetilde{Z}}$. Since $\Psi_*\xi_{\mathcal{F}}|_{\widetilde{X}} \sim_C -K_{\widetilde{X}}$, we obtain $\mu_Z^*\xi_Z|_{\widetilde{X}} \sim_C \mu_X^*(-K_X)$, which proves the assertion;

(4) Since $\widetilde{X}$ is the zero scheme of a global section of $\mathcal{O}_{\widetilde{Z}}(\Psi_*\xi_{\mathcal{F}}) \otimes r^{+*}\mathcal{G}^{\vee}$ and $\Psi_*\xi_{\mathcal{F}} = \mu_Z^*\xi_Z - \mathrm{Exc}(\mu_Z)$ holds, we obtain a section $s \in H^0(Z, \mathcal{I}_{C_0/Z}(\xi_Z) \otimes r^{+*}\mathcal{G}^{\vee})$ such that the zero scheme of s is X;
(5) This assertion immediately follows from (5.4) and Theorem 5.1 (3). □

The proof of Theorem C is complete. □

6. Singular fibers of sextic del Pezzo fibrations

This section is devoted to proving Theorem D and Corollary E.

6.1. Main result of this section

One of the main purpose of this section is to classify singular fibers of sextic del Pezzo fibration $X \to C$. It follows from the assumption $\rho(X) = 2$ (or [9, (4.6)]) that every fiber of a sextic del Pezzo fibration $\varphi\colon X \to C$ is irreducible and Gorenstein. Such surfaces were studied by many persons (*e.g.* [1,6,8,12,26]). As the first step to classify singular fibers, we review the classification of irreducible Gorenstein sextic del Pezzo surfaces.

Theorem 6.1 ([1,6,8,12,26]). *Let S be an irreducible Gorenstein del Pezzo surface with $(-K_S)^2 = 6$.*
(1) *Suppose that S has only Du Val singularities. Let $\widetilde{S} \to S$ be the minimal resolution. Then there is a birational morphism $\varepsilon\colon \widetilde{S} \to \mathbb{P}^2$ such that ε is the blow-up of $\mathbb{P}^2$ at three (possibly infinitely near) points. If Σ denotes the 0-dimensional subscheme in $\mathbb{P}^2$ of length 3 corresponding to the three (possibly infinitely near) points, then the isomorphism class of S is determined by Σ as in Table 6.1 below.*

Type	Σ	Is Σ colinear?	# of lines on S	Singularity
(2,3)	reduced	non-colinear	6	smooth
(2,2)	$\mathrm{Spec}\,\mathbb{C} \sqcup \mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^2)$	non-colinear	4	A_1
(2,1)	$\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^3)$	non-colinear	2	A_2
(1,3)	reduced	colinear	3	A_1
(1,2)	$\mathrm{Spec}\,\mathbb{C} \sqcup \mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^2)$	colinear	2	$A_1 + A_1$
(1,1)	$\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^3)$	colinear	1	$A_1 + A_2$

Table 6.1. Classification of Du Val sextic del Pezzo surfaces.

We refer to [6,8,12] *for more precise details. We also refer to Lemma* 6.12 *for the reason why we employ the above notion for Du Val sextic del Pezzo surfaces.*

(2) *Suppose that the singularities of S are not Du Val and that S is a rational surface. Let $\nu\colon \overline{S} \to S$ be the the normalization. Then $\overline{S}$ is a Hirzebruch surface. The complete linear system $|\nu^*\omega_S^{-1}|$ gives an embedding $\overline{S} \hookrightarrow \mathbb{P}^7$ and $S \subset \mathbb{P}^6$ is*

the image of the projection from a point away from $\overline{S}$. *Let* $\mathcal{C} \subset \mathcal{O}_{\overline{S}}$ *be the conductor of* ν *and* $E :=$ Spec $\mathcal{O}_{\overline{S}}/\mathcal{C}$. *Then* $(\overline{S}, \nu^*\omega_S^{-1}, E, E \to \nu(E))$ *is one of the two cases in Table* 6.2.

Type	$\overline{S}$	$\nu^*\omega_S^{-1}$	E	$E \to \nu(E)$
(n2)	$\mathbb{F}_2$	$h+2f$	$E = C_0$	double cover
(n4)	$\mathbb{F}_4$	$h+f$	$E = E_1 + E_2$ where $E_1 = C_0$ and $E_2 \sim f$	$\nu(E) = \nu(E_1) = \nu(E_2)$ and $E_i \to \nu(E)$ is isomorphic

Table 6.2. Classification of non-normal rational sextic del Pezzo surfaces.

For the notation of $\mathbb{F}_n$, h, f, *and* C_0, *we refer to Section* 1.5. *For more precise details, we refer to* [1,26].

(3) *When S is not a rational surface, S is the cone over a curve* $C \subset \mathbb{P}^5$ *of degree* 6 *and arithmetic genus* 1.

Remark 6.2. Let S be the cone over a curve $C \subset \mathbb{P}^6$ of degree 6 and arithmetic genus 1. Then we have $\dim T_v S = 6$ where v is the vertex. Hence for any sextic del Pezzo fibration $\varphi\colon X \to C$, X does not contain such an S since we assume that the total space X is smooth. In other words, every φ-fiber is of type (i, j), (n2) or (n4) as in Theorem 6.1.

The main result of this section is the following theorem.

Theorem 6.3. *Let* $\varphi\colon X \to C$ *be a sextic del Pezzo fibration and* C_0 *be an arbitrary* φ*-section. Let* $(q\colon Q \to C, T)$ *be the relative double projection of* $(\varphi\colon X \to C, C_0)$ *as in Definition* 2.2. *Then the following assertions hold for any* $t \in C$.

(1) *For* $j \in \{1, 2, 3\}$, X_t *is of type* $(2, j)$ *if and only if* Q_t *is smooth and* $\#(T_t)_{\mathrm{red}} = j$;
(2) *For* $j \in \{1, 2, 3\}$, X_t *is of type* $(1, j)$ *if and only if* Q_t *is singular,* $\#(T_t)_{\mathrm{red}} = j$, *and* $\mathrm{Sing}\, Q_t \cap T_t = \varnothing$;
(3) X_t *is of type (n2) if and only if* Q_t *is singular,* $\#(T_t)_{\mathrm{red}} = 2$, *and the double point of* T_t *is supported at the vertex of* Q_t;
(4) X_t *is of type (n4) if and only if* Q_t *is singular,* $\#(T_t)_{\mathrm{red}} = 1$, *and* $\mathrm{Sing}\, Q_t = (T_t)_{\mathrm{red}}$.

6.2. Singular fibers of moderate $(\mathbb{P}^1)^3$-fibrations

First of all, we classify the singular fibers of moderate $(\mathbb{P}^1)^3$-fibrations as in the following theorem.

Theorem 6.4. *Let* $f\colon \mathbb{F} \to C$ *and* $T \subset \mathbb{F}$ *be as in the setting of Theorem* 4.1. *Let* $\varphi_Y\colon Y \to C$ *be the moderate* $(\mathbb{P}^1)^3$*-fibration which is obtained by Theorem* 4.1. *Then the following assertions hold for any* $t \in C$.

(1) $Y_t \simeq (\mathbb{P}^1)^3$ *if and only if* $\#(T_t)_{\mathrm{red}} = 3$*;*
(2) $Y_t \simeq \mathbb{P}^1 \times \mathbb{Q}^2_0$ *if and only if* $\#(T_t)_{\mathrm{red}} = 2$*, where* $\mathbb{Q}^2_0$ *denotes the* 2*-dimensional singular quadric cone;*
(3) $Y_t \simeq \mathbb{P}^{1,1,1}$ *if and only if* $\#(T_t)_{\mathrm{red}} = 1$.

For the definition of $\mathbb{P}^{1,1,1}$*, we refer to Definition* 6.6 *below.*

To define $\mathbb{P}^{1,1,1}$, we need the following lemma.

Lemma 6.5. *Let* $\varepsilon \in \mathrm{Ext}^1_{\mathbb{F}_2}(\mathcal{O}(f), \mathcal{O}(2h-f)) = H^1(\mathbb{F}_2, \mathcal{O}_{\mathbb{F}_2}(2h-2f)) = \mathbb{C}$ *be a non-zero element. This element gives the following non-trivial extension:*

$$0 \to \mathcal{O}(2h-f) \to \mathcal{E} \to \mathcal{O}(f) \to 0. \tag{6.1}$$

Then $\mathcal{E}$ *is the cokernel of an injection* $\mathcal{O}_{\mathbb{F}_2}(-h+f) \to \mathcal{O}^2 \oplus \mathcal{O}(h+f)$*. In particular,* $\mathcal{E}$ *is globally generated.*

Proof. Let $s_1, s_2 \in H^0(\mathcal{O}_{\mathbb{F}_2}(h-f))$ be sections such that $(s_i = 0) = C_0 + l_i$, where $l_i \in |f|$ and $l_1 \cap l_2 = \varnothing$. Let $t \in H^0(\mathcal{O}_{\mathbb{F}_2}(2h))$ be a general section such that $(t=0) \cap C_0 = \varnothing$. Then the cokernel of the map $v := (s_1, s_2, t)\colon \mathcal{O}_{\mathbb{F}_2}(-h+f) \to \mathcal{O}^2 \oplus \mathcal{O}(h+f)$ is locally free. Let us confirm that $\operatorname{Cok} v = \mathcal{E}$. Set $s = (s_1, s_2)\colon \mathcal{O}_{\mathbb{F}_2}(-h+f) \to \mathcal{O}^2_{\mathbb{F}_2}$. Then we have a surjection $\operatorname{Cok} v \twoheadrightarrow \operatorname{Cok} s$. Since both $(s_1 = 0)$ and $(s_2 = 0)$ contain C_0, the morphism $s\colon \mathcal{O}(-h+f) \to \mathcal{O}^2$ factors through the map $\mathcal{O}(-f) \to \mathcal{O}^2$ which is given by (l_1, l_2). Hence we have a surjection $\operatorname{Cok} s \to \mathcal{O}(f)$ and hence a surjection $\operatorname{Cok} v \to \mathcal{O}(f)$ and an exact sequence $0 \to \mathcal{O}(2h-f) \to \operatorname{Cok} v \to \mathcal{O}(f) \to 0$, which does not split since $\mathcal{O}(2h-f)$ is not globally generated but $\operatorname{Cok} v$ is. Since $\mathrm{Ext}^1(\mathcal{O}(f), \mathcal{O}(2h-f)) = \mathbb{C}$, we have $\operatorname{Cok} v \simeq \mathcal{E}$. ☐

Definition 6.6 ([8]). Let $\mathcal{E}$ be the bundle on $\mathbb{F}_2$ fitting into the exact sequence (6.1) that does not split. We define $\widetilde{\mathbb{P}^{1,1,1}} := \mathbb{P}_{\mathbb{F}_2}(\mathcal{E})$ and set $\mathbb{P}^{1,1,1} \subset \mathbb{P}^7$ be the image of the morphism defined by $|\mathcal{O}_{\mathbb{P}_{\mathbb{F}_2}(\mathcal{E})}(1)|$.

Remark 6.7 ([8]). By Lemma 6.5 and [8, (si31i), P.170], we can check that the variety $\mathbb{P}^{1,1,1}$ is the del Pezzo variety of type (si31i) in the sense of [8]. In particular, there is a very ample divisor $H_{\mathbb{P}^{1,1,1}}$ on $\mathbb{P}^{1,1,1}$ such that $K_{\mathbb{P}^{1,1,1}} + 2H_{\mathbb{P}^{1,1,1}} \sim 0$ and $H^3_{\mathbb{P}^{1,1,1}} = 6$. Moreover, the singular locus of $\mathbb{P}^{1,1,1}$ is a line and $\mathbb{P}^{1,1,1}$ has a family of Du Val A_2-singularities along the line.

Proof of Theorem 6.4. We use the diagram (4.1) and the notation in Theorem 4.1 and its proof. By Theorem B (2), there is a Cartier divisor H_{Y_t} such that $K_{Y_t} + 2H_{Y_t} \sim 0$ and $H^3_{Y_t} = 6$. Hence Y_t is a del Pezzo variety of degree 6.

(1) Assume $\#(T_t) = 3$. Since $\widetilde{\mathbb{F}}^+_t$ is a flop of $\widetilde{\mathbb{F}}_t$, $\widetilde{\mathbb{F}}^+_t$ is smooth since so is $\widetilde{\mathbb{F}}_t$. Since $\mu_{\mathbb{F}}$ is the blow-up along a section of $Y \to C$, Y_t is smooth and hence $Y_t \simeq (\mathbb{P}^1)^3$ by Theorem 4.1 (4).

(2) Assume $\#(T_t) = 2$. Then the singular locus of $\widetilde{\mathbb{F}}_t$ is a smooth rational curve l and $\widetilde{\mathbb{F}}_t$ has a family of Du Val A_1-singularities along this curve l. Since l is contained in the exceptional locus of the blow-up $\widetilde{\mathbb{F}}_t \to \mathbb{F}_t$, l is not a flopping curve on $\widetilde{\mathbb{F}}_t$. Thus $\widetilde{\mathbb{F}}_t^+$ is also normal and has a 1-dimensional family of Du Val A_1-singularities. Since $\widetilde{\mathbb{F}}_t^+$ is the blow-up of Y_t at a smooth point, Y_t is also normal and has a 1-dimensional family of Du Val A_1-singularities. Hence Y_t is of type (vi), or (si211), or (si22i) in the sense of Fujita [8]. Moreover, it follows that $\rho(Y_t) = 2$ from this construction. By the definition in [8, P.155 (respectively P.170)], we can check that the variety V of type (vi) or (si211) is of Picard rank 1 if V has a 1-dimensional family of Du Val A_1-singularities. Thus V must be of type (si22i).

By the definition in [8, P.169], we can check that the variety of type (si22i) is isomorphic to $\mathbb{P}^1 \times \mathbb{Q}^2_0$.

(3) Assume that $\#(T_t)_{\mathrm{red}} = 1$. By Remark 4.2, T_t is defined by (x, y, z^3) locally. Hence $\widetilde{\mathbb{F}}_t$ has a 1-dimensional family of Du Val A_2-singularities. It follows from the same argument as in the proof of (2) that Y_t also has a 1-dimensional family of Du Val A_2-singularities. Then by Fujita's classification, Y_t is of type (si31i) in the sense of [8], which is isomorphic to $\mathbb{P}^{1,1,1}$ by Remark 6.7. $\square$

6.3. Singular fibers of moderate $(\mathbb{P}^2)^2$-fibrations

Next, we classify the singular fibers of the moderate $(\mathbb{P}^2)^2$-fibrations. In order to state the result, we review the definition of the variety $\mathbb{P}^{2,2}$, which was introduced by Fujita [8] and Mukai [21] independently.

Definition 6.8 ([21], [8]). We define $\widetilde{\mathbb{P}^{2,2}} := \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \Omega_{\mathbb{P}^2}(2))$. Then the tautological divisor ξ is free and the linear system $|\xi|$ gives a morphism $\widetilde{\mathbb{P}^{2,2}} \to \mathbb{P}^8$ since $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2) \oplus \Omega_{\mathbb{P}^2}(2)) = 9$. We define $\mathbb{P}^{2,2}$ as the image of the morphism and $\psi \colon \widetilde{\mathbb{P}^{2,2}} \to \mathbb{P}^{2,2}$.

Remark 6.9. We can check that the variety $\mathbb{P}^{2,2}$ is the del Pezzo variety of type (vu) in the sense of [8, P.155].

Theorem 6.10 ([9, (4.6)]). *Let $q \colon Q \to C$ and T be as in the setting of Theorem 5.1. Let $\varphi_Z \colon Z \to C$ be the moderate $(\mathbb{P}^2)^2$-fibration that is obtained by Theorem 5.1. Then the following assertions hold for any $t \in C$.*

(1) *Q_t is smooth if and only if $Z_t \simeq (\mathbb{P}^2)^2$;*
(2) *Q_t is singular if and only if $Z_t \simeq \mathbb{P}^{2,2}$.*

Proof of Theorem 6.10. Set $\mathcal{F}$ as in Theorem 5.1 (1). By Theorem 5.1 (2), the blow-up $\widetilde{Z}_t$ of Z_t at a smooth point is the flop of $\mathbb{P}_{Q_t}(\mathcal{F}_{Q_t})$. Since the map Ψ in the diagram (5.2) is a family of Atiyah flops over C, we conclude that Q_t is smooth if and only if Z_t is smooth. Hence it suffices to show that $Z_t \simeq \mathbb{P}^{2,2}$ if Z_t is singular, which is known by [8] or [9, (4.6)]. $\square$

6.4. Proof of Theorem 6.3

For proving Theorem 6.3, we need three lemmas.

Lemma 6.11. *Let S be a hyperplane section of $\mathbb{P}^{1,1,1} \subset \mathbb{P}^7$. If S is non-normal, then S is of type (n4) in Theorem* 6.1.

Proof. We use the notation in Definition 6.6. Let $\pi : \widetilde{\mathbb{P}}^{1,1,1} \to \mathbb{F}_2$ be the projection and ξ a tautological divisor. As explained in [8, (si31i), P.170], the exceptional divisor of $\widetilde{\mathbb{P}}^{1,1,1} \to \mathbb{P}^{1,1,1}$ is the union of a unique member $E_1 \in |\xi + \pi^*(-2h + f)|$ and $E_2 := \pi^* C_0$. Note that ξ is the pull-back of a hyperplane section of $\mathbb{P}^{1,1,1} \subset \mathbb{P}^7$.

Let $\widetilde{S} \subset \widetilde{\mathbb{P}}^{1,1,1} = \mathbb{P}_{\mathbb{F}_2}(\mathcal{E})$ be the proper transformation of $S \subset \mathbb{P}^{1,1,1}$. Then there exist $a, b \in \mathbb{Z}_{\geq 0}$ such that $\widetilde{S} \sim \xi - (aE_1 + bE_2) = (1-a)\xi + \pi^*((2a - b)h + (2b - a)f)$. Since $\widetilde{S}$ is effective, we have $a \leq 1$.

Let us prove that $a = 1$. If $a = 0$, then $\widetilde{S} \in \xi - \pi^*(b(h - 2f))$ and hence $b = 0$ or 1 by (6.1). Assume $b = 1$. Then we have $\widetilde{S} \sim \xi - \pi^*(h - 2f)$. Let $s \colon \mathcal{O}(h - 2f) \to \mathcal{E}$ be the section corresponding to $\widetilde{S}$. Since $\widetilde{S}$ is irreducible, the zero locus of s, say Z, is 0-dimensional. Then it follows from [22, Lemma 5.4] that $\operatorname{Cok} s$ is torsion free. Hence we have an exact sequence $0 \to \mathcal{O}(h - 2f) \to \mathcal{E} \to \mathcal{I}_Z(h + 2f) \to 0$. Since $c_2(\mathcal{E}) = 2$, Z must be empty. Then we have $\mathcal{E} \simeq \mathcal{O}(h)^2$ since $\mathcal{E}$ is nef, which is a contradiction to (6.1). Assume $b = 0$, *i.e.*, $\widetilde{S} \sim \xi$. Then S does not contain the singular locus of $\mathbb{P}^{1,1,1}$. Hence $\widetilde{S}$ is also non-normal. Let $s \colon \mathcal{O} \to \mathcal{E}$ be the section corresponding to $\widetilde{S}$ and Z the zero locus of s. By a similar argument as before, we have an exact sequence $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_Z(2h) \to 0$. By this exact sequence, $\operatorname{Bl}_Z \mathbb{F}_2$ is embedded into $\mathbb{P}_{\mathbb{F}_2}(\mathcal{E}) = \widetilde{\mathbb{P}}^{1,1,1}$. Note that $\operatorname{Bl}_Z \mathbb{F}_2$ coincides with $\widetilde{S}$ over $\mathbb{F}_2 \setminus Z$. Since both of $\operatorname{Bl}_Z \mathbb{F}_2$ and $\widetilde{S}$ are integral, $\operatorname{Bl}_Z \mathbb{F}_2$ coincides with $\widetilde{S}$. Since $c_2(\mathcal{E}) = 2$, Z is of length 2 and hence $\operatorname{Bl}_Z \mathbb{F}_2 = \widetilde{S}$ is normal, which is a contradiction.

Therefore, we have $a = 1$. Then $\widetilde{S} \sim \pi^*((2-b)h + (2b-1)f)$, which implies $b = 1$ since $\widetilde{S}$ is a prime divisor. Since $\widetilde{S}$ is integral, there is a smooth rational curve $C \in |h + f|$ such that $\widetilde{S} = \pi^{-1}(C)$. Then the restriction of the sequence (6.1) to C splits, which implies $\widetilde{S} \simeq \mathbb{F}_4$. Therefore, S is of type (n4) by Theorem 6.1. □

Lemma 6.12. *Let $\varphi \colon X \to C$ be a sextic del Pezzo fibration and $t \in C$ be a point. Let $\varphi_B \colon B \to C$ and $\varphi_T \colon T \to C$ be the associated coverings. If X_t is normal, then X_t is of type $(\#B_t, \#T_t)$.*

Proof. Let $C^0 := \{t \in C \mid X_t \text{ is normal}\}$, which is an open subset of C. Set $X^0 := \varphi^{-1}(C^0)$, $B^0 := \varphi_B^{-1}(C^0)$, and $T^0 := \varphi_T^{-1}(C^0)$. Now $X^0 \to C^0$ is a Du Val family of sextic del Pezzo surfaces in the sense of Kuznetsov [16, Definition 5.1]. Kuznetsov also constructed the double covering $\mathscr{Z}_3 \to C^0$ and the triple covering $\mathscr{Z}_2 \to C^0$ in [16, Theorem 5.2]. Then it follows from [16, Corollaries 3.13 and 5.5] that X_t is of type $(\#(\mathscr{Z}_{3,t})_{\mathrm{red}}, \#(\mathscr{Z}_{2,t})_{\mathrm{red}})$, where $\mathscr{Z}_{d,t}$ is the fiber of t under $\mathscr{Z}_d \to C^0$. Hence it is enough to show that $\mathscr{Z}_3 \simeq B^0$ and $\mathscr{Z}_2 \simeq T^0$. By [16, Propositions 5.12 and 5.14], $\mathrm{Hilb}_{dt+1}(X/C) \to C$ factors through $\mathscr{Z}_d$ as the Stein factorization and $\mathscr{X}_d$ is non-singular. Thus Lemma 3.3 implies this assertion, which completes the proof. $\square$

Lemma 6.13. *Let $\varphi\colon X \to C$, C_0, $q\colon Q \to C$, and T as in the statement of Theorem* 6.3. *Then for any $t \in C$, X_t is normal if and only if* $\mathrm{Sing}(Q_t) \cap T_t = \varnothing$.

Proof. We use the same notation as in Proposition 2.1 and its proof.

First, we suppose that $\mathrm{Sing}(Q_t) \cap T_t = \varnothing$. Then it follows from Remark 4.2 that $\widetilde{Q}_t$ is normal and hence so is $\overline{X}_t$. Note that $\widetilde{X}_t$ is the blow-up of X_t at a smooth point $x := C_{0,t}$. As shown in the proof of Proposition 2.1 (1), every exceptional curve of $\psi_t\colon \widetilde{X}_t \to \overline{X}_t$ is the proper transform of a line passing through the smooth point x. Now let us assume that X_t is non-normal. Since the non-normal locus does not contain x, the non-normal locus is not contracted by ψ_t. Hence $\overline{X}_t$ is also non-normal. This is a contradiction and hence X_t is normal.

Next, we assume that $\mathrm{Sing}(Q_t) \cap T_t \neq \varnothing$ and show that X_t is non-normal. Then Q_t is singular and hence a quadric cone. Let $x \in Q_t$ be the singular point of Q_t. Then T_t contains x. Since T is a trisection of q, T_t is not reduced at x. Hence $\#(T_t)_{\mathrm{red}} \in \{1, 2\}$.

Claim 6.14. Let $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ as in Theorem 4.1. If $\mathrm{Sing}(Q_t) \cap T_t \neq \varnothing$, then $\widetilde{Q}_t = \mathrm{Bl}_{T_t} Q_t$ is non-normal along the singular locus of $\widetilde{\mathbb{F}}_t$.

Proof. Assume that $\#(T_t)_{\mathrm{red}} = 2$ (respectively 1). Let $\mathrm{Bl}_x \mathbb{F}_t$ be the blow-up of $\mathbb{F}_t = \mathbb{P}^3$ at x and E the exceptional divisor dominating x. Let $x' \in E$ (respectively $\Sigma' \subset E$) be the intersection of E and the proper transform of T_t. When $\#(T_t)_{\mathrm{red}} = 1$, we set $x' = \Sigma'_{\mathrm{red}}$. Let $\mathrm{Bl}_{x'} \mathrm{Bl}_x \mathbb{F}_t$ be the blow-up at x' and E' the exceptional divisor. Let x'' be the reduced point of T_t (respectively the point that is the intersection of E' and the proper transform of Σ'). We set $M = \mathrm{Bl}_{x''} \mathrm{Bl}_{x'} \mathrm{Bl}_x \mathbb{F}_t$ and let $E_M \subset M$ be the proper transform of E. Then we have a natural morphism $\tau\colon M \to \widetilde{\mathbb{F}}_t = \mathrm{Bl}_{T_t} \mathbb{F}_t$ by the universal property of the blow-up. Note that $C := \tau(E) \simeq \mathbb{P}^1$ (respectively $C := \tau(E) = \tau(E') \simeq \mathbb{P}^1$) and τ is a crepant divisorial contraction. Thus $\widetilde{\mathbb{F}}_t$ has a family of Du Val A_1 (respectively A_2) -singularities along its singular locus C. Let $Q_{t,M} \subset M$ be the proper transform of Q_t on M. Since Q_t contains x as an ordinary double point, we can check that $\tau|_{Q_{t,M}}\colon Q_{t,M} \to \widetilde{Q}_t = \mathrm{Bl}_{T_t} Q_t$ is finite but not isomorphic. Hence $\widetilde{Q}_t$ is non-normal along $C = \mathrm{Sing}(\widetilde{\mathbb{F}}_t)$. $\square$

By Claim 6.14, $\widetilde{Q}_t$ is non-normal along an exceptional locus of $\widetilde{Q}_t \to Q_t$. Hence the non-normal locus of $\widetilde{Q}_t$ is not contracted by the morphism $\widetilde{\psi}_t\colon \widetilde{Q}_t \to \overline{X}_t$ in the diagram (2.1). Therefore, $\overline{X}_t$ is non-normal and hence so is $\widetilde{X}_t$. Since $\widetilde{X}_t$ is the blow-up of X_t at a smooth point, X_t is non-normal. $\square$

Proof of Theorem 6.3. (1) and (2): Assume X_t is of type $(2,j)$ (respectively $(1,j)$). By Lemma 3.1 (1), $\#(B_t)_{\mathrm{red}} = 2$ if and only if Q_t is smooth. Then Lemma 6.12 shows that Q_t is smooth (respectively singular) and $\#(T_t)_{\mathrm{red}} = j$. Moreover, when X_t is of type $(1,j)$, it follows from Lemma 6.13 that $\mathrm{Sing}\, Q_t \cap (T_t)_{\mathrm{red}} = \varnothing$.

(3) Assume X_t is of type (n2). Then Lemma 6.13 shows that Q_t is singular and $\mathrm{Sing}\, Q_t \cap (T_t)_{\mathrm{red}} \neq \varnothing$. If $\#(T_t)_{\mathrm{red}} = 1$, then X_t is contained in $\mathbb{P}^{1,1,1}$ as a hyperplane section by Theorem 6.4, which contradicts Lemma 6.11. Hence $\#(T_t)_{\mathrm{red}} = 2$.

(4) Assume X_t is of type (n4). Then Lemma 6.13 shows that Q_t is singular and $\mathrm{Sing}\, Q_t \cap (T_t)_{\mathrm{red}} \neq \varnothing$. If $\#(T_t)_{\mathrm{red}} = 2$, then X_t is contained in $\mathbb{P}^1 \times \mathbb{Q}_0^2$ as a hyperplane section by Theorem 6.4. Thus we obtain a birational morphism $\mathbb{F}_4 \to \mathbb{Q}_0^2$, which is a contradiction. Hence $\#(T_t)_{\mathrm{red}} = 1$.

The proof is complete. $\square$

6.5. Proof of Theorem D and Corollary E

Finally, we prove Theorem D and Corollary E. Combining Theorems 6.4, 6.10, and 6.3, we have Theorem D. Let us show Corollary E. Let $\varphi\colon X \to C$ be a sextic del Pezzo fibration. Then (1) and (2) follow immediately from Theorem A (3). Let us show (3). By Theorem A (3), we have $(-K_{X/C})^3 = (-K_X + \varphi^* K_C)^3 = 24g(C) - (6g(B) + 4g(T) + 14)$. Let R_B and R_T denote the ramification divisor of φ_B and φ_T. Then the Hurwitz formula implies $\deg R_B = 2g(B) + 2 - 4g(C)$ and $\deg R_T = 2g(T) + 4 - 6g(C)$. Thus we have $(-K_{X/C})^3 = -(3 \deg R_B + 2 \deg R_T) \leq 0$. Hence $(-K_{X/C})^3 = 0$ if and only if $\deg R_B = \deg R_T = 0$, which is equivalent to that φ_B and φ_T are étale. By Theorem D, this is equivalent to that φ is smooth. The proof is complete. $\square$

7. Proof of Theorem F

Let us show Theorem F by presenting explicit examples of sextic del Pezzo fibrations which contain singular fibers of type $(2, j)$ for $j = 1, 2, 3$ (see Example 7.4), $(1, j)$ for $j = 1, 2, 3$ (see Example 7.5), (n2), or (n4) (see Example 7.6).

Step 1. We start our construction of examples of sextic del Pezzo fibrations from the following submanifolds in $\mathbb{P}^4$:

$$\overline{T} \subset \mathbb{Q}^2 \subset \mathbb{Q}^3 \subset \mathbb{P}^4,$$

where $\mathbb{Q}^3$ is a smooth quadric 3-fold, $\mathbb{Q}^2 \subset \mathbb{Q}^3$ is a smooth hyperplane section, and $\overline{T} \subset \mathbb{Q}^2$ is a twisted cubic curve.

For a smooth conic $C \subset \mathbb{Q}^3$ with $\overline{T} \cap C = \varnothing$, let $\tau\colon Q := \mathrm{Bl}_C\, \mathbb{Q}^3 \to \mathbb{Q}^3$ be the blow-up along C and $T := \tau_*^{-1}\overline{T}$. Then there exists a quadric fibration $q\colon Q \to \mathbb{P}^1$ given by the linear system of hyperplane sections of $\mathbb{Q}^3$ containing C. Note that $q|_T\colon T \to \mathbb{P}^1$ is a triple covering.

Step 2.

Claim 7.1. There exists a sextic del Pezzo fibration $\varphi\colon X \to \mathbb{P}^1$ with a section s such that $(q\colon Q \to \mathbb{P}^1, T)$ is the relative double projection of $(\varphi\colon X \to \mathbb{P}^1, s)$. Moreover, X is a weak Fano 3-fold with $(-K_X)^3 = 22$ and $-K_X.s = 0$.

Proof. Let $\sigma\colon \widetilde{Q} := \mathrm{Bl}_T\, Q \to Q$ be the blow-up. Set $H = \sigma^*\tau^*\mathcal{O}_{\mathbb{Q}^3}(1)$, $E = \mathrm{Exc}(\sigma)$, and $G = \sigma_*^{-1}\,\mathrm{Exc}(\tau)$. Note that $-K_{\widetilde{Q}} = 3H - E - G$ is free since $H - G$ and $2H - E$ are free. Using [10, Proposition 3.5], which is an inverse transformation of Proposition 2.1, we obtain a sextic del Pezzo fibration $\varphi\colon X \to \mathbb{P}^1$ and a φ-section s satisfying $(-K_X)^3 = 22$ and $-K_X.s = 0$. Since $\widetilde{Q}$ is weak Fano, X is also weak Fano by [ibid, Proposition 3.5]. □

Remark 7.2. Let (B, T) be the coverings associated to this sextic del Pezzo fibration. Since $(-K_X)^3 = 22$, Theorem A implies that $B \simeq T \simeq \mathbb{P}^1$. Then $B \to \mathbb{P}^1$ is ramified over exactly two points and hence $Q \to \mathbb{P}^1$ has exactly two singular fibers by Lemma 3.1 (1).

We prove Theorem F by showing that for any condition in Theorem 6.3, there exists a suitable smooth conic C and a point $t \in \mathbb{P}^1$ such that the pair (Q_t, T_t) satisfies the condition.

Step 3. For a point $p \in \mathbb{Q}^3$, let $\mathbb{T}_p\mathbb{Q}^3$ denote the projective tangent space of $\mathbb{Q}^3$ at p. For two points $v_1, v_2 \in \mathbb{Q}^3$, we set

$$C(v_1, v_2) := \mathbb{Q}^3 \cap \mathbb{T}_{v_1}\mathbb{Q}^3 \cap \mathbb{T}_{v_2}\mathbb{Q}^3.$$

Claim 7.3. Let $C \subset \mathbb{Q}^3$ be a smooth conic and $\langle C\rangle \subset \mathbb{P}^4$ be the linear span of C. Then there exist two points $v_1(C), v_2(C) \in \mathbb{Q}^3$ such that $C(v_1(C), v_2(C)) = C$.

Moreover, there exists the following one-to-one correspondence:

$$\begin{array}{ccc} \left\{ \{v_1, v_2\} \subset \mathbb{Q}^3 \,\middle|\, C(v_1, v_2) \text{ is smooth} \right\} & \longleftrightarrow & \{ \text{smooth conics in } \mathbb{Q}^3\} \\ \uplus & & \uplus \\ \{v_1, v_2\} & \longmapsto & C(v_1, v_2) \\ \{v_1(C), v_2(C)\} & \longleftarrow\!\!\shortmid & C. \end{array}$$

Proof. The quadric fibration $q\colon Q \to \mathbb{P}^1$ has exactly two singular fibers $Q_{1,C}$ and $Q_{2,C}$ as we saw in Remark 7.2. Then $Q_i := \tau(Q_{i,C}) \subset \mathbb{Q}^3$ is a singular quadric cone for each i. Set $v_i(C) := \sigma(\mathrm{Sing}\, Q_i)$. Then a hyperplane H in $\mathbb{P}^4$ containing $\langle C\rangle$ is tangent to $\mathbb{Q}^3$ if and only if $H \cap \mathbb{Q}^3 = Q_i$ and $H = \mathbb{T}_{v_i(C)}\mathbb{Q}^3$ for some $i \in \{1, 2\}$. Therefore, we have $\mathbb{T}_{v_1(C)}\mathbb{Q}^3 \cap \mathbb{T}_{v_2(C)}\mathbb{Q}^3 = \langle C\rangle$.

Let us confirm the one-to-one correspondence. Take two points $v_1, v_2 \in \mathbb{Q}^3$ such that $C(v_1, v_2)$ is smooth conic. Then $\{v_1(C(v_1, v_2)), v_2(C(v_1, v_2))\}$ is the set of vertices of the cones $\{\mathbb{T}_{v_1}\mathbb{Q}^3 \cap \mathbb{Q}^3, \mathbb{T}_{v_2}\mathbb{Q}^3 \cap \mathbb{Q}^3\}$, which is nothing but $\{v_1, v_2\}$. Hence we are done. □

Step 4. We finish the proof by presenting suitable examples as follows.

Example 7.4 (Singular fiber of type (2, j) for $j = 1, 2, 3$). Fix $j \in \{1, 2, 3\}$. We can take a smooth conic $C \subset \mathbb{Q}^2$ such that $\#(C \cap \overline{T})_{\mathrm{red}} = j$. Let Q_1 be a smooth hyperplane section of $\mathbb{Q}^3$ such that $C = Q_1 \cap \mathbb{Q}^2$. Then we have $Q_1 \cap \overline{T} = C \cap \overline{T}$. Let $Q_2 \subset \mathbb{Q}^3$ be a general hyperplane section such that $Q_2 \cap Q_1 \cap \overline{T} = \varnothing$. The pencil $\mathrm{L} = \{\lambda Q_1 + \mu Q_2 \mid [\lambda : \mu]\}$ induces a quadric fibration $q\colon \mathrm{Bl}_{Q_1 \cap Q_2} \mathbb{Q}^3 \to \mathbb{P}^1$. Let $f\colon \mathrm{Bl}_{Q_1 \cap Q_2} \mathbb{Q}^3 \to \mathbb{Q}^3$ be the blow-up. If we set $\widetilde{Q_i} := f_*^{-1} Q_i$, then $\widetilde{Q_i}$ is a q-fiber. Then we obtain $T \cap \widetilde{Q_1} \simeq \overline{T} \cap C$ since $Q_1 \cap Q_2 \cap \overline{T} = \varnothing$. Set $t = q(\widetilde{Q_1})$. Seeing the fiber $(Q_t, \overline{T}_t) = (\widetilde{Q_1}, T \cap \widetilde{Q_1})$ and using Theorem 6.3, we obtain an example of a sextic del Pezzo fibration having fibers of type $(2, j)$.

Example 7.5 (Singular fiber of type (1, j) for $j = 1, 2, 3$). Fix $j \in \{1, 2, 3\}$. We can take a smooth conic $C \subset \mathbb{Q}^2$ such that $\#(C \cap \overline{T})_{\mathrm{red}} = j$. By Claim 7.3, we can take a point $v_1, v' \in \mathbb{Q}^3$ such that $C = C(v_1, v')$. Then we have $\mathbb{T}_{v_1}\mathbb{Q}^3 \cap \mathbb{Q}^2 = C$.

Let $v_2 \in \mathbb{Q}^3$ be a point of $\mathbb{Q}^3$ such that $\langle v_1, v_2 \rangle \not\subset \mathbb{Q}^3$ and $\mathbb{T}_{v_2}\mathbb{Q}^3 \cap C \cap \overline{T} = \varnothing$. Set $Q_i = \mathbb{T}_{v_i}\mathbb{Q}^3 \cap \mathbb{Q}^3$ and $\widetilde{Q_i} \subset \mathrm{Bl}_{C(v_1, v_2)} \mathbb{Q}^3$ be the proper transform of Q_i for $i = 1, 2$. Then by Claim 7.3, $\widetilde{Q_1}, \widetilde{Q_2}$ are the singular fibers of the quadric fibration $\mathrm{Bl}_{C(v_1, v_2)} \mathbb{Q}^3 \to \mathbb{P}^1$. Recall that $Q_1 \cap \mathbb{Q}^2 = C$ is a smooth conic. Hence $Q_1 \cap \mathbb{Q}^2$ does not contain the vertex of Q_1 and hence we have $C \cap \overline{T} \simeq \widetilde{Q_1} \cap T$. Therefore, T does not pass through the vertex of Q_1. Set $t = q(\widetilde{Q_1})$. Seeing the fiber $(Q_t, \overline{T}_t) = (\widetilde{Q_1}, T \cap \widetilde{Q_1})$ and using Theorem 6.3, we obtain an example of a sextic del Pezzo fibration having fibers of type $(1, j)$.

Example 7.6 (Singular fibers of type (n2) and (n4)). We fix an isomorphism $\mathbb{Q}^2 \simeq \mathbb{P}^1_a \times \mathbb{P}^1_b$ and let l_a and l_b be the two rulings. We may assume that $\overline{T} \in |l_a + 2l_b|$. Let $g_a\colon \overline{T} \to \mathbb{P}^1_a$ be the restriction of the first projection to $\overline{T}$. Take a point $p_1 \in \overline{T}$. Then we have $\mathbb{T}_{p_1}\mathbb{Q}^2 = l_a + l_b$ and hence $\mathbb{T}_{p_1}\mathbb{Q}^3 \cap \overline{T} = \mathbb{T}_{p_1}\mathbb{Q}^3 \cap \mathbb{Q}^2 \cap \overline{T} = (l_a + l_b) \cap \overline{T} = p_1 + g_a^{-1}(g_a(p_1))$ as effective Cartier divisors on $\overline{T}$.

Let $p_2 \in \mathbb{Q}^3$ be a general point such that $C(p_1, p_2)$ is smooth conic. Let $Q_i := \mathbb{T}_{p_i}\mathbb{Q}^3 \cap \mathbb{Q}^3$, $q\colon \underline{Q} = \mathrm{Bl}_C \mathbb{Q}^3 \to \mathbb{P}^1$ the quadric fibration, and $\widetilde{Q_i}$ the proper transform of Q_i. Then $\widetilde{Q_1}$ is a singular q-fiber with the vertex p_1 and $\widetilde{Q_1} \cap T \simeq \overline{T} \cap (l_a + l_b) = p_1 + g_a^{-1}(g_a(p_1))$.

If p_1 is a unramified (respectively ramified) point of $\overline{T} \to \mathbb{P}^1_a$, then we obtain an example of a sextic del Pezzo fibration having fibers of type (n2) (respectively (n4)) by seeing the fiber $(\widetilde{Q_1}, T \cap \widetilde{Q_1})$ and using Theorem 6.3.

The proof of Theorem F is complete. □

Appendix

A. Relative universal extensions of sheaves

This appendix is devoted to proving Theorem 3.9.

First of all, we recall the notion of relative Ext sheaves and organize some basic properties. Let $f\colon X \to Y$ be a proper morphism between noetherian schemes X and Y. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X and $\mathcal{E}$ a coherent sheaf on Y. We denote the i-th cohomology of the right derived functor of $f_*\mathcal{H}om(\mathcal{F}, -)$ by $\mathcal{E}xt^i_f(\mathcal{F}, -)$. We call this sheaf $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ the *relative Ext sheaf* (*cf.* [14]).

Note that $\mathcal{E}xt^i_f(\mathcal{O}_X, \mathcal{G}) = R^i f_*\mathcal{G}$ by the definition. Moreover, composing natural canonical morphisms

$$f_*\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E} \to f_*\left(\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes f^*\mathcal{E}\right) \to f_*\mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}),$$

we obtain a natural morphism $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}) \to \mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}$. By the projection formula, this is isomorphic when $\mathcal{E}$ is locally free.

Consider the following three spectral sequences:

$$\begin{aligned}
&R^i f_*\mathcal{E}xt^j(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}) \Rightarrow \mathcal{E}xt^{i+j}_f(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}),\\
&H^i(Y, \mathcal{E}xt^j_f(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E})) \Rightarrow \mathrm{Ext}^{i+j}(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}),\\
&H^i(X, \mathcal{E}xt^j(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E})) \Rightarrow \mathrm{Ext}^{i+j}(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}).
\end{aligned} \tag{A.1}$$

These spectral sequences give the following three natural morphisms:

$$\begin{aligned}
&\alpha\colon \mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}) \to f_*\mathcal{E}xt^1(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}),\\
&\beta'\colon\ \mathrm{Ext}^1(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}) \to H^0(Y, \mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E})),\\
&\gamma'\colon\ \mathrm{Ext}^1(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}) \to H^0(X, \mathcal{E}xt^1(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E})).
\end{aligned} \tag{A.2}$$

Additionally, we define $\widetilde{\beta}$, β, and γ as in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathcal{F},\mathcal{G} \otimes f^*\mathcal{E}) \arrow[r, "\beta'"] \arrow[rr, bend left, "\gamma'"] \arrow[dr, bend right, "\widetilde{\beta}"] \arrow[ddr, bend right, "\beta"'] \arrow[dddrr, rounded corners, "\gamma"'] & H^0(Y,\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G} \otimes f^*\mathcal{E})) \arrow[r, "H^0(\alpha)"] \arrow[d] & H^0(Y,f_*\mathcal{E}xt^1(\mathcal{F},\mathcal{G}\otimes f^*\mathcal{E})) \arrow[d] \\
 & H^0(Y, \mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}) \arrow[r] \arrow[d] & H^0(Y, f_*\mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}) \arrow[d] \\
 & \mathrm{Hom}(\mathcal{E}^\vee, \mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})) \arrow[r, "\alpha\circ-"'] & \mathrm{Hom}(\mathcal{E}^\vee, f_*\mathcal{E}xt^1(\mathcal{F}, \mathcal{G})) \arrow[d, "\text{adjoint}"', "\simeq"] \\
 & & \mathrm{Hom}(f^*\mathcal{E}^\vee, \mathcal{E}xt^1(\mathcal{F}, \mathcal{G})),
\end{tikzcd} \tag{A.3}$$

where all vertical arrows are the natural morphisms. Note that all of the vertical arrows are isomorphic when $\mathcal{E}$ is locally free.

Remark A.1. For an element $t \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E})$, it is easy to verify that the composite morphism

$$f^*\mathcal{E}^\vee \xrightarrow{f^*\beta(t)} f^*\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G}) \xrightarrow{f^*\alpha} f^*f_*\mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \xrightarrow{\varepsilon} \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \tag{A.4}$$

is nothing but $\gamma(t) \in \mathrm{Hom}(f^*\mathcal{E}^\vee, \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}))$, where $\varepsilon\colon f^*f_*\mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{G})$ is the natural morphism.

Definition A.2. Put $\mathcal{E} = \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee$. Consider the composition

$$\begin{aligned}\tau\colon \operatorname{Ext}^1(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee) &\xrightarrow{\widetilde{\beta}} H^0(Y, \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee) \\ &\xrightarrow{\theta} \operatorname{Hom}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G}), \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})),\end{aligned} \tag{A.5}$$

where $\widetilde{\beta}$ is the morphism in (A.3) and θ is the natural morphism.

We say that an element $t \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{G} \otimes f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$ is *universal* if $\tau(t) = \mathrm{id}$. If t is universal, then we say that the corresponding extension

$$0 \to \mathcal{G} \otimes f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee \to \mathcal{H}_t \to \mathcal{F} \to 0 \tag{A.6}$$

is an *f-universal extension* of $\mathcal{F}$ by $\mathcal{G}$. When $Y = \operatorname{Spec} k$ for a field k, $\mathcal{H}_t$ is just called a *universal extension* of $\mathcal{F}$ by $\mathcal{G}$.

The following lemma is a criterion for the existence of a locally free f-universal extension of $\mathcal{F}$ by $\mathcal{G}$.

Lemma A.3. *Suppose the following conditions hold:*

(1) *$\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ is locally free;*
(2) *$H^2(Y, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee) = 0$;*
(3) *X is regular, $\mathcal{G}$ is locally free, and* $\operatorname{hd}(\mathcal{F}_x) \leq 1$ *for any $x \in X$;*
(4) *$\varepsilon \circ f^*\alpha\colon f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{G})$ is surjective (see* (A.4)*).*

Then there exists a locally free f-universal extension $\mathcal{H}$ of $\mathcal{F}$ by $\mathcal{G}$.

Proof. First, we prove that the conditions (1) and (2) imply the existence of an f-universal extension. By (A.5), it is enough to see that θ and $\widetilde{\beta}$ are surjective. By (1), the morphism θ in (A.5) is surjective. Moreover, the surjectivity of $\widetilde{\beta}$ is equivalent to that of β', which is defined in (A.2) by the spectral sequence (A.1). Using this spectral sequence, (1), and (2), we deduce that β' is surjective. Therefore, there exists an f-universal extension $\mathcal{H}_t$.

Now $\mathcal{H}_t$ fits into the exact sequence (A.6). Since we assume that X is regular and $\mathcal{G}$ is locally free in (3), $\mathcal{H}_t$ is locally free if and only if $\mathcal{E}xt^i(\mathcal{H}_t, \mathcal{G}) = 0$ for any $i \geq 1$. By taking $\mathcal{H}om(-, \mathcal{G})$ of the sequence (A.6), we obtain an exact sequence

$$\mathcal{H}om(\mathcal{G} \otimes f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee, \mathcal{G}) \xrightarrow{\delta} \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{H}_t, \mathcal{G}) \to 0.$$

Since $\operatorname{hd}(\mathcal{F}_x) \leq 1$ for all $x \in X$, we have $\mathcal{E}xt^i(\mathcal{H}_t, \mathcal{G}) = 0$ for any $i \geq 2$. Hence it is enough to show that δ is surjective. Let $\nu\colon f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{G} \otimes f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee, \mathcal{G})$ be the natural map.

Recall $\gamma(t) \in \operatorname{Hom}(f^*\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G}), \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}))$ as in (A.3). Then it holds that $\delta \circ \nu = \gamma(t)$. Hence it suffices to show that $\gamma(t)$ is surjective. To see this fact, we recall the diagram (A.4). By (1) and the universality, the morphism $f^*\beta(t)$ is isomorphic. By (4), $\varepsilon \circ f^*\alpha$ is surjective. Then by Remark A.1, $\gamma(t)$ is surjective, which completes the proof. $\square$

The following lemma is an important property of a universal extension.

Lemma A.4. *Let X be a proper geometrically connected geometrically reduced scheme over a field. Let $\mathcal{F}$ and $\mathcal{L}$ be a coherent sheaf and an invertible sheaf on X respectively. Suppose that there exists a universal extension of $\mathcal{F}$ by $\mathcal{L}$:*

$$0 \to \mathcal{L} \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{L})^\vee \to \mathcal{H} \to \mathcal{F} \to 0. \tag{A.7}$$

If there are no surjections $\mathcal{F} \to \mathcal{L}$, then there are no surjections $\mathcal{H} \to \mathcal{L}$.

Proof. To obtain a contradiction, assume that there are no surjections $\mathcal{F} \to \mathcal{L}$ and there is a surjection $a\colon \mathcal{H} \to \mathcal{L}$. We consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{L})^\vee \arrow[r] \arrow[d, "b"'] & \mathcal{H} \arrow[r] \arrow[d, "a"'] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r, equal] & \mathcal{L} \arrow[r] & 0, &
\end{tikzcd}$$

where b is the composition of the morphisms $\mathcal{L} \otimes \mathrm{Ext}^1(\mathcal{F}, \mathcal{L})^\vee \to \mathcal{H}$ and a. Since there are no surjections $\mathcal{F} \to \mathcal{L}$, we have $b \neq 0$. Hence b splits. Letting V be the linear subspace of $\mathrm{Ext}^1(\mathcal{F}, \mathcal{L})^\vee$ such that $\mathrm{Ker}\, b = \mathcal{L} \otimes V$, we obtain an exact sequence $0 \to \mathcal{L} \otimes V \to \mathrm{Ker}\, a \to \mathcal{F} \to 0$. Then the exact sequence (A.7) is the push-forward of this exact sequence, which implies that id: $\mathrm{Ext}^1(\mathcal{F}, \mathcal{L}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{L})$ factors through V, which is a contradiction. Hence we are done. □

Proof of Theorem 3.9. Let $f\colon X \to Y$ and $Z \subset X$ be as in Theorem 3.9. We first prove that there is a locally free f-universal extension $\mathcal{H}$ of $\mathcal{I}_Z$ by ω_f. It is enough to check that (1) – (4) in Lemma A.3 hold for $\mathcal{F} = \mathcal{I}_Z$ and $\mathcal{G} = \omega_f$. (2) and (3) immediately hold from our assumption. We show (1). Considering the higher direct images of the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$, we have the exact sequence $0 \to \mathcal{O}_Y \to f_*\mathcal{O}_Z \to R^1 f_*\mathcal{I}_Z \to R^1 f_*\mathcal{O}_X \to 0$. Hence $R^1 f_*\mathcal{I}_Z$ is locally free since so is $R^1 f_*\mathcal{O}_X$ and $f|_Z$ is finite flat by our assumption. Since every fiber of f and $f|_Z$ are Cohen-Macaulay, [14, Theorem (21)] gives the following:

$$\begin{tikzcd}
0 \arrow[r] & (R^1 f_*\mathcal{O}_X)^\vee \arrow[r] & (R^1 f_*\mathcal{I}_Z)^\vee \arrow[r, "\widehat{\alpha}"] & (f_*\mathcal{O}_Z)^\vee \arrow[r] & (f_*\mathcal{O}_X)^\vee \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}xt^1_f(\mathcal{O}_X, \omega_f) \arrow[r] \arrow[u, "\simeq"] & \mathcal{E}xt^1_f(\mathcal{I}_Z, \omega_f) \arrow[r] \arrow[u, "\simeq"] & \mathcal{E}xt^2_f(\mathcal{O}_Z, \omega_f) \arrow[r] \arrow[u, "\simeq"] & \mathcal{E}xt^2_f(\mathcal{O}_X, \omega_f). \arrow[u, "\simeq"] &
\end{tikzcd} \tag{A.8}$$

Hence (1) follows. We show (4). When we identify $(f_*\mathcal{O}_Z)^\vee$ with $f_*\omega_{Z/Y}$, the natural map α in (A.4) can be identified with $\widehat{\alpha}$ in (A.8). Set $\mathcal{E} := \mathrm{Cok}(\mathcal{O}_Y \to f_*\mathcal{O}_Z)$. Then we have $\Im\widehat{\alpha} = \mathcal{E}^\vee$ by (A.8). By [4, Theorem 2.1 (ii)], the composition of the natural maps $f^*\mathcal{E}^\vee \to f^* f_*\omega_{Z/Y} \to \omega_{Z/Y}$ is surjective, which implies (4).

Then Lemma A.3 gives a locally free f-universal extension $\mathcal{H}$ of $\mathcal{I}_Z$ by ω_f. Tensoring $\mathcal{O}(-K_X)$, we obtain a locally free f-universal extension $\mathcal{F}$ of $\mathcal{I}_Z(-K_X)$ by $\mathcal{O}_X$. Identifying $\mathcal{E}xt^1_f(\mathcal{I}_Z, \omega_f)^\vee$ with $R^1 f_*\mathcal{I}_Z$, we obtain the exact sequence (3.5). For each $y \in Y$, the restriction of the exact sequence (3.5) to X_y is also exact since f and $f|_Z$ are flat. Since $\mathrm{Ext}^2(\mathcal{I}_{Z\cap X_y}, \omega_{X_y}) = H^0(X_y, \mathcal{I}_{Z\cap X_y}) = 0$ holds, the natural morphism $\mathcal{E}xt^1_f(\mathcal{I}_Z, \omega_f) \otimes k(y) \to \mathrm{Ext}^1(\mathcal{I}_{Z\cap X_y}, \omega_{X_y})$ is isomorphic by [3, Satz 3]. Thus $\mathcal{F}|_{X_y}$ is also a universal extension of $\mathcal{I}_{Z\cap X_y}(-K_{X_y})$ by $\mathcal{O}_{X_y}$. Since there are no surjections from $\mathcal{I}_{Z\cap X_y}(-K_{X_y})$ to $\mathcal{O}_{X_y}$, the property (2) of Theorem 3.9 follows from Lemma A.4 if X_y is reduced. The proof is complete. $\square$

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