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Applications of interpolation methods and Morrey spaces to elliptic PDEs

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Abstract. We study abstract classes of Morrey spaces generated by the Calderón-Lozanovskii product and investigate their interpolation properties. We also establish the stability of isomorphisms on interpolation scales of upper Calderón complex interpolation spaces. These studies are motivated by applications to elliptic differential equations which involve generalized Morrey spaces.

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1. Introduction

Morrey spaces play an important role in nonlinear potential analysis and harmonic analysis (see [1,2]). These Banach spaces were used for the first time by Morrey in [23] to prove that certain systems of partial differential equations have Hölder continuous solutions. We point out that Morrey spaces are also widely used in the study of the local behavior of solutions of partial differential equations including the Navier-Stokes equations (see [15,22,32]). Recently, more and more devices related to Morrey spaces are invented to investigate various problems in analysis. We notice that they are often quite close to Morrey spaces. The aim of this paper is to obtain interpolation results whose structure has something in common with Morrey spaces and then to apply the results to elliptic differential equations.

We recall that for $1 \leq q \leq p \leq \infty$, the Morrey space $\mathcal{M}_q^p := \mathcal{M}_q^p(\mathbb{R}^n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$ is defined as the space of all q-locally integrable functions f on $\mathbb{R}^n$ ($f \in L^q_{\rm loc}$ for short) such that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q \, dy\right)^{1/q} < \infty.$$

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Here $|A|$ stands for the measure of a Lebesgue measurable set A in $\mathbb{R}^n$, and $B(x,r)$ denotes the open ball in $\mathbb{R}^n$ centred at x of radius $r > 0$. In particular, by the Lebesgue Differentiation Theorem $\mathcal{M}_q^\infty = L^\infty$ with identical norms. In what follows, for simplicity of notation, we abbreviate $B(0,r)$ to $B(r)$.

Stampacchia [30], Campanato and Murthy [6] and Peetre [26] obtained some interpolation properties of the classical Morrey spaces as early as the 1960's. Lemarié-Rieusset [15,17] and Yuan, Sickel and Yang [33] studied complex interpolation of Morrey spaces. We mention that Lemarié-Rieusset [17] pointed out that if

$$1 \le q_j < p_j < \infty,\ j \in \{0,1\}, \quad \begin{aligned} 1/p &= (1-\theta)/p_0 + \theta/p_1, \\ 1/q &= (1-\theta)/q_0 + \theta/q_1, \end{aligned} \tag{1.1}$$

then for every $\theta \in (0,1)$,

$$[\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1}]_\theta \neq \mathcal{M}_q^p.$$

In the case where $q_0/p_0 = q_1/p_1$, Lemarié-Rieusset [17] obtained that

$$[\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1}]^\theta = \mathcal{M}_q^p$$

with identical norms. Here and below, $[\,\cdot\,]_\theta$ and $[\,\cdot\,]^\theta$ denote the lower and upper Calderón complex methods of interpolation, respectively.

Lemarié-Rieusset [16] also studied real interpolation of Morrey spaces. Moreover under condition (1.1), we have

$$(\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1})_{\theta,q} \hookrightarrow \mathcal{M}_q^p$$

with continuous inclusion (see [16, Theorem 3]). So, we may expect that the opposite inclusion is available once we choose p suitably in the above. However, Lemarié-Rieusset [16] also showed that

$$\mathcal{M}_q^p \hookrightarrow (\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1})_{\theta,\infty}$$

if and only if $q_0/p_0 = q_1/p_1$ (see [16,17,33] for more details).

We point out that for now there is no a complete description of the complex interpolation spaces between Morrey spaces for general parameters. This motivates the challenging question of describing these spaces. To answer this question we will need new ideas which we apply to more general abstract settings. In Section 2 we define abstract Morrey spaces generated by the Calderón-Lozanovskii product. We name them *abstract Morrey product type spaces*. We prove some general embedding properties between them. Section 3 contains applications of our results to the interpolation of Morrey spaces. In Section 4 we study the stability of isomorphic embeddings and surjections for the upper complex method $[\,\cdot\,]^\theta$ of interpolation for $\theta \in (0,1)$. Under some mild assumptions we estimate the modulus of injectivity. Combining these results, we prove the stability of invertible operators between these interpolation scales. The study of these stability properties is motivated by a problem arising in PDE's. In Section 5 we present applications of our results to elliptic differential equations, which involve generalized Morrey spaces and amalgam spaces (uniformly local Lebesgue spaces). We will work in non-homogeneous function spaces, which allows us to consider $\nabla(1-\Delta)^{-1/2}$ instead of $\partial_j(-\Delta)^{-1/2}$. Since $\nabla(1-\Delta)^{-1/2}$ is a sort of local singular integral operator, our function spaces will fall under the scope of the a priori estimates we will obtain here.

Throughout the paper we employ standard notation, in particular, for a Banach space E, we denote by $B(E)$ the closed unit ball. Given two nonnegative functions f and g defined on the same set A, we write $f \prec g$ or $g \succ f$, if there is a constant $c > 0$ such that $f(x) \leq cg(x)$ for all $x \in A$, while $f \asymp g$ means that both conditions $f \prec g$ and $g \prec f$ hold. If X and Y are topological linear spaces, then $X \hookrightarrow Y$ means that $X \subset Y$ and that the inclusion map is continuous. Let E and F be Banach spaces. For simplicity of notation, we write $E = F$ if $E \hookrightarrow F$ and $F \hookrightarrow E$. If $E = F$ with identical norms, then we write $E \cong F$.

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2. Abstract Morrey spaces

We use standard notions from interpolation theory from [4]. A mapping $\mathcal{F}$ acting on the class of all couples of Banach lattices is called a *positive interpolation functor* if for every couple $\vec{X} = (X_0, X_1)$ of Banach lattice $\mathcal{F}(\vec{X})$ is an intermediate Banach lattice with respect to $\vec{X}$ (*i.e.*, $X_0 \cap X_1 \subset \mathcal{F}(\vec{X}) \subset X_0 + X_1$), and we denote $T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})$ if $T\colon \vec{X} \to \vec{Y}$ is a positive operator between couples of Banach lattices (meaning that $T\colon X_0 + X_1 \to Y_0 + Y_1$ is linear and its restrictions $T\colon X_j \to Y_j$, $j \in \{0, 1\}$ are defined and are positive operators). If, in addition there is a constant $C > 0$ such that for every $T\colon \vec{X} \to \vec{Y}$

$$\|T\|_{\mathcal{F}(\vec{X})\to\mathcal{F}(\vec{Y})} \leq C \max\{\|T\|_{X_0\to Y_0}, \|T\|_{X_1\to Y_1}\},$$

then $\mathcal{F}$ is called *bounded* (and *exact* if $C = 1$).

We employ the Calderón-Lozanovskii spaces. Recall that if $\vec{X} = (X_0, X_1)$ is a couple of Banach lattices on $(\mathcal{X}, \mathcal{A}, \mu)$ and $\psi \in \mathcal{U}$ (*i.e.*, $\psi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is a positively homogeneous and concave function), then the Calderón-Lozanovskii space $\psi(\vec{X}) = \psi(X_0, X_1)$ consists of all $f \in L^0(\mu)$ such that $|f| \leq \lambda\psi(|f_0|, |f_1|)$ μ-a.e on $\mathcal{X}$ for some $\lambda > 0$ and some some $f_j \in B(X_j)$, $j \in \{0, 1\}$. The space $\psi(\vec{X})$ is a Banach lattice under the norm (see [19])

$$\|f\|_{\psi(\vec{X})} := \inf\big\{\lambda > 0 \,:\; |f| \leq \lambda\psi(|f_0|, |f_1|),\;\, f_0 \in B(X_0),\;\, f_1 \in B(X_1)\big\}.$$

In the case of the power function $\psi(s,t) := s^{1-\theta}t^{\theta}$ for all $s,t \geq 0$ with $0 < \theta < 1$, $\psi(\vec{X})$ is the well-known Calderón space which is denoted by $X_0^{1-\theta}X_1^{\theta}$ (see [5]). It should be noted that $\mathcal{F}$ is an exact positive interpolation functor [28].

Here we work on abstract Morrey spaces. Fix a measure space $(\mathcal{X}, \mathcal{A}, \mu)$, and let $\mathcal{D}$ be a countable family of subsets of a given index set. Often, $\mathcal{D}$ is used to denote the set of all dyadic cubes, but in our general setting $\mathcal{D}$ merely stands for a covering of some index set.

Assume that $\tau\colon \mathcal{D} \to \mathcal{A}$ is a monotone increasing set function with respect to inclusion, in the sense that $\tau(Q) \subset \tau(R)$ whenever $Q, R \in \mathcal{D}$ satisfy $Q \subset R$. Assume also that

$$\mathcal{X} = \bigcup_{Q\in\mathcal{D}} \tau(Q).$$

For every $Q \in \mathcal{D}$, suppose that we are given a Banach function lattice E^Q on $(\mathcal{X}, \mathcal{A}, \mu)$. We will write $E = \{E^Q\}_{Q\in\mathcal{D}}$.

We define the *abstract Morrey space* $\mathcal{M}^{\tau}(\{E^Q\}_{Q\in\mathcal{D}})$ (for simplicity of notation, we write $\mathcal{M}^{\tau}(\{E^Q\})$ for short) to be the space of all $f \in L^0(\mu)$ endowed with the norm

$$\|f\|_{\mathcal{M}^{\tau}(\{E^Q\}_{Q\in\mathcal{D}})} = \sup_{Q\in\mathcal{D}} \|f\chi_{\tau(Q)}\|_{E^Q}.$$

If $\mathcal{D} \subset \mathcal{A}$ and $\tau = \mathrm{id}_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{A}$ is the embedding, then we write $\mathcal{M}(\{E^Q\}_{Q\in\mathcal{D}})$ instead of $\mathcal{M}^{\tau}(\{E^Q\}_{Q\in\mathcal{D}})$.

Let $\Delta\colon \mathcal{D}\times\mathcal{D} \to \mathcal{A}$ be a function satisfying $\Delta(Q_1, Q_2) \subset \Delta(R_1, R_2)$ whenever the sets $Q_1, Q_2, R_1, R_2 \in \mathcal{D}$ satisfy $Q_1 \subset R_1$ and $Q_2 \subset R_2$. Assume that

$$\mathcal{X} = \bigcup_{Q_0, Q_1\in\mathcal{D}} \Delta(Q_0, Q_1).$$

Define the *abstract Morrey product type space* $\mathcal{M}_{\psi}^{\Delta}(\{E_0^Q\}_{Q\in\mathcal{D}}, \{E_1^Q\}_{Q\in\mathcal{D}})$ as the set of all $f \in L^0(\mu)$ for which there exist $\lambda > 0$, $\{f_0^Q\}_{Q\in\mathcal{D}} \in \prod_{Q\in\mathcal{D}} B(E_0^Q)$ and $\{f_1^Q\}_{Q\in\mathcal{D}} \in \prod_{Q\in\mathcal{D}} B(E_1^Q)$ such that

$$|f(x)|\chi_{\Delta(Q_0,Q_1)}(x) \leq \lambda\psi\big(|f_0^{Q_0}(x)|, |f_1^{Q_1}(x)|\big) \tag{2.1}$$

for μ-a.e. $x \in \mathcal{X}$. The quantity $\|f\|_{\mathcal{M}_{\psi}^{\Delta}(\{E_0^Q\}_{Q\in\mathcal{D}},\{E_1^Q\}_{Q\in\mathcal{D}})}$ stands for the infimum over all possible λ. If $\mathcal{D}$ and Δ satisfy $\Delta(Q_0, Q_1) = Q_0 \cap Q_1 \in \mathcal{D}$ for all $Q_0, Q_1 \in \mathcal{D}$, then we omit Δ in $\mathcal{M}_{\psi}^{\Delta}(\{E_0^Q\}_{Q\in\mathcal{D}}, \{E_1^Q\}_{Q\in\mathcal{D}})$ to write $\mathcal{M}_{\psi}(\{E_0^Q\}_{Q\in\mathcal{D}}, \{E_1^Q\}_{Q\in\mathcal{D}})$.

The case where $\mathcal{D}$ is a disjoint family and $\Delta(Q_1, Q_2) = Q_1 \cap Q_2$ is noteworthy. Then condition (2.1) above reads as: there exist $\lambda > 0$, $\{f_0^Q\}_{Q\in\mathcal{D}} \in \prod_{Q\in\mathcal{D}} B(E_0^Q)$ and $\{f_1^Q\}_{Q\in\mathcal{D}} \in \prod_{Q\in\mathcal{D}} B(E_1^Q)$ such that

$$|f|\chi_Q \leq \lambda\psi\big(|f_0^Q|, |f_1^Q|\big) \quad \mu - a.e.$$

for all $Q \in \mathcal{D}$. Since we can describe the Calderón-Lozanovskii product, we will be able to compute the complex interpolation between the function spaces generated by a disjoint family $\mathcal{D}$.

We have the following observation:

Proposition 2.1. *Let $\mathcal{M}^\tau(\{E_j^Q\}) := \mathcal{M}^\tau(\{E_j^Q\}_{Q\in\mathcal{D}})$ be an abstract Morrey space for $j \in \{0,1\}$. Then for any positive exact interpolation functor $\mathcal{F}$ the following continuous inclusion holds*:

$$\mathcal{F}(\mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}), \mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}})) \hookrightarrow \mathcal{M}^\tau(\{\mathcal{F}(E_0^Q, E_1^Q)\}_{Q\in\mathcal{D}}),$$

with norm of the inclusion map less than or equal to 1.

Proof. For a given $R \in \mathcal{D}$ we define a positive linear operator T_R by

$$T_R(f) := \chi_{\tau(R)} f, \quad f \in \mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}) + \mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}}).$$

It is clear that $T_R\colon (\mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}), \ \mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}})) \to (E_0^R, E_1^R)$ with $\|T_R\|_{\mathcal{M}^\tau(\{E_j^Q\})\to E_j^R} \le 1$ for each $j \in \{0,1\}$. Thus, by interpolation,

$$T_R\colon \mathcal{F}(\mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}), \mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}})) \to \mathcal{F}(E_0^R, E_1^R)$$

is bounded with norm less than or equal to 1. Hence

$$\|\chi_{\tau(R)} f\|_{\mathcal{F}(E_0^R,E_1^R)} \le \|f\|_{\mathcal{F}(\mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}),\mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}}))}$$

for all $R \in \mathcal{D}$. Since $R \in \mathcal{D}$ is arbitrary, it follows that

$$\begin{aligned}\|f\|_{\mathcal{M}^\tau(\{\mathcal{F}(E_0^Q,E_1^Q)\}_{Q\in\mathcal{D}})} &= \sup_{R\in\mathcal{D}} \|\chi_{\tau(R)} f\|_{\mathcal{F}(E_0^R,E_1^R)} \\ &\le \|f\|_{\mathcal{F}(\mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}),\mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}}))}. \end{aligned}$$ □

It should be noted that Proposition 2.1 generalizes the result due to Yuan, Sickel and Yang [33, page 1836], which states: Let $0 < \theta < 1$, $1 \le q \le p < \infty$, $1 \le q_0 \le p_0 < \infty$ and $1 \le q_1 \le p_1 < \infty$ satisfy (1.1), and let $\mathcal{F}$ be any interpolation functor of exponent θ such that $\mathcal{F}(L^{q_0}, L^{q_1}) \subset L^q$. Then T maps $\mathcal{F}(X_0, X_1)$ to $\mathcal{M}_q^p$ for any linear operator T bounded from X_0 to $\mathcal{M}_{q_0}^{p_0}$ and from X_1 to $\mathcal{M}_{q_1}^{p_1}$.

The following theorem is our originating point:

Theorem 2.2. *Let $\tau_j\colon \mathcal{D} \to \mathcal{A}$, $j = 0, 1$, be such that $\tau_j(Q) \subset \tau_j(R)$ for all $Q, R \in \mathcal{D}$ with $Q \subset R$. Assume that $\bigcup_{Q\in\mathcal{D}} \tau_0(Q) = \bigcup_{Q\in\mathcal{D}} \tau_1(Q) = \mathcal{X}$. Write $\Delta(Q,R) := \tau_0(Q) \cap \tau_1(R)$ for $Q, R \in \mathcal{D}$. Then*

$$\psi\big(\mathcal{M}^{\tau_0}(\{E_0^Q\}_{Q\in\mathcal{D}}), \mathcal{M}^{\tau_1}(\{E_1^Q\}_{Q\in\mathcal{D}})\big) \cong \mathcal{M}_\psi^\Delta\big(\{E_0^Q\}_{Q\in\mathcal{D}}, \{E_1^Q\}_{Q\in\mathcal{D}}\big).$$

In particular, when $\tau\colon \mathcal{D} \to \mathcal{D}$ is a mapping such that $\{\tau(Q)\}_{Q\in\mathcal{D}}$ forms a disjoint family, then

$$\psi\big(\mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}), \mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}})\big) \cong \mathcal{M}^\tau\big(\{\psi(E_0^Q, E_1^Q)\}_{Q\in\mathcal{D}}\big).$$

Proof. Let $f \in \mathcal{M}_\psi^\Delta\big(\{E_0^Q\}_{Q\in\mathcal{D}}, \{E_1^Q\}_{Q\in\mathcal{D}}\big)$ with norm less than 1. Then for each $Q \in \mathcal{D}$ there exist $f_0^Q \in E_0^Q$ and $f_1^Q \in E_1^Q$ such that for all $Q_0, Q_1 \in \mathcal{D}$,

$$|f|\chi_{\tau_0(Q_0)\cap\tau_1(Q_1)} \le \psi\big(|f_0^{Q_0}|, |f_1^{Q_1}|\big)$$

with $f_0^{Q_0}\chi_{\tau_0(Q_0)} \in B(E_0^{Q_0})$ and $f_1^{Q_1}\chi_{\tau_1(Q_1)} \in B(E_1^{Q_1})$. Define

$$f_0(x) := \inf\big\{|f_0^Q(x)| \,:\, Q \in \mathcal{D}, \quad x \in \tau_0(Q)\big\},$$
$$f_1(x) := \inf\big\{|f_1^Q(x)| \,:\, Q \in \mathcal{D}, \quad x \in \tau_1(Q)\big\}$$

for all $x \in \mathcal{X}$. Then our hypothesis $\bigcup\limits_{Q\in\mathcal{D}} \tau_0(Q) = \bigcup\limits_{Q\in\mathcal{D}} \tau_1(Q) = \mathcal{X}$ implies

$$|f| \le \psi(|f_0|, |f_1|), \qquad Q \in \mathcal{D},$$

and that

$$\|f_0\chi_{\tau_0(Q)}\|_{E_0^Q} \le \|f_0^Q\chi_{\tau_0(Q)}\|_{E_0^Q} \le 1, \qquad \|f_1\chi_{\tau_1(Q)}\|_{E_1^Q} \le \|f_1^Q\chi_{\tau_1(Q)}\|_{E_1^Q} \le 1.$$

Thus, it follows that $f \in B(\psi\big(\mathcal{M}^\tau(\{E_0^Q\}_{Q\in\mathcal{D}}), \mathcal{M}^\tau(\{E_1^Q\}_{Q\in\mathcal{D}})\big))$.

To prove the opposite continuous inclusion we fix

$$f \in \psi\big(\mathcal{M}^{\tau_0}(\{E_0^Q\}_{Q\in\mathcal{D}}), \mathcal{M}^{\tau_1}(\{E_1^Q\}_{Q\in\mathcal{D}})\big)$$

with norm less than or equal 1. Then we can find $f_0 \in B(\mathcal{M}^{\tau_0}(\{E_0^Q\}_{Q\in\mathcal{D}}))$ and $f_1 \in B(\mathcal{M}^{\tau_1}(\{E_1^Q\}_{Q\in\mathcal{D}}))$ such that

$$|f| \le \psi(|f_0|, |f_1|).$$

By letting $f_0^Q = f_0\chi_{\tau_0(Q)}$ and $f_1^Q = f_1\chi_{\tau_1(Q)}$ for every $Q \in \mathcal{D}$, we obtain

$$|f|\chi_{\Delta(Q_0,Q_1)} = |f|\chi_{\tau_0(Q_0)\cap\tau_1(Q_1)} \le \psi\big(|f_0^{Q_0}|, |f_1^{Q_1}|\big)$$

and $\|f_0^Q\|_{E_0^Q} = \|f_0\chi_{\tau_0(Q)}\|_{E_0^Q} \le 1$ and $\|f_1^Q\|_{E_1^Q} = \|f_1\chi_{\tau_1(Q)}\|_{E_1^Q} \le 1$. Thus, we conclude that

$$f \in \mathcal{M}_\psi^\Delta\big(\{E_0^Q\}_{Q\in\mathcal{D}}, \{E_1^Q\}_{Q\in\mathcal{D}}\big)$$

with norm less than or equal to 1. □

3. Application to the interpolation of abstract Morrey spaces

We present applications of Theorem 2.2. We check that Morrey spaces fall under the scope of this framework. Let $1 \leq q \leq p < \infty$. To show that Theorem 2.2 covers the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$, we denote by $\mathcal{D}$ the set of all dyadic cubes in $\mathbb{R}^n$. A prominent example of $E = \{E^Q\}_{Q\in\mathcal{D}}$, which recovers the classical Morrey space $\mathcal{M}_q^p$, is:

$$E^Q = E^Q(p,q) = \left\{ f \in L^q \,:\, \mathrm{supp}(f) \subset Q \right\}$$

and the norm is given by

$$\|f\|_{E^Q} = |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q}, \quad f \in E^Q.$$

In this case we have

$$\mathcal{M}_q^p = \mathcal{M}(\{E^Q(p,q)\}_{Q\in\mathcal{D}}).$$

We note that the set $E^Q(p,q)$ is independent of p as a set. However, its norm depends on p.

Based on this observation, we generalize the above observation as follows: Here and below in this section we once again work on a measure space $(\mathcal{X}, \mathcal{A}, \mu)$, where we are given a countable collection $\mathcal{D}$ of sets. The *abstract Morrey space* $\mathcal{M}_q^p(\mathcal{D})$ is the set of all $f \in L^0(\mu)$ for which

$$\|f\|_{\mathcal{M}_q^p(\mathcal{D})} = \sup_{Q\in\mathcal{D}} \mu(Q)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L^q} < \infty.$$

The following theorem explains why the output of the interpolation

$$(\mathcal{M}_{q_0}^{p_0}(\mathcal{D}))^{1-\theta}(\mathcal{M}_{q_1}^{p_1}(\mathcal{D}))^{\theta}$$

can be described within the framework of Morrey spaces when $p_0/q_0 = p_1/q_1$, which was proved in the case of classical Morrey spaces over Euclidean spaces [16, 17] and Morrey spaces over metric measure spaces [20]:

Theorem 3.1. *Let* $0 < \theta < 1$, $1 \leq q \leq p < \infty$, $1 \leq q_0 \leq p_0 < \infty$ *and* $1 \leq q_1 \leq p_1 < \infty$ *satisfy*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{q_0}{p_0} = \frac{q_1}{p_1}.$$

Then

$$(\mathcal{M}_{q_0}^{p_0}(\mathcal{D}))^{1-\theta}(\mathcal{M}_{q_1}^{p_1}(\mathcal{D}))^{\theta} = \mathcal{M}_q^p(\mathcal{D}).$$

We show that the above theorem is a consequence of Theorem 2.2.

Proof. By Theorem 2.2, it follows that the unit ball $B((\mathcal{M}^{p_0}_{q_0}(\mathcal{D}))^{1-\theta}(\mathcal{M}^{p_1}_{q_1}(\mathcal{D}))^{\theta})$ consists of all $f \in L^0(\mu)$ for which there exist $f_0^Q \in B(E^Q(p_0, q_0))$ and $f_1^Q \in B(E^Q(p_1, q_1))$ with

$$|f|\chi_{Q_0\cap Q_1} \le \left|f_0^{Q_0}\right|^{1-\theta}\left|f_1^{Q_1}\right|^{\theta}, \quad Q_0, Q_1 \in \mathcal{D}.$$

Let $r := \frac{p_0}{q_0} = \frac{p_1}{q_1} \in [1, \infty)$.

Now observe that the unit ball $B((\mathcal{M}^{p_0}_{q_0}(\mathcal{D}))^{1-\theta}(\mathcal{M}^{p_1}_{q_1}(\mathcal{D}))^{\theta})$ consists of all $f \in L^0(\mu)$ such that

$$|f|\chi_{Q_0\cap Q_1} \le \left|g_0^{Q_0}\right|^{\frac{1-\theta}{q_0}}\left|g_1^{Q_1}\right|^{\frac{\theta}{q_1}}, \quad Q_0, Q_1 \in \mathcal{D}.$$

for some $g_0^Q \in B(E^Q(r, 1))$ and $g_1^Q \in B(E^Q(r, 1))$. If we consider the minimum of the functions, we learn that this set coincides with the set of all $f \in L^0(\mu)$ for which there exist $g_0, g_1 \in L^0(\mu)$ with $g_0\chi_Q, g_1\chi_Q \in B(E^Q(r, 1))$ such that

$$|f|\chi_{Q_0\cap Q_1} \le |g_0\chi_{Q_0}|^{\frac{1-\theta}{q_0}}|g_1\chi_{Q_1}|^{\frac{\theta}{q_1}}, \quad Q_0, Q_1 \in \mathcal{D}.$$

But this set of functions consists of all $f \in L^0(\mu)$ for which we can find $g_0, g_1 \in L^0(\mu)$ with $g_0\chi_Q, g_1\chi_Q \in B(E^Q(r, 1))$ and

$$|f| \le \left|g_0\right|^{\frac{1-\theta}{q_0}}\left|g_1\right|^{\frac{\theta}{q_1}}.$$

To conclude we need only to observe that the above set coincides with of all $f \in L^0(\mu)$ such that

$$|f| \le |g|^{\frac{1-\theta}{q_0}+\frac{\theta}{q_1}}$$

for some $g \in L^0(\mu)$ such that $g\chi_Q \in B(E^Q(r, 1))$ for all $Q \in \mathcal{D}$. Obviously, this set is the unit ball $B(\mathcal{M}^p_q(\mathcal{D}))$, as required. □

If we consider the interpolation between Morrey spaces and L^∞, then we have the following result:

Theorem 3.2. *Suppose that we are given a collection of Banach spaces $\{E^Q\}_{Q\in\mathcal{D}}$. Then*

$$\psi(\mathcal{M}(\{E^Q\}_{Q\in\mathcal{D}}), \{L^\infty\}_{Q\in\mathcal{D}}) = \mathcal{M}(\{\psi(E^Q, L^\infty)\}_{Q\in\mathcal{D}})$$

with identical norms.

Proof. Based on Theorem 2.2, we calculate that:

$$\begin{aligned} &B\big(\psi\big(\mathcal{M}(\{E^Q\}_{Q\in\mathcal{D}}), \{L^\infty\}_{Q\in\mathcal{D}}\big)\big) \\ &= \bigcap_{Q\in\mathcal{D}} \left\{f \in L^0(\mu) \ : \ |f|\chi_Q \le \psi(f_0^Q, 1) \ \text{ for some } f_0^Q \in B(E_Q)\right\} \\ &= B\big(\mathcal{M}(\{\psi\big(E^Q, L^\infty)\}_{Q\in\mathcal{D}}\big)\big). \end{aligned}$$

□

We have the following corollaries which relate the Calderón-Lozanovskii product and generalized Orlicz–Morrey spaces.

Corollary 3.3. *Suppose that the parameters p, q satisfy $1 \le q \le p < \infty$. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be an Orlicz function* (i.e., $\Phi(0) = 0$ *and Φ is a non-zero convex function*), *and let $\psi(s,t) := t\Phi^{-1}(s/t)$, $t > 0$ and $\psi(s,t) := 0$ if $t = 0$, where Φ^{-1} is the right continuous inverse of Φ. Then we have*

$$\psi(\mathcal{M}^p_q(\mathcal{D}), L^\infty) = \bigcup_{\lambda>0} \left\{ f \in L^0(\mu) \,:\, \mu(Q)^{\frac{q}{p}-1} \int_Q \Phi(\lambda^{-1}|f(x)|)^q \, d\mu(x) \le 1,\, Q \in \mathcal{D} \right\}$$

and the norm is given by

$$\begin{aligned}\|f\|_{\psi(\mathcal{M}^p_q(\mathcal{D}),L^\infty)} &= \sup_{Q\in\mathcal{D}} \|\chi_Q f\|_{\psi(E^Q,L^\infty)} \\ &= \sup_{Q\in\mathcal{D}} \left[\inf\left\{ \lambda > 0 : \mu(Q)^{\frac{q}{p}-1} \int_Q \Phi(\lambda^{-1}|f(x)|)^q \, d\mu(x) \le 1 \right\} \right].\end{aligned}$$

Proof. Fix $Q \in \mathcal{D}$. Let $E^Q := L^q(Q)$ be the space of all q-integrable functions f with support Q, where the norm is given by $\|f\|_{E^Q} = \mu(Q)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q}$. Then we have

$$\begin{aligned}\|\chi_Q f\|_{\psi(E^Q,L^\infty)} &= \inf\left\{\lambda > 0 : \ \lambda^{-1}|f|\chi_Q \le \psi(|f_Q|, 1), \quad f_Q \in B(E^Q)\right\} \\ &= \inf\left\{\lambda > 0 : \ \Phi(\lambda^{-1}|f|\chi_Q) \le |f_Q|, \quad f_Q \in B(E^Q)\right\} \\ &= \inf\left\{\lambda > 0 : \ \Phi(\lambda^{-1}|f|\chi_Q) \in B(E^Q)\right\} \\ &= \inf\left\{\lambda > 0 : \ |Q|^{\frac{q}{p}-1} \int_Q \Phi(\lambda^{-1}|f(x)|)^q \, d\mu(x) \le 1\right\}. \qquad \square\end{aligned}$$

We point out that to the best knowledge of the authors, there are three classes of generalized Orlicz-Morrey spaces. Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space, and let $\Phi\colon [0,\infty) \times \mathcal{X} \to [0,\infty)$ and $\phi\colon \mathcal{D} \to (0,\infty)$ be suitable functions. For $Q \in \mathcal{D}$ define the (ϕ, Φ)-average over Q of $f \in L^0(\mu)$ by

$$\|f\|_{(\phi,\Phi):Q} := \inf\left\{\lambda > 0 : \ \frac{\phi(Q)}{\mu(Q)} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}, x\right) d\mu(x) \le 1\right\}.$$

Define the *generalized Orlicz-Morrey space* $\mathcal{L}_{\phi,\Phi}(\mathbb{R}^n)(\mu)$ of the first kind to be the Banach space of all $f \in L^0(\mu)$ such that $\|f\|_{\mathcal{L}_{\phi,\Phi}(\mathbb{R}^n)(\mu)} := \sup\{\|f\|_{(\phi,\Phi):Q} : Q \in \mathcal{D}\} < \infty$. For $Q \in \mathcal{D}$ define the Φ-*average* over Q of f by

$$\|f\|_{\Phi;Q} := \inf\left\{\lambda > 0 : \ \frac{1}{\mu(Q)} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}, x\right) d\mu(x) \le 1\right\}.$$

Define the *generalized Orlicz-Morrey space* $\widetilde{\mathcal{M}}_{\phi,\Phi}(\mathbb{R}^n)(\mathcal{D})$ of the second kind as the Banach space of all $f \in L^0(\mu)$ such hat

$$\|f\|_{\widetilde{\mathcal{M}}_{\phi,\Phi}(\mathbb{R}^n)(\mu)} := \sup\{\phi(Q)\|f\|_{\Phi;Q} : Q \in \mathcal{D}\} < \infty.$$

Assume that Φ is independent of x. Write $\Phi(t) = \Phi(t,x)$ for $t \geq 0$ and $x \in \mathcal{X}$.

The *generalized Orlicz-Morrey space* $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ of the third kind is defined as the set of all measurable functions f equipped with the norm:

$$\|f\|_{\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(\ell(Q))}\Phi^{-1}\left(\frac{1}{\mu(Q)}\right)\|f\|_{L^\Phi(Q)}.$$

In $\mathbb{R}^n$ the generalized Orlicz-Morrey spaces $\mathcal{L}_{\phi,\Phi}(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_{\phi,\Phi}(\mathbb{R}^n)$ and $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ are studied in [24, 25]. We point out that $\mathcal{L}_{\phi,\Phi}(\mathbb{R}^n)$ and $\widetilde{\mathcal{M}}_{\phi,\Phi}(\mathbb{R}^n)$ as well as $\widetilde{\mathcal{M}}_{\phi,\Phi}(\mathbb{R}^n)$ and $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ are different spaces in general (see [9]).

We can generalize Corollary 3.3 to an even wider class of spaces. We do not have to work on metric measure spaces and we can assume that ψ is a function of $s, t > 0$ and $x \in \mathcal{X}$. Namely for a given measure space $(\mathcal{X}, \mathcal{A}, \mu)$ we consider a function $\psi\colon [0,\infty) \times [0,\infty) \times \mathcal{X} \to [0,\infty)$ such that, $\psi(\cdot,\cdot,x) \in \mathcal{U}$ for every $x \in \mathcal{X}$ and every non-negative $\psi(s,t,\cdot) \in L^0(\mu)$. For example,

$$\psi(s,t,x) = s^{1-\theta(x)}t^{\theta(x)}, \quad (s,t,x) \in [0,\infty) \times [0,\infty) \times \mathcal{X},$$

where $\theta \in L^0(\mu)$ with $0 \leq \theta \leq 1$. We can define $\psi(X_0, X_1; x)$ analogously to $\psi(X_0, X_1)$.

Corollary 3.4. *Suppose that the parameters p, q satisfy $1 \leq q \leq p < \infty$. Define a bijective function $\Phi\colon [0,\infty) \times \mathcal{X} \to [0,\infty)$ so that for each $x \in \mathcal{X}$ the generalized inverse $\Phi^{-1}(\cdot, x)$ satisfies*

$$\Phi^{-1}(t,x) = \psi(t,1,x), \quad t \geq 0.$$

Then we have

$$\begin{aligned}&\psi(\mathcal{M}^p_q(\mathcal{D}), L^\infty; x)\\ &= \bigcup_{\lambda>0}\left\{f \in L^0(\mu) : \mu(Q)^{\frac{q}{p}-1}\int_Q \Phi(\lambda^{-1}|f(x)|,x)^q\,d\mu(x) \leq 1,\ Q \in \mathcal{D}\right\}\end{aligned}$$

and the norm is given by

$$\begin{aligned}\|f\|_{\psi(\mathcal{M}^p_q(\mathcal{D}),L^\infty;x)} &= \sup_{Q\in\mathcal{D}} \|\chi_Q f\|_{\psi(E^Q,L^\infty;x)}\\ &= \sup_{Q\in\mathcal{D}}\left[\inf\left\{\lambda > 0 :\ \mu(Q)^{\frac{q}{p}-1}\int_Q \Phi(\lambda^{-1}|f(x)|,x)^q\,d\mu(x) \leq 1\right\}\right].\end{aligned}$$

4. Stability of isomorphisms between upper complex spaces

The question of stability of isomorphisms when one changes the parameters that determine the lower complex interpolation space was first considered by Shneiberg [29]. In this section we study the stability of isomorphisms on interpolation scales of upper complex interpolation spaces. These studies are motivated by applications, in the next section, to elliptic differential equations which involve generalized Morrey spaces.

We will use the complex methods of interpolation introduced by Calderón in his fundamental paper [5]. Let $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ be an open strip in the complex plane. For a given $\theta \in (0,1)$ and any couple $\vec{X} = (X_0, X_1)$ we denote by $\mathcal{F}(\vec{X})$ the Banach space of all continuous functions $f \colon \overline{S} \to X_0 + X_1$ on the closure $\overline{S}$ that are analytic on S, and for which $\mathbb{R} \ni t \mapsto f(j+it) \in X_j$ is a bounded continuous function, for $j = 0, 1$. The space is endowed with the norm

$$\|f\|_{\mathcal{F}(\vec{X})} := \max_{j=0,1} \sup_{t\in\mathbb{R}} \|f(j+it)\|_{X_j}.$$

The lower complex interpolation space is defined by $[\vec{X}]_\theta := \{f(\theta) : \ f \in \mathcal{F}(\vec{X})\}$. It is equipped with the norm

$$\|x\|_{[\vec{X}]_\theta} := \inf\{\|f\|_{\mathcal{F}(\vec{X})} : \ f \in \mathcal{F}(\vec{X}),\, f(\theta) = x\}.$$

Since $[\vec{X}]_\theta$ is isometrically isomorphic with the quotient of $\mathcal{F}(\vec{X})$ by the closed subspace $\{f \in \mathcal{F}(\vec{X}) : f(\theta) = 0\}$, it is a Banach space.

Calderón defined a different interpolation method as follows. Let $\mathcal{G}(\vec{X})$ the Banach space of all continuous functions $g \colon \bar{S} \to X_0 + X_1$ that are analytic on S and grow no faster than $C(1+|z|)$ for some $C > 0$. We endow $\mathcal{G}(\vec{X})$ with the norm

$$\|g\|_{\mathcal{G}(\vec{X})} := \max_{j=0,1} \left\{ \sup_{-\infty<s<t<\infty} \frac{\|g(j+is) - g(j+it)\|_{X_j}}{|s-t|} \right\}.$$

The upper complex interpolation space is defined by $\big[\vec{X}\big]^\theta := \big\{g'(\theta) : g \in \mathcal{G}(\vec{X})\big\}$ and it is equipped with the quotient norm.

Throughout the paper when the complex methods are applied to a couple (X_0, X_1) of Banach lattices, we mean that $X_j := X_j(\mathbb{C})$ is a complexification of X_j for each $j = 0, 1$.

We recall that the *Gagliardo completion* or the *relative completion* of an intermediate with respect to $\vec{X}$ is the Banach space $X^\sim$ of all limits in $X_0{+}X_1$ of sequences that are bounded in X and endowed with the norm $\|x\|_{X^\sim} := \inf \sup_{n\geq 1} \|x_n\|_X$, where the infimum is taken over all bounded sequences $\{x_n\}$ in X whose limit in $X_0 + X_1$ equals x.

We need the following lemma from [21].

Lemma 4.1. *Let $\vec{X} = (X_0, X_1)$ be a complex Banach couple, and let $\theta \in (0,1)$.*

(i) $[X_0, X_1]^\theta \hookrightarrow [X_0, X_1]_\theta^\sim$ *with the norm of the continuous inclusion less or equal than* 1.

(ii) $[X_0, X_1]^\theta \cong [X_0, X_1]_\theta^\sim$ *if and only if* $B([X_0, X_1]^\theta)$ *is closed in* $X_0 + X_1$.

It is well known that the mapping $d\colon S \to \mathbb{D}$ defined by $d(z) := \tan\left(\frac{\pi}{2}\left(z - \frac{1}{2}\right)\right)$ for all $z \in S$ is a conformal map of the strip S onto the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. We are also going to need the following key estimate.

Theorem 4.2. *Let $\vec{X} = (X_0, X_1)$ be a complex Banach couple. Write*

$$q(z, \xi) := \frac{|d(z) - d(\xi)|}{|1 - \overline{d(z)}\, d(\xi)|}, \qquad z, \xi \in \mathbb{D}. \tag{4.1}$$

Then for all $g \in \mathcal{G}(\vec{X}) \setminus \{0\}$ and all $s, t \in (0, 1)$,

$$\|g'(t)\|_{[\vec{X}]^t} \geq \|g\|_{\mathcal{G}(\vec{X})} \frac{\|g'(s)\|_{[\vec{X}]_s^\sim} - q(s,t)\|g\|_{\mathcal{G}(\vec{X})}}{\|g\|_{\mathcal{G}(\vec{X})} - q(s,t)\|g'(s)\|_{[\vec{X}]_s^\sim}}.$$

Proof. Clearly, we need to consider the case $s < t$. In what follows we employ the following facts: if $a > 0$ and $q := q(s, t) \in (0, 1)$, then the function ρ_1 given by

$$(qa, \infty) \ni x \mapsto x \frac{a - qx}{x - qa}$$

is decreasing and that if $b > 0$ and $q := q(s, t)$, then the function ρ_2 given by

$$[0, b] \ni x \mapsto \frac{x - qb}{b - qx}$$

is increasing.

Invoking the invariant form of Schwarz's Lemma from complex analysis, it is shown in [28] that, for every $f \in \mathcal{F}(\vec{X}) \setminus \{0\}$ and all $s, t \in (0, 1)$,

$$\|f(t)\|_{[\vec{X}]_t} \geq \|f\|_{\mathcal{F}(\vec{X})} \frac{\|f(s)\|_{[\vec{X}]_s} - q(s,t)\|f\|_{\mathcal{F}(\vec{X})}}{\|f\|_{\mathcal{F}(\vec{X})} - q(s,t)\|f(s)\|_{[\vec{X}]_s}}. \tag{$*$}$$

Fix any $g \in \mathcal{G}(\vec{X}) \setminus \{0\}$. For a given $\varepsilon > 0$ we can find $f \in \mathcal{G}(\vec{X})$ such that $f'(t) = g'(t)$ and

$$\|f\|_{\mathcal{G}(\vec{X})} < \|g'(t)\|_{[\vec{X}]^t} + \varepsilon \leq \|g\|_{\mathcal{G}(\vec{X})} + \varepsilon.$$

For each positive integer n, let $f_n\colon \overline{S} \to \mathbb{C}$ be given by

$$f_n(z) = n(f(z + i/n) - f(z)), \qquad z \in \overline{S}.$$

Since $f \in \mathcal{G}(\vec{X})$ and $f_n \in \mathcal{F}(\vec{X})$ for each n,

$$\|f_n(s)\|_{[\vec{X}]_s} \le \|f_n\|_{\mathcal{F}(\vec{X})} \le \|f\|_{\mathcal{G}(\vec{X})} \le \|g\|_{\mathcal{G}(\vec{X})} + \varepsilon.$$

Combining all the above estimates with the monotonicity of ρ_1 yields

$$\begin{aligned}\varepsilon + \|g'(t)\|_{[\vec{X}]^t} &\ge \|f_n\|_{\mathcal{F}(\vec{X})} \frac{\|f_n(s)\|_{[\vec{X}]_s} - q(s,t)\|f_n\|_{\mathcal{F}(\vec{X})}}{\|f_n\|_{\mathcal{F}(\vec{X})} - q(s,t)\|f_n(s)\|_{[\vec{X}]_s}} \\ &\ge \big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big) \frac{\|f_n(s)\|_{[\vec{X}]_s} - q(s,t)\big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big)}{\big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big) - q(s,t)\|f_n(s)\|_{[\vec{X}]_s}}.\end{aligned}$$

Since $R := \sup_{n\in\mathbb{N}} \|f_n\|_{[\vec{X}]_s} < \infty$, we conclude from the monotonicity of ρ_2 that

$$\varepsilon + \|g'(t)\|_{[\vec{X}]^t} \ge \big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big) \frac{R - q(s,t)\big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big)}{\big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big) - q(s,t)R}.$$

From the definition of $g'(s)$, it follows that $\|f_n(s) - g'(s)\|_{X_0+X_1} \to 0$ and so $g'(s) \in [\vec{X}]_s^{\sim}$. This fact, combined with the above inequality and $\|g'(s)\|_{[\vec{X}]_s^{\sim}} \le R$, yields,

$$\varepsilon + \|g'(t)\|_{[\vec{X}]^t} \ge \big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big) \frac{\|g'(s)\|_{[\vec{X}]_s^{\sim}} - q(s,t)\big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big)}{\big(\|g\|_{\mathcal{G}(\vec{X})} + \varepsilon\big) - q(s,t)\|g'(t)\|_{[\vec{X}]_s^{\sim}}}.$$

Since $\varepsilon > 0$ was arbitrary, the desired estimate follows. □

We will obtain two corollaries.

Corollary 4.3. *Let $\vec{X} = (X_0, X_1)$ be a complex Banach couple. Then for all $g \in \mathcal{G}(\vec{X})$ and all $s, t \in (0,1)$,*

$$\|g'(t)\|_{[\vec{X}]^t} \ge \|g'(s)\|_{[\vec{X}]_s^{\sim}} - q(s,t)\|g\|_{\mathcal{G}(\vec{X})}.$$

Proof. We may assume $\|g\|_{\mathcal{G}(\vec{X})} > 0$ and $\|g'(s)\|_{[\vec{X}]_s^{\sim}} - q(s,g)\|g\|_{\mathcal{G}(\vec{\mathcal{X}})} > 0$; otherwise the conclusion is trivial. It follows from Theorem 4.2 that for all $s, t \in (0,1)$ we have

$$\|g'(t)\|_{[\vec{X}]^t} \ge \|g\|_{\mathcal{G}(\vec{X})} \frac{\|g'(s)\|_{[\vec{X}]_s^{\sim}} - q(s,t)\|g\|_{\mathcal{G}(\vec{X})}}{\|g\|_{\mathcal{G}(\vec{X})} - q(s,t)\|g'(s)\|_{[\vec{X}]_s^{\sim}}}.$$

Since $\|g\|_{\mathcal{G}(\vec{X})} - q(s,t)\|g'(s)\|_{[\vec{X}]_s^{\sim}} \ge 0$, we obtain the desired result. □

Corollary 4.4. *Assume that (X_0, X_1) is a complex Banach couple such that the $B([\vec{X}]^s)$ is closed in $X_0 + X_1$. Then for all $g \in \mathcal{G}(\vec{X}) \setminus \{0\}$ and all $s, t \in (0,1)$,*

$$\|g'(t)\|_{[\vec{X}]^t} \ge \|g\|_{\mathcal{G}(\vec{X})} \frac{\|g'(s)\|_{[\vec{X}]^s} - q(s,t)\|g\|_{\mathcal{G}(\vec{X})}}{\|g\|_{\mathcal{G}(\vec{X})} - q(s,t)\|g'(s)\|_{[\vec{X}]^s}}.$$

Proof. If we use Lemma 4.1, then we have $[\vec{X}]^s \cong [\vec{X}]_s^{\sim}$, and hence Theorem 4.2 applies. □

We recall that if $S\colon E \to F$ is a linear operator between Banach spaces, then the *modulus of injectivity* of S is defined by

$$j(S\colon E \to F) := \inf_{x\in B(E)} \|Sx\|_F.$$

Theorem 4.5. *Let $\theta_0, \theta \in (0,1)$. Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be couples of Banach spaces, and let $T\colon \vec{X} \to \vec{Y}$ be a bounded linear operator. Let $j_{\theta_0}(T) := j\big(T\colon [\vec{X}]^{\theta_0} \to [\vec{Y}]_{\theta_0}^{\sim}\big)$, and let $M := \|T\|_{\vec{X}\to\vec{Y}}$. Assume that $B([\vec{X}]^{\theta_0})$ and $B([\vec{X}]^{\theta})$ are closed in $X_0 + X_1$ and that $B([\vec{Y}]^{\theta_0})$ and $B([\vec{Y}]^{\theta})$ are closed in $Y_0 + Y_1$. Then*

$$j\big(T\colon [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) \geq M \max\left\{\frac{j_{\theta_0}(T) - q(\theta,\theta_0)M}{M - q(\theta,\theta_0)\, j_{\theta_0}(T)},\ 0\right\}.$$

Proof. Let $x \in B([\vec{X}]^{\theta})$. For a given $\varepsilon > 0$ we can find $g \in \mathcal{G}(\vec{X})$ such that $\|g\|_{\mathcal{G}(\vec{X})} \leq 1 + \varepsilon$ and $g'(\theta) = x$. From Theorem 4.2 and the monotonicity of ρ_1 mentioned in its proof, we deduce that

$$\|g'(\theta_0)\|_{[\vec{X}]^{\theta_0}} \geq \delta(\varepsilon) := (1+\varepsilon)\,\frac{1 - q(\theta,\theta_0)(1+\varepsilon)}{1 + \varepsilon - q(\theta,\theta_0)}.$$

We put $f(z) := T(g(z))$ for all $z \in \overline{S}$. Then we have $f \in \mathcal{G}(Y_0, Y_1)$ with $\|f\|_{\mathcal{G}(\vec{Y})} \leq (1+\varepsilon)M$. This implies (by $f'(z) = T(g'(z))$ for all $z \in S$),

$$\|f'(\theta_0)\|_{[\vec{Y}]_{\theta_0}^{\sim}} \geq j_{\theta_0}(T)\|g'(\theta_0)\|_{[\vec{X}]^{\theta_0}} \geq j_{\theta_0}(T)\delta(\varepsilon).$$

Applying Lemma 4.1 and Corollary 4.4, we obtain

$$\|f'(\theta)\|_{[\vec{Y}]^{\theta}} \geq (1+\varepsilon)M\,\frac{\|f'(\theta_0)\|_{[\vec{Y}]_{\theta_0}^{\sim}} - q(\theta,\theta_0)\,(1+\varepsilon)M}{(1+\varepsilon)M - q(\theta,\theta_0)\|f'(\theta_0)\|_{[\vec{Y}]_{\theta_0}^{\sim}}}.$$

Combining this with $\|f'(\theta_0)\|_{[\vec{Y}]_{\theta_0}^{\sim}} \geq j_{\theta_0}(T)\delta(\varepsilon)$ yields

$$\|f'(\theta)\|_{[\vec{Y}]^{\theta}} \geq M(1+\varepsilon)\,\frac{j_{\theta_0}(T)\delta(\varepsilon) - q(\theta,\theta_0)M(1+\varepsilon)}{M(1+\varepsilon) - q(\theta,\theta_0)\, j_{\theta_0}(T)\delta(\varepsilon)}.$$

Since $\delta(\varepsilon) \to 1$ as $\varepsilon \downarrow 0$, we deduce that

$$\|Tx\|_{[\vec{Y}]^{\theta}} = \|f'(\theta)\|_{[\vec{Y}]^{\theta}} \geq M\frac{j_{\theta_0}(T) - q(\theta,\theta_0)M}{M - q(\theta,\theta_0)\, j_{\theta_0}(T)},$$

which completes the proof. □

We apply Theorem 4.5 to see that T is injective.

Corollary 4.6. *Assume that $B([\vec{X}]^{\theta_0})$ and $B([\vec{X}]^{\theta})$ are closed in $X_0 + X_1$ and that $B([\vec{Y}]^{\theta_0})$ and $B([\vec{Y}]^{\theta})$ are closed in $Y_0 + Y_1$. Then under the assumptions of Theorem* 4.2, *the condition $j_{\theta_0}(T) > Mq(\theta, \theta_0)$ implies that $j\big(T : [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) > 0$,* i.e.*, $T : [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}$ is an isomorphic embedding.*

Observe that if $\theta_0 \in (0, 1)$, then $q(\theta, \theta_0) \to 0$ as $\theta \to \theta_0$. Thus we deduce $j_{\theta_0}(T) > 0$ implies that there exists $\varepsilon > 0$ such that $j\big(T : [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) > 0$ for all θ in (0, 1) with $|\theta - \theta_0| < \varepsilon$. Suppose that $B([\vec{X}]^{\frac{1}{2}})$ and $B([\vec{X}]^{\theta})$ are closed in $X_0 + X_1$ and that $B([\vec{Y}]^{\frac{1}{2}})$ and $B([\vec{Y}]^{\theta})$ are closed in $Y_0 + Y_1$. If $j_{1/2}(T) > 0$, then by

$$q(\theta, 1/2) = \left|\tan\left[\frac{\pi}{2}\Big(\theta - \frac{1}{2}\Big)\right]\right|,$$

we obtain a variant of Shneiberg's result [29] (proved for the lower complex method) for the upper method, which states that $j\big(T : [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) > 0$ provided that

$$\left|\theta - \frac{1}{2}\right| < \frac{2}{\pi}\arctan\left(\frac{j_{1/2}(T)}{M}\right).$$

To state the next result, we still need the following well-known technical result that can be found in [14], which is a part of the standard proof of the Open Mapping Theorem.

Lemma 4.7. *Let X and Y be Banach spaces and $T : X \to Y$ a linear operator. Suppose that there exist constants $M > 0$ and $0 < \varepsilon < 1$ with the following property: For every $y \in B(Y)$, there exists $x \in X$ with $\|x\|_X \le M$ and $\|Tx - y\|_Y < \varepsilon$. Then T is onto.*

Now we are ready to state the next result on the stability of isomorphisms.

Theorem 4.8. *Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be Banach couples such that, for all $s \in (0, 1)$, $B([\vec{X}]^{s})$ and $B([\vec{Y}]^{s})$ are closed in $X_0 + X_1$ and $Y_0 + Y_1$, respectively. Assume that $T : \vec{X} \to \vec{Y}$ is such that $T : [\vec{X}]^{\theta_0} \to [\vec{Y}]^{\theta_0}$ is an invertible operator. Then there exists a $\delta > 0$ such that $|\theta - \theta_0| < \delta$ implies that $T : [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}$ is an invertible operator.*

Proof. We considered injectivity in Theorem 4.5; we concentrate on surjectivity. Fix $y \in B([Y_0, Y_1]^{\theta})$. Then we can find $g \in \mathcal{G}(\vec{Y})$ such that $y = g'(\theta)$ with

$$\|g\|_{\mathcal{G}(\vec{Y})} \le 2\|y\|_{[\vec{Y}]^{\theta}} \le 2.$$

In particular this implies

$$\|g'(\theta_0)\|_{[\vec{Y}]^{\theta_0}} \le 2.$$

Since $T\colon [\vec{X}]^{\theta_0} \to [\vec{Y}]^{\theta_0}$ is an invertible operator by hypothesis, there exists $x_0 \in [\vec{X}]^{\theta_0}$ such that $Tx_0 = g'(\theta_0)$ with

$$\|x_0\|_{[\vec{X}]^{\theta_0}} \le \|T^{-1}\|_{[\vec{Y}]^{\theta_0} \to [\vec{X}]^{\theta_0}} \|g'(\theta_0)\|_{[\vec{Y}]^{\theta_0}} \le 2C,$$

where $C := \|T^{-1}\|_{[\vec{Y}]^{\theta_0} \to [\vec{X}]^{\theta_0}}$.

We choose $f \in \mathcal{G}(\vec{X})$ such that $x_0 = f'(\theta_0)$ with

$$\|f\|_{\mathcal{G}(\vec{X})} \le 2\|x_0\|_{[\vec{X}]^{\theta_0}}. \tag{4.2}$$

Now let $h := Tf - g \in \mathcal{G}(\vec{Y})$. Then $h'(z) = T(f'(z)) - g'(z)$ for all $z \in S$ and hence $h'(\theta_0) = 0$. Thus we obtain, by Corollary 4.3,

$$0 = \|h'(\theta_0)\|_{[\vec{Y}]^{\theta_0}} \ge \|h'(\theta)\|_{[\vec{Y}]^{\theta}} - d(\theta, \theta_0)\|g\|_{\mathcal{G}(\vec{Y})}.$$

Combining the above estimates gives that

$$\begin{aligned}\|T(f'(\theta)) - y\|_{[\vec{Y}]^{\theta}} &= \|T(f'(\theta)) - g'(\theta)\|_{[\vec{Y}]^{\theta}} = \|h'(\theta)\|_{[\vec{Y}]^{\theta}} \\ &\le d(\theta, \theta_0)\|g\|_{\mathcal{G}(\vec{Y})} \le 2d(\theta, \theta_0)\|y\|_{[\vec{Y}]^{\theta}} \le 2d(\theta, \theta_0).\end{aligned}$$

Since $d(\theta, \theta_0) \to 0$ as $\theta \to \theta_0$, we deduce that there exists $\delta > 0$ such that $|\theta - \theta_0| < \delta$ implies that

$$\|T(f'(\theta)) - y\|_{[\vec{Y}]^{\theta}} \le \frac{1}{2}.$$

To conclude, we observe that $f \in \mathcal{G}(\vec{X})$, combined with (4.2), imply that for all $\theta \in (0, 1)$ we have

$$\|f'(\theta)\|_{[\vec{X}]^{\theta}} \le 4\|T^{-1}\|_{[\vec{Y}]^{\theta_0} \to [\vec{X}]^{\theta_0}}$$

and so Lemma 4.7 applies with $M = 4\|T^{-1}\|_{[\vec{Y}]^{\theta_0} \to [\vec{X}]^{\theta_0}}$ and $\varepsilon = 1/2$. □

Corollary 4.9. *Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be couples of Banach lattices with the Fatou property. Assume that $T\colon \vec{X} \to \vec{Y}$ is such that $T\colon [\vec{X}]^{\theta_0} \to [\vec{Y}]^{\theta_0}$ is an invertible operator. Then there exists $\delta > 0$ such that $T\colon [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}$ is an invertible operator whenever $|\theta - \theta_0| < \delta$.*

Proof. It was shown in [21, Corollary 3.3] that for any couple $\vec{X} = (X_0, X_1)$ of Banach lattices and all $\theta \in (0, 1)$ we have that $[X_0, X_1]^{\theta} \cong X_0^{1-\theta} X_1^{\theta}$ and $X_0^{1-\theta} X_1^{\theta}$ is a maximal Banach lattice. In particular this implies that the unit ball of $[X_0, X_1]^{s}$ is closed in $X_0 + X_1$ for all $s \in (0, 1)$. To conclude it is enough to apply Theorem 4.8. □

5. Application to elliptic PDE with non-smooth coefficients

In recent years a lot of activity has been devoted to the boundary-value problems and layer potential operators associated to the elliptic differential operator $-\mathrm{div} A\nabla u$, where A is an elliptic matrix; that is, there are some real numbers $\nu_1 \geq \nu_0 > 0$ such that, if $\eta, \xi \in \mathbb{C}^{n+1}$ and if $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, then

$$\nu_0|\eta|^2 \leq \mathrm{Re}\,(\overline{\eta} \cdot A(x,t)\eta)\,, \quad |\xi \cdot A(x,t)\eta| \leq \nu_1|\xi||\eta|.$$

The following Dirichlet problem

$$\mathrm{div} A\nabla u = 0, \quad \text{in } \mathbb{R}^{n+1}_+ \text{ with } \mathrm{Tr}\, u = f \text{ on } \mathbb{R}^n,$$

where Tr denotes the trace operator, and the Neumann problem

$$\mathrm{div} A\nabla u = 0, \quad \text{in } \mathbb{R}^{n+1}_+ \text{ with } \nu \cdot A\nabla u = f \text{ on } \mathbb{R}^n$$

were studied by many authors with data in a special type of spaces (here we identify $\mathbb{R}^n$ with $\partial\mathbb{R}^{n+1}_+$).

In this section we will apply our generalized Morrey spaces to elliptic differential equations with non-smooth coefficients. We do not employ the heat kernel to solve this problem. A natural starting point in the study of problems mentioned above is when the coefficients $A(x,t) = A(x)$ are independent of the $(n+1)$st coordinate, often called the *t-coordinate*.

The celebrated solution of the Kato conjecture in [3] motivated, in recent years, the study of the above Dirichlet and Neumann problems with data in L^p and in Sobolev W^p spaces. We mention here that the Kato conjecture asserts that

$$\mathrm{Dom}\big(\sqrt{L_0}\big) = H^1(\mathbb{R}^n), \quad \big\|\sqrt{L_0} f\big\|_{L^2} \asymp \|\nabla f\|_{(L^2)^n},$$

where

$$L_0 := -\,\mathrm{div} A\nabla = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right).$$

We consider an elliptic differential operator L with non-smooth coefficients generated by a symmetric matrix $A = [a_{ij}]_{i,j=1}^n \in (L^\infty(\mathbb{R}^n))^{n^2}$ given by

$$L := I + L_0 = I - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right).$$

Here and below I stands for the identity operator in a Banach space X. For given $1 < q \leq p < \infty$ we define the *non-local Morrey space* $M^p_q(\mathbb{R}^n)$ to be the space of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ endowed with the norm given by

$$\|f\|_{M^p_q} := \sup_{x\in\mathbb{R}^n, r\geq 1} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B(x,r)} |f(y)|^q\, dy\right)^{\frac{1}{q}}.$$

When $1 < q < p < \infty$, one has that $\mathcal{M}^p_q(\mathbb{R}^n)$ is a proper subspace of $M^p_q(\mathbb{R}^n)$ as the example of the function $f(x) = |x|^{-\frac{1}{2p}-\frac{1}{2q}}$ shows.

We observe that $M_q^p(\mathbb{R}^n)$ falls under the scope of generalized Morrey spaces [27]. In fact we have $M_q^p(\mathbb{R}^n) = \mathcal{M}_q^\varphi(\mathbb{R}^n)$ with $\varphi(t) = \min\{t^{n/q}, t^{n/p}\}$ for all $t > 0$, where for a given function $\varphi\colon (0,\infty) \to (0,\infty)$, $\mathcal{M}_q^\varphi$ is defined to be the Banach space of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ endowed with the norm

$$\|f\|_{\mathcal{M}_q^\varphi} := \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)\left(\frac{1}{|Q(x,r)|}\int_{Q(x,r)} |f(y)|^q\,dy\right)^{\frac{1}{q}}.$$

Let $s \in \mathbb{R}$ and $1 < p < \infty$. Throughout this section we define $W^s M_2^p(\mathbb{R}^n)$ to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $(I-\Delta)^{\frac{s}{2}} f \in M_2^p(\mathbb{R}^n)$. One defines

$$\|f\|_{W^s M_2^p} := \|(1-\Delta)^{\frac{s}{2}} f\|_{M_2^p}, \qquad f \in W^s M_2^p(\mathbb{R}^n).$$

This type of generalized Sobolev space falls under the scope of [13, 18], so that $W^s M_q^p(\mathbb{R}^n)$ is a Banach space such that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow W^s M_q^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

We notice that singular integral operators such as the Riesz transform are bounded on $M_q^p(\mathbb{R}^n) = \mathcal{M}_q^\varphi(\mathbb{R}^n)$ according to the criterion in [11] and so

$$\|f\|_{W^1 M_q^p} \prec \|f\|_{M_q^p} + \|\nabla f\|_{(M_q^p)^n}$$

for all $f \in M_q^p(\mathbb{R}^n)$ with $\nabla f \in M_q^p(\mathbb{R}^n)^n$.

Another observation is that for any given $a > 1$ and all $f \in M_q^p(\mathbb{R}^n)$, we have

$$\|f\|_{M_q^p} \asymp \|f\|_{M_q^p;a} := \sup_{x\in\mathbb{R}^n, r\ge a} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B(x,r)} |f(y)|^q\,dy\right)^{\frac{1}{q}}$$

for all measurable functions f. It is easy to show that $u \in W^1 M_2^p(\mathbb{R}^n) \mapsto Lu \in W^{-1} M_2^p(\mathbb{R}^n)$ is bounded. We note that $W^s M_q^p(\mathbb{R}^n)$ is a variant studied extensively in [33].

Remark that L maps $W^1 M_2^p(\mathbb{R}^n)$ boundedly to $W^{-1} M_2^p(\mathbb{R}^n)$. In fact, for any $u \in W^1 M_2^p(\mathbb{R}^n)$, we have $\nabla u \in (M_2^p(\mathbb{R}^n))^n$. Thus, $A \cdot \nabla u \in (M_2^p(\mathbb{R}^n))^n$, so it remains to combine this with the boundedness of $M_q^p(\mathbb{R}^n) \ni f \mapsto \nabla f \in W^{-1} M_q^p(\mathbb{R}^n)^n$.

We will need the estimate established in the following lemma:

Lemma 5.1. *Let $p \ge 2$. Then for all $u \in W^1 M_2^p(\mathbb{R}^n)$*

$$\|u\|_{M_2^p} + \|Du\|_{(M_2^p)^n} \prec \|Lu\|_{W^{-1} M_2^p}.$$

Proof. Fix $\varepsilon \in (0, 3^{-n}\nu_0)$. It suffices to show that

$$\|u\|_{M_2^p;\varepsilon^{-1}} + \|Du\|_{(M_2^p;\varepsilon^{-1})^n} \prec \|Lu\|_{W^{-1} M_2^p;\varepsilon^{-1}}$$

for all $u \in W^1 M_2^p(\mathbb{R}^n)$. Let ψ be a bump function that equals 1 on a cube Q of volume ε^{-n} and vanishes outside its triple $3Q$ and satisfies $\|\nabla\psi\|_\infty \le M_1\varepsilon$. We put $f := Lu \in W^{-1}M_2^p(\mathbb{R}^n)$. Then we have

$$\langle f, \psi u\rangle = \int_{\mathbb{R}^n}\bigg(u(x)^2\psi(x) + \psi(x)\sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j u(x) \\ + \sum_{i,j=1}^n a_{ij}(x)u(x)\partial_i u(x)\partial_j\psi(x)\bigg)\,dx.$$

Thus we have

$$\begin{aligned}&\nu_0\bigg(\|u\|_{L^2(Q)}^2 + \sum_{j=1}^n \|\partial_j u\|_{L^2(Q)}^2\bigg) - \varepsilon\sum_{j=1}^n \|\partial_j u\|_{L^2(3Q)}^2 - \varepsilon\|u\|_{L^2(3Q)}^2 \\ &\le \int_{\mathbb{R}^n}\psi(x)\bigg(u(x)^2 + \sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j u(x)\bigg)\,dx \\ &\quad + \int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x)u(x)\partial_i u(x)\partial_j\psi(x)\,dx = \langle f, \psi u\rangle.\end{aligned}$$

Denote by M the Hardy–Littlewood maximal operator. Recall that for $\theta \in (-1, 1)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we have $(Mf)^\theta \in A_2$ and the A_2-constant depends only θ; see [8]. Then we have

$$\begin{aligned}&\nu_0\bigg(\|u\|_{L^2(Q)}^2 + \sum_{j=1}^n \|\partial_j u\|_{L^2(Q)}^2\bigg) - 3^n\varepsilon\sup_{\substack{R\in\mathcal{D}\\ |R|=\varepsilon^{-n}}}\bigg(\sum_{j=1}^n \|\partial_j u\|_{L^2(R)}^2 + \|u\|_{L^2(R)}^2\bigg) \\ &\prec \|(M\chi_Q)^\theta(1-\Delta)^{-1/2}f\|_{L^2}\|(M\chi_Q)^{-\theta}(1-\Delta)^{1/2}(\psi u)\|_{L^2} \\ &\prec \|f\|_{W^{-1}M_2^p;\varepsilon}\sum_{j=1}^n \|(M\chi_Q)^{-\theta}\partial_j(1-\Delta)^{-1/2}\partial_j(\psi u)\|_{L^2}.\end{aligned}$$

Recall that $\partial_j(1-\Delta)^{-\frac{1}{2}}$ is a singular integral operator considered in [8]. Thus,

$$\|(M\chi_Q)^{-\theta}\partial_j(1-\Delta)^{-1/2}\partial_j(\psi u)\|_{L^2} \prec \|(M\chi_Q)^{-\theta}\partial_j(\psi u)\|_{L^2}.$$

Using this estimate, we obtain

$$\begin{aligned}&\nu_0\bigg(\|u\|_{L^2(Q)}^2 + \sum_{j=1}^n \|\partial_j u\|_{L^2(Q)}^2\bigg) - 3^n\varepsilon\sup_{\substack{R\in\mathcal{D}\\ |R|=\varepsilon^{-n}}}\bigg(\sum_{j=1}^n \|\partial_j u\|_{L^2(R)}^2 + \|u\|_{L^2(R)}^2\bigg) \\ &\prec \|f\|_{W^{-1}M_2^p;\varepsilon}\sum_{j=1}^n \|(M\chi_Q)^{-\theta}\partial_j(\psi u)\|_{L^2} \\ &\prec \|f\|_{W^{-1}M_2^p;\varepsilon}\left(\|(M\chi_Q)^{-\theta}u\nabla\psi\|_{L^2} + \|(M\chi_Q)^{-\theta}\psi\nabla u\|_{L^2}\right) \\ &\prec \|f\|_{W^{-1}M_2^p;\varepsilon}\Big(\|u\|_{L^2(\mathrm{supp}(\psi))} + \|\nabla u\|_{L^2(\mathrm{supp}(\psi))}\Big).\end{aligned}$$

Thus, we obtain

$$(\nu_0 - 3^n\varepsilon) \sup_{\substack{R\in\mathcal{D}\\ |R|=\varepsilon^{-n}}} \left(\sum_{j=1}^{n} \|\partial_j u\|_{L^2(R)}^2 + \|u\|_{L^2(R)}^2 \right)$$

$$\prec \|f\|_{W^{-1}M_2^p;\varepsilon} \left(\|u\|_{L^2(\mathrm{supp}(\psi))} + \|\nabla u\|_{L^2(\mathrm{supp}(\psi))} \right).$$

By taking the supremum over Q and divide both sides by $\|u\|_{L^2(\mathrm{supp}(\psi))} + \|\nabla u\|_{L^2(\mathrm{supp}(\psi))}$, we obtain the desired result. □

As an application of Lemma 5.1, we obtain the bijectivity of L as long as ν_0 and ν_1 in the assumption given in the beginning of this section are not so different.

Corollary 5.2. *Let $\nu_0 > 0$ and $p \geq 2$ be fixed. If $|\nu_1 - \nu_0| < \delta$ for a sufficiently small $\delta > 0$, Then $(1-\Delta)^{-1/2}L(1-\Delta)^{-1/2} \colon M_2^p(\mathbb{R}^n) \to M_2^p(\mathbb{R}^n)$ is an invertible operator.*

Proof. If $L = 1 - \nu_0\Delta$, then this is a consequence of the fact that singular integral operators are bounded in $M_2^p(\mathbb{R}^n)$. In general, $\|(1-\Delta)^{-1/2}L(1-\Delta)^{-1/2}f\|_{M_2^p} \prec \|f\|_{M_2^p}$, where the implicit constant depends on ν_0, ν_1. As is in [10], by connecting $1 - \nu_0\Delta$ and L by a segment, we obtain the desired result. □

Corollary 5.3. *Let $A = [a_{ij}]_{i,j}^n$ be a symmetric elliptic matrix with $a_{ij} \in L^\infty(\mathbb{R}^n)$. Let L be an elliptic differential operator with non-smooth coefficients $A = [a_{ij}]_{i,j}^n$. Suppose that the parameters $\theta \in (0,1)$, $1 \leq q_0 \leq p_0 < \infty$ and $1 \leq q_1 \leq p_1 < \infty$ satisfy*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{2} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \text{and} \quad \frac{q_0}{q_1} = \frac{p_0}{p_1}.$$

Then there exists an open set U containing $(1/2, 1/2)$ such that $(1-\Delta)^{-1/2}L(1-\Delta)^{-1/2}$ is an isomorphism from $M_q^p(\mathbb{R}^n)$ to $M_q^p(\mathbb{R}^n)$ whenever $(1/p, 1/q) \in U \setminus \{(1/p, 1/2) :\ p \geq 2\}$.

Proof. It is easy to verify that $M_q^p(\mathbb{R}^n)$ is a maximal Banach lattice. We have

$$M_2^p(\mathbb{R}^n) \cong M_{q_0}^{p_0}(\mathbb{R}^n)^{1-\theta} M_{q_1}^{p_1}(\mathbb{R}^n)^\theta \cong [M_{q_0}^{p_0}(\mathbb{R}^n), M_{q_1}^{p_1}(\mathbb{R}^n)]^\theta$$

by Theorem 2.2 and [21, Corollary 3.3] (or by [12, Theorem 2]). Thus the statement follows from Corollary 4.6. □

A similar argument works for the amalgam space $L_{\mathrm{uloc}}^q(\mathbb{R}^n)$. Recall that the amalgam space (the uniformly local Lebesgue space) $L_{\mathrm{uloc}}^q(\mathbb{R}^n)$ is the set of all $f \in L_{\mathrm{loc}}^q(\mathbb{R}^n)$ for which

$$\|f\|_{L_{\mathrm{uloc}}^q} = \sup_{m\in\mathbb{Z}^n} \|f\|_{L^q(m+[0,1]^n)}$$

is finite. It should be noted for any $\varepsilon > 0$

$$\|f\|_{L^q_{\rm uloc}} \asymp \sup_{m\in\mathbb{Z}^n} \|f\|_{L^q(\varepsilon m+[0,\varepsilon]^n)}$$

and that

$$\|f\|_{L^q_{\rm uloc}} \asymp \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{\min(r^n,1)}{|Q(x,r)|}\int_{Q(x,r)} |f(y)|^q\,dy\right)^{\frac{1}{q}}.$$

Although singular integral operators fail to be bounded in $L^q_{\rm uloc}(\mathbb{R}^n)$ [11], we have the following result:

Proposition 5.4. *Let* $1 < q < \infty$. *Then*

$$\|\partial_j(1-\Delta)^{-1/2}f\|_{L^q_{\rm uloc}} \prec \|f\|_{L^q_{\rm uloc}}$$

for all $f \in L^q_{\rm uloc}(\mathbb{R}^n)$. *In particular, for* $g \in L^q_{\rm uloc}(\mathbb{R}^n)$, $(1-\Delta)^{1/2}g \in L^q_{\rm uloc}(\mathbb{R}^n)$ *if and only if* $\nabla g \in L^q_{\rm uloc}(\mathbb{R}^n)^n$. *Furthermore for such* g,

$$\|(1-\Delta)^{1/2}g\|_{L^q_{\rm uloc}} \asymp \|g\|_{L^q_{\rm uloc}} + \|\nabla g\|_{(L^q_{\rm uloc})^n}.$$

Proof. The second inequality is an consequence of the first inequality. We invoke an equality from [31]. The operator $(1-\Delta)^{-\frac{1}{2}}$ is a convolution operator with kernel

$$K(x) = \frac{1}{2\pi}\int_0^\infty \exp\left(-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}\right) t^{\frac{1}{2}(-n+1)}\frac{dt}{t},$$

from which we can easily deduce that

$$|\nabla K(x)| \prec |x|^{-n-1}, \quad x \in \mathbb{R}^n.$$

With this in mind, we prove at first the required estimate

$$\|\partial_j(1-\Delta)^{-1/2}f\|_{L^q_{\rm uloc}} \prec \|f\|_{L^q_{\rm uloc}}, \quad f \in L^q_{\rm uloc}(\mathbb{R}^n).$$

Let $m \in \mathbb{Z}^n$ be fixed. Then we have

$$\begin{aligned}
&\|\partial_j(1-\Delta)^{-1/2}f\|_{L^q(m+[0,1]^n)} \\
&\le \sum_{l\in\mathbb{Z}^n} \|\partial_j(1-\Delta)^{-1/2}[\chi_{l+[-1,2]^n}f]\|_{L^q(m+[0,1]^n)} \\
&\prec \|f\|_{L^q(m+[-1,2]^n)} + \sum_{l\in\mathbb{Z}^n} \|(|\cdot - l|+1)^{-n-1} * f\|_{L^q(m+[0,1]^n)} \\
&\prec \|f\|_{L^q_{\rm uloc}}.
\end{aligned}$$

□

Let $\alpha \in \mathbb{R}$ and $1 < q < \infty$. The space $W^\alpha L^q_{\rm uloc}(\mathbb{R}^n)$ consists of $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{W^\alpha L^q_{\rm uloc}} = \|(1-\Delta)^{\alpha/2} f\|_{L^q_{\rm uloc}}$ is finite. Going through the same argument as before, we obtain the following conclusion:

Lemma 5.5. *For all* $u \in W^1 L^2_{\rm uloc}(\mathbb{R}^n)$

$$\|u\|_{L^2_{\rm uloc}} + \|Du\|_{(L^2_{\rm uloc})^n} \prec \|Lu\|_{W^{-1}L^2_{\rm uloc}}.$$

Combining these results with the latter half of Theorem 2.2 in a similar manner, we obtain the following corollary:

Corollary 5.6. *Let* q *be sufficiently close to* 2.

Then $(1-\Delta)^{-1/2} L (1-\Delta)^{-1/2} \colon L^q_{\rm uloc}(\mathbb{R}^n) \to L^q_{\rm uloc}(\mathbb{R}^n)$ *is an invertible operator, or equivalently,* $L \colon W^1 L^q_{\rm uloc}(\mathbb{R}^n) \to W^{-1} L^q_{\rm uloc}(\mathbb{R}^n)$ *is a linear isomorphism.*

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