

Interplay of nonlinear diffusion, initial tails and Allee effect on the speed of invasions

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Abstract. We focus on the spreading properties of solutions of monostable equations with nonlinear diffusion. We consider both the porous medium diffusion and the fast diffusion regimes. Initial data may have heavy tails, which tends to accelerate the invasion phenomenon. On the other hand, the nonlinearity may involve a weak Allee effect, which tends to slow down the process. We study the balance between these three effects (nonlinear diffusion, initial tail, KPP nonlinearity/Allee effect), revealing the separation between “no acceleration” and “acceleration”. In most of the cases where acceleration occurs, we also give an accurate estimate of the position of the level sets.

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1. Introduction

In this paper we are concerned with the *spreading properties* of $u(t, x)$ the solution of the nonlinear monostable reaction-diffusion equation

$$\partial_t u = \partial_{xx}(u^m) + f(u), \quad t > 0, x \in \mathbb{R}. \quad (1.1)$$

We consider both the *porous medium diffusion* regime $m > 1$ and the *fast diffusion* regime $0 < m < 1$, the *linear diffusion* case $m = 1$ being already well understood. The typical nonlinearity f we have in mind is $f(s) = rs^\beta(1-s)$, with $r > 0$ and either $\beta = 1$ (Fisher-KPP) or $\beta > 1$ (*Allee effect*). Equation (1.1) is supplemented with a nonnegative initial data which is *front-like* and may have a *heavy tail*, say $u_0(x) \sim \frac{1}{x^\alpha}$ for some $\alpha > 0$ as $x \rightarrow +\infty$. Our main goal is to understand the interplay between nonlinear diffusion (measured by $m > 0$), the initial tail (measured by $\alpha \in (0, +\infty]$) and the behavior of the nonlinearity at 0 (measured by $\beta \geq 1$), and to determine if propagation occurs by accelerating or not. In the former case, we also aim at estimating the position of the level sets of $u(t, \cdot)$ as $t \rightarrow +\infty$, typically revealing that they travel exponentially or polynomially fast.

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Nonlinear diffusion. The well-posedness of the Cauchy problem for the degenerate diffusion equation

$$\partial_t u = \Delta(u^m), \quad t > 0, x \in \mathbb{R}^N,$$

is now well understood as long as $m > 1$ or $m_c := \max(0, 1 - \frac{2}{N}) < m < 1$. Precise statements and results can be found in [13, 23, 24, 32, 33, 35] and the references therein. The main feature of the regime $m > 1$ is that the equation degenerates at the points where $u = 0$. Hence, the so-called *Barenblatt self-similar solutions* exhibit a free boundary, a loss of regularity of solutions occurs and disturbances propagate with finite speed. This is in sharp contrast with the infinite speed of propagation of solutions of the heat equation ($m = 1$) and of the fast diffusion equation ($m < 1$). See, e.g., [6] and [7].

From the population dynamics point of view, let us briefly explain the role of introducing nonlinear effects into the dispersal behavior of a species. After the observation of arctic ground squirrels migrating from densely populated areas into sparsely populated ones (even if the latter is less favorable) in [10], Gurney and Nisbet [19], Gurtin and MacCamy [20] introduced porous medium diffusion ($m > 1$) in models. On the other hand, in order to adapt to low density mate distributions, individuals may need to travel extreme distances, thus leading to accelerating range expansion, for which fast diffusion ($m < 1$) may be pertinent: see [27] and the references therein for the propagation of early Palaeoindian hunter-gatherers.

KPP nonlinearities. In some population dynamics models, a common assumption is that the growth is only slowed down by the intra-specific competition, so that the growth per capita is maximal at small densities. This leads to consider reaction-diffusion equations with nonlinearities f of the Fisher-KPP type, namely

$$f(0) = f(1) = 0, \quad \text{and} \quad 0 < f(s) \leq f'(0)s, \quad \forall s \in (0, 1).$$

The simplest example $f(s) = rs(1 - s)$, $r > 0$, was first introduced by Fisher [15] and Kolmogorov, Petrovsky and Piskunov [28] to model the spreading of advantageous genetic features in a population.

In such situations, it is well known that the way the front-like initial data — in the sense of Assumption 2.1 — approaches zero at $+\infty$ is of crucial importance on the propagation, that is the invasion for large times of the unstable steady state $u \equiv 0$ by the stable steady state $u \equiv 1$.

Let us start with the linear diffusion case $m = 1$. For initial data with an exponentially bounded tail (or light tail) at $+\infty$, there is a *spreading speed* $c \geq c^* := 2\sqrt{f'(0)}$ which is selected by the rate of decay of the tail. There is a large literature on such results and improvements, and we only mention the seminal works [5, 15, 21, 25, 28, 29, 31]. On the other hand, Hamel and Roques [22] recently considered the case of initial data with heavy (or not exponentially bounded) tail, namely

$$\lim_{x \rightarrow +\infty} u_0(x)e^{\varepsilon x} = +\infty, \quad \forall \varepsilon > 0.$$

They proved that, for any $\lambda \in (0, 1)$, the λ -level set of $u(t, \cdot)$ travels infinitely fast as $t \rightarrow +\infty$, thus revealing an acceleration phenomenon. Also, the location of these level sets is estimated in terms of the decaying rate to zero of the initial data.

Much less is known on the propagation of solutions to (1.1) in the nonlinear diffusion case $m \neq 1$. Concerning the porous medium regime $m > 1$, we mention the works [6, 26] where propagation at constant speed is analyzed. In this paper, we further consider some cases where acceleration occurs because of a heavy initial tail. Concerning the fast diffusion regime $0 < m < 1$, let us mention the works [7, 27] where acceleration (already induced by diffusion, whatever the initial tail) is investigated. In this paper, we refine those results and provide precise estimates on the location of the accelerating level sets.

Allee effect. In population dynamics, due for instance to the difficulty to find mates or to the lack of genetic diversity at low density, the KPP assumption is unrealistic in some situations. In other words, the growth per capita is no longer maximal at small densities, which is referred to as an Allee effect.

In this Allee effect context, if $f'(0) > 0$ the situation — even if more complicated — is more or less comparable to the KPP situation. On the other hand, much less is known in the degenerate situation where $f'(0) = 0$, for which typical nonlinearities take the form

$$f(s) = rs^\beta(1 - s), \quad r > 0, \quad \beta > 1.$$

In the linear diffusion case $m = 1$, propagation at constant speed in presence of an Allee effect was for instance studied in [5, 8, 36, 37]. Very recently the balance between the strength of the Allee effect (which tends to slow down the invasion process) and heavy tails (which tend to accelerate it) was studied in [2]. For algebraic tails, the exact separation between acceleration or not (depending on the strength of the Allee effect) was obtained. Also, when acceleration occurs, the location of the level sets of the solution was precisely estimated.

To the best of our knowledge there are very few results on the propagation of solutions to (1.1) combining nonlinear diffusion $m \neq 1$ and an Allee effect (say $\beta > 1$). Let us mention the work [30] which, for $m > 1$ and Heaviside type initial data, proves propagation at constant speed. In this paper, we prove both “no acceleration” and “acceleration” results, depending on the parameters of diffusion $m > 0$, the initial tail $\alpha \in (0, +\infty]$, and the behavior of f at zero $\beta \geq 1$. Also, when acceleration occurs, we provide sharp estimates of the level sets of the solution.

Remark 1.1 (Nonlocal diffusion). Related results exist for the integro-differential equation of the KPP type

$$\partial_t u = J * u - u + f(u), \quad (1.2)$$

where the kernel J allows to take into account rare long-distance dispersal events. Here, the initial data is typically compactly supported and this is the tail of the

dispersion kernel J that determines how fast is the invasion. If the kernel is exponentially bounded, then propagation occurs at a constant speed, as can be seen in [11, 12, 34] among others. More recently, Garnier [16] proved an acceleration phenomenon for kernels which are not exponentially bounded, so that (1.2) is an accurate model to explain the Reid's paradox of rapid plant migration (see [16] for references on this issue). As far as the integro-differential equation (1.2) with an Allee effect is concerned, we refer to [1, 3] for results on the balance between the Allee effect and dispersion kernels with heavy tails.

To conclude on acceleration phenomena in Fisher-KPP type equations, let us mention the case when the Laplacian is replaced by the generator of a Feller semi-group, a typical example being

$$\partial_t u = -(-\partial_{xx})^\alpha u + f(u), \quad 0 < \alpha < 1, \quad (1.3)$$

where $-(-\partial_{xx})^\alpha$ stands for the Fractional Laplacian, whose symbol is $|\xi|^{2\alpha}$. In this context, it was proved by Cabré and Roquejoffre [9] that, for a compactly supported initial data, acceleration always occurs, due to the algebraic tails of the Fractional Laplacian. Last, notice that the question of acceleration or not in the nonlocal equation (1.3) with an Allee effect has been recently solved by Gui and Huan [18].

2. Assumptions and main results

Through this work, and even if not recalled, we always make the following assumption on the initial condition.

Assumption 2.1 (Initial condition). The initial condition $u_0 : \mathbb{R} \rightarrow [0, 1]$ is uniformly continuous and asymptotically front-like, in the sense that

$$u_0 > 0 \text{ in } \mathbb{R}, \quad \liminf_{x \rightarrow -\infty} u_0(x) > 0, \quad \lim_{x \rightarrow +\infty} u_0(x) = 0.$$

As far as the nonlinearity f is concerned, we always assume the following.

Assumption 2.2 (Monostable nonlinearity). The nonlinearity $f : [0, 1] \rightarrow \mathbb{R}$ is of the class C^1 , and is of the monostable type, in the sense that

$$f(0) = f(1) = 0, \quad f' > 0 \text{ in } (0, 1).$$

Notice that, in each result, we clearly state the decay of the tail of the initial data as well as the behavior of $f(u)$ as $u \rightarrow 0$ (Fisher KPP vs. Allee effect), which therefore we did not include in the above assumptions. The simplest examples of nonlinearities satisfying Assumption 2.2 are given by $f(s) = rs^\beta(1-s)$, $r > 0$, with $\beta = 1$ (Fisher-KPP) or with $\beta > 1$ (Allee effect).

In the sequel, we always denote by $u(t, x)$ the solution of (1.1) with initial condition u_0 . From the above assumptions and the comparison principle, one gets $0 \leq u(t, x) \leq 1$ and even

$$0 < u(t, x) < 1, \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}.$$

The strict upper bound can be inferred from the strong maximum principle, but the degenerate diffusion at zero (when $m \neq 1$) prevents such an argument for the strict lower bound, which however follows from [32, Corollary 4.4] when $m > 1$ (recall that u_0 is continuous and positive) and from [32, Theorem 4.6] when $0 < m < 1$.

Since the initial data is front-like, it is very expected that the state $u \equiv 1$ does invade the whole line \mathbb{R} as $t \rightarrow +\infty$: there is $c_0 > 0$ such that

$$\lim_{t \rightarrow +\infty} \inf_{x \leq c_0 t} u(t, x) = 1, \quad (2.1)$$

meaning that propagation is at least linear. We also have

$$\lim_{x \rightarrow +\infty} u(t, x) = 0, \quad \forall t \geq 0. \quad (2.2)$$

For the sake of completeness, these preliminary facts (2.1) and (2.2) will be proved in Section 3.

In order to state our results we define, for any $\lambda \in (0, 1)$ and $t \geq 0$,

$$E_\lambda(t) := \{x \in \mathbb{R} : u(t, x) = \lambda\}$$

the λ -level set of $u(t, \cdot)$. In view of (2.1) and (2.2), for any $\lambda \in (0, 1)$, there is a time $t_\lambda > 0$ such that

$$\emptyset \neq E_\lambda(t) \subset (c_0 t, +\infty), \quad \forall t \geq t_\lambda. \quad (2.3)$$

Our first main result is concerned with the acceleration phenomenon that occurs for any $m > 0$ as soon as the nonlinearity is of the Fisher-KPP type. For linear diffusion $m = 1$ this was proved in [22]. We extend the result to porous medium diffusion $m > 1$ and fast diffusion $0 < m < 1$.

Theorem 2.3 (Acceleration in the Fisher-KPP case). *Let $m > 0$ and $\alpha > 0$ be given. Assume that there are $C > 0$, $\bar{C} > 0$ and $x_0 > 1$ such that*

$$\frac{\bar{C}}{x^\alpha} \geq u_0(x) \geq \frac{C}{x^\alpha}, \quad \forall x \geq x_0, \quad (2.4)$$

as well as $r > 0$ and $s_0 \in (0, 1)$ such that

$$f(s) \geq rs, \quad \forall 0 \leq s \leq s_0. \quad (2.5)$$

Select $\bar{r} > 0$ such that

$$\bar{r}s \geq f(s), \quad \forall 0 \leq s \leq 1. \quad (2.6)$$

Then, for any $\lambda \in (0, 1)$, any small $\varepsilon > 0$, there is a time $T_{\lambda, \varepsilon} \geq t_\lambda$ such that

$$E_\lambda(t) \subset (x^-(t), x^+(t)), \quad \forall t \geq T_{\lambda, \varepsilon}, \quad (2.7)$$

where

$$x^-(t) := e^{(r-\varepsilon)\Gamma t}, \quad x^+(t) := e^{(\bar{r}+\varepsilon)\Gamma t}, \quad \Gamma := \max\left(\frac{1-m}{2}, \frac{1}{\alpha}\right).$$

The above result indicates that the level sets of the solution travel exponentially fast. For any $m > 1$, we have $\Gamma = \frac{1}{\alpha}$ (independent on m) for all $\alpha > 0$ and the estimate is the same as that of [22] when $m = 1$. On the other hand if $0 < m < 1 - \frac{2}{\alpha}$, we have $\Gamma = \frac{1-m}{2}$ and, due to fast diffusion, the estimate is in contrast with that of [22].

When $m \geq 1$ the heavy tail assumption, *i.e.* the lower bound in (2.4), is crucial for the acceleration results in Theorem 2.3 to hold. On the other hand, when $0 < m < 1$ this is not necessary since the fast diffusion equation makes the tail of the solution (at least) algebraically heavy at any positive time [23], see Subsection 6.1 for details. To shed light on this phenomenon, we state the following corollary which is a rather straightforward consequence of Theorem 2.3 and Subsection 6.1.

Corollary 2.4 (Acceleration in the Fisher-KPP case, $0 < m < 1$). *Let $0 < m < 1$ be given. Assume that there are $\bar{C} > 0$ and $x_0 > 1$ such that*

$$\frac{\bar{C}}{x^{\frac{2}{1-m}}} \geq u_0(x), \quad \forall x \geq x_0, \quad (2.8)$$

as well as $r > 0$, $\bar{r} > 0$ and $s_0 \in (0, 1)$ such that (2.5) and (2.6) hold. Then the conclusions of Theorem 2.3 are valid with $\Gamma = \frac{1-m}{2}$.

From now on, we consider an Allee effect by letting $f(s)$ behave like s^β , $\beta > 1$, as $s \rightarrow 0$. Our next theorem shows that the acceleration phenomenon disappears when β is large enough.

Theorem 2.5 (No acceleration regime). *Let $m > 0$, $\alpha > 0$ and $\beta > 1$ be such that*

$$\beta \geq \max \left(1 + \frac{1}{\alpha}, 2 - m \right). \quad (2.9)$$

Assume that there are $\bar{C} > 0$ and $x_0 > 1$ such that

$$u_0(x) \leq \frac{\bar{C}}{x^\alpha}, \quad \forall x \geq x_0, \quad (2.10)$$

as well as $\bar{r} > 0$ and $s_0 \in (0, 1)$ such that

$$f(s) \leq \bar{r}s^\beta, \quad \forall 0 \leq s \leq s_0. \quad (2.11)$$

Then, there is a speed $c > 0$ such that, for any $\lambda \in (0, 1)$, there is a time $T_\lambda \geq t_\lambda$ such that

$$\emptyset \neq E_\lambda(t) \subset (c_0 t, ct), \quad \forall t \geq T_\lambda. \quad (2.12)$$

Next, we go back to the acceleration regime and look at the intermediate values of β . We need to distinguish the $m > 1$ porous medium regime from the $0 < m < 1$ fast diffusion regime. The former is sharply solved by the following, which indicates strong similarities with the linear diffusion regime $m = 1$ studied in [2].

Theorem 2.6 (Localization of the accelerating level sets, $m > 1$). *Let $m > 1$, $\alpha > 0$ and $\beta > 1$ be such that*

$$\beta < 1 + \frac{1}{\alpha}.$$

Assume that (2.4) holds, and that there are $r > 0$, $\bar{r} > 0$, and $s_0 \in (0, 1)$ such that

$$f(s) \geq rs^\beta, \quad \forall 0 \leq s \leq s_0, \quad (2.13)$$

and

$$\bar{r}s^\beta \geq f(s), \quad \forall 0 \leq s \leq 1. \quad (2.14)$$

Then, for any $\lambda \in (0, 1)$, any small $\varepsilon > 0$, there is a time $T_{\lambda, \varepsilon} \geq t_\lambda$ such that

$$E_\lambda(t) \subset (x^-(t), x^+(t)), \quad \forall t \geq T_{\lambda, \varepsilon}, \quad (2.15)$$

where

$$x^-(t) := \left((r - \varepsilon)C^{\beta-1}(\beta - 1)t \right)^{\frac{1}{\alpha(\beta-1)}},$$

$$x^+(t) := \left((\bar{r} + \varepsilon)\bar{C}^{\beta-1}(\beta - 1)t \right)^{\frac{1}{\alpha(\beta-1)}}.$$

The similarity with the linear diffusion regime can be understood from the fact that, when $m > 1$, diffusion is slower at low values of u and therefore it is not expected to play the driving role in the acceleration phenomenon. The picture is now completely understood in the case $m > 1$ and is summarized in Figure 2.1.

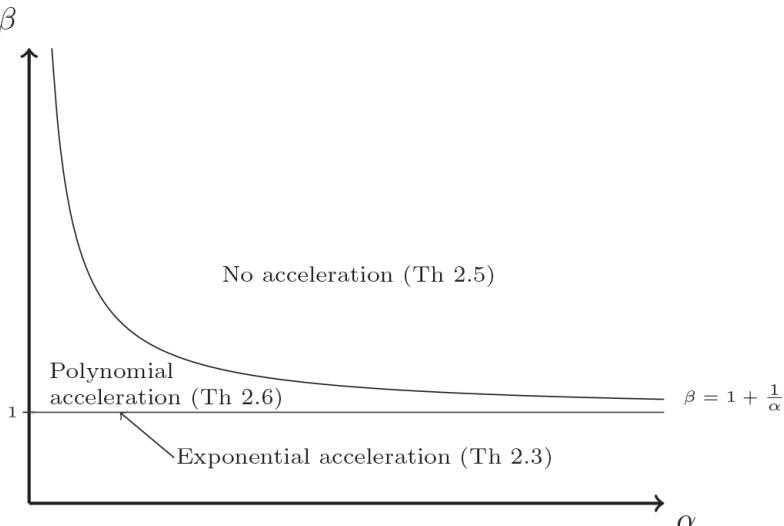


Figure 2.1. Summary of our results in the porous medium diffusion case, $m > 1$.

On the other hand, the situation is different when $m \in (0, 1)$, as the fast diffusion may then overcome the reactive growth. Let us thus turn to the situation when $m \in (0, 1)$ and β is in the intermediate range where we expect acceleration.

Theorem 2.7 (Localization of the accelerating level sets, $0 < m < 1$). *Let $m \in (0, 1)$, $\alpha \in (0, \frac{2}{1-m})$ and $\beta > 1$ be such that*

$$\beta < \min \left(1 + \frac{1}{\alpha}, m + \frac{2}{\alpha} \right). \quad (2.16)$$

Assume that the initial condition u_0 and the nonlinearity f satisfy (2.4), (2.13) and (2.14).

Then, for any $\lambda \in (0, 1)$, any small $\varepsilon > 0$, there is a time $T_{\lambda, \varepsilon} \geq t_\lambda$ such that

$$E_\lambda(t) \subset (x^-(t), x^+(t)), \quad \forall t \geq T_{\lambda, \varepsilon}, \quad (2.17)$$

where

$$x^-(t) := \left((r - \varepsilon) C^{\beta-1} (\beta - 1) t \right)^{\frac{1}{\alpha(\beta-1)}}, \quad x^+(t) := \left((\bar{r} + \varepsilon) \bar{C}^{\beta-1} (\beta - 1) t \right)^{\frac{1}{\alpha(\beta-1)}}.$$

Let us comment the assumptions on the parameters α and β of the above theorem. First of all, as already mentioned above and as will be detailed in Subsection 6.1, fast diffusion increases the tail of solutions so that the range $\alpha > \frac{2}{1-m}$ is, in some sense, irrelevant. As a consequence the assumption $\alpha \in (0, \frac{2}{1-m})$ actually only rules out the critical case $\alpha = \frac{2}{1-m}$.

Moreover, (2.16) means that β lies below two hyperbolae. The first one already appeared in Theorem 2.5. The second one, which is relevant only in the regime $\frac{1}{1-m} < \alpha \leq \frac{2}{1-m}$, seems to appear for technical reasons. Indeed in the region $\frac{1}{1-m} < \alpha \leq \frac{2}{1-m}$ and $m + \frac{2}{\alpha} \leq \beta < 2 - m$ (which is covered neither by Theorem 2.5 nor by Theorem 2.7), even if we cannot precisely localize the position of the level sets, we can still prove acceleration.

Theorem 2.8. (Acceleration in the remaining “parameters region”, and even more). *Let $m \in (0, 1)$ and $1 < \beta < 2 - m$ be given. Assume that there are $r > 0$ and $s_0 \in (0, 1)$ such that (2.13) holds.*

(i) *Then, for any $c > 0$, we have*

$$\lim_{t \rightarrow +\infty} \inf_{x \leq ct} u(t, x) = 1.$$

(ii) *If furthermore (2.4) holds with $\frac{1}{1-m} < \alpha \leq \frac{2}{1-m}$ and $m + \frac{2}{\alpha} \leq \beta < 1 + \frac{1}{\alpha}$, then for any $\lambda \in (0, 1)$, any small $\varepsilon > 0$, there is a time $T_{\lambda, \varepsilon} \geq t_\lambda$ such that*

$$E_\lambda(t) \subset (x^-(t), +\infty), \quad \forall t \geq T_{\lambda, \varepsilon}, \quad (2.18)$$

where

$$x^-(t) := \left((r - \varepsilon) C^{\beta-1} (\beta - 1) t \right)^{\frac{1}{\alpha(\beta-1)}}.$$

If instead (2.8) holds, then the same conclusion follows when replacing α by $\frac{2}{1-m}$ in the definition of $x^-(t)$.

The first part of Theorem 2.8 shows that the speed of propagation is infinite when $1 < \beta < 2 - m$ (we already treated the case $\beta = 1$), requiring only Assumption 2.1 on the initial datum. In other words, when the Allee effect is small with respect to fast diffusion, acceleration occurs regardless of the initial tail.

The second part of Theorem 2.8 deals with the situation when $m + \frac{2}{\alpha} \leq \beta < 1 + \frac{1}{\alpha}$. Indeed, in this situation and when the diffusion is of the porous medium type, we have shown in Theorem 2.6 that polynomial acceleration occurs. However, in the fast diffusion case, this parameter range was missing from Theorem 2.7. The reason is that we are not able to accurately locate the level sets. Still, Theorem 2.8 shows that the acceleration phenomenon when $0 < m < 1$ is at least as strong as in the case $m \geq 1$, as one may have expected.¹

Remark 2.9. In the regime $0 < m < 1$, $\beta > 1$ and $m + \frac{2}{\alpha} \leq \beta < 2 - m$ (see Figure 2.2), assuming (2.14) and the upper bound in (2.4), that is $u_0(x) \leq \frac{\bar{C}}{x^\alpha}$, a monomial type upper bound on the level sets can be obtained. Indeed, we can use a comparison argument with the case where α is replaced by a smaller α_0 which falls into the results of Theorem 2.7. Hence, for any small $\varepsilon > 0$, we let $\alpha_0 := \frac{2}{\beta-m+\varepsilon}$ and deduce from Theorem 2.7 the upper bound

$$x^+(t) := \left((\bar{r} + \varepsilon) \bar{C}^{\beta-1} (\beta - 1) t \right)^{\frac{\beta-m+\varepsilon}{2(\beta-1)}}.$$

For the sake of conciseness and because it is a simple consequence of our previous theorems, we omitted this upper bound in Theorem 2.8.

Using the fact that “fast diffusion increases the tail of solutions” (see Subsection 6.1), we can summarize our results for the case $m \in (0, 1)$ in Figure 2.2. Notice that the position of the level sets is not yet completely understood in the regions covered by Theorem 2.8. Indeed, in the region covered by Theorem 2.8 (ii) we are equipped with lower and upper monomial type estimates that typically differ (see Remark 2.9), whereas in the region covered by Theorem 2.8 (i) a monomial type lower bound is not even available. Such difficulties previously arose in a integro-differential equation [3], where the balance between the Allee effect and the tails of the dispersal kernel was studied.

The paper is organized as follows. In Section 3, we prove the preliminary results (2.1) and (2.2) together with the statement (i) of Theorem 2.8. In Section 4, we

¹ We point out that, between the initial acceptance of the present work and its publication, the authors have obtained refined estimates on the position of level sets in the parameter range considered in Theorem 2.8. We refer to [4] for more details in this special case.

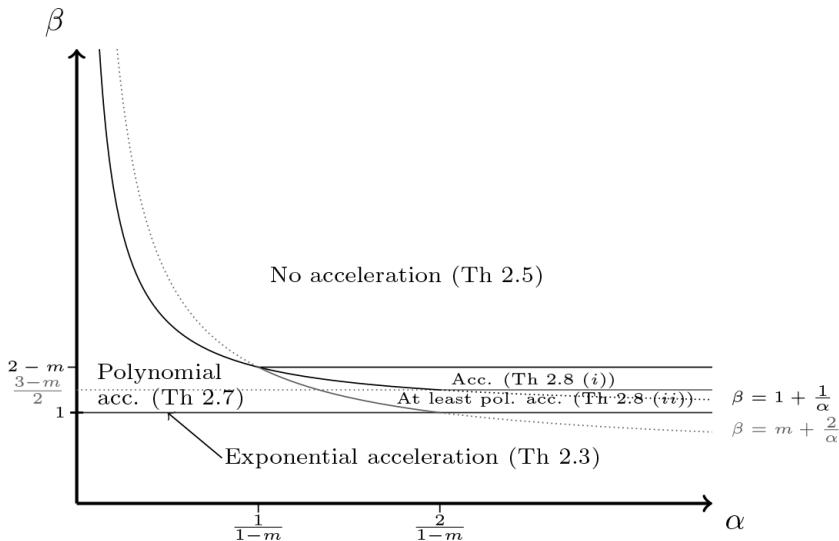


Figure 2.2. Summary of our results in the fast diffusion case, $m \in (0, 1)$. The different parameter regions are delimited by the solid lines.

consider the regime where there is no acceleration, that is we prove Theorem 2.5. Next, in Section 5, we focus on the porous medium regime: we prove simultaneously Theorem 2.3 (case $m > 1$) and Theorem 2.6. Let us point out that, while we stated those results separately because of the different conclusions (exponential vs. algebraic estimates), the proofs will be almost identical. Then, in Section 6, we turn to the fast diffusion regime: we complete the proof of Theorem 2.3 (case $0 < m < 1$), and prove Theorem 2.7. Last, in Appendix A, we complete the proof of Theorem 2.8 by dealing with statement (ii).

3. Positive and infinite speeds of propagation

In this section, we show propositions which provide subsolutions with a support bounded from above and moving with a constant speed $c > 0$, depending on the parameters of the equation (1.1). This enables us not only to prove statement (i) of Theorem 2.8 (infinite speed of propagation when $m \in (0, 1)$, $1 \leq \beta < 2 - m$), but also to show that level sets always move to the right at least linearly, namely estimate (2.1) which will be used several times throughout this paper. After that, we also prove the preliminary result (2.2).

The subsequent propositions all rely on a similar argument. First, since we will look at traveling front type solutions or subsolutions (*i.e.* whose shape remains constant in time in an appropriate moving frame), the equation (1.1) reduces to a nonlinear ordinary differential equation. By a change of variables introduced by Engler [14], see also [17], we are able to further reduce the problem to a situation where diffusion is linear and where we use a phase plane analysis.

We start with the case where $g(s) := mf(s)s^{m-1}$ has “infinite slope at zero”, meaning that $\lim_{s \rightarrow 0} \frac{g(s)}{s} = +\infty$. This covers the case $m \in (0, 1)$ and $1 < \beta < 2 - m$ of Theorem 2.8.

Proposition 3.1 (Material for subsolutions traveling at any speed $c > 0$).

Let $m \in (0, 1)$ be given. Assume that $g(s) := mf(s)s^{m-1}$ is such that $\lim_{s \rightarrow 0} \frac{g(s)}{s} = +\infty$. Then for any $\delta \in (0, 1)$, any $c > 0$, there are $x_c > 0$ and a decreasing function $U_c : [0, x_c] \rightarrow [0, \delta]$ which solves

$$(U_c^m)'' + cU_c' + f(U_c) = 0 \quad \text{on } (0, x_c), \quad (3.1)$$

as well as the boundary conditions

$$U_c(0) = \delta, \quad U_c'(0) = 0, \quad U_c(x_c) = 0. \quad (3.2)$$

Proof. Let $\delta \in (0, 1)$ and $c > 0$ be given. We first introduce the equation

$$V'' + cV' + g(V) = 0, \quad (3.3)$$

where $g(s) := mf(s)s^{m-1}$ is of the class C^1 on $(0, 1]$ but not on $[0, 1]$. Denote by $V_c(y)$ its solution starting from $V_c(0) = \delta$, $V_c'(0) = 0$. By a phase plane analysis, one infers that one of the three following statements holds:

- (i) V_c remains positive on $(0, +\infty)$, V_c' remains negative on $(0, +\infty)$, $(V_c(y), V_c'(y)) \rightarrow (0, 0)$ as $y \rightarrow +\infty$, and there exists a sequence $y_k \rightarrow +\infty$ such that

$$cV_c'(y_k) + g(V_c(y_k)) \leq 0. \quad (3.4)$$

- (ii) there is $y_c > 0$ such that V_c remains positive on $(0, y_c)$, V_c' negative on $(0, y_c)$, $(V_c(y_c), V_c'(y_c)) = (0, 0)$, and there exists an increasing sequence $y_k \rightarrow y_c$ such that (3.4) holds.

- (iii) there is $y_c > 0$ such that $V_c(y_c) = 0$ and $V_c' < 0$ on $(0, y_c]$.

Note that Cauchy-Lipschitz theorem does not apply here, so that case (ii) cannot be immediately excluded. We will now show that (iii) necessarily occurs.

Let us assume (i) and derive a contradiction. Integrating the inequality $-cV_c' \geq V_c''$ from y to $+\infty$ we get $cV_c(y) \geq -V_c'(y)$. Hence, in view of (3.4), we obtain

$$g(V_c(y_k)) \leq c^2 V_c(y_k) \quad \text{for } k \text{ large enough,}$$

which contradicts the fact that $\lim_{s \rightarrow 0} \frac{g(s)}{s} = +\infty$. In a similar fashion, in the case (ii), integrating the inequality $-cV_c' \geq V_c''$ between y and y_c leads to $cV_c(y) \geq -V_c'(y)$. This again results in a contradiction, and we conclude that the solution V_c satisfies (iii).

Now define

$$U_c(x) := V_c(\varphi^{-1}(x)), \quad (3.5)$$

where

$$\varphi(y) := \int_0^y m V_c^{m-1}(s) ds.$$

Note that φ is indeed a bijection between $[0, y_c]$ and $[0, x_c]$, with

$$x_c := \int_0^{y_c} m V_c^{m-1}(s) ds.$$

Here we used the fact $V_c'(y_c) < 0$ and $m - 1 \in (-1, 0)$, hence V_c^{m-1} is integrable on $(0, y_c)$.

Then one can compute that (3.3) rewrites as

$$\begin{aligned} & m^2 U_c^{2m-2}(\varphi(y)) U_c''(\varphi(y)) + m^2(m-1) U_c^{2m-3}(\varphi(y)) (U_c'(\varphi(y)))^2 \\ & + mc U_c^{m-1}(\varphi(y)) U_c'(\varphi(y)) + mf(U(\varphi(y))) U(\varphi(y))^{m-1} = 0, \end{aligned}$$

thus

$$m U_c^{m-1} U_c'' + m(m-1) U_c^{m-2} (U_c')^2 + c U_c' + f(U_c) = 0,$$

which is exactly (3.1). The monotonicity and the boundary conditions (3.2) are straightforward, and the proposition is proved. \square

We are now in the position to prove Theorem 2.8 (i), whose assumptions $m \in (0, 1)$, $1 < \beta < 2 - m$ and (2.13) allow to use the above proposition.

Proof of Theorem 2.8 (i). Let us fix $c > 0$ and choose $\delta < \liminf_{x \rightarrow -\infty} u_0(x)$. We now extend U_c by δ when $x \leq 0$, and by 0 when $x \geq x_c$. For convenience, we still denote the resulting function from \mathbb{R} to \mathbb{R} by U_c . We also find some $x_0 \in \mathbb{R}$ so that $U_c(\cdot + x_0) \leq u_0(\cdot)$. From Proposition 3.1, one can check that $U_c(x - ct + x_0)$ is a subsolution of (1.1) in the domain $\{(t, x) : t > 0, x < x_c - x_0 + ct\}$. Recalling that $u(t, x) > 0$ for all $t > 0, x \in \mathbb{R}$, we deduce from the maximum principle that $U_c(x - ct + x_0) \leq u(t, x)$ for all $t > 0, x \in \mathbb{R}$. We thus infer that

$$\liminf_{t \rightarrow +\infty} \inf_{x \leq c't} u(t, x) \geq \delta,$$

for any $c' < c$.

Let us now proceed by contradiction and assume that there exist sequences $t_n \rightarrow +\infty$ and x_n such that $x_n \leq c't_n$ and

$$\lim_{n \rightarrow +\infty} u(t_n, x_n) < 1.$$

By standard parabolic estimates (thanks to the lower bound on u above, the degeneracy at 0 raises no issue here), we find that $u(t + t_n, x + x_n)$ converges locally uniformly to an entire (*i.e.* for all $t \in \mathbb{R}$) solution $u_\infty \geq \delta$ of (1.1). Since f is of the monostable type, it follows that $u_\infty \geq 1$, which contradicts our choice of t_n and x_n .

Since c , hence c' , could be chosen arbitrarily large, this proves that the solution propagates with infinite speed when $m \in (0, 1)$ and $1 < \beta < 2 - m$. \square

For the next result, we only require Assumption 2.2 concerning the nonlinearity f .

Proposition 3.2 (Material for subsolutions traveling at some speed $c_0 > 0$).

Let $m > 0$ be given. Then for any $\delta \in (0, 1)$, there exists $c_0 > 0$ small enough such that the conclusions of Proposition 3.1 hold.

Proof. Let $\delta \in (0, 1)$ be given. Since $g(s) := mf(s)s^{m-1}$ may be non Lipschitz continuous, we first consider a smaller Lipschitz continuous ignition type nonlinearity. Precisely, we consider a Lipschitz continuous function \tilde{g} such that $0 \leq \tilde{g} \leq g$, $\tilde{g} = 0$ on $[0, \frac{\delta}{2}] \cup \{1\}$, $\tilde{g} > 0$ on $(\frac{\delta}{2}, 1)$. The extensions of g , \tilde{g} by zero outside $[0, 1]$ are still denoted by g , \tilde{g} . From a phase plane analysis we see that $\tilde{V}_0(y)$ the solution of the Cauchy problem

$$V'' + \tilde{g}(V) = 0, \quad V(0) = \delta, \quad V'(0) = 0,$$

is global, and that there is $\tilde{y}_0 > 0$ such that $\tilde{V}_0(\tilde{y}_0) = 0$ and $\tilde{V}'_0 < 0$ on $(0, \tilde{y}_0]$. From the continuous dependance of solutions to the Cauchy problem

$$V'' + cV' + \tilde{g}(V) = 0, \quad V(0) = \delta, \quad V'(0) = 0, \quad (3.6)$$

with respect to parameter c , we infer that for $c_0 > 0$ small enough the solution satisfies statement (iii) of the proof of Proposition 3.1. Since $\tilde{g} \leq g$, we deduce by comparison (in the phase plane, trajectories associated with g are below those associated with \tilde{g}) that the same conclusion holds for the Cauchy problem (3.6) with g in place of \tilde{g} . Next we define (3.5) and the end of the proof is similar to that of Proposition 3.1. \square

We are now in the position to prove (2.1) and (2.2).

Proof of (2.1). The proof uses the above proposition and proceeds as that of Theorem 2.8. Let us again notice that our assumption $u_0 > 0$ implies $u(t, x) > 0$ for all $t > 0$, $x \in \mathbb{R}$ so that the degeneracy/singularity at 0 (depending on the diffusion regime) raises no issue and we can apply the maximum principle on the moving truncated domain $\{(t, x) : t > 0, x < x_{c_0} - x_0 + c_0 t\}$. Details are omitted.

We conclude that, for any $m > 0$, there exists $c_0 > 0$ small enough such that, as announced in the introduction, $\lim_{t \rightarrow +\infty} \inf_{x \leq c_0 t} u(t, x) = 1$, \square

Proof of (2.2). From Assumption 2.2, there is $\bar{r} > 0$ such that (2.6) holds. By comparison, it is thus enough to consider the solution of

$$\partial_t u = \partial_{xx}(u^m) + \bar{r}u, \quad u(0, \cdot) = u_0.$$

Next, observe that letting (see [6, 7])

$$v(\tau, x) := e^{-\bar{r}\tau} u(t, x), \quad (3.7)$$

with

$$\tau(t) := \begin{cases} \frac{1}{(1-m)\bar{r}} (1 - e^{-(1-m)\bar{r}t}) & \text{if } 0 < m < 1 \\ t & \text{if } m = 1 \\ \frac{1}{(m-1)\bar{r}} (e^{(m-1)\bar{r}t} - 1) & \text{if } m > 1, \end{cases}$$

we see that $v(\tau, x)$ solves —on the time interval $(0, \tau_\infty)$ with $\tau_\infty = +\infty$ if $m \geq 1$, $\tau_\infty = \frac{1}{(1-m)\bar{r}}$ if $0 < m < 1$ —

$$\partial_\tau v = \partial_{xx}(v^m), \quad v(0, \cdot) = u_0. \quad (3.8)$$

Now consider $\varepsilon \in (0, 1)$. Select x_0 such that $u_0(x) \leq \varepsilon$ for all $x \geq x_0$. In order to construct a supersolution to (3.8), define $w(\tau, x) := \min(1, \varepsilon + e^{-\mu(x-x_0-\tau)})$, where $\mu > 0$ is to be selected. Clearly $u_0 \leq w(0, \cdot)$. For the points (τ, x) such that $w(\tau, x) < 1$, we have that

$$\begin{aligned} \partial_\tau w - \partial_{xx}(w^m) &= \mu e^{-\mu(x-x_0-\tau)} - \mu^2 m e^{-\mu(x-x_0-\tau)} w^{m-1} \\ &\quad - \mu^2 m(m-1) e^{-2\mu(x-x_0-\tau)} w^{m-2} \end{aligned}$$

is positive if $\mu > 0$ is sufficiently small (recall $\varepsilon \leq w < 1$). Since $v(\tau, x) < 1$, we can apply a comparison principle and deduce that $v(\tau, x) \leq w(\tau, x)$ for all $\tau \in (0, \tau_\infty)$, $x \in \mathbb{R}$. Hence, for a given $\tau_0 \in (0, \tau_\infty)$, we have $\limsup_{x \rightarrow +\infty} v(\tau_0, x) \leq \varepsilon$. Since $\varepsilon \in (0, 1)$ could be chosen arbitrarily small, we get $\lim_{x \rightarrow +\infty} v(\tau_0, x) = 0$, which in view of (3.7), yields (2.2). \square

4. No acceleration regime

In this short section, we prove Theorem 2.5. The formal argument is very simple: in order to avoid acceleration, we aim at finding a speed $c > 0$ and a power $p > 0$ such that $w(z) := \frac{1}{z^p}$ is a supersolution of the associated traveling wave equation for $z \gg 1$, that is

$$(w^m)''(z) + cw'(z) + f(w(z)) \leq 0.$$

In view of (2.11) this is enough to have

$$\frac{mp(mp+1)}{z^{mp+2}} - \frac{cp}{z^{p+1}} + \frac{\bar{r}}{z^{p\beta}} \leq 0 \quad \text{for large } z \gg 1,$$

which requires $p+1 \leq p\beta$ and $p+1 \leq mp+2$. The former condition is never satisfied if $\beta = 1$, and we recast it $\frac{1}{\beta-1} \leq p$; the latter condition is always true when $m \geq 1$ but reduces to $p \leq \frac{1}{1-m}$ when $m < 1$.

On the other hand we also need the ordering at initial time, which in view of (2.10), requires $p \leq \alpha$. Putting these conditions together one needs $\beta \geq \max(1 +$

$\frac{1}{\alpha}, 2 - m$), so that the condition (2.9) arises naturally. Let us now make this formal argument precise.

We define

$$p := \frac{1}{\beta - 1}, \quad w(z) := \frac{K}{z^p} \quad \text{for } z \geq z_0 := K^{1/p},$$

where $K > 1$.

Lemma 4.1 (Supersolutions traveling at constant speed). *Let assumptions (2.9) and (2.11) of Theorem 2.5 hold. Then, for any $K > 1$, there is $c > 0$ such that*

$$(w^m)''(z) + cw'(z) + f(w(z)) \leq 0, \quad \forall z \geq z_0.$$

Proof. In view of (2.11), if $z \geq z_1 := \left(\frac{K}{s_0}\right)^{1/p} > z_0$ then $w(z) = \frac{K}{z^p} \leq s_0$ so that

$$(w^m)''(z) + cw'(z) + f(w(z)) \leq \frac{K^m mp(mp+1)}{z^{mp+2}} - \frac{cKp - \bar{r}K^{\beta}}{z^{p+1}}.$$

Recalling that $p + 1 \leq mp + 2$ and choosing $c > \bar{r}(\beta - 1)K^{\beta-1}$, the above is clearly negative for z large enough, say $z \geq z_2$. Last, on the remaining compact region $z_0 \leq z \leq z_2$, we have

$$(w^m)''(z) + cw'(z) + f(w(z)) \leq \frac{K^m mp(mp+1)}{z_0^{mp+2}} - \frac{cKp}{z_2^{p+1}} + \|f\|_{L^\infty(0,1)}$$

which is nonpositive by enlarging c if necessary. \square

Proof of Theorem 2.5. We select $K = \max(1, \bar{C})$, where $\bar{C} > 0$ is the constant that appears in (2.10), and $c > 0$ the associated speed given by the above lemma. We then introduce

$$v(t, x) := \min(1, w(x - x_0 + 1 - ct)),$$

so that

$$v(0, x) = \min\left(1, \frac{K}{(x - x_0 + 1)^p}\right) \geq u_0(x),$$

in view of $u_0 \leq 1$, the assumption on the tail (2.10), $K \geq \bar{C}$ and $p = \frac{1}{\beta-1} \leq \alpha$.

Recall that $u < 1$ for positive times. Also from the above lemma we have $(\partial_t v - \partial_{xx}(v^m) - f(v))(t, x) = (-cw' - (w^m)'' - f(w))(z) \geq 0$ in the region where $v(t, x) < 1$, that is $z := x - x_0 + 1 - ct > z_0$. In other words, v is a (generalized) supersolution of equation (1.1), and applying a comparison principle on a right half-domain we get that

$$u(t, x) \leq v(t, x) = \min(1, w(x - x_0 + 1 - ct)).$$

Now, let $\lambda \in (0, 1)$ be given. In view of (2.3), for $t \geq t_\lambda$, we can pick $x \in E_\lambda(t)$, and the above inequality enforces

$$x \leq x_0 - 1 + \left(\frac{K}{\lambda} \right)^{\beta-1} + ct \leq (c + 1)t,$$

for all $t \geq T_\lambda$, if $T_\lambda \geq t_\lambda$ is sufficiently large. This proves the upper bound in (2.12). The lower bound in (2.12) being known since (2.3), this completes the proof of Theorem 2.5. \square

5. Acceleration regime for porous medium diffusion

In this section, we prove both Theorem 2.3 in the case $m > 1$, and Theorem 2.6. Throughout this section, we are thus equipped with $m > 1$, $\alpha > 0$, as well as $1 \leq \beta < 1 + \frac{1}{\alpha}$, and we assume that (2.4), (2.13) and (2.14) hold.

5.1. Lower bound on the level sets in (2.7) and (2.15)

Notice that, in view of (2.4) and the comparison principle, we only need to consider the case where

$$u_0(x) = \frac{C}{x^\alpha}, \quad \forall x \geq x_0. \quad (5.1)$$

By another comparison argument, we can also assume without loss of generality that $u_0 \in C^2(\mathbb{R})$.

An accelerating small bump as a subsolution. The main difficulty is to construct a subsolution which has the form of a small bump and travels to the right while accelerating. In this $m > 1$ regime it actually turns out that the ones constructed in [22] for $m = 1$, $\beta = 1$, and in [2] for $m = 1$, $\beta > 1$ still work. We start with some preparations.

Let $\varepsilon > 0$ small be given. We first introduce $\eta > 0$ such that

$$\beta < 1 + \eta.$$

Then we select a $\rho > 0$ such that

$$\max \left(\frac{r\beta}{1 + \eta}, r - \varepsilon \right) < \rho < r.$$

Now define $w(t, x)$ the solution of

$$\partial_t w(t, x) = \rho w^\beta(t, x), \quad w(0, x) = u_0(x).$$

Then depending on β , we have either

$$w(t, x) := u_0(x)e^{\rho t}, \quad (5.2)$$

if $\beta = 1$, or

$$w(t, x) := \frac{1}{\left(\frac{1}{u_0^{\beta-1}(x)} - \rho(\beta-1)t\right)^{\frac{1}{\beta-1}}} \quad \text{for } 0 \leq t < T(x) := \frac{1}{\rho(\beta-1)u_0^{\beta-1}(x)}, \quad (5.3)$$

if $\beta > 1$.

Remark 5.1. A straightforward computation shows that in both cases, the level sets of w are consistent with the conclusions of Theorems 2.3 and 2.6. Notice also that, in the case $\beta > 1$, the interval of existence $(0, T(x))$ of the solution $w(t, x)$ becomes large as $x \rightarrow +\infty$. Indeed, in view of (5.1),

$$T(x) = \frac{x^{\alpha(\beta-1)}}{\rho(\beta-1)C^{\beta-1}}, \quad \forall x \geq x_0.$$

In particular, since $0 < \alpha(\beta-1) < 1$, we will be able to observe acceleration (a large time phenomenon) in the subdomain where w is well defined.

We define

$$\varphi(x) := \frac{u_0'(x)}{u_0^{\beta}(x)}, \quad (5.4)$$

and in view of (5.1) we have

$$\varphi'(x) = \frac{\alpha(1-\alpha(\beta-1))}{C^{\beta-1}x^{2-\alpha(\beta-1)}}, \quad \varphi^2(x) = \left(\frac{\alpha}{C^{\beta-1}x^{1-\alpha(\beta-1)}}\right)^2, \quad \forall x \geq x_0.$$

Since $\beta < 1 + \frac{1}{\alpha}$ both $\varphi'(x)$ and $\varphi^2(x)$ tend to zero as $x \rightarrow +\infty$. Let us therefore select $x_1 > x_0$ such that

$$\begin{aligned} m|\varphi'(x)| &\leq r - \rho \quad \text{and} \quad m|\varphi'(x)| + m(2m + \beta + \eta - 1)\varphi^2(x) \\ &\leq \rho - \frac{r\beta}{1+\eta}, \quad \forall x \geq x_1. \end{aligned} \quad (5.5)$$

Now, Assumption 2.1 implies that

$$\kappa := \inf_{x \in (-\infty, x_1)} u_0(x) \in (0, 1]. \quad (5.6)$$

Last, we select $A > 0$ large enough so that $A > \frac{1}{\kappa^\eta}$, and

$$\frac{\eta}{1+\eta} \frac{1}{(A(1+\eta))^{1/\eta}} \leq s_0, \quad (5.7)$$

where s_0 is as in (2.13). Equipped with the above material, we are now in the position to construct the desired subsolution.

Lemma 5.2 (An accelerating subsolution). *Let $m > 1$ and the assumptions of either Theorem 2.3 or Theorem 2.6 hold. Further assume that (5.1) holds.*

Define

$$v(t, x) := \max(0, w(t, x) - Aw^{1+\eta}(t, x)).$$

Then

$$v(t, x) \leq u(t, x), \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R}. \quad (5.8)$$

Proof. Clearly $v(0, x) \leq u_0(x)$. Recalling that $u(t, x) > 0$ for all $t > 0, x \in \mathbb{R}$, it is thus enough to consider the points (t, x) for which $v(t, x) > 0$. We therefore need to show

$$\begin{aligned} \mathcal{L}v(t, x) &:= \partial_t v(t, x) - \partial_{xx}(v^m)(t, x) - f(v(t, x)) \leq 0 \quad \text{when} \\ v(t, x) &= w(t, x) - Aw^{1+\eta}(t, x) > 0, \end{aligned}$$

and the conclusion follows from the maximum principle on the corresponding sub-domain.

Note that $v(t, x) > 0$ implies in particular that $w(t, x) < 1/A^{1/\eta} < \kappa$ so that $u_0(x) = w(0, x) \leq w(t, x) < \kappa$ since $t \mapsto w(t, x)$ is increasing. In view of the definition of κ in (5.6), this enforces $x \geq x_1$. As a result estimates (5.5) are available. On the other hand $v(t, x) \leq \max_{0 \leq w \leq A^{-1/\eta}} w - Aw^{1+\eta} = \frac{\eta}{1+\eta} \frac{1}{(A(1+\eta))^{1/\eta}} \leq s_0$ by (5.7). Hence, it follows from (2.13) that

$$f(v(t, x)) \geq r v^\beta(t, x) = r w^\beta(t, x) (1 - Aw^\eta(t, x))^\beta.$$

Then the convexity inequality $(1 - Aw^\eta)^\beta \geq 1 - A\beta w^\eta$ yields

$$f(v(t, x)) \geq r w^\beta(t, x) - r A\beta w^{\beta+\eta}(t, x). \quad (5.9)$$

Next, we compute

$$\begin{aligned} \partial_t v(t, x) &= \partial_t w(t, x) - A(1 + \eta) \partial_t w(t, x) w^\eta(t, x) \\ &= \rho w^\beta(t, x) - A\rho(1 + \eta) w^{\beta+\eta}(t, x) \end{aligned} \quad (5.10)$$

and, omitting variables,

$$\begin{aligned} \partial_{xx}(v^m) &= \left(m\varphi' w^{m+\beta-1} + m(m+\beta-1)\varphi^2 w^{m+2\beta-2} \right) (1 - Aw^\eta)^m \\ &\quad + 2m\varphi w^{m+\beta-1} \left(-Am\eta\varphi w^{\beta+\eta-1} (1 - Aw^\eta)^{m-1} \right) \\ &\quad + w^m \left(-Am\eta\varphi' w^{\beta+\eta-1} (1 - Aw^\eta)^{m-1} \right. \\ &\quad \left. - Am\eta\varphi(\beta + \eta - 1) w^{\beta+\eta-2} \varphi w^\beta (1 - Aw^\eta)^{m-1} \right. \\ &\quad \left. - Am\eta\varphi w^{\beta+\eta-1} (m-1) (-A\eta w^{\eta-1} \varphi w^\beta) (1 - Aw^\eta)^{m-2} \right). \end{aligned}$$

Dropping some negative terms, using $m > 1$ and the crude estimate $(1 - Aw^\eta)^{m-1} \leq 1$ yield

$$\begin{aligned} -\partial_{xx}(v^m) &\leq m|\varphi'|w^\beta + 2Am^2\eta\varphi^2w^{\beta+\eta} + Am\eta|\varphi'|w^{\beta+\eta} \\ &\quad + Am\eta(\beta + \eta - 1)\varphi^2w^{\beta+\eta}. \end{aligned} \quad (5.11)$$

Combining (5.10), (5.11) and (5.9), we arrive at

$$\begin{aligned} \mathcal{L}v(t, x) &\leq w^\beta(t, x) [\rho - r + m|\varphi'(x)|] \\ &\quad + A(1 + \eta)w^{\beta+\eta}(t, x) \left[-\rho + \frac{r\beta}{1 + \eta} + m|\varphi'(x)| \right. \\ &\quad \left. + m(2m + \beta + \eta - 1)\varphi^2(x) \right]. \end{aligned}$$

Thanks to (5.5), both bracket terms are nonpositive and we conclude that $\mathcal{L}v(t, x) \leq 0$. Lemma 5.2 is proved. \square

The rest of the proof of the lower bounds in (2.7) and (2.15) is now identical to [2].

Proof of the lower bound for small λ . Equipped with the above subsolution, whose role is to “lift” the solution $u(t, x)$ on intervals that enlarge with acceleration, we first prove the lower bound on the level sets $E_\lambda(t)$ when λ is small.

Let us fix $0 < \theta < \min\{C, 1/A^{1/\eta}\}$, where C is as in (5.1). We claim that, for any $t \geq 0$, there is a unique $y_\theta(t) \in \mathbb{R}$ such that $w(t, y_\theta(t)) = \theta$, and moreover $y_\theta(t)$ is given by either

$$y_\theta(t) := \left(\frac{C}{\theta} \right)^{1/\alpha} e^{\frac{\rho}{\alpha}t}, \quad (5.12)$$

when $\beta = 1$, or

$$y_\theta(t) := \left(\left(\frac{C}{\theta} \right)^{\beta-1} + \rho C^{\beta-1}(\beta-1)t \right)^{\frac{1}{\alpha(\beta-1)}}, \quad (5.13)$$

when $\beta > 1$.

Indeed, since $\theta < 1/A^{1/\eta} < \kappa = \inf_{x \in (-\infty, x_1)} u_0(x)$ and since $w(t, x) \geq w(0, x) = u_0(x)$, for $w(t, y) = \theta$ to hold one needs $y \geq x_1$. But, when $y \geq x_1 > x_0$, one can use formula (5.1) and then solve equation $w(t, y) = \theta$, thanks to expression (5.2) or (5.3), to find the unique solution (5.12) or (5.13).

Let us now define the open set

$$\Omega := \{(t, x) : t > 0, x < y_\theta(t)\}.$$

Let us evaluate $u(t, x)$ on the boundary $\partial\Omega$. For $t > 0$, it follows from (5.8) that

$$u(t, y_\theta(t)) \geq w(t, y_\theta(t)) - Aw^{1+\eta}(t, y_\theta(t)) = \theta - A\theta^{1+\eta} > 0.$$

On the other hand, for $t = 0$ and $x \leq y_\theta(0) = (\frac{C}{\theta})^{1/\alpha}$, we have

$$u(0, x) \geq \inf_{x \leq (\frac{C}{\theta})^{1/\alpha}} u_0(x) > 0,$$

in view of Assumption 2.1. As a result $\Theta := \inf_{(t,x) \in \partial\Omega} u(t, x) > 0$. Since $\Theta > 0$ is a subsolution for equation (1.1), it follows from the comparison principle that

$$u(t, x) \geq \Theta, \quad \forall t \geq 0, \forall x \leq y_\theta(t). \quad (5.14)$$

This implies in particular that, for any $0 < \lambda < \Theta$, we have, for all $t \geq t_\lambda$,

$$\emptyset \neq E_\lambda(t) \subset (y_\theta(t), +\infty) \subset (x_\rho^-(t), +\infty),$$

where either $x_\rho^-(t) := e^{\frac{\rho}{\alpha}t}$, or $x_\rho^-(t) := (\rho C^{\beta-1}(\beta-1)t)^{\frac{1}{\alpha(\beta-1)}}$, depending on whether $\beta = 1$ or $\beta > 1$. Since $\rho > r - \varepsilon$ this implies the lower bounds in (2.7) and (2.15) for $0 < \lambda < \Theta$. \square

Proof of the lower bound for any $\lambda \in (0, 1)$. Let us now turn to the case where λ is larger than Θ . Let $\Theta \leq \lambda < 1$ be given. Let us denote by $v(t, x)$ the solution of (1.1) with initial data

$$v_0(x) := \begin{cases} \Theta & \text{if } x \leq -1 \\ -\Theta x & \text{if } -1 < x < 0 \\ 0 & \text{if } x \geq 0. \end{cases} \quad (5.15)$$

By using a subsolution based on Proposition 3.2 as in the proof of (2.1) in Section 3 (or by a straightforward extension of [30, Theorem 4.1] which dealt with Heaviside type initial data), we see that $\lim_{t \rightarrow +\infty} \inf_{x \leq c_0 t} v(t, x) = 1$, for some $c_0 > 0$. In particular there is a time $\tau_{\lambda, \varepsilon} > 0$ (this time depends on Θ and therefore on ε from the above construction of the small bump subsolution) such that

$$v(\tau_{\lambda, \varepsilon}, x) > \lambda, \quad \forall x \leq 0. \quad (5.16)$$

On the other hand, it follows from (5.14) and the definition (5.15) that

$$u(T, x) \geq v_0(x - y_\theta(T)), \quad \forall T \geq 0, \forall x \in \mathbb{R},$$

so that the comparison principle yields

$$u(T + \tau, x) \geq v(\tau, x - y_\theta(T)), \quad \forall T \geq 0, \forall \tau \geq 0, \forall x \in \mathbb{R}.$$

In view of (5.16), this implies that

$$u(T + \tau_{\lambda, \varepsilon}, x) > \lambda, \quad \forall T \geq 0, \forall x \leq y_\theta(T).$$

Hence, for any $t \geq T_{\lambda, \varepsilon}^1 := \max(\tau_{\lambda, \varepsilon}, t_\lambda)$, if we pick a $x \in E_\lambda(t)$ then the above implies $x > y_\theta(t - \tau_{\lambda, \varepsilon})$.

In particular, if $\beta = 1$ then

$$x > \left(\frac{C}{\theta} \right)^{1/\alpha} e^{\frac{\rho}{\alpha}(t - \tau_{\lambda, \varepsilon})} \geq e^{\frac{r-\varepsilon}{\alpha}t}$$

for all $t \geq T_{\lambda, \varepsilon}$, with $T_{\lambda, \varepsilon} > 0$ sufficiently large (recall that $\rho > r - \varepsilon$). This concludes the proof of the lower bound in (2.7).

On the other hand, if $\beta > 1$ then

$$\begin{aligned} x &> \left(\left(\frac{C}{\theta} \right)^{\beta-1} - \rho C^{\beta-1}(\beta-1)\tau_{\lambda, \varepsilon} + \rho C^{\beta-1}(\beta-1)t \right)^{\frac{1}{\alpha(\beta-1)}} \\ &\geq \left((r-\varepsilon)C^{\beta-1}(\beta-1)t \right)^{\frac{1}{\alpha(\beta-1)}}, \end{aligned}$$

for all t large enough, which also concludes the proof of the lower bound in (2.15). \square

5.2. Upper bound on the level sets in (2.7) and (2.15)

Let $\lambda \in (0, 1)$ and $\varepsilon > 0$ small be given. Up to enlarging $x_0 > 1$ which appears in (2.4), we can assume without loss of generality that

$$m \frac{\alpha(1 - \alpha(\beta - 1))}{\bar{C}^{\beta-1} x_0^{2-\alpha(\beta-1)}} + m(m + \beta - 1) \left(\frac{\alpha}{\bar{C}^{\beta-1} x_0^{1-\alpha(\beta-1)}} \right)^2 \leq \frac{\varepsilon}{2}, \quad (5.17)$$

and $\frac{\bar{C}}{x_0^\alpha} < 1$. This is possible since $0 < 1 - \alpha(\beta - 1) < 2 - \alpha(\beta - 1)$.

Now in view of (2.4) and the comparison principle, it is enough to prove the upper bound in (2.7) and in (2.15) when

$$u_0(x) = \frac{\bar{C}}{x^\alpha}, \quad \forall x \geq x_0. \quad (5.18)$$

Let us select

$$\rho := \bar{r} + \frac{\varepsilon}{2}.$$

We then define

$$\psi(t, x) := \min(1, w(t, x)),$$

where $w(t, x)$ is defined by either (5.2) or (5.3) depending on whether $\beta = 1$ or $\beta > 1$. Since $\inf_{x \leq x_0} u_0(x) > 0$, there exists $T > 0$ large enough so that

$$\psi(t, x) = 1, \quad \forall t \geq T, \forall x \leq x_0.$$

We claim that ψ is a (generalized) supersolution for equation (1.1) in the domain $(T, +\infty) \times \mathbb{R}$. As in Section 4, since 1 solves (1.1), it suffices to consider the points (t, x) where $\psi(t, x) = w(t, x) < 1$. From our choice of T , this implies that $x > x_0$. In view of

$$\partial_t w(t, x) = \rho w^\beta(t, x) = \left(\bar{r} + \frac{\varepsilon}{2}\right) w^\beta(t, x),$$

and inequality (2.14), some straightforward computations yield

$$\begin{aligned} & \partial_t w(t, x) - \partial_{xx}(w^m)(t, x) - f(w(t, x)) \\ & \geq \frac{\varepsilon}{2} w^\beta(t, x) - m\varphi'(x)w^{m+\beta-1}(t, x) \\ & \quad - m(m+\beta-1)\varphi^2(x)w^{m+2\beta-2}(t, x) \\ & \geq w^\beta(t, x) \left(\frac{\varepsilon}{2} - m|\varphi'(x)| - m(m+\beta-1)\varphi^2(x) \right), \end{aligned} \tag{5.19}$$

since $0 < w(t, x) < 1$, $m > 1$, and where $\varphi(x) = \frac{u'_0(x)}{u_0^\beta(x)}$ already appeared in (5.4).

In view of (5.17) and (5.18), since $0 < 1 - \alpha(\beta - 1) < 2 - \alpha(\beta - 1)$, some further computations yield that, for any $x \geq x_0$, the right-hand term in (5.19) is non-negative, which proves our claim that ψ is a supersolution of (1.1) in $(T, +\infty) \times \mathbb{R}$.

Finally, since w is increasing in time and $w(0, \cdot) \equiv u_0(\cdot)$, we have that $u_0(\cdot) \leq w(T, \cdot)$, and by assumption $u_0 \leq 1$. Therefore we can apply the comparison principle and conclude that

$$u(t, x) \leq \psi(t + T, x) \leq w(t + T, x), \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R}.$$

From this one can conclude the proof of the upper bound in (2.7) (when $m > 1$) and in (2.15). \square

6. Acceleration regime for fast diffusion

In this section, we end the proof of Theorem 2.3 by now considering the case $0 < m < 1$, and also prove Theorem 2.7. Throughout this section, we will thus take $0 < m < 1$, $\alpha > 0$, $\beta \geq 1$ (further assumptions will come below), and assume that (2.4), (2.13) and (2.14) hold.

6.1. Heavier tail by fast diffusion

We start by some short preliminary on the size of the tail of the solution in the fast diffusion regime, which partly explains qualitative differences with the linear diffusion and porous medium diffusion cases.

If $\alpha < \frac{2}{1-m}$ then we are satisfied with the lower estimate of (2.4), namely $u_0(x) \geq \frac{C}{x^\alpha}$ for $x \geq x_0 > 1$.

On the other hand, if $\alpha \geq \frac{2}{1-m}$ then the fast diffusion equation immediately makes the tail heavier (with respect to both the exponent and the multiplicative constant) at positive time. More precisely, by comparison, we have $u(t, x) \geq v(t, x)$ where $v(t, x)$ solves the fast diffusion equation

$$\partial_t v = \partial_{xx}(v^m), \quad t > 0, x \in \mathbb{R}.$$

A lower bound for the long time behavior of v is provided by [23, Theorem 2.4]: for a given $T > 0$, there is $C(T) > 0$ and $x(T) > 0$ such that

$$v(T, x) \geq \frac{C(T)}{x^{\frac{2}{1-m}}}$$

for $x \geq x(T)$. Moreover $C(T) \rightarrow +\infty$ as $T \rightarrow +\infty$.

As a result, up to shifting time by T large enough and by a comparison argument, it is enough (as far as the lower bounds of the level sets are concerned) to consider the case of a smooth decreasing data such that

$$u_0(x) = \frac{C}{x^\gamma}, \quad \forall x \geq x_0, \quad \gamma := \min\left(\alpha, \frac{2}{1-m}\right), \quad (6.1)$$

for some $x_0 > 1$ arbitrarily large. Also, as explained above, if $\alpha \geq \frac{2}{1-m}$ then we can enlarge the constant C without loss of generality.

6.2. Lower bound on the level sets in (2.7) and (2.17)

In this subsection, we will mostly assume that

$$1 \leq \beta < \min\left(1 + \frac{1}{\gamma}, m + \frac{2}{\gamma}\right).$$

Comparing this assumption to those of Theorem 2.3 and Theorem 2.7, this precludes the case $\beta = 1$ and $\gamma = \frac{2}{1-m}$ which we consider separately at the end of this subsection.

An accelerating subsolution. Let $\varepsilon > 0$ small be given, and $\eta > \beta - 1 \geq 0$. For convenience and to make some of our computations simpler, we will also add the condition

$$\frac{\eta}{1 + \eta} > \frac{1}{2}. \quad (6.2)$$

Then let $\rho > 0$ be such that

$$\max\left(\frac{r\beta}{1 + \eta}, r - \varepsilon\right) < \rho < r. \quad (6.3)$$

We again define $w(t, x)$ as the solution of

$$\partial_t w(t, x) = \rho w^\beta(t, x), \quad w(0, x) = u_0(x),$$

which is given by either (5.2) if $\beta = 1$, or (5.3) if $\beta > 1$. In this fast diffusion regime ($0 < m < 1$) a one side compactly supported subsolution $\max(0, w(t, x) - Aw^{1+\eta}(t, x))$ in the spirit of [22] or of Section 5 would not work because of terms like $(1 - Aw^\eta(t, x))^{m-1}$, $(1 - Aw^\eta(t, x))^{m-2}$ which become infinite as $w(t, x)$ approaches $(1/A)^{1/\eta}$. The idea is to cut by a small constant “on the left” rather than by zero. Let us make this precise.

First, we increase x_0 without loss of generality so that the three following inequalities hold:

$$m \frac{\gamma C^{m-\beta}(\gamma + 1 - \gamma\beta)}{x_0^{2+(m-\beta)\gamma}} \leq r - \rho, \quad (6.4)$$

$$\begin{aligned} 2^{1-m}m\eta \times \frac{\gamma C^{m-\beta}}{x_0^{2+(m-\beta)\gamma}} \left(2m\gamma + 1 + \gamma\eta + 2(1-m)\frac{\eta}{1+\eta}\gamma \right) \\ \leq \rho(1+\eta) - r\beta, \end{aligned} \quad (6.5)$$

$$2^{1-m}m\eta \left[\frac{\gamma C^{m-\beta}}{x_0^{2+(m-\beta)\gamma}} (\gamma + 1 - \gamma\beta) + \theta \frac{\gamma^2 C^{2-2\beta}}{x_0^{2+2\gamma(1-\beta)}} \right] \leq \rho(1+\eta) - r\beta, \quad (6.6)$$

where $\theta := 2m + \beta + \eta - 1 + (1-m)\frac{2\eta}{1+\eta}$. This is possible thanks to (6.3), using also the fact that $2 + (m - \beta)\gamma > 0$ and $1 + \gamma(1 - \beta) > 0$. We will see below that the rather tedious left-hand terms of these inequalities arise when computing $\partial_{xx}(v^m)$, with $v = w(1 - Aw^\eta)$ our accelerating subsolution.

Next, we select $A > 1$ large enough so that

$$A > \frac{1}{\kappa^\eta(1+\eta)}, \quad \text{and} \quad \left(\frac{1}{A(1+\eta)} \right)^{1/\eta} \frac{\eta}{1+\eta} \leq s_0,$$

where s_0 is as in (2.13) and $\kappa := \inf_{x \in (-\infty, x_0)} u_0(x) \in (0, 1]$. Now we define $X(t) \in \mathbb{R}$ such that

$$w(t, X(t)) = \left(\frac{1}{A(1+\eta)} \right)^{1/\eta}.$$

Since $w^\eta(t, x) \geq u_0^\eta(x) \geq \kappa^\eta > \frac{1}{A(1+\eta)}$ for all $x \leq x_0$, we have that $X(t) > x_0$. Using the explicit expressions for $w(t, \cdot)$ and u_0 on $(x_0, +\infty)$ provided by (6.1), it is then straightforward that such an $X(t)$ is uniquely defined. Moreover, $w(t, x) < (\frac{1}{A(1+\eta)})^{1/\eta}$ if and only if $x > X(t)$.

Lemma 6.1 (An accelerating subsolution). *Let the assumptions of either Theorem 2.3 (with $0 < m < 1$) or Theorem 2.7 hold. Further assume that (6.1) holds.*

Define

$$v(t, x) := \begin{cases} \left(\frac{1}{A(1+\eta)}\right)^{1/\eta} \frac{\eta}{1+\eta} & \text{if } x \leq X(t) \\ w(t, x)(1 - Aw^\eta(t, x)) & \text{if } x > X(t). \end{cases}$$

Then $v(t, x) \leq u(t, x)$ for all $t > 0, x \in \mathbb{R}$.

Proof. Clearly $v(0, x) < u_0(x)$. Let us already note that v is smooth in both subdomains $\{x < X(t)\}$ and $\{x > X(t)\}$. Also, it is continuous in $[0, +\infty) \times \mathbb{R}$ as well as C^1 with respect to x at the junction point $X(t)$. This means that a comparison principle is applicable provided that v satisfies

$$\mathcal{L}v(t, x) := \partial_t v(t, x) - \partial_{xx}(v^m)(t, x) - f(v(t, x)) \leq 0 \quad (6.7)$$

in both these subdomains. Indeed, if v is a positive subsolution on the half-domain $\{(t, x) \in (0, +\infty) \times (-\infty, X(t))\}$, then either $v(t, x) < u(t, x)$ for all $t > 0$ and $x \leq X(t)$, or there exists a first time $T > 0$ such that $u(T, X(T)) = v(T, X(T)) > 0$. In the latter case, by Hopf lemma we have $\partial_x u(T, X(T)) < \partial_x v(T, X(T))$, which due to the C^1 -regularity of u and v in the x -variable contradicts the comparison principle on the right half-domain $\{(t, x) \in (0, T) \times (X(t), +\infty)\}$. Therefore, the inequality $v(t, x) \leq u(t, x)$ holds for all $t \geq 0$ and $x \leq X(t)$, and by the comparison principle it also holds for all $t \geq 0$ and $x > X(t)$.

Let us now check that v satisfies (6.7) on both subdomains.

Since $\left(\frac{1}{A(1+\eta)}\right)^{1/\eta} \frac{\eta}{1+\eta}$ is obviously a subsolution to (1.1), we only need to check this inequality when $x > X(t)$. Recall that $X(t) > x_0$, hence (6.1) is available. On the other hand it is straightforward that $v(t, x) \leq \max_{w \geq 0} w(1 - Aw^\eta) = \left(\frac{1}{A(1+\eta)}\right)^{1/\eta} \frac{\eta}{1+\eta} \leq s_0$. It then follows from (2.13) that (5.9) and (5.10) still hold. Also, by the same computations as in Section 5,

$$\begin{aligned} \partial_{xx}(v^m) &\geq m\varphi' w^{m+\beta-1}(1 - Aw^\eta)^m - 2Am^2\eta\varphi^2 w^{m+2\beta+\eta-2}(1 - Aw^\eta)^{m-1} \\ &\quad - Am\eta\varphi' w^{m+\beta+\eta-1}(1 - Aw^\eta)^{m-1} \\ &\quad - Am\eta(\beta + \eta - 1)\varphi^2 w^{m+2\beta+\eta-2}(1 - Aw^\eta)^{m-1} \\ &\quad - (1 - m)A^2m\eta^2\varphi^2 w^{m+2\beta+2\eta-2}(1 - Aw^\eta)^{m-2}, \end{aligned}$$

where $\varphi = \frac{u'_0}{u_0^\beta}$. Comparing with our argument in Section 5, we will have here an additional term due to the fact that $m < 1$ now. Using $0 < m < 1$ and $1 \geq$

$1 - Aw^\eta(t, x) \geq \frac{\eta}{1+\eta} \geq \frac{1}{2}$ (recall (6.2)), we then obtain that

$$\begin{aligned} -\partial_{xx}(v^m) &\leq +w^{\beta+\eta} \times 2^{1-m} Am\eta|\varphi'| \left(\frac{w}{u_0}\right)^{m-1} u_0^{m-1} \\ &\quad + w^{\beta+\eta} \times 2^{1-m} Am\eta(\beta + \eta - 1)\varphi^2 \left(\frac{w}{u_0}\right)^{m+\beta-2} u_0^{m+\beta-2} \\ &\quad + w^{\beta+\eta} \times 2^{2-m}(1-m)Am\eta^2(Aw^\eta)\varphi^2 \left(\frac{w}{u_0}\right)^{m+\beta-2} u_0^{m+\beta-2} \end{aligned} \quad (6.8)$$

as well as

$$\begin{aligned} -\partial_{xx}(v^m) &\leq w^\beta m|\varphi'| \left(\frac{w}{u_0}\right)^{m-1} u_0^{m-1} + w^{\beta+\eta} \times 2^{2-m} Am^2\eta\varphi^2 w^{m+\beta-2} \\ &\quad + w^{\beta+\eta} \times 2^{1-m} Am\eta|\varphi'| \left(\frac{w}{u_0}\right)^{m-1} u_0^{m-1} \\ &\quad + w^{\beta+\eta} \times 2^{1-m} Am\eta(\beta + \eta - 1)\varphi^2 w^{m+\beta-2} \\ &\quad + w^{\beta+\eta} \times 2^{2-m}(1-m)Am\eta^2(Aw^\eta)\varphi^2 w^{m+\beta-2}. \end{aligned} \quad (6.9)$$

We distinguish below the two cases $\beta \leq 2 - m$ and $\beta > 2 - m$. This is rather natural since $2 - m$ is the value which both hyperbolae $1 + \frac{1}{\gamma}$ and $m + \frac{2}{\gamma}$ take at their intersection point $\gamma = \frac{1}{1-m}$.

• Let us first consider the case $\beta \leq 2 - m$ for which we will take advantage of (6.8). Since $m < 1$ and $w(t, \cdot) \geq u_0(\cdot)$ for any $t \geq 0$, we have that

$$\left(\frac{w}{u_0}\right)^{m-1}, \left(\frac{w}{u_0}\right)^{m+\beta-2} \leq 1.$$

Recall that $x \geq x_0$ and $u_0(x) = \frac{C}{x^\gamma}$, and also that $2 + (m - \beta)\gamma > 0$. Thus a straightforward computation leads from (6.8) to

$$\begin{aligned} -\partial_{xx}(v^m) &\leq w^\beta \left[m \frac{\gamma C^{m-\beta}(\gamma + 1 - \gamma\beta)}{x_0^{2+(m-\beta)\gamma}} \right] \\ &\quad + w^{\beta+\eta} \times 2^{1-m} Am\eta \left[\frac{\gamma C^{m-\beta}}{x_0^{2+(m-\beta)\gamma}} \left(2m\gamma + 1 \right. \right. \\ &\quad \left. \left. + \gamma\eta + 2(1-m)\frac{\eta}{1+\eta}\gamma \right) \right]. \end{aligned}$$

By (6.4) and (6.5), we conclude that

$$-\partial_{xx}(v^m) \leq (r - \rho)w^\beta + A(\rho(1 + \eta) - r\beta)w^{\beta+\eta}. \quad (6.10)$$

Together with (5.9), (5.10), it now follows that

$$\partial_t v(t, x) - \partial_{xx}(v^m)(t, x) - f(v(t, x)) \leq 0, \quad (6.11)$$

for all $t \geq 0$ and $x > X(t)$. This proves Lemma 6.1 when $\beta \leq 2 - m$.

- Let us now turn to the case $\beta > 2 - m$. Here we have that

$$\left(\frac{w}{u_0}\right)^{m-1}, w^{m+\beta-2} \leq 1.$$

Another computation then leads from (6.9) to

$$\begin{aligned} -\partial_{xx}(v^m) &\leq w^\beta \left[m \frac{\gamma C^{m-\beta}(\gamma + 1 - \gamma\beta)}{x_0^{2+(m-\beta)\gamma}} \right] \\ &\quad + w^{\beta+\eta} \times 2^{1-m} Am\eta \left[\frac{\gamma C^{m-\beta}}{x_0^{2+(m-\beta)\gamma}} (\gamma + 1 - \gamma\beta) + \theta \frac{\gamma^2 C^{2-2\beta}}{x_0^{2+2\gamma(1-\beta)}} \right], \end{aligned}$$

where $\theta = 2m + \beta + \eta - 1 + (1 - m) \frac{2\eta}{1+\eta}$. By (6.4) and (6.6), we see that (6.10) still holds and again obtain (6.11). This concludes the proof of Lemma 6.1. \square

The case $\beta = 1$ and $\gamma = \frac{2}{1-m}$. The only difference is that it is not sufficient to enlarge x_0 in order to obtain (6.4) and (6.5), since the x_0 -exponent $2 + (m - \beta)\gamma$ is now null. However, as it has been explained in Subsection 6.1, in this case and up to a finite shift in time, we can assume without loss of generality that the constant C is arbitrarily large. Then, $C^{m-\beta}$ can be made arbitrarily small so that (6.4) and (6.5) again hold. The rest of the proof remains unchanged.

Proof of the lower bound on the level sets. Now that we have an explicit accelerating subsolution, the proof is the same as that in Subsection 5.1 for the $m > 1$ case. We omit the details and conclude that the lower bounds in (2.7) (when $m < 1$) and (2.17) are proved. \square

6.3. Upper bound on the level sets in (2.7) and (2.17)

Let $\lambda \in (0, 1)$ and $\varepsilon > 0$ small be given. Recall that $\gamma = \min(\alpha, \frac{2}{1-m})$ and assume that

$$1 \leq \beta < \min\left(1 + \frac{1}{\gamma}, m + \frac{2}{\gamma}\right). \quad (6.12)$$

As in Subsection 6.2, this precludes the case $\beta = 1$ and $\gamma = \frac{2}{1-m}$, which we again consider separately at the end of this subsection.

From (2.4), $\gamma \leq \alpha$ and the comparison principle, it is enough (to prove the upper estimate on the level sets) to consider the case where

$$u_0(x) = \frac{\bar{C}}{x^\gamma}, \quad \forall x \geq x_0 > 1.$$

Furthermore, without loss of generality we assume that x_0 is large enough so that $\frac{\bar{C}}{x_0^\gamma} < 1$ and

$$\begin{aligned} & \max \left(m \frac{\gamma \bar{C}^{m-\beta} (\gamma+1-\gamma\beta)}{x_0^{2+(m-\beta)\gamma}}, m(m+\beta-1) \frac{\gamma^2 \bar{C}^{m-\beta}}{x_0^{2+(m-\beta)\gamma}}, m(m+\beta-1) \frac{\gamma^2 \bar{C}^{2-2\beta}}{x_0^{2\gamma(1-\beta)+2}} \right) \\ & \leq \frac{\varepsilon}{4}. \end{aligned} \quad (6.13)$$

Such an x_0 exists thanks to (6.12).

Let us select $\rho := \bar{r} + \frac{\varepsilon}{2}$. Similarly as in Section 5, we define $w(t, x)$ by either (5.2) or (5.3), so that it solves

$$\partial_t w(t, x) = \rho w^\beta(t, x), \quad w(0, x) = u_0(x).$$

We prove below that $\psi(t, x) := \min(1, w(t, x))$ is a supersolution for equation (1.1) after some large enough time $T > 0$. More precisely, let $T > 0$ be large enough so that

$$\psi(t, x) = 1, \quad \forall t \geq T, \forall x \leq x_0,$$

which is possible thanks to the fact that the infimum of u_0 on $(-\infty, x_0]$ is positive. Since 1 solves (1.1), we only need to check that $\psi(t, x)$ is a supersolution at the points (t, x) where $\psi(t, x) = w(t, x) < 1$. In view of (2.14), we get, as in (5.19),

$$\begin{aligned} & \partial_t w(t, x) - \partial_{xx}(w^m)(t, x) - f(w(t, x)) \\ & \geq \frac{\varepsilon}{2} w^\beta(t, x) - m\varphi'(x)w^{m+\beta-1}(t, x) - m(m+\beta-1)\varphi^2(x)w^{m+2\beta-2}(t, x), \end{aligned}$$

where we recall that $\varphi(x) = \frac{u'_0(x)}{u_0^\beta(x)}$. Similarly as before, one must use $(w/u_0)^{m-1} \leq 1$ and either $(w/u_0)^{m+\beta-2} \leq 1$ or $w^{m+\beta-2} \leq 1$, depending on whether $\beta \leq 2-m$ or $\beta > 2-m$. Then one can show that (6.13) implies that the right-hand term above is nonnegative, and therefore ψ is a supersolution in $(T, +\infty) \times \mathbb{R}$. We omit the details.

The upper bound in (2.7) (when $0 < m < 1$) easily follows since $\frac{1}{\gamma} = \Gamma$, as well as in (2.17) since $\gamma = \alpha$ when $0 < \alpha < \frac{2}{1-m}$.

The case $\beta = 1$ and $\gamma = \frac{2}{1-m}$. Again, the argument only slightly differs due to the fact that the exponents $2+(m-\beta)\gamma$ of x_0 appearing in (6.13) now vanish. Hence, in order for (6.13) to hold, we need to enlarge not only x_0 but also \bar{C} (notice in (6.13) the term $\bar{C}^{m-\beta} = \bar{C}^{m-1}$ which is small when \bar{C} is large). This has no incidence because in the Fisher-KPP case, the constant \bar{C} does not appear in the upper bound (2.7) on the level sets. The remainder of the argument is as above. \square

Appendix

A. Lower acceleration bound (2.18) for fast diffusion

Here we prove statement (ii) of Theorem 2.8, which provides a (a priori not optimal) polynomial lower bound on the level sets of solutions in the fast diffusion case $0 < m < 1$. Throughout this section we assume that (2.4) holds with

$$\frac{1}{1-m} < \alpha \leq \frac{2}{1-m},$$

and also that

$$\beta > 1, \quad m + \frac{2}{\alpha} \leq \beta < 1 + \frac{1}{\alpha}.$$

Let us note that the case when (2.8) holds easily follows from the space asymptotics of solutions of the fast diffusion equation, as explained in Subsection 6.1. Thus we omit this situation here.

As before, the proof relies on the construction of an accelerating subsolution. This subsolution has the same shape than that of Section 6, however technical details will differ. Let $\varepsilon > 0$ arbitrarily small be given, and $\eta > \beta + 2$. Then let $\rho > 0$ be such that

$$\max\left(\frac{r\beta}{1+\eta}, r - \varepsilon\right) < \rho < r. \quad (\text{A.1})$$

By comparison and without loss of generality, we assume that $u_0(x) = \frac{C}{x^\alpha}$ for all $x \geq x_0$. Let us also introduce an auxiliary initial datum $\tilde{u}_0(x) := \frac{C-\varepsilon}{C}u_0(x)$, so that for all $x \geq x_0$,

$$\tilde{u}_0(x) = \frac{C-\varepsilon}{x^\alpha}. \quad (\text{A.2})$$

Similarly as before, we define $w(t, x)$ as the solution of $\partial_t w(t, x) = \rho w^\beta(t, x)$ with $w(0, x) = \tilde{u}_0(x)$, which is given by (5.3), up to replacing $u_0(x)$ by $\tilde{u}_0(x)$.

Now select $A > 1$ large enough so that

$$A > \frac{1}{\kappa^\eta(1+\eta)}, \quad \left(\frac{1}{A(1+\eta)}\right)^{1/\eta} \frac{\eta}{1+\eta} \leq s_0,$$

where s_0 is as in (2.13) and $\kappa := \inf_{x \in (-\infty, x_0)} \tilde{u}_0(x) \in (0, 1]$.

Then, as in Subsection 6.2, we define $X(t) \in \mathbb{R}$ such that $w(t, X(t)) = \left(\frac{1}{A(1+\eta)}\right)^{1/\eta}$. From our choice of A , we still have that $X(t) > x_0$ and, from the explicit expressions for $w(t, \cdot)$ and \tilde{u}_0 on $(x_0, +\infty)$, it is straightforward that $X(t)$ is uniquely defined and $w(t, x) < \left(\frac{1}{A(1+\eta)}\right)^{1/\eta}$ if and only if $x > X(t)$. Let us also note that $X(t)$ is increasing with respect to time and tends to $+\infty$ as $t \rightarrow +\infty$.

Now let also δ be small enough so that

$$\delta < \left(\frac{1}{A(1+\eta)}\right)^{1/\eta}, \quad (\text{A.3})$$

and

$$\frac{A\delta^\eta}{1-A\delta^\eta} \times \left(2m\eta + \eta(\beta + \eta - 1) + (1-m)A\eta^2 \frac{\delta^\eta}{1-A\delta^\eta} \right) < m + \beta - 1. \quad (\text{A.4})$$

We are now ready to construct a subsolution.

Lemma A.1. *Under the above assumptions, define*

$$v(t, x) := \begin{cases} \left(\frac{1}{A(1+\eta)}\right)^{1/\eta} \frac{\eta}{1+\eta} & \text{if } x \leq X(t) \\ w(t, x)(1 - Aw^\eta(t, x)) & \text{if } x > X(t). \end{cases}$$

Then there exists $T \geq 0$ such that $v(t, x)$ is a (generalized) subsolution for all $t \geq T$ and $x \in \mathbb{R}$.

By generalized subsolution, we mean that v is C^1 with respect to x and satisfies

$$\mathcal{L}v(t, x) := \partial_t v(t, x) - \partial_{xx}(v^m)(t, x) - f(v(t, x)) \leq 0$$

on both subdomains $\{x < X(t)\}$ and $\{x > X(t)\}$. This is enough to apply a comparison principle, as we explained in Subsection 6.2.

Proof. The C^1 -regularity of v is straightforward, as $w \mapsto w(1 - Aw^\eta)$ reaches its maximum when $w = (\frac{1}{A(1+\eta)})^{1/\eta}$. Moreover, any positive constant is a subsolution, so that we only need to check the parabolic inequality $\mathcal{L}v \leq 0$ on the right subdomain, *i.e.* when $x > X(t)$ (and $t \geq T$ to be chosen below).

First, recall that $X(t) > x_0$ for any $t \geq 0$. In particular, (A.2) is available. Similarly as before, we define $\varphi = \frac{\tilde{u}'_0}{\tilde{u}_0^\beta}$, and compute

$$\varphi^2(x) = \frac{\alpha^2}{(C - \varepsilon)^{2\beta-2} x^{2(\alpha+1-\alpha\beta)}}, \quad \varphi'(x) = \frac{\alpha(\alpha+1-\alpha\beta)}{(C - \varepsilon)^{\beta-1} x^{\alpha+2-\alpha\beta}}.$$

We know by assumption that $\alpha\beta < \alpha + 1$. Since $X(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, we can choose T large enough so that, for all $t \geq T$ and $x \geq X(t)$, we have φ^2 and φ' small enough so that

$$0 < \varphi^2 A^{-\frac{m+\beta-2}{\eta}} \left(2m + \beta + \eta - 1 + 2(1-m)\eta \right) + \varphi' A^{-\frac{m-1}{\eta}} \leq \frac{r - \rho}{2^{1-m} m \eta}. \quad (\text{A.5})$$

We are now ready to compute $\mathcal{L}v$. By the same computations as in Section 5,

$$\begin{aligned} \partial_{xx}(v^m) &= \partial_{xx}(w^m)(1 - Aw^\eta)^m + 2\partial_x(w^m)\partial_x((1 - Aw^\eta)^m) \\ &\quad + w^m \partial_{xx}((1 - Aw^\eta)^m) \\ &= w^{m+\beta-1} \varphi' \left[m(1 - Aw^\eta)^m - Am\eta w^\eta (1 - Aw^\eta)^{m-1} \right] \\ &\quad + w^{m+2\beta-2} \varphi^2 \left[m(m + \beta - 1)(1 - Aw^\eta)^m - 2Am^2\eta w^\eta (1 - Aw^\eta)^{m-1} \right. \\ &\quad \left. - Am\eta(\beta + \eta - 1)w^\eta (1 - Aw^\eta)^{m-1} - (1-m)A^2m\eta^2 w^{2\eta} (1 - Aw^\eta)^{m-2} \right], \end{aligned}$$

where again $\varphi = \frac{\tilde{u}'_0}{\tilde{u}_0^\beta}$.

Let us first assume that $w(t, x) \leq \delta$, which is equivalent to $x \geq X_1(t)$ for some well-chosen $X_1(t)$. Since δ satisfies (A.3) and (A.4), and using also the fact that φ' is positive, it is straightforward to check that $\partial_{xx}(v^m) \geq 0$. Then, thanks to our choice of A , (2.13) and (A.1), one finds that

$$\mathcal{L}v \leq (\rho - r)w^\beta + (rA\beta - A\rho(1 + \eta))w^{\beta+\eta} \leq 0.$$

It remains to check the parabolic inequality when $(\frac{1}{A(1+\eta)})^{1/\eta} \geq w(t, x) \geq \delta$, *i.e.* when $X(t) \leq x \leq X_1(t)$; this is where the above choice of T matters. We go back to the computation of $\partial_{xx}(v^m)$. First, we remove some positive terms and find that

$$\begin{aligned} \partial_{xx}(v^m) &\geq -Am\eta\varphi'w^{m+\eta+\beta-1}(1-Aw^\eta)^{m-1} \\ &\quad +w^{m+2\beta+\eta-2} \left[-2Am^2\eta\varphi^2(1-Aw^\eta)^{m-1} \right. \\ &\quad \left. -Am\eta(\beta+\eta-1)\varphi^2(1-Aw^\eta)^{m-1} \right. \\ &\quad \left. -(1-m)A^2m\eta^2\varphi^2w^\eta(1-Aw^\eta)^{m-2} \right]. \end{aligned}$$

Now, using $0 < m < 1$ and $1 \geq 1 - Aw^\eta(t, x) \geq \frac{\eta}{1+\eta} \geq \frac{1}{2}$ (recall that $\eta > \beta + 2$), we obtain that

$$\begin{aligned} -\partial_{xx}(v^m) &\leq w^\beta \left[2^{1-m}Am\eta\varphi'w^{m+\eta-1} + 2^{2-m}Am^2\eta\varphi^2w^{m+\eta+\beta-2} \right. \\ &\quad \left. + 2^{1-m}Am\eta(\beta+\eta-1)\varphi^2w^{m+\eta+\beta-2} \right. \\ &\quad \left. + 2^{2-m}(1-m)A^2m\eta^2\varphi^2w^{m+2\eta+\beta-2} \right]. \end{aligned}$$

But the exponents $m + \eta - 1$ and $m + \eta + \beta - 2$ are positive, and we also know that $w \leq \frac{1}{A^{1/\eta}}$. Thus, using also (A.5),

$$\begin{aligned} -\partial_{xx}(v^m) &\leq w^\beta \left[2^{1-m}m\eta\varphi'A^{-\frac{m-1}{\eta}} + 2^{2-m}m^2\eta\varphi^2A^{-\frac{m+\beta-2}{\eta}} \right. \\ &\quad \left. + 2^{1-m}m\eta(\beta+\eta-1)\varphi^2A^{-\frac{m+\beta-2}{\eta}} \right. \\ &\quad \left. + 2^{2-m}(1-m)m\eta^2\varphi^2A^{-\frac{m+\beta-2}{\eta}} \right], \\ &\leq (r - \rho)w^\beta. \end{aligned}$$

We conclude that $\mathcal{L}v \leq 0$ and Lemma A.1 is proved. \square

Notice that, because T may be large and the subsolution v of Lemma A.1 is increasing in time, one may not compare directly $v(T, \cdot)$ with the initial condition u_0 . To circumvent this, one can instead prove that there exists a spatial shift $X > 0$ such that

$$u_0(\cdot - X) \geq v(T, \cdot).$$

This mostly follows from the asymptotics of $v(T, x)$ and u_0 as $x \rightarrow +\infty$. The proof of the lower bound on the level sets then proceeds as in Subsection 5.1, which ends the proof of Theorem 2.8 (ii). \square

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