Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXI (2020), 1023-1061

Existence and concentration of nontrivial solutions for a fractional magnetic Schrödinger-Poisson type equation

VINCENZO AMBROSIO

Abstract. We consider the following fractional Schrödinger-Poisson type equation with magnetic fields

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u + V(x)u + \varepsilon^{-2t}(|x|^{2t-3} * |u|^2)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3,$$

where $\varepsilon > 0$ is a parameter, $s \in (\frac{3}{4}, 1)$, $t \in (0, 1)$, $(-\Delta)^s_A$ is the fractional magnetic Laplacian, $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth magnetic potential, $V : \mathbb{R}^3 \to \mathbb{R}$ is a positive continuous electric potential and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth. Using suitable variational methods, we show the existence of a family of nontrivial solutions which concentrates around global minima of the potential $V(x)$ as $\varepsilon \to 0$.

Mathematics Subject Classification (2010): 35A15 (primary); 35R11, 35S05 (secondary).

1. introduction

In this paper we are interested in the existence of nontrivial solutions $u : \mathbb{R}^3 \to \mathbb{C}$ for the following fractional nonlinear Schrödinger-Poisson type equation

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u + V(x)u + \varepsilon^{-2t}(|x|^{2t-3} * |u|^2)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, $s \in (\frac{3}{4}, 1)$, $t \in (0, 1)$, $A : \mathbb{R}^3 \to \mathbb{R}^3 \in C^{0,\alpha}$, with $\alpha \in (0, 1]$, is a magnetic potential, and $(-\Delta)^s_A$ is the so called fractional magnetic Laplacian which can be defined by setting

$$(-\Delta)^s_A u(x) := c_s \lim_{r\to 0} \int_{B^c_r(x)} \frac{u(x) - e^{\imath(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{3+2s}}dy, \quad c_s := \pi^{-\frac{3}{2}} 2^{2s} \frac{\Gamma\left(\frac{3+2s}{2}\right)}{-\Gamma(-s)},$$

Received April 21, 2018; accepted in revised form May 02, 2019.
Published online December 2020.

for any $u\in C_c^\infty(\mathbb{R}^3,\mathbb{C})$; see [19,32] for more details. As showed in [47], when $s\to 1$, the previous operator reduces to the magnetic Laplacian $-\Delta_A := \left(\frac{1}{\imath}\nabla - A\right)^2$ (see [35,37]) given by

$$-\Delta_A u = -\Delta u - \frac{2}{\imath}A(x)\cdot\nabla u + |A(x)|^2 u - \frac{1}{\imath}u\,\mathrm{div}(A(x)),$$

which appears in the study of the following Schrödinger equation with magnetic fields

$$-\Delta_A u + V(x)u = f(x,|u|^2)u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Equation (1.2) has been widely investigated by several authors in the last thirty years; see for instance [1,9,13,15,23,34].

Along the paper, we assume that $V:\mathbb{R}^3\to\mathbb{R}$ is a continuous potential satisfying the following del Pino-Felmer type assumptions [20]:

(V_1) $V_0 := \inf_{x\in\mathbb{R}^3} V(x) > 0$;
(V_2) there exists a bounded domain $\Lambda\subset\mathbb{R}^3$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad M := \{x\in\Lambda : V(x)=V_0\} \neq \emptyset. \tag{1.3}$$

Without of loss of generality, we may assume that $0\in M$. The nonlinearity $f:\mathbb{R}\to\mathbb{R}$ is a continuous function fulfilling the following conditions:

(f_1) $f(t)=0$ for $t\le 0$ and $\displaystyle\lim_{t\to 0}\frac{f(t)}{t}=0$;
(f_2) there exist $q\in(4,2_s^*)$, where $2_s^* := \frac{6}{3-2s}$, such that

$$\lim_{t\to\infty}\frac{f(t)}{t^{\frac{q-2}{2}}}=0;$$

(f_3) there exists $\theta\in(4,2_s^*)$ such that $0<\frac{\theta}{2}F(t)\le tf(t)$ for any $t>0$, where $F(t) := \int_0^t f(\tau)d\tau$;
(f_4) $t\mapsto\frac{f(t)}{t}$ is increasing for $t>0$.

Let us state our main theorem:

Theorem 1.1. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold. Then there exists $\varepsilon_0>0$ such that, for any $\varepsilon\in(0,\varepsilon_0)$, problem (1.1) has a nontrivial solution. Moreover, if u_ε denotes one of these solutions and x_ε is a global maximum point of $|u_\varepsilon|$, then we have*

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = V_0$$

and there exists a constant $C>0$ (independent of ε) such that

$$|u_\varepsilon(x)| \le \frac{C\varepsilon^{3+2s}}{C\varepsilon^{3+2s}+|x-x_\varepsilon|^{3+2s}} \quad \forall x\in\mathbb{R}^3.$$

The above result is motivated by some works appeared in the last years concerning fractional Schrödinger equations with magnetic fields of the type

$$\varepsilon^{2s}(-\Delta)^s_A u + V(x)u = f(x,|u|^2)u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

For instance, in the unperturbed case (that is $\varepsilon = 1$), d'Avenia and Squassina [19] studied via a constrained minimization argument the existence of solutions (1.4), V is constant and f is a subcritical or critical nonlinearity. Fiscella *et al.* [27] obtained a multiplicity result for a fractional magnetic problem with homogeneous boundary conditions. When $\varepsilon > 0$ is small, Zhang *et al.* [53] focused on a fractional magnetic Schrödinger equation involving critical frequency and critical growth. Recently, the author and d'Avenia [8] dealt with the existence and the multiplicity of solutions to (1.4) for small $\varepsilon > 0$, when the potential V satisfies the global condition due to Rabinowitz [44] and f has a subcritical growth.

In absence of a magnetic field (that is $A = 0$), the fractional magnetic Laplacian $(-\Delta)^s_A$ coincides with the fractional Laplacian $(-\Delta)^s$ and equation (1.4) becomes the well-known fractional Schrödinger equation (see [36])

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N, \tag{1.5}$$

for which the existence and concentration phenomena of positive solutions have been considered by many mathematicians. For example, Dávila *et al.* [18] used a Lyapunov-Schmidt variational reduction to prove that (1.5) has a multi-peak solution when $V \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ is a positive potential and f is a subcritical nonlinearity; see also [17] in which a concentration result has been established for a nonlocal problem with Dirichlet datum. Fall *et al.* [25] showed that the concentration points of the solutions of (1.5) must be the critical points for V, as ε goes to zero. Alves and Miyagaki [2] (see also [4,5]) used the penalization method in [20] to study the existence and concentration of positive solutions of (1.5) requiring that f satisfies (f_1)-(f_4) and V fulfills (V_1)-(V_2).

On the other hand, in these last years, several authors investigated fractional Schrödinger-Poisson systems of the type

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)\phi u = g(x,u) & \text{in } \mathbb{R}^3 \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

which can be seen as the nonlocal counterpart of the well-known Schrödinger-Poisson systems appearing in quantum mechanics models [11] and in semiconductor theory [39]. Such systems have been introduced in [10] to describe systems of identical charged particles interacting each other in the case that effects of magnetic field could be ignored and its solution represents, in particular, a standing wave for such a system. We refer to [16, 29, 30, 45, 50, 54] for some interesting existence and multiplicity results for classical perturbed and unperturbed Schrödinger-Poisson systems.

Concerning (1.6), Giammetta [28] considered the local and global well-posedness of a one dimensional fractional Schrödinger-Poisson system in which

$\varepsilon = 1$ and the fractional diffusion appears only in the Poisson equation. Zhang *et al.* [52] dealt with the existence of positive solutions to (1.6) with $\varepsilon = 1$, $V(x) = \mu > 0$ and g is a general nonlinearity having subcritical or critical growth. Murcia and Siciliano [42] proved that, for suitably small ε, the number of positive solutions to a doubly singularly perturbed fractional Schrödinger-Poisson system is estimated below by the Ljusternick-Schnirelmann category of the set of minima of the potential. Liu and Zhang [38] studied multiplicity and concentration of solutions to (1.6) involving the critical exponent and under a global condition on the potential V. In [6] the author improved the results in [38] by assuming (V_1)-(V_2) and considering continuous nonlinearities. Teng [49], inspired by [30], used the penalization method due to Byeon and Wang [12] to analyze the concentration phenomenon for (1.6) under conditions (V_1)-(V_2) and $g(u)$ is a C^1 subcritical nonlinearity.

Particularly motivated by [2,4–6,8,30,49,50], in this paper we investigate the existence and concentration behavior of nontrivial solutions to (1.1) with $A \neq 0$ and under assumptions (V_1)-(V_2) and (f_1)-(f_4). We note that when $s = t = 1$ in (1.1), the multiplicity and concentration for a Schrödinger-Poisson type equation with magnetic field and under a local condition on V, has been established in [55] by using some ideas developed in [1]. Anyway, their arguments work for C^1-Nehari manifolds and we can not apply them in our situation because we are assuming the only continuity of f.

Since we do not have any information on the behavior of V at infinity, we adapt the penalization argument developed by del Pino and Felmer in [20], which consists in making an appropriate modification on f, solving a modified problem and then check that, for ε small enough, the solutions of the modified problem are indeed solutions of the original one. We point out that the penalization argument developed here is different from the one used in [49], in which the author does not assume the suplinear-4 growth on f but has to require $f \in C^1$ to apply the techniques developed in [12,30]. The existence of nontrivial solutions for the modified problem is obtained by using the mountain pass theorem [3] to the functional J_ε associated with the modified problem. We note that the main issue in the study of J_ε concerns the verification of the Palais-Smale compactness condition. Indeed, the presence of the fractional magnetic Laplacian and the convolution term $(|x|^{2t-3} * |u|^2)$, make our analysis more complicated and intriguing, and some suitable arguments will be needed to achieve our purpose; see Lemma 3.2. The next step is to show that if u_ε is a solution of the modified problem, then u_ε is also a solution of the original one (1.1). In the case $A = 0$ (see [2,49]), this is proved taking into account some fundamental estimates established in [26] concerning the Bessel operator. In the case $A \neq 0$, we do not have similar informations for the following fractional equation

$$(-\Delta)^s_A u + V_0 u = h(|u|^2)u \text{ in } \mathbb{R}^3. \tag{1.7}$$

For the above reason, we use an approximation argument which allows us to deduce that if u_ε is a solution to the modified problem, then $|u_\varepsilon|$ is a subsolution for an autonomous fractional Schrödinger equation without magnetic field, and then we apply a comparison argument to deduce informations on the behavior at infinity of

$|u_\varepsilon|$; see Lemma 4.1. We point out that, in the case $s = 1$, a similar reasoning works (see [14,34]) in view of the following distributional Kato's inequality [33]

$$-\Delta|u| \leq \Re(sign(u)(-\Delta_A u)).$$

Recently, in [31], a distributional Kato's inequality has been established for some magnetic relativistic Schrödinger operators which also include $(-\Delta)_A^{1/2}$. We suspect that a fractional Kato's inequality is available for the operator $(-\Delta)_A^s$ with any fractional power $s \in (0, 1)$ (indeed it is easily seen that a pointwise Kato's inequality holds for smooth functions), but we are not able to prove it. Again, we can not repeat the iteration done in [1] to obtain L^∞-estimates on the modulus of solutions, due to the nonlocal character of $(-\Delta)_A^s$. Anyway, in the present paper, we develop some appropriate arguments which we believe can be useful to face other problems like (1.4). Now we give a sketch of our idea. Firstly, we show that the (translated) sequence $|u_n|$ of solutions of the modified problem is bounded in $L^\infty(\mathbb{R}^3, \mathbb{R})$ uniformly in $n \in \mathbb{N}$, by applying an appropriate Moser iteration scheme [41]. After that, we prove that $|u_n|$ verifies

$$(-\Delta)^s|u_n| + V_0|u_n| \leq g(\varepsilon x, |u_n|^2)|u_n| \text{ in } \mathbb{R}^3,$$

by using $\frac{u_n}{u_{\delta,n}}\varphi$ as test function in the modified problem, where $u_{\delta,n} = \sqrt{|u_n|^2 + \delta^2}$ and φ is a real smooth nonnegative function with compact support in $\mathbb{R}^3$, and then we take the limit as $\delta \to 0$. In some sense, we are going to prove a fractional Kato's inequality for the solutions of the modified problem.. At this point, by comparison, we can show that $|u_n(x)| \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$, taking into account the power type decay of solutions of autonomous fractional Schrödinger equations; see [26]. As far as we know, the results presented here are new.

The paper is organized as follows. In Section 2 we give some results on fractional magnetic Sobolev spaces and we recall some useful lemmas. In Section 3, we introduce the modified problem and we show that the corresponding functional satisfies the assumptions of the mountain pass theorem. In the last section we give the proof of Theorem 1.1.

Remark 1.2. Arguing as in [7], we can replace the condition $V_0 < \min_{\partial\Lambda} V$ in (V_2) by the more general condition $\inf_\Lambda V < \min_{\partial\Lambda} V$; see proof of [7, Lemma 3.2]. In view of this observation, we deduce the existence of a family of solutions which concentrates around a local minimum of V as $\varepsilon \to 0$.

ACKNOWLEDGEMENTS. The author would like to express his sincere gratitude to the referee for many insightful comments and valuable suggestions, which enabled us to improve this version of the manuscript.

2. Preliminaries and functional setting

Let us consider the fractional Sobolev space

$$H^s(\mathbb{R}^3,\mathbb{R})=\{u\in L^2(\mathbb{R}^3,\mathbb{R}):[u]<\infty\},$$

where

$$[u]^2:=\frac{c_s}{2}\iint_{\mathbb{R}^6}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy,$$

endowed with the standard norm $\|u\|_{H^s(\mathbb{R}^3)}:=\sqrt{[u]^2+\|u\|_{L^2(\mathbb{R}^3)}}$.
It is well-known (see [21, 40]) that the embedding $H^s(\mathbb{R}^3,\mathbb{R})\subset L^q(\mathbb{R}^3,\mathbb{R})$ is continuous for all $q\in[2,2^*_s)$ and locally compact for all $q\in[1,2^*_s)$.

Let $L^2(\mathbb{R}^3,\mathbb{C})$ be the space of complex-valued functions such that $\int_{\mathbb{R}^3}|u|^2\,dx<\infty$ endowed with the inner product $\langle u,v\rangle_{L^2}=\Re\int_{\mathbb{R}^3}u\bar{v}\,dx$, where the bar denotes complex conjugation.

Let us denote by

$$[u]^2_A:=\frac{c_s}{2}\iint_{\mathbb{R}^6}\frac{|u(x)-e^{\imath(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{3+2s}}\,dxdy,$$

and we define

$$D^s_A(\mathbb{R}^3,\mathbb{C}):=\left\{u\in L^{2^*_s}(\mathbb{R}^3,\mathbb{C}):[u]_A<\infty\right\}.$$

In order to study our problem, for any $\varepsilon>0$ we introduce the Hilbert space

$$H^s_\varepsilon:=\left\{u\in D^s_{A_\varepsilon}(\mathbb{R}^3,\mathbb{C}):\int_{\mathbb{R}^3}V(\varepsilon x)|u|^2\,dx<\infty\right\}$$

endowed with the scalar product

$$\langle u,v\rangle_\varepsilon:=\Re\int_{\mathbb{R}^3}V(\varepsilon x)u\bar{v}dx+\frac{c_s}{2}\Re\iint_{\mathbb{R}^6}\frac{(u(x)-e^{\imath(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y))\overline{(v(x)-e^{\imath(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}v(y))}}{|x-y|^{3+2s}}dxdy$$

and we set

$$\|u\|_\varepsilon:=\sqrt{\langle u,u\rangle_\varepsilon}.$$

The space H^s_ε satisfies the following fundamental properties; see [8, 19] for more details.

Lemma 2.1 ([8,19]). *The space H^s_ε is complete and $C^\infty_c(\mathbb{R}^3,\mathbb{C})$ is dense in H^s_ε.*

Theorem 2.2 ([8,19]). *The space H^s_ε is continuously embedded in $L^r(\mathbb{R}^3,\mathbb{C})$ for all $r\in[2,2^*_s]$, and compactly embedded in $L^r_{\rm loc}(\mathbb{R}^3,\mathbb{C})$ for all $r\in[1,2^*_s)$.*

Lemma 2.3 ([19]). *If* $u \in H^s_\varepsilon(\mathbb{R}^3, \mathbb{C})$ *then* $|u| \in H^s(\mathbb{R}^3, \mathbb{R})$ *and we have*

$$[|u|] \le [u]_{A_\varepsilon}.$$

Lemma 2.4 ([8]). *If* $u \in H^s(\mathbb{R}^3, \mathbb{R})$ *and u has compact support, then* $w = e^{\iota A(0)\cdot x} u \in H^s_\varepsilon$.

We also recall the following vanishing lemma [26] which will be useful for our study:

Lemma 2.5 ([26]). *Let* $q \in [2, 2^*_s)$. *If* (u_n) *is a bounded sequence in* $H^s(\mathbb{R}^3, \mathbb{R})$ *and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^q dx = 0$$

for some $R > 0$, *then* $u_n \to 0$ *in* $L^r(\mathbb{R}^3, \mathbb{R})$ *for all* $r \in (2, 2^*_s)$.

Now, let $s, t \in (0, 1)$ be such that $4s + 2t \ge 3$. Since $H^s(\mathbb{R}^3, \mathbb{R}) \subset L^q(\mathbb{R}^3, \mathbb{R})$ for all $q \in [2, 2^*_s)$, we can deduce that

$$H^s(\mathbb{R}^3, \mathbb{R}) \subset L^{\frac{12}{3+2t}}(\mathbb{R}^3, \mathbb{R}). \tag{2.1}$$

For any $u \in H^s_\varepsilon$, we know that $|u| \in H^s(\mathbb{R}^3, \mathbb{R})$ in view of Lemma 2.3, and then we consider the linear functional $\mathcal{L}_{|u|} : D^{t,2}(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R}$ given by

$$\mathcal{L}_{|u|}(v) = \int_{\mathbb{R}^3} |u|^2 v \, dx,$$

where

$$D^{t,2}(\mathbb{R}^3, \mathbb{R}) := \left\{ u \in L^{2^*_t}(\mathbb{R}^3, \mathbb{R}) : \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2t}} dxdy < \infty \right\}.$$

Using the Hölder inequality and (2.1) we can see that

$$|\mathcal{L}_{|u|}(v)| \le \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx \right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{2^*_t} dx \right)^{\frac{1}{2^*_t}} \le C \|u\|^2_{D^{s,2}} \|v\|_{D^{t,2}},$$

where

$$\|v\|^2_{D^{t,2}} := \frac{c_t}{2} \iint_{\mathbb{R}^6} \frac{|v(x) - v(y)|^2}{|x-y|^{3+2t}} dxdy,$$

and this shows that $\mathcal{L}_{|u|}$ is well defined and continuous. Applying the Lax-Milgram Theorem, there exists a unique $\phi^t_{|u|} \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$ such that

$$(-\Delta)^t \phi^t_{|u|} = |u|^2 \text{ in } \mathbb{R}^3. \tag{2.2}$$

Then we have the following t-Riesz formula (see Chapter V in [48])

$$\phi^t_{|u|}(x) = c'_t \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|^{3-2t}} dy \quad (x \in \mathbb{R}^3), \quad c'_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3-2t}{2})}{\Gamma(t)}. \tag{2.3}$$

In the sequel, we will omit the constants $c_s/2$ and c'_t in order to lighten the notation. Finally, we prove some properties on the convolution term.

Lemma 2.6. *Let us assume that* $4s + 2t \geq 3$ *and* $u \in H^s_\varepsilon$. *Then we have:*

(1) $\phi^t_{|u|} : H^s(\mathbb{R}^3, \mathbb{R}) \to D^{t,2}(\mathbb{R}^3, \mathbb{R})$ *is continuous and maps bounded sets into bounded sets;*
(2) *if* $u_n \rightharpoonup u$ *in* H^s_ε *then* $\phi^t_{|u_n|} \rightharpoonup \phi^t_{|u|}$ *in* $D^{t,2}(\mathbb{R}^3, \mathbb{R})$*;*
(3) $\phi^t_{|ru|} = r^2 \phi^t_{|u|}$ *for all* $r \in \mathbb{R}$ *and* $\phi^t_{|u(\cdot+y)|}(x) = \phi^t_{|u|}(x + y)$*;*
(4) $\phi^t_{|u|} \geq 0$ *for all* $u \in H^s_\varepsilon$*, and we have*

$$\|\phi^t_{|u|}\|_{D^{t,2}} \leq C\|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \leq C\|u\|^2_\varepsilon \text{ and } \int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx \leq C\|u\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \leq C\|u\|^4_\varepsilon.$$

Proof. (1) Since $\phi^t_{|u|} \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$ satisfies (2.2), that is

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi^t_{|u|} (-\Delta)^{\frac{t}{2}} v \, dx = \int_{\mathbb{R}^3} |u|^2 v \, dx$$

for all $v \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$, we can see that $\mathcal{L}_{|u|}$ is such that $\|\mathcal{L}_{|u|}\|_{\mathcal{L}(D^{t,2},\mathbb{R})} = \|\phi^t_{|u|}\|_{D^{t,2}}$ for all $u \in H^s_\varepsilon$. Hence, in order to prove the continuity of $\phi^t_{|u|}$, it is enough to show that the map $u \mapsto \mathcal{L}_{|u|}$ is continuous. Let $u_n \to u$ in H^s_ε. Using Lemma 2.3 and Theorem 2.2 we deduce that $|u_n| \to |u|$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. Hence, for all $v \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$ we have

$$\begin{aligned}
&|\mathcal{L}_{|u_n|}(v) - \mathcal{L}_{|u|}(v)| \\
&= \left| \int_{\mathbb{R}^3} (|u_n|^2 - |u|^2) v \, dx \right| \\
&\leq \left(\int_{\mathbb{R}^3} ||u_n|^2 - |u|^2|^{\frac{6}{3+2t}} \, dx \right)^{\frac{3+2t}{6}} \|v\|_{L^{\frac{6}{3-2t}}(\mathbb{R}^3)} \\
&\leq C \left[\left(\int_{\mathbb{R}^3} ||u_n| - |u||^{\frac{12}{3+2t}} \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} ||u_n| + |u||^{\frac{12}{3+2t}} \, dx \right)^{\frac{1}{2}} \right]^{\frac{3+2t}{6}} \|v\|_{D^{t,2}} \\
&\leq C \||u_n| - |u|\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \|v\|_{D^{t,2}},
\end{aligned}$$

which implies that $\|\phi^t_{|u_n|} - \phi^t_{|u|}\|_{D^{t,2}} = \|\mathcal{L}_{|u_n|} - \mathcal{L}_{|u|}\|_{\mathcal{L}(D^{t,2},\mathbb{R})} \to 0$ as $n \to \infty$.
(2) If $u_n \rightharpoonup u$ in H^s_ε, then Lemma 2.3 and Theorem 2.2 yield $|u_n| \to |u|$ in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R})$ for all $q \in [1, 2^*_s)$. Hence, for all $v \in C^\infty_c(\mathbb{R}^3, \mathbb{R})$ we get

$$\begin{aligned}
\langle \phi^t_{|u_n|} - \phi^t_{|u|}, v \rangle &= \int_{\mathbb{R}^3} (|u_n|^2 - |u|^2) v \, dx \\
&\leq \left(\int_{supp(v)} ||u_n| - |u||^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} ||u_n| + |u||^2 \, dx \right)^{\frac{1}{2}} \|v\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C \||u_n| - |u|\|_{L^2(supp(v))} \|v\|_{L^\infty(\mathbb{R}^3)} \to 0.
\end{aligned}$$

(3) and (4) are easily obtained by applying the Hardy-Littlewood-Sobolev inequality (see Theorem 4.3 in [37]), Hölder inequality and Sobolev embedding. □

3. The modified problem

Using the change of variable $x \mapsto \varepsilon x$, we can see that the study of (1.1) is equivalent to consider the following problem

$$(-\Delta)^s_{A_\varepsilon} u + V_\varepsilon(x)u + (|x|^{2t-3} * |u|^2)u = f(|u|^2)u \text{ in } \mathbb{R}^3, \tag{3.1}$$

where $A_\varepsilon(x) := A(\varepsilon x)$ and $V_\varepsilon(x) := V(\varepsilon x)$.

As in [2,20], we fix $\kappa > \frac{\theta}{\theta-2}$ and $a > 0$ such that $f(a) = \frac{V_0}{\kappa}$, and we introduce the function

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \le a \\ \frac{V_0}{\kappa} & \text{if } t > a. \end{cases}$$

Then we define the penalized nonlinearity $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x,t) = \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t),$$

where χ_Λ is the characteristic function on Λ, and we set $G(x,t) = \int_0^t g(x,\tau)\,d\tau$.

From assumptions (f_1)-(f_4) it is standard to check that g verifies the following properties:

(g_1) $\lim\limits_{t\to 0} \dfrac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;

(g_2) $\lim_{t\to\infty} \frac{g(x,t)}{t^{\frac{q-2}{2}}} = 0$ uniformly in $x \in \mathbb{R}^3$;

(g_3) (i) $0 < \frac{\theta}{2} G(x,t) \le g(x,t)t$ for any $x \in \Lambda$ and $t > 0$,
(ii) $0 \le G(x,t) \le g(x,t)t \le \frac{V(x)}{\kappa} t$ and $0 \le g(x,t) \le \frac{V(x)}{\kappa}$ for any $x \in \Lambda^c$ and $t > 0$;

(g_4) $t \mapsto \frac{g(x,t)}{t}$ is increasing for all $x \in \Lambda$ and $t > 0$.

Then, we consider the following modified problem

$$(-\Delta)^s_{A_\varepsilon} u + V_\varepsilon(x)u + \phi^t_{|u|} u = g_\varepsilon(x, |u|^2)u \text{ in } \mathbb{R}^3, \tag{3.2}$$

where $g_\varepsilon(x,t) = g(\varepsilon x, t)$ and $\phi^t_{|u|}$ is given by (2.3).

Let us note that if u is a solution of (3.2) such that

$$|u(x)| \le \sqrt{a} \text{ for all } x \in \Lambda^c_\varepsilon, \tag{3.3}$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$, then u is also a solution of the original problem (3.1).

In order to find weak solutions to (3.2), we look for critical points of the Euler-Lagrange functional $J_\varepsilon : H^s_\varepsilon \to \mathbb{R}$ defined as

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} G_\varepsilon(x, |u|^2)\, dx.$$

We also consider the scalar limiting problem associated with (3.1), that is

$$(-\Delta)^s u + V_0 u + \phi^t_{|u|} u = f(u^2)u \text{ in } \mathbb{R}^3, \tag{3.4}$$

and we denote by $I_0 : H^s(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R}$ the corresponding energy functional

$$\begin{aligned} I_0(u) &= \frac{1}{2}[u]^2 + \frac{V_0}{2}\|u\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|u|} u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)\, dx \\ &= \frac{1}{2}\|u\|^2_0 + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|u|} u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)\, dx \end{aligned}$$

where $\|u\|_0 := \sqrt{[u]^2 + V_0\|u\|^2_{L^2(\mathbb{R}^3)}}$ is a norm in $H^s(\mathbb{R}^3, \mathbb{R})$ equivalent to the standard one.

In what follows, we show that J_ε verifies the assumptions of the mountain pass theorem [3].

Lemma 3.1. *The functional J_ε possesses a mountain pass geometry:*

(i) $J_\varepsilon(0) = 0$*;*
(ii) *there exist $\alpha, \rho > 0$ such that $J_\varepsilon(u) \geq \alpha$ for any $u \in H^s_\varepsilon$ such that $\|u\|_\varepsilon = \rho$;*
(iii) *there exists $e \in H^s_\varepsilon$ with $\|e\|_\varepsilon > \rho$ such that $J_\varepsilon(e) < 0$.*

Proof. The condition (i) is obvious. Using (g_1), (g_2), and Theorem 2.2, we can see that for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$J_\varepsilon(u) \geq \frac{1}{2}\|u\|^2_\varepsilon - \delta C\|u\|^4_\varepsilon - C_\delta\|u\|^q_\varepsilon.$$

Choosing $\delta > 0$ sufficiently small, we can see that (ii) holds. Regarding (iii), we can note that in view of (g_3), we have for any $u \in H^s_\varepsilon \setminus \{0\}$ with $supp(u) \subset \Lambda_\varepsilon$ and $T > 1$

$$\begin{aligned} J_\varepsilon(Tu) &\leq \frac{T^2}{2}\|u\|^2_\varepsilon + \frac{T^4}{4}\int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx - \frac{1}{2}\int_{\Lambda_\varepsilon} G_\varepsilon(x, T^2|u|^2)\, dx \\ &\leq \frac{T^4}{2}\left(\|u\|^2_\varepsilon + \int_{\mathbb{R}^3} \phi^t_{|u|}|u|^2 dx\right) - CT^\theta \int_{\Lambda_\varepsilon} |u|^\theta\, dx + C, \end{aligned}$$

which together with $\theta > 4$ implies that $J_\varepsilon(Tu) \to -\infty$ as $T \to \infty$. □

Lemma 3.2. *Let $c \in \mathbb{R}$. Then J_ε satisfies the Palais-Smale condition at the level c.*

Proof. Let $(u_n) \subset H^s_\varepsilon$ be a $(PS)_c$ sequence. Then (u_n) is bounded in H^s_ε. Indeed, using (g_3) we have

$$\begin{aligned}
C(1+\|u_n\|_\varepsilon) &\geq J_\varepsilon(u_n) - \frac{1}{\theta}\langle J'_\varepsilon(u_n), u_n\rangle \\
&= \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2_\varepsilon + \left(\frac{1}{4}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2 dx \\
&\quad + \frac{1}{\theta}\int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^3} G_\varepsilon(x,|u_n|^2)\,dx \\
&\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2_\varepsilon + \frac{1}{\theta}\int_{\Lambda_\varepsilon}\left(g_\varepsilon(x,|u_n|^2)|u_n|^2 - \frac{\theta}{2}G_\varepsilon(x,|u_n|^2)\right)dx \\
&\quad + \frac{1}{\theta}\int_{\Lambda^c_\varepsilon} g_\varepsilon(x,|u_n|^2)|u_n|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^3}\frac{V_\varepsilon(x)}{\kappa}|u_n|^2 dx \\
&\geq \frac{1}{2}\left(\frac{\theta-2}{\theta}-\frac{1}{\kappa}\right)\|u_n\|^2_\varepsilon,
\end{aligned}$$

and recalling that $\kappa > \frac{\theta}{\theta-2}$ we get the thesis. Now, we show that for any $\xi > 0$ there exists $R = R_\xi > 0$ such that $\Lambda_\varepsilon \subset B_{R/2}$ and

$$\limsup_{n\to\infty}\int_{B^c_R}\left[\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}dy\right] + V_\varepsilon(x)|u_n|^2\,dx \leq \xi. \tag{3.5}$$

Assume for the moment that the above claim holds, and we show how this information can be used. Using $u_n \rightharpoonup u$ in H^s_ε, Theorem 2.2 and (g_1)-(g_2), it is easy to see that

$$\begin{aligned}
&(u_n,\psi)_\varepsilon \to (u,\psi)_\varepsilon \text{ and } \Re\left(\int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|^2)u_n\bar{\psi}\,dx\right) \\
&\qquad \to \Re\left(\int_{\mathbb{R}^3} g_\varepsilon(x,|u|^2)u\bar{\psi}\,dx\right) \quad \text{for all } \psi \in C^\infty_c(\mathbb{R}^3,\mathbb{C}).
\end{aligned} \tag{3.6}$$

Moreover, by (3.5) and Theorem 2.2 we can see that for all $\xi > 0$ there exists $R = R_\xi > 0$ such that for any n large enough

$$\begin{aligned}
\|u_n-u\|_{L^q(\mathbb{R}^3)} &= \|u_n-u\|_{L^q(B_R)} + \|u_n-u\|_{L^q(B^c_R)} \\
&\leq \|u_n-u\|_{L^q(B_R)} + (\|u_n\|_{L^q(B^c_R)} + \|u\|_{L^q(B^c_R)}) \\
&\leq \xi + C\xi,
\end{aligned}$$

where $q \in [2, 2^*_s)$, which gives

$$u_n \to u \text{ in } L^q(\mathbb{R}^3,\mathbb{C}) \quad \forall q \in [2,2^*_s). \tag{3.7}$$

Since $||u_n|-|u|| \leq |u_n-u|$ and $\frac{12}{3+2t} \in (2,2^*_s)$, we have that $|u_n| \to |u|$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3,\mathbb{R})$.

Then, recalling that $\phi_{|u|}: L^{\frac{12}{3+2t}}(\mathbb{R}^3,\mathbb{R}) \to D^{t,2}(\mathbb{R}^3,\mathbb{R})$ is continuous (see Lemma 2.6), we can deduce that

$$\phi^t_{|u_n|} \to \phi^t_{|u|} \text{ in } D^{t,2}(\mathbb{R}^3,\mathbb{R}). \tag{3.8}$$

Putting together (3.7), (3.8), Hölder's inequality and Theorem 2.2 we obtain

$$\begin{aligned}
&\Re\left(\int_{\mathbb{R}^3}(\phi^t_{|u_n|}u_n-\phi^t_{|u|}u)\bar{\psi}\,dx\right)\\
&=\Re\left(\int_{\mathbb{R}^3}\phi^t_{|u_n|}(u_n-u)\bar{\psi}+\int_{\mathbb{R}^3}(\phi^t_{|u_n|}-\phi^t_{|u|})u\bar{\psi}\,dx\right)\\
&\le \|\phi^t_{|u_n|}\|_{L^{\frac{6}{3+2t}}(\mathbb{R}^3)}\|u_n-u\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\|\psi\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\\
&\quad+\|\phi^t_{|u_n|}-\phi^t_{|u|}\|_{\frac{6}{3+2t}}\|u\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}\|\psi\|_{\frac{12}{3+2t}}\\
&\le C\|u_n-u\|_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)}+C\|\phi^t_{|u_n|}-\phi^t_{|u|}\|_{D^{t,2}}\to 0.
\end{aligned} \tag{3.9}$$

Now, we show that

$$\int_{\mathbb{R}^3}\phi^t_{|u_n|}|u_n|^2dx \to \int_{\mathbb{R}^3}\phi^t_{|u|}|u|^2dx. \tag{3.10}$$

Let us start by proving that

$$|\mathbb{D}(u_n)-\mathbb{D}(u)| \le \sqrt{\mathbb{D}(||u_n|^2-|u|^2|^{1/2})}\sqrt{\mathbb{D}(||u_n|^2+|u|^2|^{1/2})},$$

where

$$\mathbb{D}(u)=\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u(y)|^2dxdy.$$

Indeed, taking into account $|x|^{-(3-2t)}$ is even and Theorem 9.8 in [37] (see Remark after Theorem 9.8 and recall that $-3<-(3-2t)<0$) we have

$$\begin{aligned}
&|\mathbb{D}(u_n)-\mathbb{D}(u)|\\
&=\left|\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u_n(x)|^2|u_n(y)|^2dxdy-\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u(y)|^2dxdy\right|\\
&=\Bigg|\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u_n(x)|^2|u_n(y)|^2dxdy+\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u_n(x)|^2|u(y)|^2dxdy\\
&\quad-\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u_n(y)|^2dxdy-\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}|u(x)|^2|u(y)|^2dxdy\Bigg|\\
&=\left|\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}(|u_n(x)|^2-|u(x)|^2|)(|u_n(y)|^2+|u(y)|^2)dxdy\right|\\
&\le\iint_{\mathbb{R}^6}|x-y|^{-(3-2t)}||u_n(x)|^2-|u(x)|^2|||u_n(y)|^2+|u(y)|^2|dxdy\\
&\le C\sqrt{\mathbb{D}(||u_n|^2-|u|^2|^{1/2})}\sqrt{\mathbb{D}(||u_n|^2+|u|^2|^{1/2})}.
\end{aligned}$$

Thus, using the Hardy-Littlewood-Sobolev inequality (see Theorem 4.3 in [37]), Hölder's inequality, the boundedness of $(|u_n|)$ in $H^s(\mathbb{R}^3, \mathbb{R})$ and $|u_n| \to |u|$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3, \mathbb{R})$ we can see that

$$
\begin{aligned}
|\mathbb{D}(u_n) - \mathbb{D}(u)|^2 &\leq C \| | |u_n|^2 - |u|^2 |^{1/2} \|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \| | |u_n|^2 + |u|^2 |^{1/2} \|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \\
&\leq C \| |u_n| - |u| \|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \to 0.
\end{aligned}
$$

Therefore, by $\langle J'_\varepsilon(u_n), \psi \rangle = o_n(1)$ for all $\psi \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$, (3.6), (3.9) and exploiting Lemma 2.1, we can check that $J'_\varepsilon(u) = 0$. In particular,

$$
\|u\|_\varepsilon^2 + \int_{\mathbb{R}^3} \phi^t_{|u|} |u|^2 dx = \int_{\mathbb{R}^3} g_\varepsilon(x, |u|^2)|u|^2 \, dx. \tag{3.11}
$$

Now, we know that $\langle J'_\varepsilon(u_n), u_n \rangle = o_n(1)$ is equivalent to

$$
\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3} \phi^t_{|u_n|} |u_n|^2 dx = \int_{\mathbb{R}^3} g_\varepsilon(x, |u_n|^2)|u_n|^2 \, dx + o_n(1). \tag{3.12}
$$

By (g_1)-(g_2) and (3.5) we deduce that

$$
\int_{\mathbb{R}^3} g_\varepsilon(x, |u_n|^2)|u_n|^2 \, dx \to \int_{\mathbb{R}^3} g_\varepsilon(x, |u|^2)|u|^2 \, dx. \tag{3.13}
$$

Then, taking into account (3.10), (3.11), (3.12) and (3.13) we can infer that

$$
\lim_{n \to \infty} \|u_n\|_\varepsilon^2 = \|u\|_\varepsilon^2.
$$

It remains to prove that (3.5) holds. Let $\eta_R \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be such that $0 \leq \eta_R \leq 1$, $\eta_R = 0$ in $B_{\frac{R}{2}}$, $\eta_R = 1$ in B_R^c and $|\nabla \eta_R| \leq \frac{C}{R}$ for some $C > 0$ independent of R. Since $(u_n \eta_R)$ is bounded, we can see that $\langle J'_\varepsilon(u_n), u_n \eta_R \rangle = o_n(1)$, that is

$$
\begin{aligned}
&\Re \Bigg(\iint_{\mathbb{R}^6} \\
&\times \frac{(u_n(x) - u_n(y) e^{\imath A_\varepsilon(\frac{x+y}{2}) \cdot (x-y)}) \overline{(u_n(x)\eta_R(x) - u_n(y)\eta_R(y) e^{\imath A_\varepsilon(\frac{x+y}{2}) \cdot (x-y)})}}{|x-y|^{3+2s}} \, dxdy \Bigg) \\
&+ \int_{\mathbb{R}^3} \phi^t_{|u_n|} |u_n|^2 \eta_R dx + \int_{\mathbb{R}^3} V_\varepsilon(x) \eta_R |u_n|^2 \, dx = \int_{\mathbb{R}^3} g_\varepsilon(x, |u_n|^2)|u_n|^2 \eta_R \, dx + o_n(1).
\end{aligned}
$$

From

$$\Re\left(\iint_{\mathbb{R}^6} \times \frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})\overline{(u_n(x)\eta_R(x)-u_n(y)\eta_R(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})}}{|x-y|^{3+2s}}\,dxdy\right)$$
$$=\Re\left(\iint_{\mathbb{R}^6}\overline{u_n(y)}e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})(\eta_R(x)-\eta_R(y))}{|x-y|^{3+2s}}dxdy\right)$$
$$+\iint_{\mathbb{R}^6}\eta_R(x)\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\,dxdy,$$

and using (g_3)-(ii) and Lemma 2.6-(4), it follows that

$$\iint_{\mathbb{R}^6}\eta_R(x)\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\,dxdy+\int_{\mathbb{R}^3}V_\varepsilon(x)\eta_R|u_n|^2\,dx$$
$$\leq -\Re\left(\iint_{\mathbb{R}^6}\overline{u_n(y)}e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)} \times\frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})(\eta_R(x)-\eta_R(y))}{|x-y|^{3+2s}}\,dxdy\right) \tag{3.14}$$
$$+\frac{1}{\kappa}\int_{\mathbb{R}^3}V_\varepsilon(x)\eta_R|u_n|^2\,dx+o_n(1).$$

Now, by the Hölder inequality and the boundedness of (u_n) in H^s_ε we get

$$\left|\Re\left(\iint_{\mathbb{R}^6}\overline{u_n(y)}e^{-\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\frac{(u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)})(\eta_R(x)-\eta_R(y))}{|x-y|^{3+2s}}dxdy\right)\right|$$
$$\leq\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac{1}{2}} \tag{3.15}$$
$$\times\left(\iint_{\mathbb{R}^6}|\overline{u_n(y)}|^2\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac{1}{2}}$$
$$\leq C\left(\iint_{\mathbb{R}^6}|u_n(y)|^2\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{\frac{1}{2}}.$$

In what follows, we show that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}|u_n(y)|^2\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy=0. \tag{3.16}$$

Let us note that

$$\begin{aligned}\mathbb{R}^6&=((\mathbb{R}^3\setminus B_{2R})\times(\mathbb{R}^3\setminus B_{2R}))\cup((\mathbb{R}^3\setminus B_{2R})\times B_{2R})\cup(B_{2R}\times\mathbb{R}^3)\\&=:X_R^1\cup X_R^2\cup X_R^3.\end{aligned}$$

Accordingly,

$$\begin{aligned}&\iint_{\mathbb{R}^6}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2dxdy\\&=\iint_{X_R^1}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2dxdy\\&\quad+\iint_{X_R^2}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2dxdy\\&\quad+\iint_{X_R^3}\frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}|u_n(x)|^2dxdy.\end{aligned} \tag{3.17}$$

Since $\eta_R=1$ in $\mathbb{R}^3\setminus B_{2R}$, we can see that

$$\iint_{X_R^1}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}dxdy=0. \tag{3.18}$$

Now, fix $k>4$, and we observe that

$$X_R^2=(\mathbb{R}^3\setminus B_{2R})\times B_{2R}\subset((\mathbb{R}^3\setminus B_{kR})\times B_{2R})\cup((B_{kR}\setminus B_{2R})\times B_{2R})$$

If $(x,y)\in(\mathbb{R}^3\setminus B_{kR})\times B_{2R}$, then

$$|x-y|\ge|x|-|y|\ge|x|-2R>\frac{|x|}{2}.$$

Therefore, using the above fact, $0\le\eta_R\le1$, $|\nabla\eta_R|\le\frac{C}{R}$ and applying the Hölder inequality we obtain

$$\begin{aligned}
&\iint_{X_R^2} \frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}dxdy \\
&= \int_{\mathbb{R}^3\setminus B_{kR}}\int_{B_{2R}} \frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}dxdy \\
&\quad + \int_{B_{kR}\setminus B_{2R}}\int_{B_{2R}} \frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}dxdy \\
&\le C\int_{\mathbb{R}^3\setminus B_{kR}}\int_{B_{2R}} \frac{|u_n(x)|^2}{|x|^{3+2s}}\,dxdy \\
&\quad + \frac{C}{R^2}\int_{B_{kR}\setminus B_{2R}}\int_{B_{2R}} \frac{|u_n(x)|^2}{|x-y|^{3+2(s-1)}}\,dxdy \\
&\le CR^3\int_{\mathbb{R}^3\setminus B_{kR}} \frac{|u_n(x)|^2}{|x|^{3+2s}}\,dx + \frac{C}{R^2}(kR)^{2(1-s)}\int_{B_{kR}\setminus B_{2R}}|u_n(x)|^2dx \\
&\le CR^3\left(\int_{\mathbb{R}^3\setminus B_{kR}}|u_n(x)|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}\left(\int_{\mathbb{R}^3\setminus B_{kR}}\frac{1}{|x|^{\frac{3^2}{2s}+3}}\,dx\right)^{\frac{2s}{3}} \\
&\quad + \frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\setminus B_{2R}}|u_n(x)|^2dx \\
&\le \frac{C}{k^3}\left(\int_{\mathbb{R}^3\setminus B_{kR}}|u_n(x)|^{2_s^*}dx\right)^{\frac{2}{2_s^*}} + \frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\setminus B_{2R}}|u_n(x)|^2dx \\
&\le \frac{C}{k^3}+\frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\setminus B_{2R}}|u_n(x)|^2dx.
\end{aligned} \tag{3.19}$$

Take $\xi \in (0,1)$, and we obtain

$$\begin{aligned}
&\iint_{X_R^3} \frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&\le \int_{B_{2R}\setminus B_{\xi R}}\int_{\mathbb{R}^3} \frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&\quad + \int_{B_{\xi R}}\int_{\mathbb{R}^3} \frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy.
\end{aligned} \tag{3.20}$$

Since

$$\int_{B_{2R}\setminus B_{\xi R}}\int_{\mathbb{R}^3\cap\{y:|x-y|<R\}}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}dxdy \le \frac{C}{R^{2s}}\int_{B_{2R}\setminus B_{\xi R}}|u_n(x)|^2dx$$

and

$$\int_{B_{2R}\setminus B_{\xi R}}\int_{\mathbb{R}^3\cap\{y:|x-y|\geq R\}}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\leq\frac{C}{R^{2s}}\int_{B_{2R}\setminus B_{\xi R}}|u_n(x)|^2dx,$$

we can see that

$$\int_{B_{2R}\setminus B_{\xi R}}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\leq\frac{C}{R^{2s}}\int_{B_{2R}\setminus B_{\xi R}}|u_n(x)|^2dx. \quad (3.21)$$

On the other hand, from the definition of η_R, $\xi\in(0,1)$, and $0\leq\eta_R\leq 1$ we obtain

$$\begin{aligned}
&\int_{B_{\xi R}}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\\
&=\int_{B_{\xi R}}\int_{\mathbb{R}^3\setminus B_R}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\\
&\leq C\int_{B_{\xi R}}\int_{\mathbb{R}^3\setminus B_R}\frac{|u_n(x)|^2}{|x-y|^{3+2s}}\,dxdy\\
&\leq C\int_{B_{\xi R}}|u_n|^2dx\int_{(1-\xi)R}^{\infty}\frac{1}{r^{1+2s}}dr\\
&=\frac{C}{[(1-\xi)R]^{2s}}\int_{B_{\xi R}}|u_n|^2dx,
\end{aligned} \quad (3.22)$$

where we used the fact that if $(x,y)\in B_{\xi R}\times(\mathbb{R}^3\setminus B_R)$ then $|x-y|>(1-\xi)R$. Then (3.20), (3.21) and (3.22) yield

$$\begin{aligned}
&\iint_{X_R^3}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\\
&\leq\frac{C}{R^{2s}}\int_{B_{2R}\setminus B_{\xi R}}|u_n(x)|^2dx+\frac{C}{[(1-\xi)R]^{2s}}\int_{B_{\xi R}}|u_n(x)|^2dx.
\end{aligned} \quad (3.23)$$

In view of (3.17), (3.18), (3.19) and (3.23) we can infer

$$\begin{aligned}
&\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\\
&\leq\frac{C}{k^3}+\frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\setminus B_{2R}}|u_n(x)|^2dx\\
&\quad+\frac{C}{R^{2s}}\int_{B_{2R}\setminus B_{\xi R}}|u_n(x)|^2dx+\frac{C}{[(1-\xi)R]^{2s}}\int_{B_{\xi R}}|u_n(x)|^2dx.
\end{aligned} \quad (3.24)$$

Since $(|u_n|)$ is bounded in $H^s(\mathbb{R}^3,\mathbb{R})$, using the Sobolev embeddings in Theorem 2.2, we may assume that $|u_n|\to|u|$ in $L^2_{\rm loc}(\mathbb{R}^3,\mathbb{R})$. Letting $n\to\infty$ in (3.24) we find

$$\begin{aligned}
&\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\\
&\leq \frac{C}{k^3}+\frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\setminus B_{2R}}|u(x)|^2dx\\
&\quad+\frac{C}{R^{2s}}\int_{B_{2R}\setminus B_{\xi R}}|u(x)|^2dx+\frac{C}{[(1-\xi)R]^{2s}}\int_{B_{\xi R}}|u(x)|^2dx\\
&\leq \frac{C}{k^3}+Ck^2\left(\int_{B_{kR}\setminus B_{2R}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}+C\left(\int_{B_{2R}\setminus B_{\xi R}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}\\
&\quad+C\left(\frac{\xi}{1-\xi}\right)^{2s}\left(\int_{B_{\xi R}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}},
\end{aligned}$$

where in the last passage we used the Hölder inequality. Since $|u|\in L^{2^*_s}(\mathbb{R}^3,\mathbb{R})$, $k>4$ and $\xi\in(0,1)$, we can see that

$$\limsup_{R\to\infty}\int_{B_{kR}\setminus B_{2R}}|u(x)|^{2^*_s}dx=\limsup_{R\to\infty}\int_{B_{2R}\setminus B_{\xi R}}|u(x)|^{2^*_s}dx=0.$$

Thus, taking $\xi=\frac{1}{k}$, we have

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{3+2s}}\,dxdy\\
&\leq \lim_{k\to\infty}\limsup_{R\to\infty}\left[\frac{C}{k^3}+Ck^2\left(\int_{B_{kR}\setminus B_{2R}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}+C\left(\int_{B_{2R}\setminus B_{\frac{R}{k}}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}\right.\\
&\quad\left.+C\left(\frac{1}{k-1}\right)^{2s}\left(\int_{B_{\frac{R}{k}}}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}\right]\\
&\leq \lim_{k\to\infty}\frac{C}{k^3}+C\left(\frac{1}{k-1}\right)^{2s}\left(\int_{\mathbb{R}^3}|u(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}}=0,
\end{aligned}$$

which implies that (3.16) holds true. Putting together (3.14), (3.15) and (3.16) we can deduce that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\int_{B_R^c}\left[\int_{\mathbb{R}^3}\frac{|u_n(x)-u_n(y)e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}|^2}{|x-y|^{3+2s}}dy\right] + \left(1-\frac{1}{\kappa}\right)V_\varepsilon(x)|u_n|^2dx=0,$$

and this completes the proof of (3.5). □

In view of Lemma 3.1, we can define the mountain pass level by

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} J_\varepsilon(\gamma(t)),$$

where

$$\Gamma_\varepsilon := \{\gamma\in C([0,1],H^s_\varepsilon) : \gamma(0)=0 \text{ and } J_\varepsilon(\gamma(1))<0\}.$$

By applying the mountain pass theorem [3], we can see that there exists $u_\varepsilon \in H^s_\varepsilon \setminus \{0\}$ such that $J_\varepsilon(u_\varepsilon)=c_\varepsilon$ and $J'_\varepsilon(u_\varepsilon)=0$. In a similar fashion, one can prove that also I_0 has a mountain pass geometry, and we denote by d_0 the mountain pass value of I_0.

Now, let us introduce the Nehari manifold associated with J_ε, that is

$$\mathcal{N}_\varepsilon := \{u\in H^s_\varepsilon\setminus\{0\} : \langle J'_\varepsilon(u),u\rangle = 0\},$$

and we denote by $\mathcal{M}_0$ the Nehari manifold associated with I_0.
It is standard to verify (see [51]) that c_ε can be also characterized as follows:

$$c_\varepsilon = \inf_{u\in H^s_\varepsilon\setminus\{0\}}\sup_{t\geq 0} J_\varepsilon(tu) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Next, we prove the existence of a ground state solution to (3.4).

Lemma 3.3. *Let $(u_n)\subset\mathcal{M}_0$ be a sequence satisfying $I_0(u_n)\to d_0$. Then, up to subsequences, the following alternative holds:*

(i) *(u_n) strongly converges in $H^s(\mathbb{R}^3,\mathbb{R})$;*
(ii) *there exists a sequence $(\tilde{y}_n)\subset\mathbb{R}^3$ such that, up to a subsequence, $v_n(x) := u_n(x+\tilde{y}_n)$ converges strongly in $H^s(\mathbb{R}^3,\mathbb{R})$.*

In particular, there exists a minimizer $w\in H^s(\mathbb{R}^3,\mathbb{R})$ for I_0 with $I_0(w)=d_0$.

Proof. Since I_0 has a mountain pass geometry, we can use a version of the mountain pass theorem without (PS) condition (see [51]), and we may suppose that (u_n) is a $(PS)_{d_0}$ sequence for I_0. Arguing as in Lemma 3.2, it is easy to check that (u_n) is bounded in $H^s(\mathbb{R}^3,\mathbb{R})$ and thus, up to a subsequence, we may assume that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3,\mathbb{R})$. The weak convergence is enough to deduce that $I'_0(u)=0$. Now, we suppose that $u \neq 0$. Since $u \in \mathcal{M}_0$, we can use (f_3) and Fatou's Lemma to see that

$$\begin{aligned} d_0 &\leq I_0(u) - \frac{1}{4}\langle I_0'(u), u\rangle \\ &= \frac{1}{4}\|u\|_\mu^2 + \frac{1}{2}\int_{\mathbb{R}^3} \frac{1}{2} f(u^2)u - F(u^2)\, dx \\ &\leq \liminf_{n\to\infty}\left[I_0(u_n) - \frac{1}{4}\langle I_0'(u), u\rangle\right] = d_0, \end{aligned}$$

which implies that $I_0(u) = d_0$.

Let us consider the case $u = 0$. Since $d_0 > 0$ and I_0 is continuous, we can see that $\|u_n\|_0 \not\to 0$. Then, in view of Lemma 2.5 and (f_1)-(f_2), it is standard to prove that there are a sequence $(y_n) \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^4 dx \geq \beta > 0.$$

Let us define $v_n := u_n(\cdot + y_n)$, and we note that v_n has a nontrivial weak limit v in $H^s(\mathbb{R}^3, \mathbb{R})$. It is also clear that (v_n) is a $(PS)_{d_0}$ sequence for I_0, and arguing as before we can deduce that $I_0(v) = d_0$. In conclusion, problem (3.4) admits a ground state solution.

Now, let u be a ground state for (3.4). Using $\varphi = u^- := \min\{u, 0\}$ as test function in $\langle I_0'(u), \varphi\rangle = 0$, it is easy to check that $u \geq 0$ in $\mathbb{R}^3$. In particular, observing that $\phi_u^t \geq 0$ and f has a subcritical growth, we can argue as in Proposition 5.1.1 in [22] to see that $u \in L^\infty(\mathbb{R}^3, \mathbb{R})$. In particular, we have

$$\begin{aligned} \phi_u^t(x) &= \int_{|y-x|\geq 1} \frac{|u(y)|^2}{|x-y|^{3-2t}} dy + \int_{|y-x|<1} \frac{|u(y)|^2}{|x-y|^{3-2t}} dy \\ &\leq \|u\|_{L^2(\mathbb{R}^3)}^2 + \|u\|_{L^\infty(\mathbb{R}^3)}^2 \int_{|y-x|<1} \frac{1}{|x-y|^{3-2t}} dy \leq C, \end{aligned}$$

so that $g(x) = f(u^2)u - \mu u - \phi_u^t u \in L^\infty(\mathbb{R}^3, \mathbb{R})$. By Proposition 2.9 in [46] and $s \in (\frac{3}{4}, 1)$, we can deduce that $u \in C^{1,\gamma}(\mathbb{R}^3, \mathbb{R})$ for any $\gamma < 2s - 1$. Using the maximum principle (see Corollary 3.4 in [24]) we can see that $u > 0$ in $\mathbb{R}^3$. Since $u \in C^{1,\gamma}(\mathbb{R}^3, \mathbb{R}) \cap L^p(\mathbb{R}^3, \mathbb{R})$ for all $p \in [2, \infty]$, we can deduce that $u(x) \to 0$ as $|x| \to \infty$, so we can find $R > 0$ such that $(-\Delta)^s u + \frac{V_0}{2} u \leq 0$ in $|x| > R$. By Lemma 4.3 in [26] we know that there exists a positive continuous function w such that for $|x| > R$ (taking R larger if necessary) it holds $(-\Delta)^s w + \frac{V_0}{2} w = 0$ and $w(x) \leq \frac{C_0}{|x|^{3+2s}}$, for some $C_0 > 0$. In view of the continuity of u and w there exists some constant $C_1 > 0$ such that $z := u - C_1 w \leq 0$ on $|x| = R$. Moreover, we can see that $(-\Delta)^s z + \frac{V_0}{2} z \geq 0$ in $|x| \geq R$. Then, it follows by the maximum principle that $z \leq 0$ in $|x| \geq R$, that is $0 < u(x) \leq C_1 w(x) \leq \frac{C_2}{|x|^{3+2s}}$ for all $|x|$ big enough. □

Now we prove the following interesting relation between c_ε and d_0.

Lemma 3.4. *The numbers c_ε and d_0 satisfy the following inequality*

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq d_0.$$

Proof. Let $w \in H^s(\mathbb{R}^3, \mathbb{R})$ be a positive ground state to the autonomous problem (3.4) (see Lemma 3.3), so $I_0'(w) = 0$ and $I_0(w) = d_0$. We recall that $w \in C^{1,\gamma}(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$ and that satisfies the following decay estimate:

$$0 < w(x) \leq \frac{C}{|x|^{3+2s}} \text{ for all } |x| > 1. \tag{3.25}$$

Let $\eta \in C_c^\infty(\mathbb{R}^3, [0, 1])$ be a cut-off function such that $\eta = 1$ in a neighborhood of zero $B_{\frac{\delta}{2}}$ and $\mathrm{supp}(\eta) \subset B_\delta \subset \Lambda$ for some $\delta > 0$.

Let us define $w_\varepsilon(x) := \eta_\varepsilon(x)w(x)e^{\imath A(0)\cdot x}$, with $\eta_\varepsilon(x) := \eta(\varepsilon x)$ for $\varepsilon > 0$, and we observe that $|w_\varepsilon| = \eta_\varepsilon w$, and $w_\varepsilon \in H^s_\varepsilon$ in virtue of Lemma 2.4. Now we prove that

$$\lim_{\varepsilon\to 0} \|w_\varepsilon\|^2_\varepsilon = \|w\|^2_0 \in (0, \infty). \tag{3.26}$$

Since it is clear that $\int_{\mathbb{R}^3} V_\varepsilon(x)|w_\varepsilon|^2 dx \to \int_{\mathbb{R}^3} V_0|w|^2 dx$, we only need to show that

$$\lim_{\varepsilon\to 0}[w_\varepsilon]^2_{A_\varepsilon} = [w]^2. \tag{3.27}$$

By Lemma 5 in [43] we know that

$$[\eta_\varepsilon w] \to [w] \text{ as } \varepsilon \to 0. \tag{3.28}$$

On the other hand,

$$\begin{aligned}
&[w_\varepsilon]^2_{A_\varepsilon} \\
&= \iint_{\mathbb{R}^6} \frac{|e^{\imath A(0)\cdot x}\eta_\varepsilon(x)w(x) - e^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}e^{\imath A(0)\cdot y}\eta_\varepsilon(y)w(y)|^2}{|x-y|^{3+2s}} dxdy \\
&= [\eta_\varepsilon w]^2 + \iint_{\mathbb{R}^6} \frac{\eta^2_\varepsilon(y)w^2(y)|e^{\imath[A_\varepsilon(\frac{x+y}{2})-A(0)]\cdot(x-y)} - 1|^2}{|x-y|^{3+2s}} dxdy \\
&\quad +2\Re\iint_{\mathbb{R}^6} \frac{(\eta_\varepsilon(x)w(x)-\eta_\varepsilon(y)w(y))\eta_\varepsilon(y)w(y)(1-e^{-\imath[A_\varepsilon(\frac{x+y}{2})-A(0)]\cdot(x-y)})}{|x-y|^{3+2s}} dxdy \\
&=: [\eta_\varepsilon w]^2 + X_\varepsilon + 2Y_\varepsilon.
\end{aligned}$$

Then, in view of $|Y_\varepsilon| \leq [\eta_\varepsilon w]\sqrt{X_\varepsilon}$ and (3.28), it is suffices to prove that $X_\varepsilon \to 0$ as $\varepsilon \to 0$ to deduce that (3.27) holds. Let us note that for $0 < \beta < \alpha/(1+\alpha-s)$,

$$\begin{aligned} X_\varepsilon &\leq \int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|\geq\varepsilon^{-\beta}} \frac{|e^{\imath[A_\varepsilon(\frac{x+y}{2})-A(0)]\cdot(x-y)}-1|^2}{|x-y|^{3+2s}}dx \\ &\quad + \int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{|e^{\imath[A_\varepsilon(\frac{x+y}{2})-A(0)]\cdot(x-y)}-1|^2}{|x-y|^{3+2s}}dx \\ &=: X_\varepsilon^1 + X_\varepsilon^2. \end{aligned} \tag{3.29}$$

Using $|e^{\imath t}-1|^2 \leq 4$ and $w \in H^s(\mathbb{R}^3, \mathbb{R})$, we get

$$X_\varepsilon^1 \leq C \int_{\mathbb{R}^3} w^2(y)dy \int_{\varepsilon^{-\beta}}^{\infty} \rho^{-1-2s} d\rho \leq C\varepsilon^{2\beta s} \to 0. \tag{3.30}$$

Since $|e^{\imath t}-1|^2 \leq t^2$ for all $t \in \mathbb{R}$, $A \in C^{0,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ for $\alpha \in (0,1]$, and $|x+y|^2 \leq 2(|x-y|^2+4|y|^2)$, we have

$$\begin{aligned} X_\varepsilon^2 &\leq \int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{|A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)|^2}{|x-y|^{3+2s-2}}dx \\ &\leq C\varepsilon^{2\alpha} \int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{|x+y|^{2\alpha}}{|x-y|^{3+2s-2}}dx \\ &\leq C\varepsilon^{2\alpha} \bigg(\int_{\mathbb{R}^3} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{1}{|x-y|^{3+2s-2-2\alpha}}dx \\ &\qquad\qquad + \int_{\mathbb{R}^3} |y|^{2\alpha} w^2(y)dy \int_{|x-y|<\varepsilon^{-\beta}} \frac{1}{|x-y|^{3+2s-2}}dx \bigg) \\ &=: C\varepsilon^{2\alpha}(X_\varepsilon^{2,1} + X_\varepsilon^{2,2}). \end{aligned} \tag{3.31}$$

Then

$$X_\varepsilon^{2,1} = C \int_{\mathbb{R}^3} w^2(y)dy \int_0^{\varepsilon^{-\beta}} \rho^{1+2\alpha-2s} d\rho \leq C\varepsilon^{-2\beta(1+\alpha-s)}. \tag{3.32}$$

On the other hand, using (3.25), we infer that

$$\begin{aligned} X_\varepsilon^{2,2} &\leq C \int_{\mathbb{R}^3} |y|^{2\alpha} w^2(y)dy \int_0^{\varepsilon^{-\beta}} \rho^{1-2s} d\rho \\ &\leq C\varepsilon^{-2\beta(1-s)} \left[\int_{B_1} w^2(y)dy + \int_{B_1^c} \frac{1}{|y|^{2(3+2s)-2\alpha}}dy \right] \\ &\leq C\varepsilon^{-2\beta(1-s)}. \end{aligned} \tag{3.33}$$

Taking into account (3.29), (3.30), (3.31), (3.32) and (3.33) we can conclude that $X_\varepsilon \to 0$. Therefore (3.26) holds. Moreover, since $\eta_\varepsilon w$ strongly converges to w in $H^s(\mathbb{R}^3, \mathbb{R})$, we can use Lemma 2.4-(5) in [38] to see that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \phi^t_{|w_\varepsilon|} |w_\varepsilon|^2 dx = \int_{\mathbb{R}^3} \phi^t_w w^2 dx. \tag{3.34}$$

Now, let $t_\varepsilon > 0$ be the unique number such that

$$J_\varepsilon(t_\varepsilon w_\varepsilon) = \max_{t\geq 0} J_\varepsilon(t w_\varepsilon).$$

Then t_ε satisfies

$$\begin{aligned} \|w_\varepsilon\|^2_\varepsilon + t^2_\varepsilon \int_{\mathbb{R}^3} \phi^t_{|w_\varepsilon|} |w_\varepsilon|^2 dx &= \int_{\mathbb{R}^3} g_\varepsilon(x, t^2_\varepsilon |w_\varepsilon|^2)|w_\varepsilon|^2 dx \\ &= \int_{\mathbb{R}^3} f(t^2_\varepsilon |w_\varepsilon|^2)|w_\varepsilon|^2 dx \end{aligned} \tag{3.35}$$

where we used $supp(\eta) \subset \Lambda$ and $g = f$ on Λ.
Let us prove that $t_\varepsilon \to 1$ as $\varepsilon \to 0$. Since $\eta = 1$ in $B_{\frac{\delta}{2}}$ and that w is a continuous positive function, we can see that (f_4) yields

$$\frac{1}{t^2_\varepsilon}\|w_\varepsilon\|^2_\varepsilon + \int_{\mathbb{R}^3} \phi^t_{|w_\varepsilon|} |w_\varepsilon|^2 dx \geq \frac{f(t^2_\varepsilon \alpha^2_0)}{t^2_\varepsilon \alpha^2_0} \int_{B_{\frac{\delta}{2}}} |w|^2 dx,$$

where $\alpha_0 := \min_{\bar{B}_{\frac{\delta}{2}}} w > 0$. Hence, if $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, we can use (f_3), (3.34) and (3.26) to deduce that $\int_{\mathbb{R}^3} \phi^t_{|w|}|w|^2 dx = \infty$, that is a contradiction. On the other hand, if $t_\varepsilon \to 0$ as $\varepsilon \to 0$, we can use the growth assumptions on g, (3.34), (3.26) to infer that $\|w\|^2_0 = 0$ which gives an absurd. Therefore, $t_\varepsilon \to t_0 \in (0, \infty)$ as $\varepsilon \to 0$. Now, taking the limit as $\varepsilon \to 0$ in (3.35) and using (3.34), (3.26), we can deduce that

$$\frac{1}{t^2_0}\|w\|^2_0 + \int_{\mathbb{R}^3} \phi^t_{|w|}|w|^2 dx = \int_{\mathbb{R}^3} \frac{f(t^2_0|w|^2)}{(t^2_0|w|^2)}|w|^4 dx. \tag{3.36}$$

Then $t_0 = 1$ as a consequence of $w \in \mathcal{M}_0$ and (f_4). Applying the Dominated Convergence Theorem we obtain that

$$\int_{\mathbb{R}^3} F(|t_\varepsilon w_\varepsilon|^2)\, dx \to \int_{\mathbb{R}^3} F(|w|^2)\, dx,$$

so we have $\lim_{\varepsilon\to 0} J_\varepsilon(t_\varepsilon w_\varepsilon) = I_0(w) = d_0$. Since $c_\varepsilon \leq \max_{t\geq 0} J_\varepsilon(t w_\varepsilon) = J_\varepsilon(t_\varepsilon w_\varepsilon)$, we can infer that $\limsup_{\varepsilon\to 0} c_\varepsilon \leq d_0$. □

Now, we prove the following useful compactness result:

Lemma 3.5. *Let* $\varepsilon_n \to 0$ *and* $(u_n) \subset H^s_{\varepsilon_n}$ *be such that* $J_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$ *and* $J'_{\varepsilon_n}(u_n) = 0$. *Then there exists* $(\tilde{y}_n) \subset \mathbb{R}^3$ *such that* $v_n(x) := |u_n|(x + \tilde{y}_n)$ *has a convergent subsequence in* $H^s(\mathbb{R}^3, \mathbb{R})$. *Moreover, up to a subsequence,* $y_n := \varepsilon_n \tilde{y}_n \to y_0$ *for some* $y_0 \in \Lambda$ *such that* $V(y_0) = V_0$ (i.e., $y_0 \in M$).

Proof. Since $\langle J'_{\varepsilon_n}(u_n), u_n \rangle = 0$, $J_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$ and using Lemma 3.4, we can see that (u_n) is bounded in $H^s_{\varepsilon_n}$. Then, there exists $C > 0$ (independent of n) such that $\|u_n\|_{\varepsilon_n} \le C$ for all $n \in \mathbb{N}$. Moreover, from Lemma 2.3, we also know that $(|u_n|)$ is bounded in $H^s(\mathbb{R}^3, \mathbb{R})$.

Now, we prove that there exist a sequence $(\tilde{y}_n) \subset \mathbb{R}^3$ and constants $R > 0$ and $\gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^2 \, dx \ge \gamma > 0. \tag{3.37}$$

Assume by contradiction (3.37) does not hold, so that, for all $R > 0$ we get

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \, dx = 0.$$

Using the boundedness of $(|u_n|)$ and Lemma 2.5, we know that $|u_n| \to 0$ in $L^q(\mathbb{R}^3, \mathbb{R})$ for any $q \in (2, 2^*_s)$. This fact and (g_1) and (g_2) yield

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 \, dx = 0 = \lim_{n\to\infty} \int_{\mathbb{R}^3} G_{\varepsilon_n}(x, |u_n|^2) \, dx. \tag{3.38}$$

On the other hand, $|u_n| \to 0$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3, \mathbb{R})$ and by Lemma 2.6-(4) we deduce that

$$\int_{\mathbb{R}^3} \phi^t_{|u_n|} |u_n|^2 dx \to 0. \tag{3.39}$$

Taking into account $\langle J'_{\varepsilon_n}(u_n), u_n \rangle = 0$, (3.38) and (3.39) we can infer that $\|u_n\|_{\varepsilon_n} \to 0$ as $n \to \infty$. This is impossible because (g_1), (g_2) and $\langle J'_{\varepsilon_n}(u_n), u_n \rangle = 0$ imply that there exists $\alpha_0 > 0$ such that $\|u_n\|^2_{\varepsilon_n} \ge \alpha_0$ for all $n \in \mathbb{N}$. Now, we set $v_n(x) := |u_n|(x + \tilde{y}_n)$. Then, (v_n) is bounded in $H^s(\mathbb{R}^3, \mathbb{R})$ and we may suppose that $v_n \rightharpoonup v \not\equiv 0$ in $H^s(\mathbb{R}^3, \mathbb{R})$ as $n \to \infty$. Fix $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{M}_0$. In view of Lemma 2.3 we have

$$d_0 \le I_0(\tilde{v}_n) \le \max_{t\ge 0} J_{\varepsilon_n}(t v_n) = J_{\varepsilon_n}(u_n),$$

which together with Lemma 3.4 yields $I_0(\tilde{v}_n) \to d_0$. Then, $\tilde{v}_n \nrightarrow 0$ in $H^s(\mathbb{R}^3, \mathbb{R})$. Since (v_n) and $(\tilde{v}_n)$ are bounded in $H^s(\mathbb{R}^3, \mathbb{R})$ and $\tilde{v}_n \nrightarrow 0$ in $H^s(\mathbb{R}^3, \mathbb{R})$, we deduce that $t_n \to t^* > 0$. From the uniqueness of the weak limit we can deduce that $\tilde{v}_n \rightharpoonup \tilde{v} = t^* v \not\equiv 0$ in $H^s(\mathbb{R}^3, \mathbb{R})$, and using Lemma 3.3 we can infer that

$$\tilde{v}_n \to \tilde{v} \text{ in } H^s(\mathbb{R}^3, \mathbb{R}). \tag{3.40}$$

Therefore, $v_n \to v$ in $H^s(\mathbb{R}^3, \mathbb{R})$ as $n \to \infty$.

Now, we define $y_n := \varepsilon_n \tilde{y}_n$ and we show that (y_n) admits a subsequence, still denoted by y_n, such that $y_n \to y_0$ for some $y_0 \in \Lambda$ satisfying $V(y_0) = V_0$. Firstly, we prove that (y_n) is bounded. Assume by contradiction that, up to a subsequence, $|y_n| \to \infty$ as $n \to \infty$. Take $R > 0$ such that $\Lambda \subset B_R$. Since we may suppose that $|y_n| > 2R$, we have that for any $z \in B_{R/\varepsilon_n}$

$$|\varepsilon_n z + y_n| \geq |y_n| - |\varepsilon_n z| > R.$$

Hence using $\langle J'_{\varepsilon_n}(u_n), u_n \rangle = 0$, (V_1), Lemma 2.3, Lemma 2.6 and the change of variable $x \mapsto z + \tilde{y}_n$ we obtain that

$$\begin{aligned}
[v_n]^2 + \int_{\mathbb{R}^3} V_0 v_n^2 \, dx &\leq [v_n]^2 + \int_{\mathbb{R}^3} V_0 v_n^2 \, dx + \int_{\mathbb{R}^3} \phi^t_{|v_n|} v_n^2 \, dx \\
&\leq \int_{\mathbb{R}^3} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2 dx \\
&\leq \int_{B_{\frac{R}{\varepsilon_n}}} \tilde{f}(|v_n|^2)|v_n|^2 dx \\
&\quad + \int_{\mathbb{R}^3 \setminus B_{\frac{R}{\varepsilon_n}}} f(|v_n|^2)|v_n|^2 dx.
\end{aligned} \tag{3.41}$$

Since $v_n \to v$ in $H^s(\mathbb{R}^3, \mathbb{R})$ as $n \to \infty$ and $\tilde{f}(t) \leq \frac{V_0}{\kappa}$, we can see that (3.41) yields

$$\min\left\{1, V_0\left(1 - \frac{1}{\kappa}\right)\right\} \left([v_n]^2 + \int_{\mathbb{R}^3} |v_n|^2 \, dx\right) = o_n(1),$$

that is $v_n \to 0$ in $H^s(\mathbb{R}^3, \mathbb{R})$ and this gives a contradiction. Thus, (y_n) is bounded and we may assume that $y_n \to y_0 \in \mathbb{R}^3$. If $y_0 \notin \overline{\Lambda}$, we can proceed as before to deduce that $v_n \to 0$ in $H^s(\mathbb{R}^3, \mathbb{R})$. Therefore $y_0 \in \overline{\Lambda}$. We observe that if $V(y_0) = V_0$, then $y_0 \notin \partial\Lambda$ in view of (V_2). Then, it is enough to verify that $V(y_0) = V_0$. Otherwise, if we suppose that $V(y_0) > V_0$, putting together (3.40), Fatou's Lemma, the invariance of $\mathbb{R}^3$ by translations, Lemma 2.3 and Lemma 3.4, we have

$$
\begin{aligned}
d_0 = I_0(\tilde{v}) &< \frac{1}{2}[\tilde{v}]^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(y_0)\tilde{v}^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|\tilde{v}|}\tilde{v}^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}|^2)\,dx \\
&\le \liminf_{n\to\infty}\Big[\frac{1}{2}[\tilde{v}_n]^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)|\tilde{v}_n|^2\,dx \\
&\qquad + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{|\tilde{v}_n|}|\tilde{v}_n|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}_n|^2)\,dx\Big] \\
&\le \liminf_{n\to\infty}\Big[\frac{t_n^2}{2}[|u_n|]^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^3} V(\varepsilon_n z)|u_n|^2\,dz \\
&\qquad + \frac{t_n^4}{4}\int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(|t_n u_n|^2)\,dz\Big] \\
&\le \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n) \le \liminf_{n\to\infty} J_{\varepsilon_n}(u_n) \le d_0,
\end{aligned}
$$

which is a contradiction. This ends the proof of this lemma. □

4. Proof of Theorem 1.1

This section is devoted to the proof of the main theorem of this work. Firstly, we prove the following lemma which plays a fundamental role to show that the solutions of (3.2) are indeed solutions to (3.1).

Lemma 4.1. *Let $\varepsilon_n \to 0$ and $u_n \in H^s_{\varepsilon_n}$ be a solution to* (3.2)*. Then, $v_n := |u_n|(\cdot + \tilde{y}_n)$ satisfies $v_n \in L^\infty(\mathbb{R}^3, \mathbb{R})$ and there exists $C > 0$ such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^3)} \le C \text{ for all } n \in \mathbb{N},$$

where $\tilde{y}_n$ is given by Lemma 3.5*. Moreover it holds*

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Proof. For each $n \in \mathbb{N}$ and $L > 0$, we define $u_{L,n} := \min\{|u_n|, L\} \ge 0$ and $v_{L,n} := u_{L,n}^{2(\beta-1)} u_n$, where $\beta > 1$ will be chosen later. Taking $v_{L,n}$ as test function in (3.2) we can see that

$$
\begin{aligned}
&\Re\Bigg(\iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \\
&\qquad \times \overline{(u_n(x)u_{L,n}^{2(\beta-1)}(x) - u_n(y)u_{L,n}^{2(\beta-1)}(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}\,dxdy\Bigg) \\
&= -\int_{\mathbb{R}^3} \phi^t_{|u_n|}|u_n|^2 u_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^3} g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 u_{L,n}^{2(\beta-1)}\,dx \\
&\quad - \int_{\mathbb{R}^3} V_{\varepsilon_n}(x)|u_n|^2 u_{L,n}^{2(\beta-1)}\,dx.
\end{aligned}
\tag{4.1}
$$

Let us observe that

$$\begin{aligned}
&\Re\Big[(u_n(x)-u_n(y)e^{\iota A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}) \\
&\times \overline{(u_n(x)u_{L,n}^{2(\beta-1)}(x)-u_n(y)u_{L,n}^{2(\beta-1)}(y)e^{\iota A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}\Big] \\
&=\Re\Big[|u_n(x)|^2u_{L,n}^{2(\beta-1)}(x)-u_n(x)\overline{u_n(y)}u_{L,n}^{2(\beta-1)}(y)e^{-\iota A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)} \\
&\quad -u_n(y)\overline{u_n(x)}u_{L,n}^{2(\beta-1)}(x)e^{\iota A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}+|u_n(y)|^2u_{L,n}^{2(\beta-1)}(y)\Big] \\
&\geq (|u_n(x)|^2u_{L,n}^{2(\beta-1)}(x)-|u_n(x)||u_n(y)|u_{L,n}^{2(\beta-1)}(y) \\
&\quad -|u_n(y)||u_n(x)|u_{L,n}^{2(\beta-1)}(x)+|u_n(y)|^2u_{L,n}^{2(\beta-1)}(y) \\
&=(|u_n(x)|-|u_n(y)|)(|u_n(x)|u_{L,n}^{2(\beta-1)}(x)-|u_n(y)|u_{L,n}^{2(\beta-1)}(y)),
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
&\Re\Big(\iint_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y)e^{\iota A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \\
&\quad\times\overline{(u_n(x)u_{L,n}^{2(\beta-1)}(x)-u_n(y)u_{L,n}^{2(\beta-1)}(y)e^{\iota A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}\,dxdy\Big) \qquad (4.2)\\
&\geq\iint_{\mathbb{R}^6}\frac{(|u_n(x)|-|u_n(y)|)}{|x-y|^{3+2s}}(|u_n(x)|u_{L,n}^{2(\beta-1)}(x)-|u_n(y)|u_{L,n}^{2(\beta-1)}(y))\,dxdy.
\end{aligned}$$

For all $t\geq 0$, let us define

$$\gamma(t):=\gamma_{L,\beta}(t)=tt_L^{2(\beta-1)}$$

where $t_L:=\min\{t,L\}$. Let us observe that, since γ is an increasing function, then it holds

$$(a-b)(\gamma(a)-\gamma(b))\geq 0 \quad \text{for any } a,b\in\mathbb{R}.$$

Let us define the functions

$$\Lambda(t):=\frac{|t|^2}{2} \quad\text{and}\quad \Gamma(t):=\int_0^t(\gamma'(\tau))^{\frac{1}{2}}d\tau$$

and we note that

$$\Lambda'(a-b)(\gamma(a)-\gamma(b))\geq|\Gamma(a)-\Gamma(b)|^2 \text{ for any } a,b\in\mathbb{R}. \qquad (4.3)$$

Indeed, for any $a, b \in \mathbb{R}$ such that $a < b$, and using the Jensen inequality we have

$$\begin{aligned}\Lambda'(a-b)(\gamma(a)-\gamma(b)) &= (a-b)\int_b^a \gamma'(t)dt = (a-b)\int_b^a (\Gamma'(t))^2 dt \\ &\geq \left(\int_b^a \Gamma'(t)dt\right)^2 = (\Gamma(a)-\Gamma(b))^2.\end{aligned}$$

In a similar fashion, we can prove that (4.3) holds true for any $a \geq b$.

In view of (4.3) we can deduce that

$$\begin{aligned}|\Gamma(|u_n(x)|)-\Gamma(|u_n(y)|)|^2 \leq (|u_n(x)|-|u_n(y)|)\Big(&|u_n(x)|u_{L,n}^{2(\beta-1)}(x) \\ &-|u_n(y)|u_{L,n}^{2(\beta-1)}(y)\Big).\end{aligned} \tag{4.4}$$

Putting together (4.2) and (4.4) we have

$$\begin{aligned}&\Re\Bigg(\iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}} \\ &\qquad\times \overline{(u_n(x)u_{L,n}^{2(\beta-1)}(x)-u_n(y)u_{L,n}^{2(\beta-1)}(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}\,dxdy\Bigg) \\ &\geq \iint_{\mathbb{R}^6} \frac{|\Gamma(|u_n(x)|)-\Gamma(|u_n(y)|)|^2}{|x-y|^{3+2s}}dxdy = [\Gamma(|u_n|)]^2.\end{aligned} \tag{4.5}$$

Since $\Gamma(|u_n|) \geq \frac{1}{\beta}|u_n|u_{L,n}^{\beta-1}$ and recalling that $D^{s,2}(\mathbb{R}^3,\mathbb{R}) \subset L^{2^*_s}(\mathbb{R}^3,\mathbb{R})$ (see [21]), we get

$$[\Gamma(|u_n|)]^2 \geq S_*\|\Gamma(|u_n|)\|^2_{L^{2^*_s}(\mathbb{R}^3)} \geq \left(\frac{1}{\beta}\right)^2 S_*\||u_n|u_{L,n}^{\beta-1}\|^2_{L^{2^*_s}(\mathbb{R}^3)}. \tag{4.6}$$

Taking into account (4.1), (4.5) and (4.6) we obtain

$$\begin{aligned}&\left(\frac{1}{\beta}\right)^2 S_*\||u_n|u_{L,n}^{\beta-1}\|^2_{L^{2^*_s}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V_{\varepsilon_n}(x)|u_n|^2 u_{L,n}^{2(\beta-1)}\,dx \\ &\leq \int_{\mathbb{R}^3} g_{\varepsilon_n}(x,|u_n|^2)|u_n|^2 u_{L,n}^{2(\beta-1)}\,dx.\end{aligned} \tag{4.7}$$

On the other hand, from assumptions (g_1) and (g_2), for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$g_\varepsilon(x,t^2)t^2 \leq \xi|t|^2 + C_\xi|t|^{2^*_s} \text{ for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{4.8}$$

Taking $\xi \in (0, V_0)$ and using (4.7), (4.8) and Lemma 2.6 we can infer that

$$\|w_{L,n}\|^2_{L^{2^*_s}(\mathbb{R}^3)} \leq C\beta^2 \int_{\mathbb{R}^3} |u_n|^{2^*_s} u_{L,n}^{2(\beta-1)}\,dx, \tag{4.9}$$

where we set $w_{L,n} := |u_n|u_{L,n}^{\beta-1}$. Now, take $\beta = \frac{2_s^*}{2}$ and fix $R > 0$. Observing that $0 \leq u_{L,n} \leq |u_n|$ and applying Hölder's inequality we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |u_n|^{2_s^*} u_{L,n}^{2(\beta-1)}\,dx &= \int_{\mathbb{R}^3} |u_n|^{2_s^*-2}|u_n|^2 u_{L,n}^{2_s^*-2}\,dx \\
&= \int_{\mathbb{R}^3} |u_n|^{2_s^*-2}\big(|u_n|u_{L,n}^{\frac{2_s^*-2}{2}}\big)^2\,dx \\
&\leq \int_{\{|u_n|<R\}} R^{2_s^*-2}|u_n|^{2_s^*}\,dx \\
&\quad + \int_{\{|u_n|>R\}} |u_n|^{2_s^*-2}\big(|u_n|u_{L,n}^{\frac{2_s^*-2}{2}}\big)^2\,dx \\
&\leq \int_{\{|u_n|<R\}} R^{2_s^*-2}|u_n|^{2_s^*}\,dx \\
&\quad + \left(\int_{\{|u_n|>R\}} |u_n|^{2_s^*}dx\right)^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\mathbb{R}^3} (|u_n|u_{L,n}^{\frac{2_s^*-2}{2}})^{2_s^*}dx\right)^{\frac{2}{2_s^*}}.
\end{aligned} \tag{4.10}$$

Since $(|u_n|)$ is bounded in $H^s(\mathbb{R}^3, \mathbb{R})$, we can choose R sufficiently large such that

$$\left(\int_{\{|u_n|>R\}} |u_n|^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\beta^2}. \tag{4.11}$$

In view of (4.9), (4.10) and (4.11) we can infer

$$\left(\int_{\mathbb{R}^3} \big(|u_n|u_{L,n}^{\frac{2_s^*-2}{2}}\big)^{2_s^*}\,dx\right)^{\frac{2}{2_s^*}} \leq C\beta^2 \int_{\mathbb{R}^3} R^{2_s^*-2}|u_n|^{2_s^*}\,dx < \infty$$

and letting $L \to \infty$ we obtain $|u_n| \in L^{\frac{(2_s^*)^2}{2}}(\mathbb{R}^3, \mathbb{R})$.
Now, using $0 \leq u_{L,n} \leq |u_n|$ and taking the limit as $L \to \infty$ in (4.9) we have

$$\||u_n|\|_{L^{2_s^*\beta}(\mathbb{R}^3)}^{2\beta} \leq C\beta^2 \int_{\mathbb{R}^3} |u_n|^{2_s^*+2(\beta-1)}\,dx,$$

from which we deduce that

$$\left(\int_{\mathbb{R}^3} |u_n|^{2_s^*\beta}\,dx\right)^{\frac{1}{2_s^*(\beta-1)}} \leq (C\beta)^{\frac{1}{\beta-1}} \left(\int_{\mathbb{R}^3} |u_n|^{2_s^*+2(\beta-1)}\,dx\right)^{\frac{1}{2(\beta-1)}}.$$

For $m \geq 1$ we define β_{m+1} inductively so that $2_s^* + 2(\beta_{m+1} - 1) = 2_s^*\beta_m$ and $\beta_1 = \frac{2_s^*}{2}$.

Then we can see that

$$\left(\int_{\mathbb{R}^3} |u_n|^{2_s^*\beta_{m+1}} dx\right)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leq (C\beta_{m+1})^{\frac{1}{\beta_{m+1}-1}} \left(\int_{\mathbb{R}^3} |u_n|^{2_s^*\beta_m}\right)^{\frac{1}{2_s^*(\beta_m-1)}}.$$

Let us define

$$D_m := \left(\int_{\mathbb{R}^3} |u_n|^{2_s^*\beta_m}\right)^{\frac{1}{2_s^*(\beta_m-1)}},$$

and using an iteration argument, we can find $C_0 > 0$ independent of m such that

$$D_{m+1} \leq \prod_{k=1}^{m} (C\beta_{k+1})^{\frac{1}{\beta_{k+1}-1}} D_1 \leq C_0 D_1.$$

Passing to the limit as $m \to \infty$ we find

$$\|u_n\|_{L^\infty(\mathbb{R}^3)} \leq C_0 D_1 =: K \text{ for all } n \in \mathbb{N}. \tag{4.12}$$

In what follows, we show that $|u_n|$ is a weak subsolution to

$$\begin{cases} (-\Delta)^s v + V_0 v = g(\varepsilon_n x, v^2) v & \text{in } \mathbb{R}^3 \\ v \geq 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.13}$$

Fix $\varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\varphi \geq 0$, and we take $\psi_{\delta,n} = \frac{u_n}{u_{\delta,n}}\varphi$ as test function in (3.1), where we set $u_{\delta,n} = \sqrt{|u_n|^2 + \delta^2}$ for $\delta > 0$. We note that $\psi_{\delta,n} \in H^s_{\varepsilon_n}$ for all $\delta > 0$ and $n \in \mathbb{N}$. Indeed $\int_{\mathbb{R}^3} V_{\varepsilon_n}(x)|\psi_{\delta,n}|^2 dx \leq \int_{\mathrm{supp}(\varphi)} V_{\varepsilon_n}(x)\varphi^2 dx < \infty$. On the other hand, we can observe

$$\begin{aligned}
&\psi_{\delta,n}(x) - \psi_{\delta,n}(y) e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)} \\
&= \left(\frac{u_n(x)}{u_{\delta,n}(x)}\right)\varphi(x) - \left(\frac{u_n(y)}{u_{\delta,n}(y)}\right)\varphi(y) e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)} \\
&= \left[\left(\frac{u_n(x)}{u_{\delta,n}(x)}\right) - \left(\frac{u_n(y)}{u_{\delta,n}(x)}\right) e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}\right]\varphi(x) \\
&+ [\varphi(x) - \varphi(y)] \left(\frac{u_n(y)}{u_{\delta,n}(x)}\right) e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)} \\
&+ \left(\frac{u_n(y)}{u_{\delta,n}(x)} - \frac{u_n(y)}{u_{\delta,n}(y)}\right)\varphi(y) e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}
\end{aligned}$$

which gives

$$
\begin{aligned}
&|\psi_{\delta,n}(x)-\psi_{\delta,n}(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}|^2\\
&\leq \frac{4}{\delta^2}|u_n(x)-u_n(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}|^2\|\varphi\|^2_{L^\infty(\mathbb{R}^3)}+\frac{4}{\delta^2}|\varphi(x)-\varphi(y)|^2\|u_n\|^2_{L^\infty(\mathbb{R}^3)}\\
&+\frac{4}{\delta^4}\|u_n\|^2_{L^\infty(\mathbb{R}^3)}\|\varphi\|^2_{L^\infty(\mathbb{R}^3)}|u_{\delta,n}(y)-u_{\delta,n}(x)|^2\\
&\leq \frac{4}{\delta^2}|u_n(x)-u_n(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}|^2\|\varphi\|^2_{L^\infty(\mathbb{R}^3)}+\frac{4K^2}{\delta^2}|\varphi(x)-\varphi(y)|^2\\
&+\frac{4K^2}{\delta^4}\|\varphi\|^2_{L^\infty(\mathbb{R}^3)}||u_n(y)|-|u_n(x)||^2,
\end{aligned}
$$

where we used

$$|z+w+k|^2\leq 4(|z|^2+|w|^2+|k|^2)\quad \forall z,w,k\in\mathbb{C},$$

$|e^{\imath t}|=1$ for all $t\in\mathbb{R}$, $u_{\delta,n}\geq\delta$, $|\frac{u_n}{u_{\delta,n}}|\leq 1$, (4.12) and the following inequality

$$|\sqrt{|z|^2+\delta^2}-\sqrt{|w|^2+\delta^2}|\leq ||z|-|w||\quad \forall z,w\in\mathbb{C}.$$

Since $u_n\in H^s_{\varepsilon_n}$, $|u_n|\in H^s(\mathbb{R}^3,\mathbb{R})$ (by Lemma 2.3) and $\varphi\in C^\infty_c(\mathbb{R}^3,\mathbb{R})$, we deduce that $\psi_{\delta,n}\in H^s_{\varepsilon_n}$.
Therefore

$$
\begin{aligned}
&\Re\Big[\iint_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}}\\
&\quad\times\Big(\frac{\overline{u_n(x)}}{u_{\delta,n}(x)}\varphi(x)-\frac{\overline{u_n(y)}}{u_{\delta,n}(y)}\varphi(y)e^{-\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}\Big)dxdy\Big]\\
&\quad+\int_{\mathbb{R}^3}V_{\varepsilon_n}(x)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx+\int_{\mathbb{R}^3}\phi^t_{|u_n|}\frac{|u_n|^2}{u_{\delta,n}}\varphi dx\\
&=\int_{\mathbb{R}^3}g_{\varepsilon_n}(x,|u_n|^2)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx.
\end{aligned}
\tag{4.14}
$$

Since $\Re(z) \leq |z|$ for all $z \in \mathbb{C}$ and $|e^{\imath t}| = 1$ for all $t \in \mathbb{R}$, we get

$$\begin{aligned}
&\Re\Bigg[(u_n(x) - u_n(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}) \\
&\quad \times \left(\overline{\frac{u_n(x)}{u_{\delta,n}(x)}}\varphi(x) - \overline{\frac{u_n(y)}{u_{\delta,n}(y)}}\varphi(y)e^{-\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}\right)\Bigg] \\
&= \Re\Bigg[\frac{|u_n(x)|^2}{u_{\delta,n}(x)}\varphi(x) + \frac{|u_n(y)|^2}{u_{\delta,n}(y)}\varphi(y) - \frac{u_n(x)\overline{u_n(y)}}{u_{\delta,n}(y)}\varphi(y)e^{-\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)} \\
&\quad - \frac{u_n(y)\overline{u_n(x)}}{u_{\delta,n}(x)}\varphi(x)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}\Bigg] \\
&\geq \Bigg[\frac{|u_n(x)|^2}{u_{\delta,n}(x)}\varphi(x) + \frac{|u_n(y)|^2}{u_{\delta,n}(y)}\varphi(y) - |u_n(x)|\frac{|u_n(y)|}{u_{\delta,n}(y)}\varphi(y) \\
&\quad - |u_n(y)|\frac{|u_n(x)|}{u_{\delta,n}(x)}\varphi(x)\Bigg].
\end{aligned} \tag{4.15}$$

Now, we can note that

$$\begin{aligned}
&\frac{|u_n(x)|^2}{u_{\delta,n}(x)}\varphi(x) + \frac{|u_n(y)|^2}{u_{\delta,n}(y)}\varphi(y) - |u_n(x)|\frac{|u_n(y)|}{u_{\delta,n}(y)}\varphi(y) - |u_n(y)|\frac{|u_n(x)|}{u_{\delta,n}(x)}\varphi(x) \\
&= \frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)| - |u_n(y)|)\varphi(x) - \frac{|u_n(y)|}{u_{\delta,n}(y)}(|u_n(x)| - |u_n(y)|)\varphi(y) \\
&= \left[\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)| - |u_n(y)|)\varphi(x) - \frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)| - |u_n(y)|)\varphi(y)\right] \\
&\quad + \left(\frac{|u_n(x)|}{u_{\delta,n}(x)} - \frac{|u_n(y)|}{u_{\delta,n}(y)}\right)(|u_n(x)| - |u_n(y)|)\varphi(y) \\
&= \frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)| - |u_n(y)|)(\varphi(x) - \varphi(y)) \\
&\quad + \left(\frac{|u_n(x)|}{u_{\delta,n}(x)} - \frac{|u_n(y)|}{u_{\delta,n}(y)}\right)(|u_n(x)| - |u_n(y)|)\varphi(y) \\
&\geq \frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)| - |u_n(y)|)(\varphi(x) - \varphi(y)),
\end{aligned} \tag{4.16}$$

where in the last inequality we used the fact that

$$\left(\frac{|u_n(x)|}{u_{\delta,n}(x)} - \frac{|u_n(y)|}{u_{\delta,n}(y)}\right)(|u_n(x)| - |u_n(y)|)\varphi(y) \geq 0$$

because

$$h(t) = \frac{t}{\sqrt{t^2 + \delta^2}} \text{ is increasing for } t \geq 0 \quad \text{and} \quad \varphi \geq 0 \text{ in } \mathbb{R}^3.$$

Observing that

$$\begin{aligned}&\frac{|\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}}\\&\leq\frac{||u_n(x)|-|u_n(y)||}{|x-y|^{\frac{3+2s}{2}}}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{\frac{3+2s}{2}}}\in L^1(\mathbb{R}^6),\end{aligned}$$

and $\frac{|u_n(x)|}{u_{\delta,n}(x)}\to 1$ a.e. in $\mathbb{R}^3$ as $\delta\to 0$, we can use (4.15), (4.16) and the Dominated Convergence Theorem to deduce that

$$\begin{aligned}&\limsup_{\delta\to 0}\Re\Bigg[\iint_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y)e^{\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{3+2s}}\\&\qquad\times\left(\frac{\overline{u_n(x)}}{u_{\delta,n}(x)}\varphi(x)-\frac{\overline{u_n(y)}}{u_{\delta,n}(y)}\varphi(y)e^{-\imath A_{\varepsilon_n}(\frac{x+y}{2})\cdot(x-y)}\right)dxdy\Bigg]\\&\geq\limsup_{\delta\to 0}\iint_{\mathbb{R}^6}\frac{|u_n(x)|}{u_{\delta,n}(x)}(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))\frac{dxdy}{|x-y|^{3+2s}}\\&=\iint_{\mathbb{R}^6}\frac{(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy.\end{aligned}\tag{4.17}$$

We can also see that the Dominated Convergence Theorem (we recall that $\frac{|u_n|^2}{u_{\delta,n}}\leq|u_n|$ and $\varphi\in C_c^\infty(\mathbb{R}^3,\mathbb{R})$) and Fatou's Lemma yield

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}V_{\varepsilon_n}(x)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx=\int_{\mathbb{R}^3}V_{\varepsilon_n}(x)|u_n|\varphi dx\geq\int_{\mathbb{R}^3}V_0|u_n|\varphi dx\tag{4.18}$$

$$\liminf_{\delta\to 0}\int_{\mathbb{R}^3}\phi^t_{|u_n|}\frac{|u_n|^2}{u_{\delta,n}}\varphi dx\geq\int_{\mathbb{R}^3}\phi^t_{|u|}|u|\varphi dx\geq 0\tag{4.19}$$

and

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}g_{\varepsilon_n}(x,|u_n|^2)\frac{|u_n|^2}{u_{\delta,n}}\varphi dx=\int_{\mathbb{R}^3}g_{\varepsilon_n}(x,|u_n|^2)|u_n|\varphi dx.\tag{4.20}$$

Taking into account (4.14), (4.17), (4.19), (4.18) and (4.20) we can infer that

$$\begin{aligned}&\iint_{\mathbb{R}^6}\frac{(|u_n(x)|-|u_n(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}dxdy+\int_{\mathbb{R}^3}V_0|u_n|\varphi dx\\&\leq\int_{\mathbb{R}^3}g_{\varepsilon_n}(x,|u_n|^2)|u_n|\varphi dx\end{aligned}$$

for any $\varphi\in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ such that $\varphi\geq 0$, that is $|u_n|$ is a weak subsolution to (4.13).

Now, we note that $v_n = |u_n|(\cdot + \tilde{y}_n)$ solves

$$(-\Delta)^s v_n + V_0 v_n \leq g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n^2) v_n \text{ in } \mathbb{R}^3. \tag{4.21}$$

Let us denote by $z_n \in H^s(\mathbb{R}^3, \mathbb{R})$ the unique solution to

$$(-\Delta)^s z_n + V_0 z_n = g_n \text{ in } \mathbb{R}^3, \tag{4.22}$$

where

$$g_n := g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n^2) v_n \in L^r(\mathbb{R}^3, \mathbb{R}) \quad \forall r \in [2, \infty].$$

Since (4.12) yields $\|v_n\|_{L^\infty(\mathbb{R}^3)} \leq C$ for all $n \in \mathbb{N}$, by interpolation we know that $v_n \to v$ strongly converges in $L^r(\mathbb{R}^3, \mathbb{R})$ for all $r \in [2, \infty)$, for some $v \in L^r(\mathbb{R}^3, \mathbb{R})$. From the growth assumptions on f, we have $g_n \to f(v^2)v$ in $L^r(\mathbb{R}^3, \mathbb{R})$ and $\|g_n\|_{L^\infty(\mathbb{R}^3)} \leq C$ for all $n \in \mathbb{N}$. In view of [26], we know that $z_n = \mathcal{K} * g_n$, where $\mathcal{K}$ is the Bessel kernel, and proceeding as in [2], we can infer that $|z_n(x)| \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$. Since v_n solves (4.21) and z_n verifies (4.22), it is easy to use a comparison argument to deduce that $0 \leq v_n \leq z_n$ a.e. in $\mathbb{R}^3$ and for all $n \in \mathbb{N}$. Therefore, $v_n(x) \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$. □

Now, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of Lemma 3.5, we can find $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $\varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in \Lambda$ satisfying $V(y_0) = V_0$. Then there is $r > 0$ such that, for some subsequence still denoted by itself, it holds $B_r(\tilde{y}_n) \subset \Lambda$ for all $n \in \mathbb{N}$. Thus, $B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$ for all $n \in \mathbb{N}$, and we can deduce that $\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$ for all $n \in \mathbb{N}$. By Lemma 4.1, we know that there exists $R > 0$ such that

$$v_n(x) < \sqrt{a} \quad \text{for all } |x| \geq R, n \in \mathbb{N},$$

where $v_n(x) := |u_{\varepsilon_n}|(x + \tilde{y}_n)$. Thus, $|u_{\varepsilon_n}(x)| < \sqrt{a}$ for any $x \in \mathbb{R}^N \setminus B_R(\tilde{y}_n)$ and $n \in \mathbb{N}$. On the other hand, there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ and $r/\varepsilon_n > R$ it holds

$$\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^3 \setminus B_R(\tilde{y}_n),$$

which gives $|u_{\varepsilon_n}(x)| < \sqrt{a}$ for any $x \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n}$ and $n \geq \nu$.

Therefore, there exists $\varepsilon_0 > 0$ such that problem (3.1) admits a nontrivial solution u_ε for all $\varepsilon \in (0, \varepsilon_0)$. Setting $\hat{u}_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$, we can see that $\hat{u}_\varepsilon$ is a solution to the original problem (1.1). Finally, we investigate the behavior of the maximum points of $|u_{\varepsilon_n}|$. Using (g_1), there exists $\gamma \in (0, \sqrt{a})$ small such that

$$g_\varepsilon(x, t^2)t^2 \leq \frac{V_0}{2} t^2, \text{ for all } x \in \mathbb{R}^3, |t| \leq \gamma. \tag{4.23}$$

Arguing as before, we can take $R > 0$ such that

$$\|u_{\varepsilon_n}\|_{L^\infty(B^c_R(\tilde{y}_n))} < \gamma. \tag{4.24}$$

Up to a subsequence, we may also assume that

$$\|u_{\varepsilon_n}\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{4.25}$$

Indeed, if (4.25) does not hold, we have $\|u_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^3)} < \gamma$, and using $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$, (4.23) and Lemma 2.3 we can see that

$$\begin{aligned}[][|u_{\varepsilon_n}|]^2 + \int_{\mathbb{R}^3} V_0|u_{\varepsilon_n}|^2 dx &\leq \|u_{\varepsilon_n}\|^2_{\varepsilon_n} + \int_{\mathbb{R}^3} \phi^t_{|u_{\varepsilon_n}|}|u_{\varepsilon_n}|^2 dx \\ &= \int_{\mathbb{R}^3} g_{\varepsilon_n}(x, |u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|^2\, dx \leq \frac{V_0}{2}\int_{\mathbb{R}^3} |u_{\varepsilon_n}|^2\, dx\end{aligned}$$

that is $\||u_{\varepsilon_n}|\|_{H^s(\mathbb{R}^3)} = 0$ which is a contradiction. Accordingly, (4.25) holds true. Let now p_n be a global maximum point of $|u_{\varepsilon_n}|$. In view of (4.24) and (4.25), we can see that p_n belongs to $B_R(\tilde{y}_n)$, that is $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R$. Since $\hat{u}_n(x) = u_{\varepsilon_n}(x/\varepsilon_n)$ is a solution to (1.1), we deduce that $\eta_{\varepsilon_n} = \varepsilon_n\tilde{y}_n + \varepsilon_n q_n$ is a global maximum point of $|\hat{u}_n|$. Thanks to $q_n \in B_R$, $\varepsilon_n\tilde{y}_n \to y_0$ and $V(y_0) = V_0$, we can use the continuity of V to infer that

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V_0.$$

Finally, we prove the power decay estimate of $|\hat{u}_n|$. Applying Lemma 4.3 in [26], we can find a function w such that

$$0 < w(x) \leq \frac{C}{1+|x|^{3+2s}}, \tag{4.26}$$

and

$$(-\Delta)^s w + \frac{V_0}{2} w = 0 \text{ in } B^c_{R_1} \tag{4.27}$$

for some suitable $R_1 > 0$. Invoking Lemma 4.1, we know that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$, and according to (f_1), we can find $R_2 > 0$ such that

$$h_n = g(\varepsilon_n x + \varepsilon_n\tilde{y}_n, v_n^2)v_n \leq \frac{V_0}{2} v_n \text{ in } B^c_{R_2}. \tag{4.28}$$

Let w_n be the unique solution to

$$(-\Delta)^s w_n + V_0 w_n = h_n \text{ in } \mathbb{R}^3.$$

Then, $w_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$, and by comparison $0 \le v_n \le w_n$ in $\mathbb{R}^3$. By (4.28) we can see that

$$(-\Delta)^s w_n + \frac{V_0}{2} w_n = h_n - \frac{V_0}{2} w_n \le 0 \text{ in } B^c_{R_2}.$$

Set $R_3 := \max\{R_1, R_2\}$ and we define

$$\sigma := \inf_{B_{R_3}} w > 0 \text{ and } \tilde{w}_n := (b+1)w - \sigma w_n, \tag{4.29}$$

where $b := \sup_{n\in\mathbb{N}} \|w_n\|_{L^\infty(\mathbb{R}^3)} < \infty$. Our aim is to prove that

$$\tilde{w}_n \ge 0 \text{ in } \mathbb{R}^3. \tag{4.30}$$

We first observe that

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} \tilde{w}_n(x) = 0, \tag{4.31}$$

$$\tilde{w}_n \ge b\sigma + w - b\sigma > 0 \text{ in } B_{R_3}, \tag{4.32}$$

$$(-\Delta)^s \tilde{w}_n + \frac{V_0}{2} \tilde{w}_n \ge 0 \text{ in } B^c_{R_3}. \tag{4.33}$$

Now assume by contradiction that there exists a sequence $(\bar{x}_{j,n}) \subset \mathbb{R}^3$ such that

$$\inf_{x\in\mathbb{R}^3} \tilde{w}_n(x) = \lim_{j\to\infty} \tilde{w}_n(\bar{x}_{j,n}) < 0. \tag{4.34}$$

Clearly, by (4.31), it follows that $(\bar{x}_{j,n})$ is bounded, and thus, up to subsequence, we may suppose that there exists $\bar{x}_n \in \mathbb{R}^3$ such that $\bar{x}_{j,n} \to \bar{x}_n$ as $j \to \infty$. Then, (4.34) implies that

$$\inf_{x\in\mathbb{R}^3} \tilde{w}_n(x) = \tilde{w}_n(\bar{x}_n) < 0. \tag{4.35}$$

From the minimality of $\bar{x}_n$ and the representation formula for the fractional Laplacian [21], we obtain that

$$(-\Delta)^s \tilde{w}_n(\bar{x}_n) = \frac{c_s}{2} \int_{\mathbb{R}^3} \frac{2\tilde{w}_n(\bar{x}_n) - \tilde{w}_n(\bar{x}_n + \xi) - \tilde{w}_n(\bar{x}_n - \xi)}{|\xi|^{3+2s}} d\xi \le 0. \tag{4.36}$$

In view of (4.32) and (4.34), we have $\bar{x}_n \in B^c_{R_3}$, and using (4.35) and (4.36), we can conclude that

$$(-\Delta)^s \tilde{w}_n(\bar{x}_n) + \frac{V_0}{2} \tilde{w}_n(\bar{x}_n) < 0,$$

which is impossible due to (4.33). Therefore, (4.30) holds true and using (4.26) and $v_n \le w_n$ we have

$$0 \le v_n(x) \le w_n(x) \le \frac{(b+1)}{\sigma} w(x) \le \frac{\tilde{C}}{1+|x|^{3+2s}} \text{ for all } n \in \mathbb{N}, x \in \mathbb{R}^3,$$

for some constant $\tilde{C} > 0$. Bearing in mind the definition of v_n, we can infer that

$$\begin{aligned}|\hat{u}_n|(x) = |u_{\varepsilon_n}|\left(\frac{x}{\varepsilon_n}\right) &= v_n\left(\frac{x}{\varepsilon_n} - \tilde{y}_n\right) \\ &\leq \frac{\tilde{C}}{1 + |\frac{x}{\varepsilon_n} - \tilde{y}_{\varepsilon_n}|^{3+2s}} \\ &= \frac{\tilde{C}\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \varepsilon_n \tilde{y}_{\varepsilon_n}|^{3+2s}} \\ &\leq \frac{\tilde{C}\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \eta_{\varepsilon_n}|^{3+2s}}.\end{aligned}$$

□

References

[1] C. O. ALVES, G. M. FIGUEIREDO and M. F. FURTADO, *Multiple solutions for a nonlinear Schrödinger equation with magnetic fields*, Comm. Partial Differential Equations **36** (2011), 1565–1586.

[2] C. O. ALVES and O. H. MIYAGAKI, *Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method*, Calc. Var. Partial Differential Equations **55**, art. 47, 19 (2016).

[3] A. AMBROSETTI and P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[4] V. AMBROSIO, *Multiplicity of positive solutions for a class of fractional Schrödinger equations via penalization method*, Ann. Mat. Pura Appl. (4) **196** (2017), 2043–2062.

[5] V. AMBROSIO, *Concentrating solutions for a class of nonlinear fractional Schrödinger equations in $\mathbb{R}^N$*, Rev. Mat. Iberoam. **35** (2019), 1367–1414.

[6] V. AMBROSIO, *Multiplicity and concentration results for a class of critical fractional Schrödinger-Poisson systems via penalization method*, Commun. Contemp. Math. **22** (2020), 1850078, 45 pp.

[7] V. AMBROSIO, *A local mountain pass approach for a class of fractional NLS equations with magnetic fields*, Nonlinear Anal. **190** (2020), 111622, 14 pp.

[8] V. AMBROSIO and P. D'AVENIA, *Nonlinear fractional magnetic Schrödinger equation: existence and multiplicity*, J. Differential Equations **264** (2018), 3336–3368.

[9] G. ARIOLI and A. SZULKIN, *A semilinear Schrödinger equation in the presence of a magnetic field*, Arch. Ration. Mech. Anal. **170** (2003), 277–295.

[10] V. BENCI and D. FORTUNATO, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonlinear Anal. **11** (1998), 283–293.

[11] R. BENGURIA, H. BRÉZIS and E. H. LIEB, *The Thomas-Fermi-von Weizsäker theory of atoms and molecules*, Comm. Math. Phys. **79** (1981) 167–180.

[12] J. BYEON and Z. Q. WANG, *Standing waves with a critical frequency for nonlinear Schrödinger equations, II*, Calc. Var. Partial Differential Equations, **18** (2003), 207–219.

[13] S. CINGOLANI, *Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field* J. Differential Equations **188** (2003), 52–79.

[14] S. CINGOLANI and S. SECCHI, *Semiclassical states for NLS equations with magnetic potentials having polynomial growths*, J. Math. Phys. **46** (2005), 19.

[15] S. CINGOLANI, S. SECCHI and M. SQUASSINA, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), 973–1009.

[16] T. D'APRILE and J. WEI, *On bound states concentrating on spheres for the Maxwell-Schrödinger equation*, SIAM J. Math. Anal. **37** (2005), 321–342.

[17] J. DÁVILA, M. DEL PINO, S. DIPIERRO and E. VALDINOCI, *Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum*, Anal. PDE **8** (2015), 1165–1235.

[18] J. DÁVILA, M. DEL PINO and J. WEI, *Concentrating standing waves for the fractional nonlinear Schrödinger equation*, J. Differential Equations **256** (2014), 858–892.

[19] P. D'AVENIA and M. SQUASSINA, *Ground states for fractional magnetic operators*, ESAIM Control Optim. Calc. Var. **24** (2018), 1–24.

[20] M. DEL PINO and P. L. FELMER, *Local Mountain Pass for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations, **4** (1996), 121–137.

[21] E. DI NEZZA, G. PALATUCCI and E. VALDINOCI, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[22] S. DIPIERRO, M. MEDINA and E. VALDINOCI, "Fractional Elliptic Problems with Critical Growth in the Whole of $\mathbb{R}^n$", Appunti, Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], 15. Edizioni della Normale, Pisa, 2017, viii+152 pp.

[23] M. ESTEBAN and P. L. LIONS, *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, In: "Partial Differential Equations and the Calculus of Variations", Vol. I, Progr. Nonlinear Differential Equations Appl., 1, Birkhäuser Boston, Boston, MA, 1989, 401–449.

[24] M. FALL and S. JAROHS, *Overdetermined problems with fractional Laplacian*, ESAIM Control Optim. Calc. Var. **21** (2015), 924–938.

[25] M. M. FALL, F. MAHMOUDI and E. VALDINOCI, *Ground states and concentration phenomena for the fractional Schrödinger equation*, Nonlinearity **28** (2015), 1937–1961.

[26] P. FELMER, A. QUAAS and J. TAN, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 1237–1262.

[27] A. FISCELLA, A. PINAMONTI and E. VECCHI, *Multiplicity results for magnetic fractional problems*, J. Differential Equations **263** (2017), 4617–4633.

[28] A. R. GIAMMETTA, *Fractional Schrödinger-Poisson-Slater system in one dimension*, preprint arXiv:1405.2796.

[29] X. HE, *Multiplicity and concentration of positive solutions for the Schrödinger-Poisson equations*, Z. Angew. Math. Phys. **62** (2011), 869–889.

[30] Y. HE and G. LI, *Standing waves for a class of Schrödinger-Poisson equations in $\mathbb{R}^3$ involving critical Sobolev exponents*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 729–766.

[31] F. HIROSHIMA, T. ICHINOSE and J. LÖRINCZI, *Kato's Inequality for Magnetic Relativistic Schrödinger Operators*, Publ. Res. Inst. Math. Sci. **53** (2017), 79–117.

[32] T. ICHINOSE, *Magnetic relativistic Schrödinger operators and imaginary-time path integrals*, In: "Mathematical Physics, Spectral Theory and Stochastic Analysis", Oper. Theory Adv. Appl., Vol. 232, Birkhäuser/Springer Basel AG, Basel, 2013, 247–297.

[33] T. KATO, *Schrödinger operators with singular potentials*, In: "Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972)", Israel J. Math. **13** (1973), 135–148.

[34] K. KURATA, *Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields*, Nonlinear Anal. **41** (2000), 763–778.

[35] L. D. LANDAU and E. M. LIFSHITZ, "Quantum Mechanics", Pergamon Press, 1977.

[36] N. LASKIN, *Fractional quantum mechanics and Lévy path integrals*, Phys. Letters A **268** (2000) 298–305.

[37] E. H. LIEB and M. LOSS, "Analysis", Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 1997.

[38] Z. LIU and J. ZHANG, *Multiplicity and concentration of positive solutions for the fractional Schrödinger-Poisson systems with critical growth*, ESAIM Control Optim. Calc. Var. **23** (2017), 1515–1542.

[39] P. MARKOWICH, C. RINGHOFER and C. SCHMEISER, "Semiconductor Equations", Springer-Verlag, New York, 1990.

[40] G. MOLICA BISCI, V. RĂDULESCU and R. SERVADEI, "Variational Methods for Nonlocal Fractional Problems", Cambridge University Press, Vol. 162, Cambridge, 2016.

[41] J. MOSER, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468.

[42] E. MURCIA and G. SICILIANO, *Positive semiclassical states for a fractional Schrödinger-Poisson system*, Differential Integral Equations **30** (2017), 231–258.

[43] G. PALATUCCI and A. PISANTE, *Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces*, Calc. Var. Partial Differential Equations **50** (2014), 799–829.

[44] P. H. RABINOWITZ, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.

[45] D. RUIZ, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), 655–674.

[46] L. SILVESTRE, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), 67–112.

[47] M. SQUASSINA and B. VOLZONE, *Bourgain-Brezis-Mironescu formula for magnetic operators*, C. R. Math. Acad. Sci. Paris **354** (2016), 825–831.

[48] E. M. STEIN, "Singular Integrals and Differentiability Properties of Functions", Princeton Mathematical Series, Vol. 30, Princeton University Press, Princeton, N.J. 1970.

[49] K. TENG, *Concentrating bounded states for fractional Schrödinger-Poisson system*, preprint arXiv:1710.03495.

[50] J. WANG, L. TIAN, J. XU and F. ZHANG, *Existence and concentration of positive solutions for semilinear Schrödinger-Poisson systems in* $\mathbb{R}^3$, Calc. Var. Partial Differential Equations **48** (2013) 275–276.

[51] M. WILLEM, "Minimax Theorems", Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

[52] J. ZHANG, M. DO Ó and M. SQUASSINA, *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity*, Adv. Nonlinear Stud. **16** (2016), 15–30.

[53] B. ZHANG, M. SQUASSINA and X. ZHANG, *Fractional NLS equations with magnetic field, critical frequency and critical growth*, Manuscripta Math. **155** (2018), 115–140.

[54] L. ZHAO and F. ZHAO, *On the existence of solutions for the Schrödinger-Poisson equations*, J. Math. Anal. Appl. **346** (2008), 155–169.

[55] A. ZHU and X. SUN, *Multiple solutions for Schrödinger-Poisson type equation with magnetic field*, J. Math. Phys. **56** (2015), 091504.

Dipartimento di Ingegneria Industriale
e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche, 12
60131 Ancona, Italia
v.ambrosio@univpm.it