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An Andreotti-Grauert theorem with L^r estimates

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Abstract. By a theorem of Andreotti and Grauert if ω is a (p, q) current, $q < n$, in a Stein manifold, $\bar{\partial}$ closed and with compact support, then there is a solution u to $\bar{\partial}u = \omega$ still with compact support. The main result of this work is to show that if moreover $\omega \in L^r(dm)$, where m is a suitable "Lebesgue" measure on the Stein manifold, then we have a solution u with compact support *and* in $L^r(dm)$. We prove it by estimates in L^r spaces with weights.

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1. Introduction

Let ω be a $\bar{\partial}$ closed (p, q) form in $\mathbb{C}^n$ with compact support $K := \operatorname{Supp}\omega$ and such that $\omega \in L^r(\mathbb{C}^n)$, the Lebesgue space in $\mathbb{C}^n$. Setting K in a ball $\mathbb{B} := B(0, R)$ with R big enough, we know, by a theorem of Ovrelid [14], that we have a $(p, q - 1)$ form $u \in L^r(\mathbb{B})$ such that $\bar{\partial}u = \omega$. On the other hand we also know, at least when $q < n$, that there is a current v with compact support such that $\bar{\partial}v = \omega$, by a theorem of Andreotti-Grauert [6].

So a natural question is: may we have a solution u of $\bar{\partial}u = \omega$ with compact support *and* in $L^r(\mathbb{C}^n)$?

There is a work by H. Skoda [16] who proved such a result. Let Ω be a strictly pseudo-convex bounded domain in $\mathbb{C}^n$ with smooth boundary then in [16, Corollaire page 295], H. Skoda proved that if f is a (p, q)-form with measure coefficients, $q < n$, $\bar{\partial}$ closed and with compact support in Ω, then there is a solution U to the equation $\bar{\partial}U = f$ such that $\|U\|_{L^r(\Omega)} \leq C(\Omega, r)\|f\|_1$, for any r such that $1 < r < \frac{2n+2}{2n-1}$ and U has zero boundary values in the sense of Stokes formula. This means that essentially U has compact support and, because Ω is bounded, $\|f\|_1 \lesssim \|f\|_{L^r(\Omega)}$. So he got the answer for Ω strictly pseudo-convex and $1 < r < \frac{2n+2}{2n-1}$.

We answered this question by the affirmative for any $r \in [1, \infty]$ in a joint work with S. Mongodi [5] linearly by the "method of coronas". This method asks

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for extra L^r conditions on derivatives of coefficients of ω, when $q < n$; we shall denote the set of ω verifying these conditions $\mathcal{W}_q^r(\Omega)$, as in [5].

The aim of this work is to extend this result to Stein manifolds and get rid of the extra L^r conditions $\mathcal{W}_q^r(\Omega)$. For it we use a completely different approach inspired by the Serre duality [15]. Because Hahn Banach theorem is used, this method is no longer constructive as in [5].

The basic notion we shall use here is the following:

Definition 1.1. Let X be a complex manifold equipped with a Borel σ-finite measure dm and Ω a domain in X; let $r \in [1, \infty]$, we shall say that Ω is *r regular* if for any $p, q \in \{0, ..., n\}$, $q \geq 1$, there is a constant $C = C_{p,q}(\Omega)$ such that for any (p, q) form ω, $\bar\partial$ closed in Ω and in $L^r(\Omega, dm)$ there is a $(p, q-1)$ form $u \in L^r(\Omega, dm)$ such that $\bar\partial u = \omega$ and $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$.

We shall say that Ω is *weakly r regular* if for any compact set $K \Subset \Omega$ there are 3 open sets $\Omega_1, \Omega_2, \Omega_3$ such that $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 := \Omega$ and 3 constants C_1, C_2, C_3 such that:

$$\forall j = 0, 1, 2,\ \forall p, q \in \{0, ..., n\},\ q \geq 1,\ \forall \omega \in L^r_{p,q}(\Omega_j, dm),\ \bar\partial\omega = 0,$$
$$\exists u \in L^r_{p,q-1}(\Omega_{j+1}, dm),\ \bar\partial u = \omega$$

and $\|u\|_{L^r(\Omega_{j+1})} \leq C_{j+1}\|\omega\|_{L^r(\Omega_j)}$.

I.e., we have a 3 steps chain of resolution.

Of course the r regularity implies the weak r regularity, just taking $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$.

Examples of 2 regular domains are the bounded pseudo-convex domains by Hörmander [10].

Examples of r regular domains in $\mathbb{C}^n$ are the bounded strictly pseudo-convex (s.p.c.) domains with smooth boundary by Ovrelid [14]; the polydiscs in $\mathbb{C}^n$ by Charpentier [7], finite transverse intersections of strictly pseudo-convex bounded domains in $\mathbb{C}^n$ by Menini [13]. A generalisation of the results by Menini was done in the nice work of Ma and Vassiliadou [12]: they treated also the case of intersection of q-convex sets.

Examples of r regular domains in a Stein manifold are the strictly pseudo-convex domains with smooth boundary [3]. (See the previous work for (0, 1) forms by N. Kerzman [11] and for all (p, q) forms by J-P. Demailly and C. Laurent [8, Remarque 4, page 596], but here the manifold has to be equipped with a metric with null curvature. See also [4] for the case of intersection of q-convex sets in a Stein manifold).

Let X be a Stein manifold and Ω a domain in X, *i.e.* an open connected set in X. Let $\mathcal{H}_p(\Omega)$ be the set of all $(p, 0)$ $\bar\partial$ closed forms in Ω. If $p = 0$, $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ is the set of holomorphic functions in Ω. If $p > 0$, we have, in a chart (φ, U), $h \in \mathcal{H}_p(\Omega) \Rightarrow h(z) = \sum_{|J|=p} a_J(z)dz^J$, where $dz^J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and the functions $a_J(z)$ are holomorphic in $\varphi(U) \subset \mathbb{C}^n$.

We shall denote $L^{r,c}_{p,q}(\Omega)$ the set of (p, q)-forms in $L^r(\Omega)$ with compact support in Ω.

We also use the notation r' for the conjugate exponent of r, *i.e.* $\frac{1}{r}+\frac{1}{r'}=1$.

Our main theorem is:

Theorem 1.2. *Let Ω be a weakly r' regular domain in a Stein manifold X. Then there is a $C>0$ such that for any (p,q) form ω in $L^{r,c}(\Omega),\ r>1$ with:*

- *if* $1\leq q<n,\ \bar\partial\omega=0$*;*
- *if* $q=n,\ \forall V\subset\Omega,\ \operatorname{Supp}\omega\subset V,\ \omega\perp\mathcal{H}_{n-p}(V)$*;*

there is a $(p,q-1)$ form u in $L^{r,c}(\Omega)$ such that $\bar\partial u=\omega$ as distributions and $\|u\|_{L^r(\Omega)}\leq C\|\omega\|_{L^r(\Omega)}$.

The notion of r regularity gives a good control of the support: if the support of the data ω is contained in $\Omega\backslash C$ where Ω is a weakly r' regular domain and C is a weakly r regular domain, then the support of the solution u is contained in $\Omega\backslash C'$, where C' is any relatively compact domain in C, provided that $q\geq 2$. One may observe that $\Omega\backslash C$ is *not* Stein in general even if Ω is.

There is also a result of this kind for $q=1$, see Section 3.3.

In particular the support of the solution u is contained in the intersection of all the weakly r' regular domains containing the support of ω.

The idea is to solve $\bar\partial u=\omega$ in a space $L^r(\Omega)$ with a "big weight η outside" of the support of ω; this way we shall have a "small solution u outside" of the support of ω. Then, using a sequence of such weights going to infinity "outside" of the support of ω, we shall have a u zero "outside of the support" of ω.

Comparing to my previous work [2] the results here are improved and the proofs are much simpler by a systematic use of the Hodge $*$ operator.

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2. Duality

We shall study a duality between currents inspired by the Serre duality [15].

Let X be a complex manifold of dimension n. We proceed now exactly as in Hörmander [10, page 119], by introducing a hermitian metric on differential forms locally equivalent to the usual one on any analytic coordinates system.

We define the "Lebesgue measure" still as in Hörmander's book [10, Section 5.2]: associated to this metric there is a volume measure dm and we take it for the Lebesgue measure on X. Moreover, because X is a complex manifold, it is canonically oriented.

2.1. Weighted L^r spaces

Let Ω be a domain in X. We denote also dm the volume form on X. We shall take the following notation from the book by C. Voisin [17].

To a (p,q)-form α on Ω we associate its Hodge $*$ $(n-p,n-q)$-form $*\alpha$. This gives us a pointwise scalar product and a pointwise modulus:

$$(\alpha,\beta)dm := \alpha\wedge\overline{*\beta};\quad |\alpha|^2\,dm := \alpha\wedge\overline{*\alpha,} \tag{2.1}$$

because $\alpha\wedge\overline{*\beta}$ is a (n,n)-form hence is a function time the volume form dm.

We are also given a scalar product $\langle\alpha,\beta\rangle$ on (p,q)-forms such that $\int_\Omega|\alpha|^2dm<\infty$ and the link between these notions is given by [17, Lemme 5.8, page 119]:

$$\langle\alpha,\beta\rangle=\int_\Omega\alpha\wedge\overline{*\beta}. \tag{2.2}$$

We shall define now $L^r_{p,q}(\Omega)$ to be the set of $(p,q)-$forms α defined on Ω such that

$$\|\alpha\|^r_{L^r_{p,q}(\Omega)} := \int_\Omega|\alpha(z)|^r\,dm(z)<\infty,$$

where $|\alpha|$ is defined by (2.1).

Lemma 2.1. *Let $\eta>0$ be a weight. If u is a (p,q)-current defined on $(n-p,n-q)$-forms α in $L^{r'}(\Omega,\eta)$ and such that*

$$\forall\alpha\in L^{r'}_{(n-p,n-q)}(\Omega,\eta),\ |\langle u,\ *\alpha\rangle|\le C\|\alpha\|_{L^{r'}(\Omega,\eta)},$$

then $\|u\|_{L^r_{p,q}(\Omega,\eta^{1-r})}\le C.$

Proof. We use the classical trick: set $\tilde\alpha:=\eta^{1/r'}\alpha;\ \tilde u:=\frac{1}{\eta^{1/r'}}u$ then we have

$$\langle u,*\alpha\rangle=\int_\Omega u\wedge\overline{\alpha}=\int_\Omega\tilde u\wedge\overline{\tilde\alpha}=\langle\tilde u,*\tilde\alpha\rangle$$

and $\|\tilde\alpha\|_{L^{r'}(\Omega)}=\|\alpha\|_{L^{r'}(\Omega,\eta)}$.

We notice that $\|\tilde\alpha\|_{L^{r'}(\Omega)}=\|*\tilde\alpha\|_{L^{r'}(\Omega)}$ because we have $(*\tilde\alpha,*\tilde\alpha)dm=*\tilde\alpha\wedge\overline{**\tilde\alpha}$ but $**\tilde\alpha=(-1)^{(p+q)(2n-p-q)}\tilde\alpha$, by [17, Lemma 5.5], hence, because $(*\tilde\alpha,*\tilde\alpha)$ is positive, $(*\tilde\alpha,*\tilde\alpha)=|\tilde\alpha|^2$.

By use of the duality $L^r_{p,q}(\Omega)-L^{r'}_{n-p,n-q}(\Omega)$, done in Lemma A.3, we get

$$\|\tilde u\|_{L^r_{p,q}(\Omega)}=\sup_{\alpha\in L^{r'}_{n-p,n-q}(\Omega),\ \alpha\neq0}\frac{|\langle\tilde u,*\tilde\alpha\rangle|}{\|\tilde\alpha\|_{L^{r'}(\Omega)}}.$$

But

$$\|\tilde u\|^r_{L^r_{p,q}(\Omega)}:=\int_\Omega|u|^r\,\eta^{-\frac{r}{r'}}dm=\int_\Omega|u|^r\,\eta^{1-r}dm=\|u\|^r_{L^r(\Omega,\eta^{1-r})}.$$

So we get

$$\|u\|_{L^r_{p,q}(\Omega,\eta^{1-r})} = \sup_{*\alpha\in L^{r'}_{p,q}(\Omega,\eta),\ \alpha\neq 0} \frac{|\langle u, *\alpha\rangle|}{\|\alpha\|_{L^{r'}(\Omega,\eta)}}.$$

The proof is complete. □

It may seem strange that we have such an estimate when the dual of $L^{r'}(\Omega,\eta)$ is $L^r(\Omega,\eta)$, but the reason is, of course, that in the duality current-form there is no weights. The point here is that when η is small, η^{1-r} is big for $r > 1$.

3. Solution of the $\bar\partial$ equation with compact support

3.1. *r* regular domains

As we have seen, examples of r regular domains in Stein manifolds are the relatively compact s.p.c. domains with smooth boundary. To prove that a Stein manifold Ω is weakly r regular we shall need the following lemma.

Lemma 3.1. *Let Ω be a Stein manifold. Then it contains an exhaustive sequence of open relatively compact strictly pseudo-convexs sets $\{D_k\}_{k\in\mathbb{N}}$ with $\mathcal{C}^\infty$ smooth boundary.*

Proof. For the case of Ω pseudo-convex in $\mathbb{C}^n$, the proof was already done explicitely in the proof of [9, Theorem 2.8.1, page 86].

By Theorem 5.1.6 of Hörmander [10] there exists a $\mathcal{C}^\infty$ strictly plurisubharmonic (s.p.s.h.) exhausting function φ for Ω. Take $K \Subset \Omega$ such that $d\varphi \neq 0$ on K. Because φ is s.p.s.h. then $K \neq \emptyset$. Then we use the [9, Lemma 2.12.2, page 93], to get: $\forall\epsilon > 0,\ \exists\rho_\epsilon$ s.p.s.h. $\mathcal{C}^\infty$-function on Ω such that:

- (i) $\varphi - \rho_\epsilon$ together with its first and second derivatives is less than ϵ on Ω.
- (ii) The set $\mathrm{Crit}(\rho_\epsilon) := \{z \in \Omega :: d\rho_\epsilon(z) = 0\}$ is discrete in Ω. (In a formula, the notation :: means "such that".)
- (iii) $\rho_\epsilon = \varphi$ on K.

As stated in Lemma 2.12.2 if $\varphi \in \mathcal{C}^2$ then $\rho_\epsilon \in \mathcal{C}^2$, but in fact the proof of this Lemma 2.12.2 gives $\rho_\epsilon = \varphi + \sum\chi_j$, where $\sum\chi_j$ is $\mathcal{C}^\infty$ (see [9, page 93]). Hence ρ_ϵ has the same $\mathcal{C}^k$ regularity than φ.

Fix $\epsilon > 0$, then the function $\rho := \rho_\epsilon$ is also a s.p.s.h. exhausting function for Ω, because, from $-\epsilon \le \varphi - \rho_\epsilon \le \epsilon$, we get that, for any $\alpha \in \mathbb{R}$,

$$\{z \in \Omega :: \rho_\epsilon(z) < \alpha\} \subset \{z \in \Omega :: \varphi(z) < \epsilon + \alpha\}$$

and, because φ is an exhausting function, this set is relatively compact in Ω.

Because the set of critical points of ρ is discrete in Ω, the same way as in the proof of [9, Theorem 2.8.1, page 86], we can find a sequence $c_k \in \mathbb{R},\ c_k \to \infty$, such that $D_k := \{z \in \Omega :: \rho(z) < c_k\}$ make an exhaustive sequence of open relatively compact sets in Ω, $\partial\rho \neq 0$ on ∂D_k, hence D_k is strictly pseudo-convex with $\mathcal{C}^\infty$ smooth boundary, and finally $D_k \nearrow \Omega$. The proof is complete. □

Proposition 3.2. *A Stein manifold Ω is weakly r regular.*

Proof. By Lemma 3.1 there is an exhaustive sequence of open relatively compact s.p.c. sets in Ω, $\{D_k\}_{k\in\mathbb{N}}$ with $\mathcal{C}^\infty$ smooth boundary. Let $\omega\in L^r_{p,q}(\Omega)$, $\bar\partial\omega=0$, by [3], we can solve $\bar\partial u=\omega$ in D_k with $u\in L^r_{p,q-1}(D_k)$ and

$$\|u\|_{L^r(D_k)}\leq C_k\|\omega\|_{L^r(D_k)}\leq C_k\|\omega\|_{L^r(\Omega)}.$$

Hence if Γ is a compact set in Ω, there is a D_k such that $\Gamma\Subset D_k$ and we can take $\Omega_1=\Omega_2=\Omega_3=D_k$. This proves the weak r regularity of Ω. □

3.2. The main result

Let X be a Stein manifold and Ω a domain in X. In order to simplify notation, we set the pairing for α a (p,q)-form and β a $(n-p,n-q)$-form: $\ll\alpha,\beta\gg:=\int_\Omega\alpha\wedge\beta$. With this notation we also have $\langle\alpha,\beta\rangle=\ll\alpha,\overline{*\beta}\gg$.

Let Ω be a weakly r' regular domain in X. We set $K:=\operatorname{Supp}\omega\Subset\Omega$ and, by the definition of the r' weak regularity, we get 3 open sets such that $K\Subset\Omega_3\subset\Omega_2\subset\Omega_1\subset\Omega_0=\Omega$ with: $\forall j=0,1,2,\ \forall p,q\in\{0,...,n\},\ q\geq 1,$

$$\forall\alpha\in L^r_{p,q}(\Omega_j),\ \bar\partial\alpha=0,\ \exists\varphi\in L^r_{p,q-1}(\Omega_{j+1}),\ \bar\partial\varphi=\alpha.$$

Set the weight $\eta=\eta_\epsilon:=\mathbb{1}_{\Omega_1}(z)+\epsilon\mathbb{1}_{\Omega\setminus\Omega_1}(z)$ for a fixed $\epsilon>0$.

Let $\omega\in L^{r,c}_{p,q}(\Omega)$. Suppose moreover that ω is such that $\bar\partial\omega=0$ if $1\leq q<n$ and for any open $V\Subset\Omega$, $\operatorname{Supp}\omega\Subset V$ we have $\omega\perp\mathcal{H}_{n-p}(V)\iff\forall h\in\mathcal{H}_{n-p}(V),\ \ll\omega,h\gg=0$ if $q=n$.

We shall use the following lemma, with the previous notation:

Lemma 3.3. *Let $\mathcal{E}$ be the set of $(n-p,n-q+1)$ forms $\alpha\in L^{r'}(\Omega,\eta)$, $\bar\partial$ closed in Ω. Let us define $\mathcal{L}_\omega$ on $\mathcal{E}$ as follows:*

$$\mathcal{L}_\omega(\alpha):=(-1)^{p+q-1}\ll\varphi,\omega\gg,$$

where $\varphi\in L^{r'}(\Omega_1)$ is such that $\bar\partial\varphi=\alpha$ in Ω_1. Then the form $\mathcal{L}_\omega$ is well defined and linear.

Proof. Because $\epsilon>0$ we have $\alpha\in L^{r'}(\Omega,\eta)\Rightarrow\alpha\in L^{r'}(\Omega)$ and the weak r' regularity of Ω gives a $\varphi\in L^{r'}(\Omega_1)$ with $\bar\partial\varphi=\alpha$ in Ω_1.

Let us see that $\mathcal{L}_\omega$ is well defined.

- Suppose first that $q<n$.

In order for $\mathcal{L}_\omega$ to be well defined we need

$$\forall\varphi,\psi\in L^{r'}_{(n-p,n-q)}(\Omega_1),\ \bar\partial\varphi=\bar\partial\psi=\alpha\Rightarrow\ll\varphi,\omega\gg=\ll\psi,\omega\gg .$$

This is meaningful because $\omega \in L^{r,c}(\Omega)$, $r > 1$, $\operatorname{Supp}\omega \Subset \Omega_1$. Then we have $\bar\partial(\varphi - \psi) = 0$ in Ω_1, hence, because Ω is weakly r' regular, we can solve $\bar\partial$ in $L^{r'}(\Omega_2)$:

$$\exists\gamma \in L^{r'}_{(n-p,n-q-1)}(\Omega_2) :: \bar\partial\gamma = (\varphi - \psi).$$

So $\ll \varphi - \psi, \omega \gg = \ll \bar\partial\gamma, \omega \gg = (-1)^{p+q-1} \ll \gamma, \bar\partial\omega \gg = 0$ because ω is compactly supported in Ω_2 and $\bar\partial$ closed. Hence $\mathcal{L}_\omega$ is well defined in that case.

• Suppose now that $q = n$.

For φ, ψ $(n-p,\ 0)$ forms in Ω_1, such that $\bar\partial\varphi = \bar\partial\psi = \alpha$, we need to have $\ll \varphi, \omega \gg = \ll \psi, \omega \gg$. But then $\bar\partial(\varphi - \psi) = 0$, which means that $h := \varphi - \psi$ is a $\bar\partial$ closed $(n-p,\ 0)$ form, hence $h \in \mathcal{H}_{n-p}(\Omega_1)$. Taking $V = \Omega_1$ in the hypothesis $\omega \perp \mathcal{H}_{n-p}(V)$, we get $\ll h, \omega \gg = 0$, and $\mathcal{L}_\omega$ is also well defined in that case.

It remains to see that $\mathcal{L}_\omega$ is linear.

• Suppose first that $q < n$.

Let $\alpha = \alpha_1 + \alpha_2$, with $\alpha_j \in L^{r'}(\Omega, \eta)$, $\bar\partial\alpha_j = 0$, $j = 1, 2$; we have $\alpha = \bar\partial\varphi$, $\alpha_1 = \bar\partial\varphi_1$ and $\alpha_2 = \bar\partial\varphi_2$, with φ, φ_1, φ_2 in $L^{r'}(\Omega_1)$ so, because $\bar\partial(\varphi - \varphi_1 - \varphi_2) = 0$, we have

$$\varphi = \varphi_1 + \varphi_2 + \bar\partial\psi, \quad \text{with } \psi \text{ in } L^{r'}(\Omega_2),$$

so

$$\begin{aligned}\mathcal{L}_\omega(\alpha) &= (-1)^{p+q-1} \ll \varphi, \omega \gg = (-1)^{p+q-1} \ll \varphi_1 + \varphi_2 + \bar\partial\psi, \omega \gg = \\ &= \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2) + (-1)^{p+q-1} \ll \bar\partial\psi, \omega \gg,\end{aligned}$$

but again $\ll \bar\partial\psi, \omega \gg = 0$, hence $\mathcal{L}_\omega(\alpha) = \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2)$.

The same for $\alpha = \lambda\alpha_1$.

• Suppose now that $q = n$. We have

$$\mathcal{L}_\omega(\alpha) := (-1)^{p+n-1} \ll \varphi, \omega \gg,$$

where $\varphi \in L^{r'}(\Omega_1)$ is such that $\bar\partial\varphi = \alpha$ in Ω_1. Let $\alpha = \alpha_1 + \alpha_2$, with $\alpha_j \in L^{r'}(\Omega, \eta)$, $\bar\partial\alpha_j = 0$, $j = 1, 2$; we have $\alpha = \bar\partial\varphi$, $\alpha_1 = \bar\partial\varphi_1$ and $\alpha_2 = \bar\partial\varphi_2$, with φ, φ_1, φ_2 in $L^{r'}(\Omega_1)$ so, because $\bar\partial(\varphi - \varphi_1 - \varphi_2) = 0$, we have $\varphi - \varphi_1 - \varphi_2$ is a $(n-p, 0)$ $\bar\partial$-closed form, hence:

$$\varphi = \varphi_1 + \varphi_2 + h, \quad \text{with } h \in \mathcal{H}_{n-p}(\Omega_1).$$

So

$$\begin{aligned}\mathcal{L}_\omega(\alpha) &= (-1)^{p+q-1} \ll \varphi, \omega \gg = (-1)^{p+q-1} \ll \varphi_1 + \varphi_2 + h, \omega \gg \\ &= \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2) + (-1)^{p+q-1} \ll h, \omega \gg .\end{aligned}$$

Taking $V = \Omega_1$ in the hypothesis $\omega \perp \mathcal{H}_{n-p}(V)$, we get $\ll h, \omega \gg = 0$, hence $\mathcal{L}_\omega(\alpha) = \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2)$. The same for $\alpha = \lambda\alpha_1$. The proof is complete. □

Remark 3.4. If Ω is Stein, we can take the domain Ω_1 to be s.p.c. with $\mathcal{C}^\infty$ smooth boundary, hence also Stein. So because $K := \operatorname{Supp}\omega \subset \Omega_1 \subset \Omega$, the $A(\Omega_1)$ convex hull of K, $\hat{K}_{\Omega_1}$ is still in Ω_1, and any holomorphic function in Ω_1 can be uniformly approximated on $\hat{K}_{\Omega_1}$ by holomorphic functions in Ω.

Then for $q = n$ instead of asking $\omega \perp \mathcal{H}_{n-p}(\Omega_1)$ we need just $\omega \perp \mathcal{H}_{n-p}(\Omega)$.

Theorem 3.5. *Let Ω be a weakly r' regular domain and ω be a (p, q) form in $L^{r,c}(\Omega)$, $r > 1$. Suppose that ω is such that:*

- *if $1 \le q < n$, $\bar\partial\omega = 0$;*
- *if $q = n$, $\forall V \subset \Omega$, $\operatorname{Supp}\omega \subset V$, $\omega \perp \mathcal{H}_{n-p}(V)$.*

Then there is a $C > 0$ and a $(p, q-1)$ form u in $L^{r,c}(\Omega)$ such that $\bar\partial u = \omega$ as distributions and $\|u\|_{L^r(\Omega)} \le C\|\omega\|_{L^r(\Omega)}$.

Proof. Because Ω is weakly r' regular there is a $\Omega_1 \subset \Omega$, $\Omega_1 \supset \operatorname{Supp}\omega$ such that

$$\forall\alpha \in L^{r'}(\Omega), \bar\partial\alpha = 0,\ \exists\varphi \in L^{r'}(\Omega_1) :: \bar\partial\varphi = \alpha,\ \|\varphi\|_{L^{r'}(\Omega_1)} \le C_1\|\alpha\|_{L^{r'}(\Omega)}.$$

There is a Ω_2 such that $\operatorname{Supp}\omega \Subset \Omega_2 \subset \Omega_1 \subset \Omega$ with the same properties as Ω_1. Let us consider the weight $\eta = \eta_\epsilon := \mathbb{1}_{\Omega_1}(z) + \epsilon\mathbb{1}_{\Omega\setminus\Omega_1}(z)$ for a fixed $\epsilon > 0$ and the form $\mathcal{L}_\omega$ defined in Lemma 3.3. By Lemma 3.3 we have that $\mathcal{L}_\omega$ is a linear form on $(n-p, n-q+1)$-forms $\alpha \in L^{r'}(\Omega, \eta)$, $\bar\partial$ closed in Ω.

If α is a $(n-p, n-q+1)$-form in $L^{r'}(\Omega, \eta)$, then α is in $L^{r'}(\Omega)$ because $\epsilon > 0$.

The weak r' regularity of Ω gives that there is a $\varphi \in L^{r'}(\Omega_1) :: \bar\partial\varphi = \alpha$ which can be used to define $\mathcal{L}_\omega(\alpha)$.

We have also that $\alpha \in L^{r'}(\Omega_1)$, $\bar\partial\alpha = 0$ in Ω_1, hence, still with the weak r' regularity of Ω, we have

$$\exists\psi \in L^{r'}(\Omega_2) :: \bar\partial\psi = \alpha,\ \|\psi\|_{L^{r'}(\Omega_2)} \le C_2\|\alpha\|_{L^{r'}(\Omega_1)}.$$

• For $q < n$, we have $\bar\partial(\varphi-\psi) = \alpha-\alpha = 0$ on Ω_2 and, by the weak r' regularity of Ω, there is a $\Omega_3 \subset \Omega_2$, such that $\operatorname{Supp}\omega \subset \Omega_3 \subset \Omega_2$, and a $\gamma \in L^{r'}(\Omega_3)$, $\bar\partial\gamma = \varphi - \psi$ in Ω_3. So we get

$$\ll \varphi - \psi, \omega \gg = \ll \bar\partial\gamma, \omega \gg = (-1)^{p+q-1} \ll \gamma, \bar\partial\omega \gg = 0,$$

this is meaningful because $\operatorname{Supp}\omega \subset \Omega_3$. Hence $\mathcal{L}_\omega(\alpha) = \ll \varphi, \omega \gg = \ll \psi, \omega \gg$.

• For $q = n$, we still have $\bar\partial(\varphi - \psi) = \alpha - \alpha = 0$ on Ω_2, hence $\varphi - \psi \in \mathcal{H}_p(\Omega_2)$; this time we choose $V = \Omega_2$ and the assumption gives $\ll \varphi - \psi, \omega \gg = 0$ hence again $\mathcal{L}_\omega(\alpha) = \ll \varphi, \omega \gg = \ll \psi, \omega \gg$.

In any cases, by Hölder inequalities done in Lemma A.1,

$$|\mathcal{L}_\omega(\alpha)| \le \|\omega\|_{L^r(\Omega_1)}\|\psi\|_{L^{r'}(\Omega_2)} \le \|\omega\|_{L^r(\Omega)}\|\psi\|_{L^{r'}(\Omega_2)}.$$

But, by the weak r' regularity of Ω, there is a constant C_2 such that

$$\|\psi\|_{L^{r'}(\Omega_2)} \leq C_2\|\alpha\|_{L^{r'}(\Omega_1)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega_1)} \leq \|\alpha\|_{L^{r'}(\Omega,\ \eta)}$$

because $\eta = 1$ on Ω_1, hence

$$|\mathcal{L}_\omega(\alpha)| \leq C_2\|\omega\|_{L^r(\Omega)}\|\alpha\|_{L^{r'}(\Omega,\ \eta)}.$$

So we have that the norm of $\mathcal{L}_\omega$ is bounded on the subspace of $\bar\partial$ closed forms in $L^{r'}(\Omega,\eta)$ by $C\|\omega\|_{L^r(\Omega)}$ which is *independent* of ϵ.

We apply the Hahn-Banach theorem to extend $\mathcal{L}_\omega$ with the *same* norm to *all* $(n-p, n-q+1)$ forms in $L^{r'}(\Omega,\ \eta)$. As in the Serre Duality Theorem [15, page 20], this is one of the major ingredients in the proof.

This means, by the definition of currents, that there is a $(p, q-1)$ current u which represents the extended form $\mathcal{L}_\omega$: $\mathcal{L}_\omega(\alpha) = \ll \alpha, u \gg$. So if $\alpha := \bar\partial\varphi$ with $\varphi \in \mathcal{C}_c^\infty(\Omega)$, we get

$$\mathcal{L}(\alpha)\ = \ll \alpha, u \gg = \ll \bar\partial\varphi, u \gg =\ (-1)^{p+q-1}\ \ll \varphi, \omega \gg$$

hence $\bar\partial u = \omega$ as distributions because φ is compactly supported. And we have:

$$\sup_{\alpha\in L^{r'}(\Omega,\eta),\ \|\alpha\|=1} |\ll \alpha, u \gg| \leq C\|\omega\|_{L^r(\Omega)}.$$

By Lemma 2.1 with the weight η, this implies

$$\|u\|_{L^r(\Omega,\eta^{1-r})} \leq C\|\omega\|_{L^r(\Omega)}$$

because $|\ll \alpha, u \gg| = |\langle\alpha, \overline{*u}\rangle|$ and, as already seen,

$$\|u\|_{L^r(\Omega,\eta^{1-r})} = \|{*u}\|_{L^r(\Omega,\eta^{1-r})} = \|\overline{*u}\|_{L^r(\Omega,\eta^{1-r})}.$$

In particular $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$ because with $\epsilon < 1$ and $r > 1$, we have $\eta^{1-r} \geq 1$.

Now for $\epsilon > 0$ with $\eta_\epsilon(z) := \mathbb{1}_{\Omega_1}(z) + \epsilon\mathbb{1}_{\Omega\setminus\Omega_1}(z)$, let $u_\epsilon \in L^r(\Omega, \eta_\epsilon^{1-r})$ be the previous solution, then

$$\|u_\epsilon\|^r_{L^r(\Omega,\eta_\epsilon^{1-r})} \leq \int_\Omega |u_\epsilon|^r\ \eta^{1-r}dm \leq C^r\,\|\omega\|^r_{L^r(\Omega)}.$$

Replacing η by its value we get

$$\int_{\Omega_1} |u_\epsilon|^r\ dm + \int_{\Omega\setminus\Omega_1} |u_\epsilon|^r\ \epsilon^{1-r}dm \leq C^r\,\|\omega\|^r_{L^r(\Omega)}$$
$$\Rightarrow \int_{\Omega\setminus\Omega_1} |u_\epsilon|^r\ \epsilon^{1-r}dm \leq C^r\,\|\omega\|^r_{L^r(\Omega)}$$

hence

$$\int_{\Omega\backslash\Omega_1} |u_\epsilon|^r \, dm \leq C^r \epsilon^{r-1} \|\omega\|^r_{L^r(\Omega)}.$$

Because C and the norm of ω are independent of ϵ, we have that $\|u_\epsilon\|_{L^r(\Omega)}$ is uniformly bounded and $r > 1$ implies that $L^r_{p,q-1}(\Omega)$ is a dual by Lemma A.3, hence there is a sub-sequence $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\epsilon\}$ which converges weakly, when $\epsilon_k \to 0$, to a $(p, q-1)$ form u in $L^r_{p,q-1}(\Omega)$, still with $\|u\|_{L^r_{p,q-1}(\Omega)} \leq C\|\omega\|_{L^r_{p,q}(\Omega)}$. Let us write $u_k := u_{\epsilon_k}$.

To see that this form u is 0 $a.e.$ on $\Omega\backslash\Omega_1$ let us write the weak convergence:

$$\forall \alpha \in L^{r'}_{p,q-1}(\Omega),\ \langle u_k, \alpha\rangle = \int_\Omega u_k \wedge \overline{*\alpha} \to \langle u, \alpha\rangle = \int_\Omega u \wedge \overline{*\alpha}.$$

As usual take $\alpha := \dfrac{u}{|u|}\mathbb{1}_E$ where $E := \{|u| > 0\} \cap (\Omega\backslash\Omega_1)$ then we get

$$\int_\Omega u \wedge \overline{*\alpha} = \int_E |u|\, dm = \lim_{k\to\infty} \int_\Omega u_k \wedge \overline{*\alpha} = \lim_{k\to\infty} \int_E \frac{u_k \wedge \overline{*u}}{|u|}.$$

Now we have, by Hölder inequalities:

$$\left|\int_E \frac{u_k \wedge \overline{*u}}{|u|}\right| \leq \|u_k\|_{L^r(E)} \|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|u_k\|^r_{L^r(E)} \leq \int_{\Omega\backslash\Omega_1} |u_k|^r \, dm \leq (\epsilon_k)^{r-1} C\|\omega\|_{L^r(\Omega)} \to 0,\ k \to \infty$$

and $\|\mathbb{1}_E\|_{L^{r'}(E)} = (m(E))^{1/r'}$. Hence

$$\begin{aligned}\left|\int_E |u|\, dm\right| &= \lim_{k\to\infty} \int_E \frac{u_k \wedge \overline{*u}}{|u|} \\ &\leq \lim_{k\to\infty} C^r (m(E))^{1/r'} (\epsilon_k)^{r-1} \|\omega\|^r_{L^r(\Omega)} = 0,\end{aligned}$$

so $\int_E |u|\, dm = 0$ which implies $m(E) = 0$ because on E, $|u| > 0$.

Hence we get that the form u is 0 $a.e.$ on $\Omega\backslash\Omega_1$.

So we proved

$$\begin{aligned}&\forall \varphi \in \mathcal{D}_{n-p,n-q}(\Omega),\ (-1)^{p+q-1} \ll \varphi, \omega \gg = \ll \bar\partial\varphi, u_\epsilon \gg \to \ll \bar\partial\varphi, u \gg \\ &\quad \Rightarrow \ll \bar\partial\varphi, u \gg = (-1)^{p+q-1} \ll \varphi, \omega \gg\end{aligned}$$

hence $\bar\partial u = \omega$ in the sense of distributions. The proof is complete. □

Remark 3.6. As in remark 3.4 if Ω is Stein for $q = n$ instead of asking $\omega \perp \mathcal{H}_p(\Omega_2)$ we need just $\omega \perp \mathcal{H}_p(\Omega)$.

Remark 3.7. The condition of orthogonality to $\mathcal{H}_p(V)$ in the case $q = n$ is necessary: suppose there is a $(p, n-1)$ current u such that $\bar\partial u = \omega$ and u with compact support in an open set $V \subset \Omega$, then if $h \in \mathcal{H}_p(V)$, we have

$$h \in \mathcal{H}_p(V),\ \ll \omega, h \gg = \ll \bar\partial u, h \gg = (-1)^{n+p} \ll u, \bar\partial h \gg = 0,$$

because, u being compactly supported, there is no boundary term and

$$\ll \bar\partial u, h \gg = (-1)^{n+p} \ll u, \bar\partial h \gg .$$

This kind of condition was already seen for extension of CR functions, see [1] and the references therein.

3.3. Finer control of the support

Here we shall get a better control on the support of a solution.

Theorem 3.8. *Let Ω be a weakly r' regular domain in a Stein manifold X.*

Suppose the (p, q) form ω is in $L^{r,c}(\Omega, dm)$, $\bar\partial\omega = 0$, if $q < n$, and $\omega \perp \mathcal{H}_p(V)$ for any V such that Supp $\omega \subset V$, *if $q = n$, with* Supp $\omega \subset \Omega\backslash C$, *where C is a weakly r regular domain.*

For any open relatively compact set U in C, there is a $u \in L^{r,c}(\Omega, dm)$ such that $\bar\partial u = \omega$ and with support in $\Omega\backslash\bar U$, provided that $q \geq 2$.

Proof. Let ω be a (p, q) form with compact support in $\Omega\backslash C$ then there is a $v \in L^r_{p,q-1}(\Omega)$, $\bar\partial v = \omega$, with compact support in Ω, by theorem 3.5 or, if Ω is a polydisc in $\mathbb{C}^n$ and if $\omega \in \mathcal{W}^r_q(\Omega)$, by the theorem in [5].

Because ω has compact support outside C we have $\omega = 0$ in C; this means that $\bar\partial v = 0$ in C. Because C is weakly r regular and $q \geq 2$, we have

$$\exists C' \subset C,\ C' \supset \bar U,\ \exists h \in L^r_{p,q-2}(C') \text{ s.t. } \bar\partial h = v \text{ in } C'.$$

Let χ be a smooth function such that $\chi = 1$ in U and $\chi = 0$ near $\partial C'$; then set $u := v - \bar\partial(\chi h)$. We have that $u = v - \chi\bar\partial h - \bar\partial\chi \wedge h = v - \chi v - \bar\partial\chi \wedge h$ hence u is in $L^r(\Omega)$; moreover $u = 0$ in $\bar U$ because $\chi = 1$ in U hence $\bar\partial\chi = 0$ there. Finally $\bar\partial u = \bar\partial v - \bar\partial^2(\chi h) = \omega$ and we are done. ☐

If Ω and C are, for instance, pseudo-convex in $\mathbb{C}^n$ then $\Omega\backslash C$ is no longer pseudo-convex in general, so this theorem improves actually the control of the support.

Remark 3.9. The correcting function h is given by kernels in the case of Stein domains, hence it is linear; if the primitive solution v is also linear in ω, then the solution u is linear too. This is the case in $\mathbb{C}^n$ with the solution given in [5].

This theorem cannot be true for $q = 1$ as shown by the following example: take a holomorphic function φ in the open unit ball $B(0, 1)$ in $\mathbb{C}^n$ such that it extends to no open ball of center 0 and radius > 1. For instance $\varphi(z) := \exp(-\frac{z_1+1}{z_1-1})$. Take $R < 1$, then φ is $\mathcal{C}^\infty(\bar{B}(0, R))$ hence by a theorem of Whitney φ extends $\mathcal{C}^\infty$ to $\mathbb{C}^n$; call φ_R this extension. Let $\chi \in \mathcal{C}_c^\infty(B(0, 2))$ such that $\chi = 1$ in the ball $B(0, 3/2)$ and consider the $(0, 1)$ form $\omega := \bar\partial(\chi\varphi_R)$. Then $\operatorname{Supp}\omega \subset B(0,2)\backslash B(0, R)$, ω is $\bar\partial$ closed and is $\mathcal{C}^\infty$ hence in $L^r_{0,1}(B(0, 2))$. Moreover $B(0, R)$ is strictly pseudo-convex hence r' regular, but there is no function u such that $\bar\partial u = \omega$ and u zero near the origin because any solution u will be C.R. on $\partial B(0, R)$ and by Hartog's phenomenon will extends holomorphically to $B(0, R)$, hence cannot be identically null near 0.

Nevertheless in the case $q = 1$, we have:

Theorem 3.10. *Let Ω be a weakly r' regular domain in a Stein manifold X. Then for any $(p, 1)$ form ω in $L^{r,c}(\Omega)$, $\bar\partial\omega = 0$, with support in $\Omega_1\backslash C$ where Ω_1 is a weak r' regular domain in Ω and C is a domain such that $C \subset \Omega$ and $C\backslash\Omega_1 \neq \emptyset$; there is a $u \in L^{r,c}(\Omega)$ such that $\bar\partial u = \omega$ and with support in $\Omega\backslash C$.*

Proof. There is $u \in L^r_{p,0}(\Omega_1)$ such that $\bar\partial u = \omega$ with compact support in Ω_1, by theorem 3.5. Then $\bar\partial u = 0$ in C hence u is locally holomorphic in C. Because $C\backslash\Omega_1 \neq \emptyset$, there is an open set in $C\backslash\Omega_1 \subset \Omega\backslash\Omega_1$ where u is 0 and holomorphic, hence u is identically 0 in C, C being connected. □

Remark 3.11. If there is a $u \in L^{r,c}_{p,0}(\Omega_1)$ which is 0 in C, we have

$$\forall h \in L^{r'}_{n-p,n-1}(C) :: \operatorname{Supp}\bar\partial h \subset C,\ 0 = \ll u, \bar\partial h \gg = \ll \omega, h \gg,$$

hence the necessary condition:

$$\forall h \in L^{r'}_{n-p,n-1}(C) :: \operatorname{Supp}\bar\partial h \subset C,\ \ll \omega, h \gg = 0.$$

We proved in [5]:

Theorem 3.12. *Let $f \in \mathcal{O}(\bar{\mathbb{D}}^n)$ be a holomorphic function in a neighborhood of the closed unit polydisc in $\mathbb{C}^n$ and set $Z := f^{-1}(0)$. Then for any $(0, q)$ form ω in $L^r(\mathbb{D}^n\backslash Z) \cap \mathcal{W}^r_q(\Omega)$, $\bar\partial\omega = 0$, with compact support in $\mathbb{D}^n\backslash Z$, for any $k \in \mathbb{N}$, we can find a $(0, q-1)$-form $\beta \in L^{r,c}(\mathbb{D}^n)$ such that $\bar\partial(f^k\beta) = \omega$. Equivalently we can find a $(0, q-1)$-form $\eta = f^k\beta$ such that $\eta \in L^{r,c}(\mathbb{D}^n)$, η is 0 on Z up to order k and $\bar\partial\eta = \omega$.*

And by Remark 6.3 of this paper, the solutions are given by a *bounded linear operator.*

The following corollary will generalise strongly this result but at the price that *we have not the linearity, nor even the constructivity of the solution*.

Corollary 3.13. *Let Ω be a Stein manifold. Let f be a holomorphic function in Ω and set $Z := f^{-1}(0)$. Then for any (p,q) form ω in $L^{r,c}(\Omega\backslash Z)$, $\bar{\partial}\omega = 0$, if $1 \leq q < n$, and $\omega \perp \mathcal{H}_p(\Omega\backslash Z)$ if $q = n$, there is a $(p, q-1)$ form $u \in L^r(\Omega\backslash Z)$ such that $\bar{\partial}u = \omega$ and u has its support still in $\Omega\backslash Z$.*

Proof. We first show that $\Omega\backslash Z$ is Stein. Because $f \neq 0$ in $\Omega\backslash Z$ we have that $\varphi := \frac{1}{|f|^2}$ is plurisubharmonic in $\Omega\backslash Z$ and $\mathcal{C}^\infty(\Omega\backslash Z)$. Because Ω is Stein we have, by Theorem 5.1.6 of Hörmander [10], a strictly plurisubharmonic exhausting function ρ in $\mathcal{C}^\infty(\Omega)$. Now the function $\gamma := \varphi + \rho$ is still strictly plurisubharmonic and $\mathcal{C}^\infty$ in $\Omega\backslash Z$. Now we shall prove:

$$\forall\alpha \in \mathbb{R},\ K_\alpha := \{z \in \Omega\backslash Z :: \gamma(z) < \alpha\} \text{ is relatively compact in } \Omega\backslash Z.$$

We have $\rho(z) < \alpha - \varphi(z) < \alpha$ on K_α because $\varphi(z) \geq 0$, hence, because ρ is exhaustive in Ω, we have that K_α is contained in a compact set F in Ω. So on F, hence on K_α, we have that $\rho(z) \geq A > -\infty$ because ρ is continuous.

We also have $\varphi(z) < \alpha - \rho(z)$ on K_α *i.e.* $|f(z)|^2 > \frac{1}{\alpha - \rho(z)}$. So, on the set K_α, $\alpha > \rho(z) \geq A > -\infty$, hence $|f(z)| > \dfrac{1}{\alpha - A} > 0$ on K_α, so K_α is far away from Z, hence K_α is relatively compact in $\Omega\backslash Z$.

So we can apply [10, Theorem 5.2.10, p. 127] to get that $\Omega\backslash Z$ is a Stein manifold.

Now we are in position to apply Theorem 3.5. Let ω be a (p,q) form in $L^{r,c}(\Omega\backslash Z)$, $\bar{\partial}\omega = 0$, if $1 \leq q < n$, and $\omega \perp \mathcal{H}_p(\Omega\backslash Z)$ if $q = n$, Theorem 3.5 gives a $(p, q-1)$ form $u \in L^r(\Omega\backslash Z)$ such that $\bar{\partial}u = \omega$ and u has its compact support in $\Omega\backslash Z$. The proof is complete. □

Remark 3.14. This leaves open the question to have a linear (or a constructive) solution to this problem even in the case of the polydisc.

A. Appendix

Here we shall prove certainly known results on the duality $L^r - L^{r'}$ for (p,q)-forms in a complex manifold X. Because I was unable to find precise references for them, I prove them here.

Recall we have a pointwise scalar product and a pointwise modulus for (p,q)-forms in X:

$$(\alpha,\beta)dm := \alpha \wedge \overline{*\beta};\quad |\alpha|^2\, dm := \alpha \wedge \overline{*\alpha.}$$

By the Cauchy-Schwarz inequality for scalar products we get:

$$\forall x \in X,\ |(\alpha,\beta)(x)| \leq |\alpha(x)|\,|\beta(x)|\,.$$

This gives Hölder inequalities for (p,q)-forms:

Lemma A.1. *(Hölder inequalities) Let* $\alpha \in L^r_{p,q}(\Omega)$ *and* $\beta \in L^{r'}_{p,q}(\Omega)$*. We have*

$$|\langle \alpha, \beta\rangle| \leq \|\alpha\|_{L^r(\Omega)} \|\beta\|_{L^{r'}(\Omega)}.$$

Proof. We start with $\langle \alpha, \beta\rangle = \int_\Omega (\alpha, \beta)(x) dm(x)$ hence

$$|\langle \alpha, \beta\rangle| \leq \int_\Omega |(\alpha, \beta)(x)|\, dm \leq \int_\Omega |\alpha(x)|\, |\beta(x)|\, dm(x).$$

By the usual Hölder inequalities for functions we get

$$\int_\Omega |\alpha(x)|\, |\beta(x)|\, dm(x) \leq \left(\int_\Omega |\alpha(x)|^r\, dm\right)^{1/r} \left(\int_\Omega |\beta(x)|^{r'}\, dm\right)^{1/r'},$$

which ends the proof of the lemma. □

Lemma A.2. *Let* $\alpha \in L^r_{p,q}(\Omega)$*. Then*

$$\|\alpha\|_{L^r_{p,q}(\Omega)} = \sup_{\beta \in L^{r'}_{p,q}(\Omega),\ \beta \neq 0} \frac{|\langle \alpha, \beta\rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

Proof. We choose $\beta := \alpha\, |\alpha|^{r-2}$, then:

$$|\beta|^{r'} = |\alpha|^{r'(r-1)} = |\alpha|^r \Rightarrow \|\beta\|^{r'}_{L^{r'}(\Omega)} = \|\alpha\|^r_{L^r(\Omega)}.$$

Hence

$$\langle \alpha, \beta\rangle = \left\langle \alpha, \alpha\, |\alpha|^{r-2}\right\rangle = \int_\Omega (\alpha, \alpha)\, |\alpha|^{r-2}\, dm = \|\alpha\|^r_{L^r(\Omega)}.$$

On the other hand we have

$$\|\beta\|_{L^{r'}(\Omega)} = \|\alpha\|^{r/r'}_{L^r(\Omega)} = \|\alpha\|^{r-1}_{L^r(\Omega)},$$

so

$$\|\alpha\|_{L^r(\Omega)} \times \|\beta\|_{L^{r'}(\Omega)} = \|\alpha\|^r_{L^r(\Omega)} = \langle \alpha, \beta\rangle.$$

Hence

$$\|\alpha\|_{L^r(\Omega)} = \frac{|\langle \alpha, \beta\rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

A fortiori for any choice of β:

$$\|\alpha\|_{L^r(\Omega)} \leq \sup_{\beta \in L^{r'}(\Omega)} \frac{|\langle \alpha, \beta\rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

To prove the other direction, we use the Hölder inequalities, Lemma A.1:

$$\forall \beta \in L^{r'}_{p,q}(\Omega),\ \frac{|\langle \alpha, \beta\rangle|}{\|\beta\|_{L^{r'}(\Omega)}} \leq \|\alpha\|_{L^r(\Omega)}.$$

The proof is complete. □

Now we are in a position to state:

Lemma A.3. *The dual space of the Banach space $L^r_{p,q}(\Omega)$ is $L^{r'}_{n-p,n-q}(\Omega)$.*

Proof. Suppose first that $u \in L^{r'}_{n-p,n-q}(\Omega)$. Then consider:

$$\forall \alpha \in L^r_{p,q}(\Omega),\ \mathcal{L}(\alpha) := \int_\Omega \alpha \wedge u = \langle \alpha, \overline{*u} \rangle.$$

This is a linear form on $L^r_{p,q}(\Omega)$ and its norm, by definition, is

$$\|\mathcal{L}\| = \sup_{\alpha \in L^r(\Omega)} \frac{|\langle \alpha, \overline{*u} \rangle|}{\|\alpha\|_{L^r(\Omega)}}.$$

By use of Lemma A.2 we get

$$\|\mathcal{L}\| = \|\overline{*u}\|_{L^{r'}_{p,q}(\Omega)} = \|u\|_{L^{r'}_{n-p,n-q}(\Omega)}.$$

So we have $\left(L^r_{p,q}(\Omega)\right)^* \supset L^{r'}_{n-p,n-q}(\Omega)$ with the same norm.

Conversely take a continuous linear form $\mathcal{L}$ on $L^r_{p,q}(\Omega)$. We have, again by definition, that:

$$\|\mathcal{L}\| = \sup_{\alpha \in L^r(\Omega)} \frac{|\mathcal{L}(\alpha)|}{\|\alpha\|_{L^r(\Omega)}}.$$

Because $\mathcal{D}_{p,q}(\Omega) \subset L^r_{p,q}(\Omega)$, $\mathcal{L}$ is a continuous linear form on $\mathcal{D}_{p,q}(\Omega)$, hence, by definition, $\mathcal{L}$ can be represented by a $(n-p, n-q)$-current u. So we have:

$$\forall \alpha \in \mathcal{D}_{p,q}(\Omega),\ \mathcal{L}(\alpha) := \int_\Omega \alpha \wedge u = \langle \alpha, \overline{*u} \rangle.$$

Moreover we have, by Lemma A.2,

$$\|\mathcal{L}\| = \sup_{\alpha \in \mathcal{D}_{p,q}(\Omega)} \frac{|\langle \alpha, *\bar{u} \rangle|}{\|\alpha\|_{L^r(\Omega)}} = \|*u\|_{L^{r'}(\Omega)}$$

because $\mathcal{D}_{p,q}(\Omega)$ is dense in $L^r_{p,q}(\Omega)$. So we proved $\left(L^r_{p,q}(\Omega)\right)^* \subset L^{r'}_{n-p,n-q}(\Omega)$ with the same norm. The proof is complete. □

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