Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXI (2020), 1063-1085

An equivalence principle between polynomial and simultaneous Diophantine approximation

JOHANNES SCHLEISCHITZ

Abstract. Mahler partitioned the real numbers into S, T and U-numbers subject to the growth of the sequence of Diophantine exponents $(w_n(\zeta))_{n\geq 1}$ associated to a real number ζ. Koksma introduced a similar classification that turned out to be equivalent. We add two more equivalent definitions in terms of classical exponents of Diophantine approximation. One concerns certain natural assumptions on the decay of the sequence $(\lambda_n(\zeta))_{n\geq 1}$ related to simultaneous rational approximation to $(\zeta, \zeta^2, \dots, \zeta^n)$. Thereby we obtain a much clearer picture on simultaneous approximation to successive powers of a real number in general. The other variant of Mahler's classification deals with uniform approximation by algebraic numbers. We further provide various other applications of our underlying method to exponents of Diophantine approximation and metric theory.

Mathematics Subject Classification (2010): 11J13 (primary); 11J82, 11J83 (secondary).

1. Classical exponents of Diophantine approximation

In 1932, Mahler [19] introduced his famous partition of transcendental real numbers into S, T and U-numbers. His classification relies on the growth of the sequence of Diophantine exponents $(w_n(\zeta))_{n\geq 1}$ for a given real number ζ. Hereby the exponent $w_n(\zeta)$ is defined for $n \geq 1$ as the supremum of real numbers w such that the inequality

$$0 < |P(\zeta)| \leq H(P)^{-w}, \tag{1.1}$$

has infinitely many solutions $P \in \mathbb{Z}[T]$ of degree at most n. Here $H(P)$ denotes the naive height of P, *i.e.*, the maximum modulus of the coefficients of P. For fixed ζ, these exponents obviously form a non-decreasing sequence

$$w_1(\zeta) \leq w_2(\zeta) \leq \cdots, \tag{1.2}$$

Received February 21, 2019; accepted in revised form May 07, 2019.
Published online December 2020.

In case of ζ a real algebraic number of degree $d \geq 2$, it is known that $w_n(\zeta) = \min\{d-1, n\}$, in particular the sequence is bounded, see [8]. Otherwise, for transcendental real ζ Dirichlet's Theorem implies

$$w_n(\zeta) \geq n, \qquad n \geq 1. \tag{1.3}$$

Now according to Mahler a transcendental real number ζ is a U-number if $w_n(\zeta) = \infty$ for some $n \geq 1$. If m is the smallest index with this property then ζ is called a U_m-number, thus the set of U-numbers is the disjoint union of the sets of U_m-numbers over $m \geq 1$. Further, a number ζ is called a T-number if $w_n(\zeta) < \infty$ for all $n \geq 1$, but $\limsup_{n\to\infty} w_n(\zeta)/n = \infty$ holds. Finally, the remaining real numbers for which $\limsup_{n\to\infty} w_n(\zeta)/n < \infty$ are called S-numbers. A famous result of Sprindžuk [37] states that almost all real numbers in the sense of Lebesgue measure satisfy $w_n(\zeta) = n$ for all $n \geq 1$, in particular almost all numbers are S-numbers in Mahler's classification. Refinements by of Baker, Schmidt [5] and Bernik [7] imply that the sets of U-numbers and T-numbers both have Hausdorff dimension 0. However, they are well-known to be non-empty, see LeVeque [18] and Schmidt [32]. To determine exponents $w_n(\zeta)$ for a given ζ is a typically quite challenging, and conversely to find a number ζ with any suitable given sequence of exponents $(w_n(\zeta))_{n\geq 1}$ remains a very difficult open problem. In this context, we recall the partial assertion of the *Main problem* in [8, Section 3.4, page 61] on the *joint spectrum of* $(w_n(\zeta))_{n\geq 1}$.

Problem 1.1. Let $(w_n)_{n\geq 1}$ be a non-decreasing sequence of real numbers with $w_n \geq n$. Does there exist ζ such that $w_n(\zeta) = w_n$ simultaneously for all $n \geq 1$?

Although a positive answer is strongly expected, only special cases have been verified, see [2,3,5,8,9]. Koksma [16] introduced a similar classification of transcendental real numbers into U^*, T^* and S^*-numbers, based on approximation to real numbers by algebraic numbers of degree at most n. Let $w_n^*(\zeta)$ be the supremum of numbers w^* such that

$$0 < |\zeta - \alpha| \leq H(\alpha)^{-w^*-1}$$

has infinitely many solutions in algebraic numbers α of degree at most n, where $H(\alpha) = H(P_\alpha)$ denotes the height of the irreducible minimum polynomial P_α of α over $\mathbb{Z}[T]$. The exponents $w_n^*(\zeta)$ and $w_n(\zeta)$ are closely connected. It turned out later that indeed Mahler's and Koksma's partitions are identical, that is the sets of U, T and S-numbers equal the respective sets of U^*, T^* and S^*-numbers. This is an immediate consequence of the estimates

$$w_n^*(\zeta) \leq w_n(\zeta) \leq w_n^*(\zeta) + n - 1, \qquad n \geq 1, \tag{1.4}$$

see [8, Lemma A8]. However, we should point out that there are some remarkable differences between the respective exponents. It remains a very open problem formulated by Wirsing [39] to decide whether $w_n^*(\zeta) \geq n$ holds for any $n \geq 1$ and any transcendental real number ζ, motivated by the analogous fundamental property

(1.3) for the exponents $w_n(\zeta)$. We refer to [8] for more results on the classifications of Mahler and Koksma.

In this paper we establish two further equivalent formulations of Mahler's and Koksma's classifications of real numbers in terms of other well-studied exponents of Diophantine approximation. First we consider the exponents of simultaneous rational approximation introduced by Bugeaud and Laurent [10]. They are denoted $\lambda_n(\zeta)$ and defined as the supremum of real numbers λ such that the inequality

$$\max_{1\le i\le n} |\zeta^i x - y_i| \le x^{-\lambda}$$

has infinitely many solutions in integer vectors $(x, y_1, \dots, y_n)$. For fixed irrational real ζ, they form a non-increasing sequence

$$\lambda_1(\zeta) \ge \lambda_2(\zeta) \ge \cdots, \tag{1.5}$$

and Dirichlet's Theorem yields

$$\lambda_n(\zeta) \ge \frac{1}{n}, \qquad n \ge 1. \tag{1.6}$$

Again for ζ an algebraic numbers of degree $d \ge 2$ the sequence is ultimately constant, more precisely $\lambda_n(\zeta) = \max\{1/(d-1), 1/n\}$. Our first equivalence principle reads as follows.

Theorem 1.2 (Equivalence principle I). *Let ζ be a transcendental real number. Then ζ is a U-number if and only if*

$$\lim_{n\to\infty} \lambda_n(\zeta) > 0. \tag{1.7}$$

More precisely, if ζ is a U_m-number, then $\lambda_n(\zeta) = \frac{1}{m-1}$ for all sufficiently large $n \ge n_0(\zeta)$. Moreover, ζ is a T-number if and only if

$$\lim_{n\to\infty} \lambda_n(\zeta) = 0, \qquad \limsup_{n\to\infty} n\lambda_n(\zeta) = \infty.$$

Finally ζ is an S-number if and only if

$$\limsup_{n\to\infty} n\lambda_n(\zeta) < \infty.$$

The theorem shows that Mahler's classification can be equivalently obtained by natural assumptions on the decay of the sequence $(\lambda_n(\zeta))_{n\ge 1}$. We believe that any $n_0(\zeta) \ge m-1$ can appear as minimum value in the claim for U_m-numbers, which would in particular follow from a positive answer to Problem 1.1. For $m = 2$ this is true, see Theorem 4.5 below. The method in [2] and [26, Corollary 1.9] imply that there is no upper bound uniform in ζ for the value $n_0(\zeta)$. Conversely, for any $m \ge 2$ and the U_m-numbers ζ constructed in [3] with the property $w_{m-1}(\zeta) \le 2m-2$ (see also [33]) it follows from Corollary 3.2 below that we may take $n_0(\zeta) = 4m - 5$ as a suitable value. We remark that Khintchine's famous transference principle [15]

$$\frac{w_n(\zeta)}{(n-1)w_n(\zeta)+n} \leq \lambda_n(\zeta) \leq \frac{w_n(\zeta)-n+1}{n} \tag{1.8}$$

would only admit the conclusion $\lim_{n\to\infty} \lambda_n(\zeta) = 0$ upon $\liminf_{n\to\infty} w_n(\zeta)/n = 1$, a reasonably stronger assumption than ζ not being a U-number as in Theorem 1.2. See also Remark 3.4 below for implications from previously known results. As a first corollary we determine all limits of the sequences $(\lambda_n(\zeta))_{n\geq 1}$.

Corollary 1.3. *The set $\mathcal{S}$ of all values $\lim_{n\to\infty} \lambda_n(\zeta)$ as ζ attains any transcendental real number is precisely the countable set $\mathcal{S} = \{0, \infty\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.*

Proof. For any S-number and T-number the limit is 0 by Theorem 1.2. For a U_m-number the limit is $1/(m-1)$ again by Theorem 1.2. The claim follows. □

Remark 1.4. Previous results from [9, 26] could have settled $\{0, 1, \infty\} \subseteq \mathcal{S} \subseteq [0, 1] \cup \{\infty\}$. Indeed, the inclusion $\{0, 1, \infty\} \subseteq \mathcal{S}$ follows from Sprindžuk [37] and Bugeaud [9, Theorem 4, Corollary 2], whereas [26, Corollary 1.9] implies $(1, \infty) \cap \mathcal{S} = \emptyset$.

In our second main result, we connect Mahler's classification with exponents of uniform approximation to a real number by algebraic numbers of degree bounded by n. We need to define the uniform versions of the previously introduced exponents $w_n^*(\zeta)$. As in [10], let $\widehat{w}_n^*(\zeta)$ be the supremum of real numbers w^* such that the system

$$H(\alpha) \leq X, \qquad 0 < |\zeta - \alpha| \leq H(\alpha)^{-1} X^{-w^*} \tag{1.9}$$

has a real algebraic solution α of degree at most n, for all large X. Davenport and Schmidt [14] showed that for any transcendental real ζ the exponent is effectively bounded by

$$\widehat{w}_n^*(\zeta) \leq 2n - 1. \tag{1.10}$$

For $n = 2$, the smaller bound $(3 + \sqrt{5})/2$ established in [14] was later shown to be optimal [21] and for $n > 2$ recently slight improvements of (1.10) were made in [12] and [30]. Furthermore we want to state the inequalities

$$\widehat{w}_n^*(\zeta) \geq \frac{1}{\lambda_n(\zeta)}, \qquad w_n^*(\zeta) \geq \frac{1}{\widehat{\lambda}_n(\zeta)}, \tag{1.11}$$

that link approximation by algebraic numbers with simultaneous approximation established in [14] and [25]. We show that Mahler's and Koksma's classifications are obtained as well by imposing natural assumptions on the sequence of uniform exponents $\widehat{w}_n^*(\zeta)$.

Theorem 1.5 (Equivalence principle II). *Let ζ be a transcendental real number. Then ζ is a U-number if and only if*

$$\lim_{n\to\infty} \widehat{w}_n^*(\zeta) < \infty. \tag{1.12}$$

More precisely, if ζ is a U_m-number, then $\widehat{w}_n^*(\zeta) \in [m-1, m]$ for all sufficiently large n. Moreover, ζ is a T-number if and only if

$$\lim_{n\to\infty} \widehat{w}_n^*(\zeta) = \infty, \qquad \liminf_{n\to\infty} \frac{\widehat{w}_n^*(\zeta)}{n} = 0. \tag{1.13}$$

Finally ζ is an S-number if and only if $\liminf_{n\to\infty} \widehat{w}_n^*(\zeta)/n > 0$.

Remark 1.6. Several variants of equivalence principle II can be derived similarly. For example one can fix the degree of the algebraic numbers in (1.9) equal to n, or restrict to algebraic integers or algebraic units within the definition of the exponent $\widehat{w}_n^*(\zeta)$. See for example [13,14], or [29]. The previously known results

$$\widehat{w}_n^*(\zeta) \geq \frac{w_n(\zeta)}{w_n(\zeta) - n + 1}, \qquad w_n^*(\zeta) \geq \frac{\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta) - n + 1} \tag{1.14}$$

from [11] would imply $\lim_{n\to\infty} \widehat{w}_n^*(\zeta) = \infty$ only upon the considerably stronger condition $\liminf_{n\to\infty} w_n(\zeta)/n = 1$. Similarly, from (1.14) a uniform lower bound for the quantities $\widehat{w}_n^*(\zeta)/n$ would require a uniform upper bound on $w_n(\zeta) - n$ instead of for $w_n(\zeta)/n$.

ACKNOWLEDGEMENTS. The author warmly thanks the referee for the careful reading and suggestions to improve the exposition.

2. Effective versions of the equivalence principles

We can provide effective relations between the sequences $(w_n(\zeta))_{\geq 1}$ and $(\lambda_n(\zeta))_{n\geq 1}$. We recall the notion of the order $\tau(\zeta)$ of a T-number [8], defined as

$$\tau(\zeta) = \limsup_{n\to\infty} \frac{\log w_n(\zeta)}{\log n}.$$

We have $\tau(\zeta) \in [1, \infty]$ for any T-number ζ by (1.3). All T-numbers that have been constructed so far have order $\tau(\zeta) \geq 3$, and R. Baker [6] conversely constructed T-numbers of the given degree $\tau(\zeta) \in [3, \infty]$. See also [8, Theorem 7.2], however there seems to be a problem in the proof as in (7.28) a stronger estimate than the assumption (7.24) is used. A positive answer to Problem 1.1 would clearly imply the existence of T-numbers of any order $\tau(\zeta) \in [1,\infty]$. We propose a somehow dual order $\sigma(\zeta)$, defined as

$$\sigma(\zeta) = -\limsup_{n\to\infty} \frac{\log \lambda_n(\zeta)}{\log n}.$$

It follows from (1.5) and (1.6) that $\sigma(\zeta) \in [0,1]$ for any ζ which is not a Liouville number (*i.e.*, a U_1-number). Further define

$$\overline{w}(\zeta) := \limsup_{n\to\infty} \frac{w_n(\zeta)}{n}, \qquad \overline{\lambda}(\zeta) := \limsup_{n\to\infty} n\lambda_n(\zeta), \tag{2.1}$$

and

$$\underline{w}(\zeta) := \liminf_{n\to\infty} \frac{w_n(\zeta)}{n}, \qquad \underline{\lambda}(\zeta) := \liminf_{n\to\infty} n\lambda_n(\zeta). \tag{2.2}$$

The set of S-numbers equals the set of numbers with $\overline{w}(\zeta) < \infty$. For S-numbers and T-numbers of order $\tau(\zeta) = 1$, the quantities $\overline{w}(\zeta), \underline{w}(\zeta)$ provide a refined measure. Similarly $\overline{\lambda}(\zeta), \underline{\lambda}(\zeta)$ refine $\sigma(\zeta)$. We obtain connections between the quantities as follows.

Theorem 2.1. *For any real transcendental ζ we have*

$$\frac{(\overline{w}(\zeta)+1)^2}{4\overline{w}(\zeta)} \le \overline{\lambda}(\zeta) \le \overline{w}(\zeta)+2, \qquad \frac{(\underline{w}(\zeta)+1)^2}{4\underline{w}(\zeta)} \le \underline{\lambda}(\zeta) \le \underline{w}(\zeta)+2, \tag{2.3}$$

and moreover

$$\sigma(\zeta) = \frac{1}{\tau(\zeta)}. \tag{2.4}$$

In the theorem and generally in the sequel we always agree on $1/\infty = 0$ and $1/0 = +\infty$. There is no reason to believe that the bounds in (2.3) are optimal. It is tempting to conjecture that $\overline{w}(\zeta) = \overline{\lambda}(\zeta)$ and $\underline{w}(\zeta) = \underline{\lambda}(\zeta)$ hold for any transcendental real ζ.

Now we provide effective versions of the second equivalence principle. Define the quantities

$$\underline{w}^*(\zeta) = \liminf_{n\to\infty} \frac{w_n^*(\zeta)}{n}, \qquad \overline{w}^*(\zeta) = \limsup_{n\to\infty} \frac{w_n^*(\zeta)}{n},$$

$$\underline{\widehat{w}}^*(\zeta) = \liminf_{n\to\infty} \frac{\widehat{w}_n^*(\zeta)}{n}, \qquad \overline{\widehat{w}}^*(\zeta) = \limsup_{n\to\infty} \frac{\widehat{w}_n^*(\zeta)}{n},$$

and further let

$$\theta(\zeta) = \liminf_{n\to\infty} \frac{\log \widehat{w}_n^*(\zeta)}{\log n}.$$

By (1.10) we have $0 \le \underline{\widehat{w}}^*(\zeta) \le \overline{\widehat{w}}^*(\zeta) \le 2$ and $\theta(\zeta) \in [0,1]$. An effective version of the second equivalence principle reads as follows.

Theorem 2.2. *Let ζ be any transcendental real number. We have*

$$\frac{1}{\overline{w}(\zeta)+2} \leq \underline{\widehat{w}}^{*}(\zeta) \leq \min\left\{\underline{w}(\zeta), \frac{4}{\overline{w}(\zeta)}\right\}, \tag{2.5}$$

and

$$\frac{1}{\underline{w}(\zeta)+2} \leq \overline{\widehat{w}}^{*}(\zeta) \leq \min\left\{\overline{w}(\zeta), \frac{4}{\underline{w}(\zeta)}\right\}. \tag{2.6}$$

Moreover

$$\theta(\zeta) = \frac{1}{\tau(\zeta)} = \sigma(\zeta). \tag{2.7}$$

It turns out that for large values of $\overline{w}(\zeta)$ and $\underline{w}(\zeta)$, the respective lower and upper bound differ roughly by the same factor 4 as in Theorem 2.1, which is surprising as the proofs are unrelated. Note that Wirsing's [39] estimate $w_n^*(\zeta) \geq (w_n(\zeta)+1)/2 \geq (n+1)/2$ and (1.4) imply

$$\frac{1}{2} \leq \max\left\{\frac{\overline{w}(\zeta)}{2}, \overline{w}(\zeta)-1\right\} \leq \overline{w}^{*}(\zeta) \leq \overline{w}(\zeta),$$

and the analogous relation holds between $\underline{w}(\zeta)$, $\underline{w}(\zeta)^*$. Thus $\tau(\zeta)$ equals the order $\tau^*(\zeta)$ obtained by replacing $w_n(\zeta)$ by $w_n^*(\zeta)$. Thereby we obtain a variant of (2.7) in terms of quantities derived from $w_n^*(\zeta)$ and $\widehat{w}_n^*(\zeta)$ only. Similar to Corollary 1.3, we can ask for the set $\mathscr{W}$ of limits of the sequences $(\widehat{w}_n^*(\zeta))_{n\geq 1}$ as ζ attains every real number. We conjecture that $\mathscr{W} = \{\infty\} \cup \{1, 2, 3, \ldots\}$. However, Theorem 1.5 only admits the inclusion $\mathscr{W} \supseteq \{1, \infty\}$, and conversely we cannot even exclude $\mathscr{W} = [1, \infty]$.

2.1. Comments on related exponents

We recapitulate that we derived four equivalent definitions of the Mahler classification in terms of the sequences $(w_n(\zeta))_{n\geq 1}$, $(\lambda_n(\zeta))_{n\geq 1}$, $(w_n^*(\zeta))_{n\geq 1}$ and $(\widehat{w}_n^*(\zeta))_{n\geq 1}$ respectively. Two additional classical exponents of Diophantine approximation $\widehat{w}_n(\zeta)$ and $\widehat{\lambda}_n(\zeta)$ are have been studied. The uniform exponent $\widehat{w}_n(\zeta)$ will play a crucial role in the proofs below. It is defined by Bugeaud and Laurent [10] as the supremum of $w \in \mathbb{R}$ such that the system

$$H(P) \leq X, \qquad 0 < |P(\zeta)| \leq X^{-w}, \tag{2.8}$$

has a solution $P \in \mathbb{Z}[T]$ of degree at most n for all large X. In [10] they further define the exponent $\widehat{\lambda}_n(\zeta)$ as the supremum of $\lambda \in \mathbb{R}$ such that the system

$$1 \leq x \leq X, \qquad \max_{1\leq i\leq n} |\zeta^i x - y_i| \leq X^{-\lambda}, \tag{2.9}$$

has a solution $(x, y_1, y_2, \ldots, y_n) \in \mathbb{Z}^{n+1}$ for all large values of X. In contrast, for the ordinary exponent $w_n(\zeta)$ and $\lambda_n(\zeta)$ we require (2.8) and (2.9) respectively to have solutions for certain arbitrarily large X only. For any transcendental real ζ and $n \geq 1$, similarly to (1.2) we see from these definitions that

$$\widehat{w}_1(\zeta) \leq \widehat{w}_2(\zeta) \leq \cdots, \tag{2.10}$$

and by Dirichlet's Theorem the estimate (1.3) can be refined to

$$w_n(\zeta) \geq \widehat{w}_n(\zeta) \geq n, \qquad n \geq 1. \tag{2.11}$$

Analogous claims corresponding to (1.5) and (1.6) hold for the exponents $\widehat{\lambda}_n(\zeta)$. Besides, analogous estimates to (1.4) hold for the uniform exponents $\widehat{w}_n(\zeta)$, $\widehat{w}_n^*(\zeta)$ as well, and together with (1.10) we may comprise

$$\widehat{w}_n^*(\zeta) \leq \widehat{w}_n(\zeta) \leq \min\{2n-1, \widehat{w}_n^*(\zeta)+n-1\}. \tag{2.12}$$

It is natural to ask if the sequences $(\widehat{w}_n(\zeta))_{n\geq 1}$ and $(\widehat{\lambda}_n(\zeta))_{n\geq 1}$ can be somehow included in the picture. However, almost all S-numbers satisfy $\widehat{w}_n(\zeta) = n$ and $\widehat{\lambda}_n(\zeta) = 1/n$ for all $n \geq 1$ by a famous result of Sprindžuk [37], and any Liouville number (*i.e.*, a U_1-number) shares the same property by [26, Corollary 5.2]. The (equivalent) relations $\widehat{w}_n(\zeta) > n$ and $\widehat{\lambda}_n(\zeta) > 1/n$ appear to be too restrictive to fit into the scheme of an equivalence principle. In fact no example of such a number for any $n > 2$ has yet been found.

3. Preliminary results

Theorem 1.2 will be an immediate consequence of Theorem 3.1, Theorem 3.3 and Theorem 3.5 formulated below in this section.

3.1. Upper bounds for λ_n

The upper bounds in Theorem 1.2 and Theorem 2.1 are a consequence of the following very general Theorem 3.1. We agree on $w_0(\zeta) = 0$.

Theorem 3.1. *Let $n \geq 1$ be an integer and ζ a transcendental real number. Assume $w_n(\zeta) < \infty$. Then we have*

$$\lambda_N(\zeta) \leq \max\left\{\frac{1}{\widehat{w}_n(\zeta)}, \frac{1}{\widehat{w}_{N-n+1}(\zeta) - w_n(\zeta)}\right\}, \qquad N \geq \lceil w_n(\zeta)\rceil + n - 1. \tag{3.1}$$

Moreover, in the case of $w_n(\zeta) < 2n+1$ we have

$$\lambda_N(\zeta) \leq \max\left\{\frac{1}{\widehat{w}_n(\zeta)}, \frac{1}{\widehat{w}_{N-n+1}(\zeta) - w_{N-2n}(\zeta)}\right\}, \qquad \lfloor w_n(\zeta)\rfloor + n \leq N \leq 3n. \tag{3.2}$$

We see that in case of $w_n(\zeta) < 2n+1$, for $N < 3n$ the bound (3.2) is possibly stronger than (3.1) because of the smaller index in the right expression. The case $w_n(\zeta) < n+1$ and $N = 2n$ in (3.2) will play a crucial role for improving the upper bounds for the exponents $\widehat{\lambda}_{2n}(\zeta)$ in Section 4.1. The estimate (3.1) with a suitable choice of N yields the desired implications for the equivalence principle.

Corollary 3.2. *Let $n \geq 1$ be an integer and ζ a transcendental real number and assume $w_n(\zeta) < \infty$. Then*

$$\lambda_N(\zeta) \leq \frac{1}{n}, \qquad N \geq \lceil w_n(\zeta) \rceil + 2n - 1. \tag{3.3}$$

Thus, if ζ is not a U-number then $\lim_{n\to\infty} \lambda_n(\zeta) = 0$, *and if ζ is an S-number then* $\overline{\lambda}(\zeta) = \limsup_{n\to\infty} n\lambda_n(\zeta) < \infty$, *and more precisely*

$$\overline{\lambda}(\zeta) \leq \overline{w}(\zeta) + 2, \qquad \underline{\lambda}(\zeta) \leq \underline{w}(\zeta) + 2. \tag{3.4}$$

Proof. In view of (2.11), as soon as $N \geq w_n(\zeta) + 2n - 1$ the right hand side in (3.1) can be estimated above by

$$\max\left\{\frac{1}{\widehat{w}_n(\zeta)}, \frac{1}{\widehat{w}_{N-n+1}(\zeta) - w_n(\zeta)}\right\} \leq \max\left\{\frac{1}{n}, \frac{1}{N-n+1-w_n(\zeta)}\right\} = \frac{1}{n}.$$

Hence (3.3) follows. The claim (3.4) follows by reversing the argument. Let $\epsilon > 0$ and N be large. Let $n = \lceil N/(\overline{w}(\zeta)+2+\epsilon)\rceil$ and $n' = \lceil N/(\underline{w}(\zeta)+2+\epsilon)\rceil$. Notice that both n, n' attain all large integers as N runs through the integers $\geq N_0$. The condition in (3.3) is satisfied for arbitrarily large N and the induced n, as well as for all large N and its induced n'. We obtain $\lambda_N(\zeta) \leq 1/n = (\overline{w}(\zeta)+2)/N + \varepsilon_N$ and $\lambda_N(\zeta) \leq 1/n' = (\underline{w}(\zeta)+2)/N + \varepsilon_N$, for the respective integers N, where $\varepsilon_N \ll \epsilon$. It suffices to let ϵ tend to 0. □

The bound $1/n$ in (3.3) in general cannot be improved for any N, as follows from Theorem 1.2 by taking ζ a U_{n+1}-number.

3.2. Lower bounds for λ_n

This section deals with the lower bounds in Theorem 1.2. For this we need that large values of some $w_m(\zeta)$ imply large values of certain $\lambda_n(\zeta)$. The first result treats the extremal case $w_m(\zeta) = \infty$.

Theorem 3.3. *Let $m \geq 2$ be an integer and ζ be a U_m-number. Then*

$$\lambda_n(\zeta) \geq \frac{1}{m-1}, \qquad n \geq 1. \tag{3.5}$$

Remark 3.4. Any U_m-number ζ satisfies $\widehat{w}_n^*(\zeta) \leq m$ for all $n \geq 1$, see [12, Corollary 2.5]. Combining this with (1.11) would yield $\lambda_n(\zeta) \geq 1/m$ for any U_m-number ζ and $n \geq 1$, a weaker conclusion than (3.5).

If we agree on $1/0 = \infty$ then (3.5) is true for $n = 1$ as well, which has already been observed in [9]. For $n \leq m$, the estimate (3.5) follows from Khintchine's inequality (1.8) and (1.5), however for $n > m$ the result is new. A similar method as in the proof of Theorem 3.3 will lead to the next partial claim of Theorem 1.2.

Theorem 3.5. *For any transcendental real ζ the quantities defined in* (2.1)*,* (2.2) *satisfy*

$$\frac{(\overline{w}(\zeta)+1)^2}{4\overline{w}(\zeta)} \leq \overline{\lambda}(\zeta), \qquad \frac{(\underline{w}(\zeta)+1)^2}{4\underline{w}(\zeta)} \leq \underline{\lambda}(\zeta). \tag{3.6}$$

In particular, any T-number ζ satisfies

$$\limsup_{n\to\infty} n\lambda_n(\zeta) = \infty. \tag{3.7}$$

We notice that analogous estimates can be obtained in the same manner for similarly defined uniform exponents $\widehat{\overline{w}}(\zeta), \widehat{\underline{w}}(\zeta)$ and $\widehat{\overline{\lambda}}(\zeta), \widehat{\underline{\lambda}}(\zeta)$. These quantities are effectively bounded by $1 \leq \widehat{\underline{w}}(\zeta) \leq \widehat{\overline{w}}(\zeta) \leq 2$ and $1 \leq \widehat{\underline{\lambda}}(\zeta) \leq \widehat{\overline{\lambda}}(\zeta) \leq 2$ as stems from [14], see also [12, 27]. However, it is very doubtful that any such uniform quantity can exceed 1. The sparse present results on the exponents $\widehat{w}_n, \widehat{\lambda}_n$, allow no conclusion. Our method further yields the following estimates relating different exponents of approximation.

Theorem 3.6. *For $m \geq 0, n \geq 1$ integers and any real transcendental number ζ we have*

$$\lambda_{m+n}(\zeta) \geq \frac{w_n(\zeta) - m}{(n-1)w_n(\zeta) + m + n} \tag{3.8}$$

and

$$\widehat{\lambda}_{m+n}(\zeta) \geq \frac{\widehat{w}_n(\zeta) - m}{(n-1)\widehat{w}_n(\zeta) + m + n}. \tag{3.9}$$

In fact Badziahin and Bugeaud [4] were the first to explicitly state (3.8), with a different proof. The author then discovered that this result and the uniform variant (3.9) directly follow from the method above as well.

Notice that the special choice $m = 0$ in (3.8) leads to the left transference inequality in (1.8). Currently there is no case of $m \geq 1, n, \zeta$ known where (3.9) is non-trivial.

4. Other applications to Diophantine exponents

The equivalence principles and their proofs provide much more information on exponents of Diophantine approximation. To keep the length of this section under control, we only give a brief summary of the most striking applications and refer to the arXiv online resource [31] for a more comprehensive exposition, including detailed proofs.

4.1. Upper bounds for $\widehat{\lambda}_n(\zeta)$

Recall the uniform exponents $\widehat{\lambda}_n(\zeta)$ from Section 2.1. The problem on determining upper bounds for $\widehat{\lambda}_n(\zeta)$, although not using this notation, dates back to Davenport and Schmidt [14]. In [14] the upper bound $\widehat{\lambda}_{2n+1}(\zeta) \leq \widehat{\lambda}_{2n}(\zeta) \leq \frac{1}{n}$ was provided. These estimates have been refined by Laurent [17] for odd indices, his claim can be stated $\widehat{\lambda}_{2n}(\zeta) \leq \widehat{\lambda}_{2n-1}(\zeta) \leq \frac{1}{n}$. A significantly shorter proof of this bound using Mahler's duality, and in fact a slight refinement of the bound for $\widehat{\lambda}_{2n}(\zeta)$, was recently given by the author [27, Theorem 2.3]. In this section we further improve the bound in even dimension. The new key ingredient for the improvement relies on (3.2).

Theorem 4.1. *Let $n \geq 1$ be an integer and ζ a transcendental real number. Then we have $\widehat{\lambda}_{2n}(\zeta) \leq \Theta_n$, with Θ_n the solution of the polynomial equation*

$$P_n(x) = n^{2n}x^{2n+1} - (n+1)x + 1 = 0$$

in the interval $(\frac{1}{n+1}, \frac{1}{n})$. In the case of $\lambda_{2n}(\zeta) > \frac{1}{n}$, the stronger estimate $\widehat{\lambda}_{2n}(\zeta) \leq \frac{1}{n+1}$ holds.

When $n = 1$ we obtain the sharp bound $\widehat{\lambda}_2(\zeta) \leq \Theta_1 = (1+\sqrt{5})/2 = 0.6180\ldots$ (see [21]), for $n = 2$ we obtain $\widehat{\lambda}_4(\zeta) \leq \Theta_2 = 0.3706\ldots$. This may be compared with Roy's [23] bound for $\widehat{\lambda}_3(\zeta) \leq 0.4245\ldots$. It can be shown that $\Theta_n = \frac{1}{n} - \frac{\beta}{n^2} + O(n^{-3})$, for $\beta \in (0.796, 0.797)$ the unique positive real root of the power series

$$-1 + \sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{(k+1)!}x^k = -1 + 2x - \frac{4}{3}x^2 + \frac{2}{3}x^3 - \frac{4}{15}x^4 + \frac{4}{45}x^5 - \cdots. \quad (4.1)$$

In particular $\widehat{\lambda}_{2n}(\zeta) < 1/n$. Theorem 4.1 is the only result in Section 4 we prove here.

Proof. The estimate in [27, Theorem 2.1] with $m = n+1$ yields

$$\widehat{\lambda}_{2n}(\zeta) \leq \max\left\{\frac{1}{w_n(\zeta)}, \frac{1}{\widehat{w}_{n+1}(\zeta)}\right\}.$$

In the case of $w_n(\zeta) \geq n+1$, by (2.11) we infer $\widehat{\lambda}_{2n}(\zeta) \leq (n+1)^{-1}$, which is smaller than Θ_n. In case of $w_n(\zeta) < n+1$, if we let $N = 2n$, we may apply (3.2) and as a consequence we obtain

$$\lambda_{2n}(\zeta) \leq \frac{1}{\widehat{w}_n(\zeta)} \leq \frac{1}{n}. \quad (4.2)$$

A conjecture by Schmidt and Summerer [36] proved by Marnat and Moshchevitin [20] shows that the exponent $\widehat{\lambda}_{2n}(\zeta)$ is maximized among all ζ with a given value of $\lambda_{2n}(\zeta)$ in a special case called the regular graph. Using the implicit equation established in [28, (30)] that relates $\widehat{\lambda}_{2n}, \lambda_{2n}(\zeta)$ in case of the regular graph, together with our estimate $\lambda_{2n}(\zeta) \leq n^{-1}$ from (4.2), we derive exactly the bound Θ_n in the theorem. Reversing the proof we see that $\lambda_{2n}(\zeta) > \frac{1}{n}$ implies $w_n(\zeta) \geq n+1$, and as above we infer the bound $\widehat{\lambda}_{2n}(\zeta) \leq (n+1)^{-1}$. □

Remark 4.2. A weaker upper bound of the form $\sqrt{\left(n+\frac{1}{2n}\right)^2 - \frac{1}{n}} - n + \frac{1}{2n}$ can be derived using Schmidt and Summerer [35, (1.21)] instead of [28, (30)].

There is a well-known link between uniform approximation to successive powers of ζ and approximation to ζ by algebraic numbers/integers, already observed by Davenport and Schmidt [14, Lemma 1]. Concretely, from Theorem 4.1 we immediately derive

Corollary 4.3. *Let $n \geq 1$ be an integer and ζ a transcendental real number. Then, for any $\varepsilon > 0$, there are infinitely many algebraic integers α of degree at most $2n+1$ (and algebraic numbers α of degree at most $2n$) with the property*

$$|\zeta - \alpha| < H(\alpha)^{-1/\Theta_n - 1 + \varepsilon}. \tag{4.3}$$

Since $1/\Theta_n = n + \beta + \frac{\beta^2}{n-\beta} + O(n^{-2})$ with $\beta = 0.796\ldots$ the root of (4.1) as above, the exponent in (4.3) is of order $-n-1-\beta+o(1)$ as $n \to \infty$.

We should point out that concerning approximation by algebraic numbers slightly better lower estimates are known, the estimate $w_{2n}^*(\zeta) \geq n + 3 - o(1)$ due to Tsishchanka [38] beats our bound $n+\beta+o(1)$. However our new bound concerning algebraic integers of odd degree in Corollary 4.3 is currently best known, and again sharp for $n = 1$ by [22].

4.2. Metric theory

Now we investigate the metric problem of determining the Hausdorff dimensions

$$h_n^\lambda = \dim(H_n^\lambda), \qquad H_n^\lambda := \{\zeta \in \mathbb{R} : \lambda_n(\zeta) \geq \lambda\}$$

posed in [9, Problem 2]. For $n \in \{1, 2\}$ the problem is solved. It was further shown in [26, Corollary 1.8] that

$$h_n^\lambda = \frac{2}{(1+\lambda)n}, \qquad n \geq 1, \ \lambda > 1. \tag{4.4}$$

However, for $n \geq 3$ and parameters $\lambda \in [1/n, 1]$ the problem of determining h_n^λ is open. We only highltight a special consequence of our equivalence principle, see the online resource [31] for precise estimates, their proofs and further references of other known results on the metric problem of rational approximation to Veronese curves.

Theorem 4.4. *Let $\lambda > 0$. There exist positive constants $c_1(\lambda)$, $c_2(\lambda)$ such that*

$$\frac{c_1(\lambda)}{n} \leq h_n^{\lambda} \leq \frac{c_2(\lambda)}{n}, \qquad n \geq 1.$$

The lower bound with $c_1(\lambda) = 2/(1+\lambda)$, in agreement with (4.4), is easily implied by an estimate in [9], the upper bound is the new substance.

4.3. On U_2-numbers

For any U_2-number ζ, we can determine the sequence $(\lambda_n(\zeta))_{n\geq 1}$ depending on $\lambda_1(\zeta)$.

Theorem 4.5. *Let ζ be a U_2-number with $w_1(\zeta) = w \in [1, \infty)$. Then*

$$\lambda_n(\zeta) = \frac{w+1-n}{n}, \qquad 1 \leq n \leq \frac{w+1}{2}, \tag{4.5}$$

$$\lambda_n(\zeta) = 1, \qquad n \geq \frac{w+1}{2}. \tag{4.6}$$

In particular if $w = 1$ then $\lambda_n(\zeta) = 1$ for all $n \geq 1$. The sequences of the form

$$\left(w, \frac{w-1}{2}, \frac{w-2}{3}, \dots, \frac{w+1-\lfloor\frac{w+1}{2}\rfloor}{\lfloor\frac{w+1}{2}\rfloor}, 1, 1, 1, \dots\right), \qquad w \geq 1, \tag{4.7}$$

coincide precisely with the sequences $(\lambda_n(\zeta))_{n\geq 1}$ induced by the set of U_2-numbers ζ. In particular they all belong to the joint spectrum of $(\lambda_n)_{n\geq 1}$. Conversely, the sequences in (4.7) *with $w \in [1, \infty]$ are precisely those sequences in the joint spectrum of $(\lambda_n)_{n\geq 1}$ with $\lambda_n(\zeta) > \frac{1}{2}$ for all $n \geq 1$.*

The claims vastly generalize [9, Theorem 4 and Theorem 5]. They are obtained by a combination of the equivalence principle and results from [26], incorporating also the existence of U_2-numbers with any prescribed value $w_1(\zeta)$ as as carried out in [8] (see also [8]). See [31] for previous results and further references.

4.4. An estimate involving various exponents

With the aid of our equivalence principle we can infer a relation between the exponents $\widehat{w}_n(\zeta)$, $\widehat{w}_n^*(\zeta)$ and the exponent $w_{n+1}(\zeta)$.

Theorem 4.6. *Let ζ be a transcendental real number and assume*

$$\widehat{w}_n^*(\zeta) > n, \qquad \text{for some } n \geq 2. \tag{4.8}$$

Then we have

$$w_{n+1}(\zeta) \leq \frac{\widehat{w}_n^*(\zeta)^3 - \widehat{w}_n^*(\zeta)}{(\widehat{w}_n^*(\zeta) - n)^2} - 1.$$

Now assume ζ satisfies $\widehat{w}_2(\zeta) > 2$. Then we have

$$w_3(\zeta) \leq \frac{\widehat{w}_2(\zeta)^3 - \widehat{w}_2(\zeta)^2 + 3\widehat{w}_2(\zeta) - 4}{(\widehat{w}_2(\zeta) - 2)^2} \leq \frac{d}{(\widehat{w}_2(\zeta) - 2)^2},$$

where we may choose $d = 14.9444$.

Notice that (4.8) is (potentially) a slightly stronger assumption than $\widehat{w}_n(\zeta) > n$ (for some n). Upon some mild additional assumption we may infer the first claim with $\widehat{w}_n^*(\zeta)$ altered to $\widehat{w}_n(\zeta)$ consisitenly. Again we refer to [31] for details and all proofs. See also Adamczewski, Bugeaud [1] and the recent paper by Roy [24] for upper bounds on the sequence of exponents $w_m(\zeta)$ for ζ as in Theorem 4.6 and numbers with similar properties. Our bound for $m = n + 1$ in Theorem 4.6 is stronger, however the method does not extend to larger m. Again Theorem 3.1 can be readily applied to infer bounds on the exponents $\lambda_m(\zeta)$ in these instances, we refer the reader to [31, Section 4.3] for details.

5. Proofs

5.1. Deduction of the equivalence principles

First we deduce Theorem 1.2 from the partial results in Section 3.

Proof of Theorem 1.2. Theorem 3.3 shows that any U_m-number satisfies

$$\lim_{n\to\infty} \lambda_n(\zeta) \geq \frac{1}{m-1} > 0.$$

On the other hand in Corollary 3.2 we noticed that otherwise if ζ is not a U-number, then $\lim_{n\to\infty} \lambda_n(\zeta) = 0$. Moreover, when ζ is a U_m-number, then $w_{m-1}(\zeta) < \infty$ and again Corollary 3.2 yields that we actually have $\lambda_n(\zeta) \leq \frac{1}{m-1}$ for large n, so by the above observation there must be equality. In Theorem 3.5 we proved that for T-numbers we have $\limsup_{n\to\infty} n\lambda_n(\zeta) = \infty$, and $\lim_{n\to\infty} \lambda_n(\zeta) = 0$ was shown above. Finally the claim for S-numbers was noticed in Corollary 3.2 as well. □

We now settle the second equivalence principle Theorem 1.5 and Theorem 2.2. Lower bounds for $\overline{\widehat{w}}^*(\zeta)$ are based on Theorem 1.2 and the relations (1.11). For upper bounds we employ a recent result from [12]. It was shown in [12, Theorem 2.4] that for m, n positive integers

$$\widehat{w}_n^*(\zeta) \leq m + (n-1)\frac{\widehat{w}_n^*(\zeta)}{w_m(\zeta)}, \tag{5.1}$$

upon the condition $w_m(\zeta) > m + n - 1$.

Proof of Theorem 1.5. By (1.11) and Theorem 1.2, for (1.12) to hold, ζ must be a U-number. The refined result on U_m-numbers in Theorem 1.2 moreover implies that for ζ a U_m-number we have $\widehat{w}_n^*(\zeta) \geq m-1$ for large n. On the other hand, it was shown in [12, Corollary 2.5] that for any U_m-number we have $\widehat{w}_n^*(\zeta) \leq m$ for all $n \geq 1$. The above implies the left property of (1.13) for S and T-numbers. We next prove the right claim in (1.13) for T-numbers. For a T-number ζ and every integer N we have $w_m(\zeta) \geq N^2 m$ for some m. If we choose $n = Nm$, then the condition $w_m(\zeta) > m+n-1$ of (5.1) is satisfied when $N \geq 2$. From (2.12) and (5.1) we infer

$$\widehat{w}_{mN}^*(\zeta) \leq m + \frac{2(mN)^2}{N^2 m} \leq 3m.$$

Hence indeed $\widehat{w}_n^*(\zeta)/n = \widehat{w}_{mN}^*(\zeta)/(mN) \leq 3/N$ which tends to 0 as $N \to \infty$. Finally, for S-numbers we derive $\widehat{w}_n^*(\zeta) \geq \lambda_n(\zeta)^{-1} \geq cn$ for fixed $c > 0$ from (1.11) and Theorem 1.2, the converse follows from the above considerations. □

We remark that we can obtain the variants of Theorem 1.5 mentioned in Remark 1.6 by considering the corresponding variants of (1.11). Again relation (1.11) and a refined treatment of the argument for T-numbers leads to a proof of Theorem 2.2.

Proof of Theorem 2.2. The respective left inequalities in (2.5) and (2.6) and $\theta(\zeta) \geq \tau(\zeta)^{-1} = \sigma(\zeta)$ follow immediately from Theorem 2.1 and (1.11). Concerning the respective right inequalities in (2.5) and (2.6), the estimates $\overline{\widehat{w}}^*(\zeta) \leq \overline{\widehat{w}}(\zeta) \leq \overline{w}(\zeta)$ and $\underline{\widehat{w}}^*(\zeta) \leq \underline{\widehat{w}}(\zeta) \leq \underline{w}(\zeta)$ are an easy consequence of (2.11) and (2.12). For the remaining bounds, we refine the argument for T-numbers in the proof of Theorem 1.5. We may assume $\overline{w}(\zeta) > 2$ and $\underline{w}(\zeta) > 2$ respectively, otherwise the left bounds are smaller and the claim is obvious. So assume $\alpha > 2$ is a fixed real number and m is a large integer such that $w_m(\zeta)/m > \alpha$. If n is another integer and we define $\beta = n/m$, then in the case of $\beta \leq \alpha - 1$ the condition $w_m(\zeta) > m+n-1$ of (5.1) is satisfied. Its application and rearrangements yield

$$\frac{\widehat{w}_n^*(\zeta)}{n} \leq \frac{\alpha}{(\alpha-\beta)\beta}.$$

Let $n = \lfloor m\alpha/2 \rfloor$. Then $\beta = n/m = \alpha/2 + O(1/m)$. Since for $\alpha > 2$ we have $\alpha/2 < \alpha - 1$, the above condition $\beta \leq \alpha - 1$ is satisfied for large m. By inserting we obtain the upper bound $4/\alpha + O(1/m)$ for $\widehat{w}_n^*(\zeta)/n$. By definition we may choose α arbitrarily close to $\overline{w}(\zeta)$ for certain arbitrarily large m, and (2.6) follows. Similarly, any given large n can be written $n = \lfloor m\alpha/2 \rfloor + s$ with integers m and $0 \leq s \leq \lceil \alpha/2 \rceil$, where we may choose $\alpha = \underline{w}(\zeta) - \epsilon$ for $n \geq n_0(\epsilon)$. Since s is fixed the final estimate in (2.5) follows very similarly as well. Finally we show $\theta(\zeta) \leq \tau(\zeta)^{-1}$ to settle (2.7). Let $\epsilon > 0$ and assume $w_m(\zeta) \geq m^\gamma$ for some $\gamma > 1$. Let $n = \lceil m^{\gamma-\epsilon} \rceil$ and observe that again the condition $w_m(\zeta) > m+n-1$ is satisfied for large m. Thus by (5.1), again for large enough $m \geq m_0(\epsilon)$, we infer

$$\widehat{w}_n^*(\zeta) \leq \frac{m^{\gamma+1}}{m^{\gamma} - \lceil m^{\gamma-\epsilon} \rceil + 1} \leq 2m \leq 2n^{1/(\gamma-\epsilon)}.$$

Hence taking logarithms to base n gives $\theta(\zeta) \leq \tau(\zeta)^{-1}$ as ϵ can be chosen arbitrarily small, γ is arbitrarily close to $\tau(\zeta)$ for certain arbitrarily large m, and the induced n obviously tend to infinity. □

We place the proof of Theorem 2.1 to the end of the paper as it requires some partial results of the proof of Theorem 3.5.

5.2. Proofs of the upper bounds

Next we show Theorem 3.1. The proof is similar to [27, Theorem 2.1]. We need to define successive minima exponents that refine the classical exponents $w_n(\zeta), \widehat{w}_n(\zeta)$ and $\lambda_n(\zeta), \widehat{\lambda}(\zeta)$. For $1 \leq j \leq n+1$, let $\lambda_{n,j}(\zeta)$ and $\widehat{\lambda}_{n,j}(\zeta)$ be the supremum of λ for which (2.9) has j linearly independent integer vector solutions for arbitrarily large X and all large X, respectively. Similarly, let $w_{n,j}(\zeta)$ and $\widehat{w}_{n,j}(\zeta)$ be the supremum of w for which (2.8) has j linearly independent polynomial solutions for arbitrarily large and all large X, respectively. Obviously, for $j = 1$ we recover the corresponding classical exponents, and the relations

$$\lambda_{n,1}(\zeta) \geq \lambda_{n,2}(\zeta) \geq \cdots \geq \lambda_{n,n+1}(\zeta), \qquad \widehat{\lambda}_{n,1}(\zeta) \geq \widehat{\lambda}_{n,2}(\zeta) \geq \cdots \geq \widehat{\lambda}_{n,n+1}(\zeta),$$
$$w_{n,1}(\zeta) \geq w_{n,2}(\zeta) \geq \cdots \geq w_{n,n+1}(\zeta), \quad \widehat{w}_{n,1}(\zeta) \geq \widehat{w}_{n,2}(\zeta) \geq \cdots \geq \widehat{w}_{n,n+1}(\zeta),$$

hold. As noticed in [25], Mahler's Theorem on polar convex bodies implies

$$\lambda_{n,j}(\zeta) = \frac{1}{\widehat{w}_{n,n+2-j}(\zeta)}, \qquad \widehat{\lambda}_{n,j}(\zeta) = \frac{1}{w_{n,n+2-j}(\zeta)}. \tag{5.2}$$

Our proof for the upper bounds are based on a lower estimate for the uniform last successive minimum exponent of the dual problem $\widehat{w}_{N,N+1}(\zeta)$, for suitable N, which by (5.2) indeed tranlates into upper bounds for $\lambda_N(\zeta)$. For the proof we further recall Gelfond's Lemma, asserting that

$$H(P)H(Q) \ll_n H(PQ) \ll_n H(P)H(Q) \tag{5.3}$$

holds for any polynomials P, Q each of degree at most n. Here and elsewhere the notation $a \ll_. b$ and $a \gg_. b$ respectively mean that $a \leq Cb$ and $a \geq Cb$ for some C that depends only on the subscript parameters.

Proof of Theorem 3.1. Let n, ζ be as in the theorem and $\epsilon > 0$. By definition of $\widehat{w}_n(\zeta)$, for any large $X \geq X_0(\epsilon)$ there exists an integer polynomial $P_X = P_X(T)$ of degree at most n such that

$$H(P_X) \leq X, \qquad |P_X(\zeta)| \leq X^{-\widehat{w}_n(\zeta)+\epsilon}.$$

Now choose an integer $k \geq w_n(\zeta)$. The definition of $\widehat{w}_k(\zeta)$ similarly yields an integer polynomial $Q_X = Q_X(T)$ of degree at most k such that

$$H(Q_X) \leq X, \qquad |Q_X(\zeta)| \leq X^{-\widehat{w}_k(\zeta)+\epsilon}. \tag{5.4}$$

Write $Q_X = R_X S_X$, where $R_X = R_X(T)$ consists of the factors dividing P_X as well, and $S_X = S_X(T)$ is coprime to P_X. Let $\epsilon > 0$. We claim that, unless R_X is of small height $H(R_X) \ll_\epsilon 1$, we have

$$|R_X(\zeta)| \geq H(R_X)^{-w_n(\zeta)-\epsilon} \gg_{k,\zeta} X^{-w_n(\zeta)-\epsilon}, \tag{5.5}$$

if X was chosen sufficiently large. First notice that the corresponding estimate

$$|U_X(\zeta)| \gg H(U_X)^{-w_n(\zeta)-\epsilon}, \tag{5.6}$$

applies to any irreducible factor $U_X = U_X(T)$ of R_X. Indeed, such U_X has degree at most n as it also divides P_X, and by definition of $w_n(\zeta)$ we obtain (5.6). From (5.3) we see that this property is essentially (up to a factor depending on k only) preserved when taking arbitrary products, which indeed yields (5.5). In case of R_X of small height $H(R_X) \ll_\epsilon 1$, we can even estimate $|R_X(\zeta)| \gg_{n,\zeta} 1$ by the finiteness and since ζ is transcendental, which is stronger than (5.5) for X large enough. From (5.4) and (5.5) we deduce

$$|S_X(\zeta)| = \frac{|Q_X(\zeta)|}{|R_X(\zeta)|} \leq X^{-\widehat{w}_k(\zeta)+w_n(\zeta)+2\epsilon}.$$

Moreover, since S_X divides Q_X, Gelfond's estimate (5.3) implies $H(S_X) \ll_k H(Q_X) \leq X$. Hence we have

$$\max\{H(P_X), H(S_X)\} \ll_k X, \qquad \max\{|P_X(\zeta)|, |S_X(\zeta)|\} \leq X^{-\theta_{k,n}+2\epsilon}, \tag{5.7}$$

with

$$\theta_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_k(\zeta) - w_n(\zeta)\}.$$

Let $d_X = d \leq n$ be the degree of P_X and $e_X = e \leq k$ be the degree of S_X. Then, since P_X and Q_X are coprime, the set of polynomials

$$\mathscr{P}_X := \{P_X, TP_X, \dots, T^{e-1}P_X, S_X, TS_X, \dots, T^{d-1}S_X\}$$

is linearly independent and spans the space of polynomials of degree at most $d + e - 1 \leq k + n - 1$. In case of strict inequality $d + e - 1 < k + n - 1$ for some X, we consider

$$\mathscr{R}_X = \mathscr{P}_X \cup \{T^d S_X, T^{d+1} S_X, \dots, T^{k+n-1-e} S_X\}$$

instead of $\mathscr{P}_X$ (see also the proof of Proposition 5.1 below). Clearly $\mathscr{R}_X$ is linearly independent as well, and spans the space of polynomial of degree at most $N :=$

$k+n-1$. In any case, in view of (5.7) and since X was arbitrary and we may choose ϵ arbitrarily small, this means

$$\widehat{w}_{N,N+1}(\zeta) \geq \theta_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_{N-n+1}(\zeta) - w_n(\zeta)\}.$$

Since $\theta_{k,n} > 0$ by construction, Mahler's relation (5.2) with $j=1$ further implies $\lambda_N(\zeta) \leq 1/\theta_{k,n}$. We may choose any integer $k > w_n(\zeta)$, and the choice $k = \lceil w_n(\zeta) \rceil$ yields $N = n+k-1 = \lceil w_n(\zeta) \rceil + n - 1$. The claim (3.1) follows.

Now we prove (3.2). We now choose an integer k with strict inequality $k > w_n(\zeta)$, and again obtain (5.4) for some Q_X of degree at most k for any $X \geq X_0(\epsilon)$. We proceed as above splitting $Q_X = R_X S_X$. By a very similar argument as above, from (5.3) we derive that Q_X cannot split solely in irreducible polynomials of degree at most n. Thus it must have an irreducible factor of degree at least $n+1$, which must divide S_X. Hence $R_X = Q_X/S_X$ has degree at most $k-(n+1)$. In particular if $w_n(\zeta) < n+1$, for $k = n+1$ we infer $S_X = Q_X$ and $R_X \equiv 1$ for all large X. From the definition of $w_{k-n-1}(\zeta)$, for sufficiently large $H(R_X)$ we derive

$$|R_X(\zeta)| \geq H(R_X)^{-w_{k-n-1}(\zeta)-\epsilon} \gg_{k,\zeta} X^{-w_{k-n-1}(\zeta)-\epsilon}. \tag{5.8}$$

In case of small heights of R_X we use the argument from the proof of (3.1) again. We infer (5.7) very similarly as above with $\theta_{k,n}$ replaced by the new expression

$$\tilde{\theta}_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_k(\zeta) - w_{k-n-1}(\zeta)\}.$$

Let $N = k+n-1$ again, proceeding as above yields

$$\widehat{w}_{N,N+1}(\zeta) \geq \tilde{\theta}_{k,n} = \min\{\widehat{w}_n(\zeta), \widehat{w}_{N-n+1}(\zeta) - w_{N-2n}(\zeta)\}.$$

We may start with any integer $k > w_n(\zeta)$, or equivalently $k \geq \lfloor w_n(\zeta) \rfloor + 1$, which leads to $N \geq \lfloor w_n(\zeta) \rfloor + n$. The claim (3.2) follows from (5.2) again as soon as $\tilde{\theta}_{k,n} > 0$, which we can guarantee for $N \leq 3n$ by construction as then $w_{N-2n}(\zeta) \leq w_n(\zeta)$ whereas $\widehat{w}_{N-n+1}(\zeta) \geq N-n+1 = \lfloor w_n(\zeta) \rfloor + 1 > w_n(\zeta)$. The condition $w_n(\zeta) < 2n+1$ is only required for the set of values N in (3.2) to be non-empty. □

5.3. Parametric geometry of numbers

The proofs of Section 3.2 can be derived in a convenient, and in fact surprisingly easy way, utilizing the parametric geometry of numbers introduced by Schmidt and Summerer [34]. We recall the fundamental concepts, in a slightly modified form to fit our purposes. Let $\zeta \in \mathbb{R}$ be given and $Q > 1$ a parameter. For $n \geq 1$ and $1 \leq j \leq n+1$, define $\psi_{n,j}(Q)$ as the minimum of $\eta \in \mathbb{R}$ such that

$$|x| \leq Q^{1+\eta}, \qquad \max_{1 \leq j \leq n} |\zeta^j x - y_j| \leq Q^{-\frac{1}{n}+\eta}$$

has j linearly independent solution vectors $(x, y_1, \dots, y_n) \in \mathbb{Z}^{n+1}$. The functions $\psi_{n,j}(Q)$ can be equivalently defined via a lattice point problem, see [34]. As pointed out in [34], they have the properties

$$-1 \le \psi_{n,1}(Q) \le \psi_{n,2}(Q) \le \dots \le \psi_{n,n+1}(Q) \le \frac{1}{n}, \qquad Q > 1. \tag{5.9}$$

Let

$$\underline{\psi}_{n,j} = \liminf_{Q\to\infty} \psi_{n,j}(Q), \qquad \overline{\psi}_{n,j} = \limsup_{Q\to\infty} \psi_{n,j}(Q). \tag{5.10}$$

These values all belong to the interval $[-1, 1/n]$ by (5.9). From Dirichlet's Theorem it follows that $\psi_{n,1}(Q) \le 0$ for all $Q > 1$ and hence $\underline{\psi}_{n,1} \le 0$. Similarly, for $1 \le j \le n+1$ define the functions $\psi^*_{n,j}(Q)$ as the minimum of η such that the system

$$H(P) \le Q^{\frac{1}{n}+\eta}, \qquad |P(\zeta)| \le Q^{-1+\eta} \tag{5.11}$$

has j linearly independent integer polynomial solutions P of degree at most n. Again put

$$\underline{\psi}^*_{n,j} = \liminf_{Q\to\infty} \psi^*_{n,j}(Q), \qquad \overline{\psi}^*_{n,j} = \limsup_{Q\to\infty} \psi^*_{n,j}(Q). \tag{5.12}$$

We have

$$-\frac{1}{n} \le \psi^*_{n,1}(Q) \le \psi^*_{n,2}(Q) \le \dots \le \psi^*_{n,n+1}(Q) \le 1, \qquad Q > 1.$$

As pointed out in [34] Mahler's relations (5.2) are essentially equivalent to

$$\underline{\psi}_{n,j} = -\overline{\psi}^*_{n,n+2-j}, \qquad \overline{\psi}_{n,j} = -\underline{\psi}^*_{n,n+2-j}, \qquad 1 \le j \le n+1. \tag{5.13}$$

Schmidt and Summerer [35, (1.11)] further established the inequalities

$$j\underline{\psi}_{n,j} + (n+1-j)\overline{\psi}_{n,n+1} \ge 0, \qquad j\overline{\psi}_{n,j} + (n+1-j)\underline{\psi}_{n,n+1} \ge 0,$$

for $1 \le j \le n+1$. The dual inequalities

$$j\underline{\psi}^*_{n,j} + (n+1-j)\overline{\psi}^*_{n,n+1} \ge 0, \qquad j\overline{\psi}^*_{n,j} + (n+1-j)\underline{\psi}^*_{n,n+1} \ge 0, \tag{5.14}$$

can be obtained very similarly. Moreover, by [34, Theorem 1.4] the quantities in (5.10) and (5.12) are connected to the exponents $\lambda_{n,j}, \widehat{\lambda}_{n,j}$ and $w_{n,j}, \widehat{w}_{n,j}$ defined in Section 5.2 via the identities

$$(1+\lambda_{n,j}(\zeta))(1+\underline{\psi}_{n,j}) = (1+\widehat{\lambda}_{n,j}(\zeta))(1+\overline{\psi}_{n,j}) = \frac{n+1}{n}, \quad 1 \le j \le n+1, \tag{5.15}$$

and

$$(1+w_{n,j}(\zeta))\Big(\frac{1}{n}+\underline{\psi}^*_{n,j}\Big) = (1+\widehat{w}_{n,j}(\zeta))\Big(\frac{1}{n}+\overline{\psi}^*_{n,j}\Big) = \frac{n+1}{n}, \quad 1 \le j \le n+1. \tag{5.16}$$

In fact it was only observed for $j = 1$ in [34], but as remarked in [25] it is true as well for $2 \le j \le n+1$ for the same reason.

5.4. Proofs of the lower bounds

The following easy observation will play a crucial role in the proofs of lower bounds.

Proposition 5.1. *Let m, n be positive integers and ζ be a real transcendental number. Then*

$$w_{m+n,m+i}(\zeta) \geq w_{n,i}(\zeta), \quad \widehat{w}_{m+n,m+i}(\zeta) \geq \widehat{w}_{n,i}(\zeta), \quad 1 \leq i \leq n+1. \tag{5.17}$$

Proof. By the definition of $w_{n,i}(\zeta)$, for certain arbitrarily large X there exist linearly independent integer polynomials $P_1, \ldots, P_i$ of degree at most n with the properties

$$\max_{1\leq j\leq i} H(P_j) \leq X, \qquad \max_{1\leq j\leq i} |P_j(\zeta)| \leq H(P_j)^{-w_{n,i}(\zeta)+\epsilon}.$$

Without loss of generality assume the degree of P_1 is maximal among the P_j. For any $m \geq 1$ consider the set of polynomials

$$\mathcal{P}_{m,n,i} = \mathcal{P}_{m,n,i}(X) = \{P_1(T), TP_1(T), T^2P_1(T), \ldots, T^mP_1(T), P_2(T), \ldots, P_i(T)\}$$

It is not hard to see that $\mathcal{P}_{m,n,i}$ consists of $m+i$ polynomials of degree at most $n+m$, which are linearly independent as well, and satisfies

$$\max_{P\in\mathcal{P}_{m,n,i}} H(P) \leq X, \qquad \max_{P\in\mathcal{P}_{m,n,i}} |P(\zeta)| \leq \max\{1, |\zeta|^m\} H(P)^{-w_{n,i}(\zeta)+\epsilon}.$$

The left inequalities of (5.17) follow. The analogous right estimates are shown similarly using the definition of $\widehat{w}_{n,i}(\zeta)$ and considering any large X. □

In fact we only need the case $i = 1$. First we deduce Theorem 3.3 from the proposition.

Proof of Theorem 3.3. By (1.5) it suffices to prove (3.5) for $n \geq m$. So let $n = m+k$ with $k \geq 0$. From $w_m(\zeta) = \infty$ and Proposition 5.1 we derive $w_{n,k+1}(\zeta) = \infty$. Together with (5.16) we infer

$$\underline{\psi}^*_{n,k+1} = -\frac{1}{n}.$$

Hence (5.14) with $j = k+1$ and (5.13) yield

$$\underline{\psi}_{n,1} = -\overline{\psi}^*_{n,n+1} \leq \frac{k+1}{(n+1)-(k+1)} \underline{\psi}^*_{n,k+1} = -\frac{k+1}{(k+m)m}. \tag{5.18}$$

Inserting in (5.15) yields $\lambda_n(\zeta) \geq 1/(m-1)$ as asserted. □

Theorem 3.5 follows similarly as Theorem 3.3, with slightly more computation involved.

Proof of Theorem 3.5. Let m, n be positive integers and $C \geq 1$ a real number to be chosen later and assume we have $w_n(\zeta) \geq Cn$. Proposition 5.1 yields

$$w_{m+n,m+1}(\zeta) \geq nC.$$

With (5.16) we obtain

$$\underline{\psi}^*_{m+n,m+1} \leq \frac{m+n+1}{(m+n)(1+nC)} - \frac{1}{m+n} = \frac{m+n(1-C)}{(m+n)(1+nC)}.$$

Hence (5.14) with $j = m+1$, where n corresponds to the present $m+n$, and (5.13) imply

$$\begin{aligned} \underline{\psi}_{m+n,1} &= -\overline{\psi}^*_{m+n,m+n+1} \leq \frac{m+1}{n}\underline{\psi}^*_{m+n,m+1} \\ &= \frac{m+1}{n} \cdot \frac{m+n(1-C)}{(m+n)(1+nC)}. \end{aligned} \tag{5.19}$$

Application of (5.15) yields

$$\begin{aligned} \lambda_{m+n}(\zeta) &\geq \frac{m+n+1}{m+n} \cdot \frac{1}{1+\frac{m+1}{n}\frac{m+n(1-C)}{(m+n)(1+nC)}} - 1 \\ &= \frac{Cn-m}{m+n(1+C(n-1))}. \end{aligned} \tag{5.20}$$

Let $\epsilon > 0$ and $m = \lceil Rn \rceil$ with the optimal parameter $R = (C-1)/2$. A short computation shows

$$(m+n)\lambda_{m+n}(\zeta) \geq \left(\frac{C+1}{2}\right)^2 \frac{n}{R+1+(n-1)C} - \epsilon > \frac{(C+1)^2}{4C} - 2\epsilon, \tag{5.21}$$

for $n \geq n_0(C, \epsilon)$. We infer the left inequality in (3.6) as ϵ is arbitrarily small and as we may choose C arbitrarily close to $\overline{w}(\zeta)$ for certain arbitrarily large n. For T-numbers we may choose arbitrarily large C for certain large n, and the claim (3.7) follows. For the right inequality in (3.6), we have to be a bit more careful. Let $C = \underline{w}(\zeta) - \varepsilon$ for small $\varepsilon > 0$, so that $w_n(\zeta) \geq Cn$ for all large n. For given large N, we may write $N = m+n$ with $m = Rn + s$ with $R = (C-1)/2$ for an integer n and some $0 \leq s < R$, where n tends to infinity as N does (unless if $R = 0$ or equivalently $C = 1$, but then the claim (5.21) is clear by (1.6) anyway). Since $s \leq (\underline{w}(\zeta) - 1)/2$ is effectively bounded by $O(1)$ and we observed $n \to \infty$, estimate (5.21) can still be deduced from (5.20) by choosing ε small enough compared to ϵ. The claim follows as N was arbitrary and we let ϵ tend to 0. □

We can readily extract Theorem 3.6 from the proof above.

Proof of Theorem 3.6. The inequality (3.8) follows from (5.20), upon identifying $w_n(\zeta)$ with Cn, after a short calculation. With a very similar proof strategy, using the uniform estimates from Proposition 5.1, interchanging overline and underline in ψ-quantities throughout, and again in view of (5.14), we can derive (3.9). We skip the details. □

Finally we put the results together to prove Theorem 2.1.

Proof of Theorem 2.1. The inequalities in (2.3) have already been established in Theorem 3.5 and Corollary 3.2. The inequality $\sigma(\zeta) \leq 1/\tau(\zeta)$ follows easily from (3.3). The reverse inequality $\sigma(\zeta) \geq 1/\tau(\zeta)$ follows from (5.20) by taking $m = n^{\tau(\zeta)-\delta}$ and letting δ tend to 0, observing that $\log C/\log n$ is arbitrarily close to $\tau(\zeta) - 1$ for certain large n. □

References

[1] B. ADAMCZEWSKI and Y. BUGEAUD, *Mesures de transcendance et aspects quantitatifs de la méthode de Thue-Siegel-Roth-Schmidt*, Proc. Lond. Math. Soc. **101** (2010), 1–31.

[2] K. ALNIAÇIK, *On U_m-numbers*, Proc. Amer. Math. Soc. **85** (1982), 499–505.

[3] K. ALNIAÇIK, Y. AVCI and Y. BUGEAUD, *On U_m numbers with small transcendence measure*, Acta Math. Hungar. **99** (2003), 271–277.

[4] D. BADZIAHIN and Y. BUGEAUD, *On simultaneous rational approximation to a real number and its integral powers II*, New York J. Math. **26** (2020), 362–377.

[5] A. BAKER and W. M. SCHMIDT, *Diophantine approximation and Hausdorff dimension*, Proc. London Math. Soc. **21** (1970), 1–11.

[6] R. C. BAKER, *On approximation with algebraic numbers of bounded degree*, Mathematika **23** (1976), 18–31.

[7] V. I. BERNIK, *Application of the Hausdorff dimension in the theory of Diophantine approximations*, Acta Arith. **42** (1983), 219–253 (in Russian), English transl. in Amer. Math. Soc. Transl. **140** (1988), 15–44.

[8] Y. BUGEAUD, "Approximation by Algebraic Numbers", Cambridge Tracts in Mathematics, Vol. 160, Cambridge University Press, 2004.

[9] Y. BUGEAUD, *On simultaneous rational approximation to a real numbers and its integral powers*, Ann. Inst. Fourier (Grenoble) **60** (2010), 2165–2182.

[10] Y. BUGEAUD and M. LAURENT, *Exponents in Diophantine approximation*, In: "Diophantine Geometry", CRM Series, Vol. 4, Ed. Norm., Pisa, 2007, 101–121.

[11] Y. BUGEAUD and M. LAURENT, *Exponents of Diophantine approximation and Sturmian continued fractions*, Ann. Inst. Fourier (Grenoble) **55** (2005), 773–804.

[12] Y. BUGEAUD and J. SCHLEISCHITZ, *On uniform approximation to real numbers*, Acta Arith. **175** (2016), 255–268.

[13] Y. BUGEAUD and O. TEULIÉ, *Approximation d'un nombre réel par des nombres algebriques de degré donné*, Acta Arith. **93** (2000), 77–86.

[14] H. DAVENPORT and W. M. SCHMIDT, *Approximation to real numbers by algebraic integers*, Acta Arith. **15** (1969), 393–416.

[15] Y. A. KHINTCHINE, *Zur metrischen theorie der diophantischen approximationen*, Math. Z. **24** (1926), 706–714.

[16] J. F. KOKSMA, *Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen*, Monatsh. Math. Phys. **48** (1939), 176–189.

[17] M. LAURENT, *Simultaneous rational approximation to the successive powers of a real number*, Indag. Math. **11** (2003), 45–53.
[18] W. J. LEVEQUE, *On Mahler's U-numbers*, J. London Math. Soc. **28** (1953), 220–229.
[19] K. MAHLER, *Zur Approximation der Exponentialfunktionen und des Logarithmus I, II*, J. Reine Angew. Math. **166** (1932), 118–150.
[20] A. MARNAT and N. MOSHCHEVITIN, *An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation*, arXiv: 1802.03081.
[21] D. ROY, *Approximation to real numbers by cubic algebraic integers I*, Proc. London Math. Soc. **88** (2004), 42–62.
[22] D. ROY, *Approximation to real numbers by cubic algebraic integers II*, Ann. of Math. (2) **158** (2003), 1081–1087.
[23] D. ROY, *On simultaneous rational approximations to a real number, its square, and its cube*, Acta Arith. **133** (2008), 185–197.
[24] D. ROY, *A measure of transcendence for singular points on conics*, J. Théor. Nombres Bordeaux, to appear arXiv:1809.03642.
[25] J. SCHLEISCHITZ, *Two estimates concerning classical diophantine approximation constants*, Publ. Math. Debrecen **84** (2014), 415–437.
[26] J. SCHLEISCHITZ, *On the spectrum of Diophantine approximation constants*, Mathematika **62** (2016), 79–100.
[27] J. SCHLEISCHITZ, *On uniform approximation to successive powers of a real number*, Indag. Math. **28** (2017), 406–423.
[28] J. SCHLEISCHITZ, *Some notes on the regular graph defined by Schmidt and Summerer and uniform approximation*, JP J. Algebra Number Theory Appl. **39** (2017), 115–150.
[29] J. SCHLEISCHITZ, *Diophantine approximation in prescribed degree*, Mosc. Math. J. **18** (2018), 491–516.
[30] J. SCHLEISCHITZ *Uniform Diophantine approximation and best approximation polynomials*, Acta Arith. **185** (2018), 249–274.
[31] J. SCHLEISCHITZ, *An equivalence principle between polynomial and simultaneous Diophantine approximation*, arXiv:1704.00055.
[32] W. M. SCHMIDT, *T-numbers do exist*, In: "Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)", Academic Press, London, 3–26.
[33] W. M. SCHMIDT, *Mahler and Koksma classifications of points in $\mathbb{R}^n$ and $\mathbb{C}^n$*, Funct. Approx. Comment. Math. **35** (2006), 307–319.
[34] W. M. SCHMIDT and L. SUMMERER, *Parametric geometry of numbers and applications*, Acta Arith. **140** (2009), 67–91.
[35] W. M. SCHMIDT and L. SUMMERER, *Diophantine approximation and parametric geometry of numbers*, Monatsh. Math. **169** (2013), 51–104.
[36] W. M. SCHMIDT and L. SUMMERER, *Simultaneous approximation to three numbers*, Mosc. J. Comb. Number Theory **3** (2013), 84–107.
[37] V. G. SPRINDŽUK, "Mahler's Problem in Metric Number Theory", Izdat. "Nauka i Tehnika", Minsk, 1967 (in Russian). English translation by B. Volkmann, Translations of Mathematical Monographs, Vol. 25, American Mathematical Society, Providence, R.I., 1969.
[38] K. I. TSISHCHANKA, *On approximation of real numbers by algebraic numbers of bounded degree*, J. Number Theory **123** (2007), 290–314.
[39] E. WIRSING, *Approximation mit algebraischen Zahlen beschränkten Grades*, J. Reine Angew. Math. **206** (1961), 67–77.

Middle East Technical University
Northern Cyprus Campus
Kalkanli, Güzelyurt
johannes@metu.edu.tr
jschleischitz@outlook.com