

## The connecting solution of the Painlevé phase transition model

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**Abstract.** The second Painlevé O.D.E.  $y'' - xy - 2y^3 = 0$ ,  $x \in \mathbb{R}$ , is known to play an important role in the theory of integrable systems, random matrices, Bose-Einstein condensates and other problems. The generalized second Painlevé equation  $\Delta y - x_1 y - 2y^3 = 0$ ,  $(x_1, x_2) \in \mathbb{R}^2$ , is obtained by multiplying by  $-x_1$  the linear term  $u$  of the Allen-Cahn equation  $\Delta u = u^3 - u$ . It involves a non autonomous potential  $H(x_1, y)$  which is bistable for every fixed  $x_1 < 0$ , and thus describes as the Allen-Cahn equation a phase transition model. The scope of this paper is to construct a solution  $y$  connecting along the vertical direction  $x_2$ , the two branches of minima of  $H$  parametrized by  $x_1$ . This solution plays a similar role that the heteroclinic orbit for the Allen-Cahn equation. It is the the first to our knowledge solution of the Painlevé P.D.E. both relevant from the applications point of view (liquid crystals), and mathematically interesting.

**Mathematics Subject Classification (2010):** 35J91 (primary); 35J20, 35B40, 35B06, 35B25 (secondary).

### 1. The Allen-Cahn and Painlevé phase transition models

A standard phase transition model is given by the Allen-Cahn O.D.E.:

$$u'' = u^3 - u, \quad \text{in } \mathbb{R}, \quad (1.1)$$

that can alternatively be written  $u'' = W'(u)$ , where  $W(u) = \frac{1}{4}(u^2 - 1)^2$  is a double well potential. In this model,  $u$  describes the mass fraction of the two phases of a substance (e.g., an alloy), and takes values approximately  $+1$  or  $-1$  for the pure phases. Equation (1.1) has a variational structure. Let

$$E_{AC}(u, (a, b)) := \int_a^b \left( \frac{1}{2}|u'|^2 + \frac{1}{4}(u^2 - 1)^2 \right)$$

M. G. Clerc was partially supported by Fondecyt 1180903.

M. Kowalczyk was partially supported by Chilean research grants Fondecyt 1130126 and 1170164, Fondo Basal AFB170001 CMM-Chile.

P. Smyrnelis was partially supported by Fondo Basal AFB170001 CMM-Chile and Fondecyt postdoctoral grant 3160055.

Received January 22, 2019; accepted in revised form April 30, 2019.

Published online December 2020.

be the Allen-Cahn energy associated to (1.1). To minimize  $E_{AC}$  the right balance between the contributions of the kinetic energy  $\frac{1}{2}|u'|^2$  and the potential should be achieved. On the one hand the term  $\frac{1}{2}|u'|^2$  penalizes high variations of  $u$ , while on the other hand the potential term  $W$  forces the minimizer to be close to its global minima  $\pm 1$ . It is clear that the trivial solutions  $\pm 1$  are the two global minimizers of  $E_{AC}$ . Thus, it is more relevant to investigate instead, the existence of *local minimizers* which are also called *minimal* solutions. While solutions of (1.1) are critical points of  $E_{AC}$ , a minimal solution  $u$  of (1.1) satisfies the stronger condition:

$$E_{AC}(u, \text{supp } \phi) \leq E_{AC}(u + \phi, \text{supp } \phi),$$

for all  $\phi \in C_0^\infty(\mathbb{R})$  (*i.e.* any perturbation with compact support of  $u$  has greater or equal energy). It turns out that up to translations and change of  $x$  by  $-x$ , the only nontrivial minimal solution of (1.1) is the heteroclinic orbit  $\eta(x) = \tanh(x/\sqrt{2})$ , connecting at  $\pm\infty$  the two phases  $\pm 1$ .

A much more challenging problem is the description of all bounded solutions of the Allen-Cahn P.D.E.:

$$\Delta u = u^3 - u, \quad \text{in } \mathbb{R}^n, \quad (1.2)$$

which is associated to the functional  $E_{AC}(u, \Omega) := \int_{\Omega} \left( \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2 \right)$  (where  $\Omega \subset \mathbb{R}^n$  is bounded). De Giorgi in 1978 [17] suggested a striking analogy with minimal surface theory that led to significant developments in P.D.E. and the Calculus of Variations, by stating the following conjecture about bounded solutions on  $\mathbb{R}^n$ :

**Conjecture 1.1 (De Giorgi).** Let  $u \in C^2(\mathbb{R}^n)$  be a solution to (1.2) such that

- i)  $|u| < 1$ ;
- ii)  $\frac{\partial u}{\partial x_n} > 0$  for all  $x \in \mathbb{R}^n$ .

Is it true that all the level sets of  $u$  are hyperplanes, at least for  $n \leq 8$ ?

The relationship with the Bernstein problem for minimal graphs is the reason why  $n \leq 8$  appears in the conjecture. We refer to the expository papers of Farina and Valdinoci [18], and Savin [28] for a detailed account. The conjecture was proved by Ghoussoub and Gui in [19] for  $n = 2$ , for  $n = 3$  by Ambrosio and Cabré in [7] and for  $4 \leq n \leq 8$  by Savin in [27] under the extra requirement that

$$\lim_{x_n \rightarrow \pm\infty} u(x_1, \dots, x_n) = \pm 1. \quad (1.3)$$

If we drop the monotonicity requirement as well as (1.3) and simply ask about the structure of minimal solutions<sup>1</sup> of (1.2), then we know from [27] that, for  $n \leq 7$  any minimal solution  $u$  of (1.2) is either trivial, *i.e.*,  $u \equiv \pm 1$  or one dimensional,

<sup>1</sup> Again, we say that the solution  $u$  is minimal if  $E_{AC}(u, \text{supp } \phi) \leq E_{AC}(u + \phi, \text{supp } \phi)$ , for all  $\phi \in C_0^\infty(\mathbb{R}^n)$ .

i.e.,  $u(x) = \eta((x - x_0) \cdot v)$ , for some  $x_0 \in \mathbb{R}^n$ , and some unit vector  $v \in \mathbb{R}^n$ . Thus the heteroclinic orbit  $\eta$  of O.D.E. (1.1) plays a crucial role for entire solutions of P.D.E. (1.2).

In order to construct other connecting solutions of (1.2), one shall impose some additional requirements. For instance, when  $n = 2$ , (1.2) admits a unique *saddle* solution  $u$  (cf. [16]) satisfying the following properties:

- $u(x_1, x_2)$  has the same sign as the product  $x_1 x_2$ ;
- $u$  is odd with respect to  $x_1$  and  $x_2$ ;
- $\lim_{x_1 \rightarrow \infty} u(x_1, x_2) = \eta(x_2)$ , and  $u_{x_1}(x_1, x_2) > 0, \forall x_2 > 0$ ;
- $\lim_{\lambda \rightarrow \infty} u(\lambda \cos \theta, \lambda \sin \theta) = 1, \forall \theta \in (0, \pi/2)$ .

This example also outlines the hierarchical structure of (1.2), since by taking the limit of a solution along certain directions at infinity, lower dimensional solutions are obtained (cf. [6, Chapter 8]). For more examples of connecting maps under symmetry assumptions or in the vector case, we refer to [6, Chapters 6, 7, 9] and the references therein (in particular [4] and [29]).

The second Painlevé O.D.E.:

$$y'' - xy - 2y^3 = 0, \quad x \in \mathbb{R}, \quad (1.4)$$

is basically obtained by multiplying the linear term in the right hand side of (1.1) by  $-x$ . Alternatively, we can write (1.4) as

$$y'' = H_y(x, y), \quad x \in \mathbb{R}, \quad (1.5)$$

with  $H(x, y) := \frac{1}{2}xy^2 + \frac{1}{2}y^4$ . In contrast with  $W$  (defined below (1.1)), the potential  $H$  is non autonomous i.e. it depends both on  $x$  and  $y$ .

Equation (1.4) is known to play an important role in the theory of integrable systems [1], random matrices [11, 15, 20], Bose-Einstein condensates [2, 3, 24, 30] and other problems [5, 23, 25]. Recently [12] it has been shown that when the right hand side of (1.4) is allowed to be a constant  $\alpha \in \mathbb{R}$  then it describes local profiles of the so-called *shadow kink* in the theory of light-matter interaction of nematic liquid crystals (cf. also [31, 32]). In [8, 13, 14] further relation between other types of non topological defects (*shadow vortices*, *shadow domain walls*) and the *generalized* Painlevé equation

$$\Delta y - x_1 y - 2y^3 = 0, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \quad (1.6)$$

was established showing that their local structure is determined by special solutions of (1.6). One of the characteristics of these solutions is that they should be entire, another is that they should be *minimal*. To explain what this means, let  $\Omega \in \mathbb{R}^2$  be a bounded subset of  $\mathbb{R}^2$  and

$$E_{\text{P}_\text{II}}(u, \Omega) = \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2} x_1 u^2 + \frac{1}{2} u^4 \right],$$

be the functional associated to the generalized second Painlevé equation. By definition a solution of (1.6) is minimal if

$$E_{P_{II}}(y, \text{supp } \phi) \leq E_{P_{II}}(y + \phi, \text{supp } \phi) \quad (1.7)$$

for all  $\phi \in C_0^\infty(\mathbb{R}^2)$ . This notion of minimality is standard for many problems in which the energy of a localized solution is actually infinite due to non compactness of the domain. The study of minimal solutions of (1.4) has been recently initiated in [12] where we have showed that the Hastings-McLeod solution, denoted in this paper by  $h$ , is, up to the sign change, the only minimal solution which is bounded at  $+\infty$ . We recall (*cf.* [22]) that  $h : \mathbb{R} \rightarrow \mathbb{R}$  is positive, strictly decreasing ( $h' < 0$ ) and such that

$$\begin{aligned} h(x) &\sim Ai(x), & x \rightarrow \infty, \\ h(x) &\sim \sqrt{|x|/2}, & x \rightarrow -\infty. \end{aligned} \quad (1.8)$$

Clearly, the asymptotic behaviour of  $h$  is determined by the location of the global minima of the potential  $H(x, y)$  associated to the equation (1.4). Indeed for  $x$  fixed,  $H$  attains its global minimum equal to 0 when  $y = 0$  and  $x \geq 0$ , and equal to  $-\frac{x^2}{8}$  when  $y = \pm\sqrt{|x|/2}$  and  $x < 0$ . Thus, the global minima of  $H$  bifurcate from the origin, and the two minimal solutions  $\pm h$  of (1.4) interpolate these two branches of minima.

Equation (1.6) or equivalently  $\Delta y = H_y(x_1, y)$ , with  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $H(x_1, y) := \frac{1}{2}x_1y^2 + \frac{1}{2}y^4$  (*cf.* the expression of  $E_{P_{II}}$ ), involves a non autonomous potential which is bistable for every fixed  $x_1 < 0$ . Hence the Painlevé generalized equation (1.6) describes as the Allen-Cahn equation a phase transition model. For the latter the phase transition connects the two minima  $\pm 1$  while for the former the phase transition connects the two branches  $\pm\sqrt{(-x_1)^+/2}$  of minima of  $H$  parametrized by  $x_1$ . Note that in the Painlevé model the phase transition occurs only in the P.D.E. case, *i.e.*, when the domain is  $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$  with  $n \geq 1$ . The scope of this paper is to construct a solution  $y$  of (1.6) connecting as  $x_2 \rightarrow \pm\infty$  and  $x_1$  is fixed, the two branches of minima of  $H$  (*cf.* Theorem 1.2 below). It is the first to our knowledge example of solution of the generalized Painlevé equation both relevant from the applications point of view and mathematically interesting. The solution  $y$  has similar properties as the heteroclinic orbit  $\eta$ : it is odd and monotonous with respect to  $x_2$ , and as  $x_1 \rightarrow -\infty$  its rescaled profile is actually given by  $\eta$ . After the statement of Theorem 1.2, we will further discuss its similarities with the heteroclinic orbit  $\eta$ .

**Theorem 1.2.** *There exists a solution  $y : \mathbb{R}^2 \rightarrow \mathbb{R}$  to*

$$\Delta y - x_1 y - 2y^3 = 0, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \quad (1.9)$$

such that

- (i)  $y$  is positive in the upper-half plane and odd with respect to  $x_2$  i.e.  $y(x_1, x_2) = -y(x_1, -x_2)$ ;
- (ii)  $y$  and its derivatives are bounded in the half-planes  $[s_0, \infty) \times \mathbb{R}$ ,  $\forall s_0 \in \mathbb{R}$ ;
- (iii)  $y$  is minimal with respect to perturbations  $\phi \in C_0^\infty(\mathbb{R}^2)$  such that  $\phi(x_1, x_2) = -\phi(x_1, -x_2)$ ;
- (iv)  $\frac{|y(x_1, x_2)|}{Ai(x_1)} = O(1)$ , as  $x_1 \rightarrow \infty$  (uniformly in  $x_2$ );
- (v) For every  $x_2 \in \mathbb{R}$  fixed, let  $\tilde{y}(t_1, t_2) := \frac{\sqrt{2}}{(-\frac{3}{2}t_1)^{\frac{1}{3}}} y\left(-\left(-\frac{3}{2}t_1\right)^{\frac{2}{3}}, x_2 + t_2\left(-\frac{3}{2}t_1\right)^{-\frac{1}{3}}\right)$ .

Then

$$\lim_{l \rightarrow -\infty} \tilde{y}(t_1 + l, t_2) = \begin{cases} \tanh(t_2/\sqrt{2}) & \text{when } x_2 = 0 \\ 1 & \text{when } x_2 > 0 \\ -1 & \text{when } x_2 < 0 \end{cases} \quad (1.10)$$

for the  $C_{\text{loc}}^1(\mathbb{R}^2)$  convergence;

- (vi)  $y_{x_1}(x_1, x_2) < 0$ ,  $\forall x_1 \in \mathbb{R}$ ,  $\forall x_2 > 0$ ;
- (vii)  $y_{x_2}(x_1, x_2) > 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$ , and  $\lim_{l \rightarrow \pm\infty} y(x_1, x_2 + l) = \pm h(x_1)$  in  $C_{\text{loc}}^2(\mathbb{R}^2)$ , where  $h$  is the Hastings-McLeod solution of (1.4).

The solution provided by Theorem 1.2 has a form of a quadruple connection between the Airy function  $Ai$ , the two one dimensional Hastings-McLeod solutions  $\pm h$ , and the heteroclinic orbit  $\eta$  of the one dimensional Allen-Cahn equation. Comparing (iv) with (1.8) we see that as  $x_1 \rightarrow \infty$  the function  $y(x_1, x_2)$  behaves similarly as the Hastings-McLeod solution  $h(x_1)$ . At the same time, as  $x_2 \rightarrow \pm\infty$  we have  $y(x_1, x_2) \rightarrow \pm h(x_1)$ ,  $x_2 \rightarrow \pm\infty$ . Perhaps the most interesting aspect of the above solution  $y$  is stated in property (v), since after rescaling we obtain as  $x_1 \rightarrow -\infty$ , the convergence to the heteroclinic orbit  $\eta(x) = \tanh(x/\sqrt{2})$  of the Allen-Cahn O.D.E. (1.1). In the proof of Theorem 1.2 it is shown that a minimal solution of (1.9) rescaled as in (v), converges as  $x_1 \rightarrow -\infty$  to a minimal solution of (1.2). This deep connection of the structure of the Painlevé equation with the Allen-Cahn P.D.E., suggests that several properties of the Allen-Cahn equation should be transferred to the Painlevé equation. Although by construction the solution  $y$  is only minimal for odd perturbations, we expect that  $y$  is actually minimal for general perturbations, and plays a similar role that the heteroclinic orbit for the Allen-Cahn equation. What's more the two global minimizers  $\pm 1$  of the functional  $E_{\text{AC}}$  have their counterparts in the two minimal solutions  $\pm h$  of the Painlevé equation. Indeed, property (vii) establishes that  $y$  connects monotonically along the vertical direction  $x_2$ , the two minimal solutions  $\pm h(x_1)$ , in the same way that  $\eta$  connects monotonically the two global minimizers  $\pm 1$ . While  $\eta$  is a one dimensional object, the solution  $y(x_1, x_2)$  is two dimensional, since  $x_1$  parametrizes the branches of minima of the potential  $H$ , and only  $x_2$  is involved in the phase transition.

We believe that in higher dimension  $y : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , ( $n \geq 1$ ) the structure of solutions of (1.9) exactly mirrors that of (1.2), and going further, one may ask: is it

true that that in dimension  $n \leq 7$ , any minimal solution  $Y : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  of (1.9) is either  $Y(x_1, x_2, \dots, x_{n+1}) = \pm h(x_1)$  or  $Y(x_1, x_2, \dots, x_{n+1}) = y(x_1, (x_2, \dots, x_{n+1})) \cdot \mathbf{n} + b$ , for some constant  $b \in \mathbb{R}$ , and some unit vector  $\mathbf{n} \in \mathbb{S}^{n-1}$ ?

## 2. Odd minimizers of the Ginzburg-Landau type energy

We consider the energy functional

$$E(u) = \int_{\mathbb{R}^2} \frac{\epsilon}{2} |\nabla u|^2 - \frac{1}{2\epsilon} \mu(x) u^2 + \frac{1}{4\epsilon} u^4, \quad (2.1)$$

where  $u \in H^1(\mathbb{R}^2)$  and  $\epsilon > 0$  is a small parameter. We suppose that  $\mu \in C^\infty(\mathbb{R}^2)$  is radial, *i.e.*,  $\mu(x) = \mu_{\text{rad}}(|x|)$ , with  $\mu_{\text{rad}} \in C^\infty(\mathbb{R})$  an even function. In addition we assume that

$$\mu \in L^\infty(\mathbb{R}^2), \quad \mu'_{\text{rad}} < 0 \text{ in } (0, \infty), \quad \text{and } \mu_{\text{rad}}(\rho) = 0 \text{ for a unique } \rho > 0, \quad (2.2)$$

In the physical context described in [8] the function  $\mu$  is specific

$$\mu(x) = e^{-|x|^2} - \chi, \quad \text{with some } \chi \in (0, 1), \quad f(x) = -\frac{1}{2} \nabla \mu(x),$$

but this particular form is irrelevant here. The Euler-Lagrange equation of  $E$  is

$$\epsilon^2 \Delta u + \mu(x) u - u^3 = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (2.3)$$

and we also write its weak formulation:

$$\int_{\mathbb{R}^2} -\epsilon^2 \nabla u \cdot \nabla \psi + \mu u \psi - u^3 \psi = 0, \quad \forall \psi \in H^1(\mathbb{R}^2), \quad (2.4)$$

where  $\cdot$  denotes the inner product in  $\mathbb{R}^2$ . Note that due to the radial symmetry of  $\mu$  the energy (2.1) and equation (2.3) are invariant under orthogonal transformations in the domain, and sign change in the range. Our strategy to construct the solution of (1.9) enjoying the properties described in Theorem 1.2 is to find first an *odd* with respect to  $x_2$  minimizer  $u_\epsilon$  of  $E$  and then scaling and passing to the limit  $\epsilon \searrow 0$  recover  $y$  - this gives us existence. Second, in Section 3 we show all the properties of  $y$  stated in Theorem 1.2.

We explain, formally at the moment, the relation between (1.9) and the energy  $E$ . Looking at the energy density of  $E$  it is evident that as  $\epsilon \rightarrow 0$  the modulus of the global or odd minimizer  $u_\epsilon$  should approach a nonnegative root of the polynomial

$$-\mu(x)z + z^3 = 0,$$

or in other words,  $|u_\epsilon| \rightarrow \sqrt{\mu^+}$  as  $\epsilon \rightarrow 0$  in some, perhaps weak, sense. This function, called the Thomas-Fermi limit of the minimizer is not in  $H^1(\mathbb{R}^2)$  and

therefore the transition near the set  $\mu(x) = 0$  has to be mediated somehow. To see this let us consider a point  $\xi$  such that  $\mu(\xi) = 0$ . By (2.2)  $\xi = \rho e^{i\theta}$ . At  $\xi$  introduce local orthogonal frame  $(e^{i\theta}, ie^{i\theta})$  and coordinates  $s = (s_1, s_2)$  associated with it. Let  $u_\epsilon$  be any solution of (2.3) and

$$z(s) = \epsilon^{-1/3} u_\epsilon(\xi + \epsilon^{2/3} s).$$

Noting that  $\mu(\xi + \epsilon^{2/3} s) = \epsilon^{2/3} s_1 \mu_1 + \dots$  with  $\mu_1 < 0$  we get that  $z$  satisfies

$$\Delta_s z + s_1 \mu_1 z - z^3 = o(1), \quad \text{as } \epsilon \searrow 0.$$

The equation on the left becomes the second Painlevé equation after passing to the limit and suitable scaling. Now, suppose that  $u_\epsilon$  is the odd minimizer of  $E$ , i.e.  $u_\epsilon(x_1, x_2) = -u_\epsilon(x_1, -x_2)$ . Except for the points  $\bar{x} = (\pm\rho, 0)$  the limiting function  $z$  could be one of the Hastings-McLeod one dimensional solutions. However, at  $(\pm\rho, 0)$  we should have  $z(s_1, s_2) = -z(s_1, -s_2)$ , which means that  $z$  genuinely depends on both variables. This is the idea behind the proof of the existence part in Theorem 1.2. Showing properties of the solution is a different story and depends on rather tricky application of the moving plane method.

Our first purpose in this paper is to study qualitative properties of the global minimizers of  $E$  as  $\epsilon \searrow 0$ . In our previous work [12] we studied the following energy

$$E(u, \mathbb{R}) = \int_{\mathbb{R}} \frac{\epsilon}{2} |u_x|^2 - \frac{1}{2\epsilon} \mu(x) u^2 + \frac{1}{4\epsilon} |u|^4 - af(x)u, \quad u: \mathbb{R} \rightarrow \mathbb{R},$$

where  $a \geq 0$  is a parameter and  $f = -\frac{1}{2}\mu'$ , and in [13] we studied its analog for maps  $u: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

By proceeding as in [13], one can see that under the above assumptions there exists a global minimizer  $v$  of  $E$  in  $H^1(\mathbb{R}^2)$ , namely that  $E(v) = \min_{H^1(\mathbb{R}^2)} E$ . In addition, we show that  $v$  is a classical solution of (2.3), and  $v$  is radial. Similarly, in the class  $H_{\text{odd}}^1(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : u(x_1, x_2) = -u(x_1, -x_2)\}$  of odd functions with respect to  $x_2$ , there exists an odd minimizer  $u$  which also solves (2.3) and satisfies  $u(x_1, x_2) = u(-x_1, x_2)$ . Although in the sequel we will focus on the odd minimizer for completeness we chose to present our next result in a slightly more general framework.

**Theorem 2.1.** *For  $\epsilon \ll 1$  let  $u_\epsilon$  be a solution of (2.3) converging to 0 as  $|x| \rightarrow \infty$  (which may be the odd or global minimizer). Let  $\rho > 0$  be the zero of  $\mu_{\text{rad}}$  and let  $\mu_1 := \mu'_{\text{rad}}(\rho) < 0$ . For every  $\xi = \rho e^{i\theta}$ , we consider the local coordinates  $s = (s_1, s_2)$  in the basis  $(e^{i\theta}, ie^{i\theta})$ , and the rescaled functions:*

$$w_\epsilon(s) = 2^{-1/2} (-\mu_1 \epsilon)^{-1/3} u_\epsilon \left( \xi + \epsilon^{2/3} \frac{s}{(-\mu_1)^{1/3}} \right). \quad (2.5)$$

As  $\epsilon \rightarrow 0$ , the function  $w_\epsilon$  converges in  $C_{\text{loc}}^2(\mathbb{R}^2)$  up to subsequence, to a function  $y$  bounded in the half-planes  $[s_0, \infty) \times \mathbb{R}$ , for every  $s_0 \in \mathbb{R}$ , which is a solution of

$$\Delta y(s) - s_1 y(s) - 2y^3(s) = 0, \quad \forall s = (s_1, s_2) \in \mathbb{R}^2. \quad (2.6)$$

In particular, if we take  $u_\epsilon$  to be the odd minimizer of  $E$  and  $\xi = (\pm\rho, 0)$ , then the solution  $y$  satisfies  $y(s_1, s_2) = -y(s_1, -s_2)$ , and is minimal with respect to perturbations  $\phi \in C_0^\infty(\mathbb{R}^2)$ ,  $\phi(s_1, s_2) = -\phi(s_1, -s_2)$ . On the other hand, if we take  $u_\epsilon$  to be the global minimizer then  $y(s_1, s_2) = h(s_1)$  or  $y(s_1, s_2) = -h(s_1)$ .

We observe that as a corollary of [14, Theorem 1.1.] it can be proven that  $|v_\epsilon| \rightarrow \sqrt{\mu^+}$  in  $C_{\text{loc}}^0(D(0; \rho))$ , where  $D(0; \rho)$  is the disc of radius  $\rho$  centered at 0. Because of the analogy between the functional  $E$  and the Gross-Pitaevskii functional in the theory of Bose-Einstein condensates we will call  $\sqrt{\mu^+}$  the Thomas-Fermi limit of  $v_\epsilon$ . Theorem 2.1 gives account on how non smoothness of the limit of  $v_\epsilon$  is mediated near the circumference  $|x| = \rho$ , where  $\mu$  changes sign, through the solution of (2.6). We should mention here that detailed description of the minimizers for yet more general setting of the energy can be found in [13, 14].

Before proving the theorem we gather general results for minimizers and solutions that are valid for any values of the parameters  $\epsilon > 0$ . For the rest of this paper  $v$  or  $v_\epsilon$  will be the global minimizer and  $u$  or  $u_\epsilon$  will be the odd minimizer or a critical point of  $E$ . We first prove the existence of global and odd minimizers.

**Lemma 2.2.** *For every  $\epsilon > 0$  there exists  $v \in H^1(\mathbb{R}^2)$  such that  $E(v) = \min_{H^1(\mathbb{R}^2)} E$ . As a consequence,  $v$  is a  $C^\infty$  classical solution of (2.3). Moreover, for  $\epsilon \ll 1$  the global minimizer  $v$  is unique up to change of  $v$  by  $-v$ , and it can be written as  $v(x) = v_{\text{rad}}(|x|)$ , with  $v_{\text{rad}} \in C^\infty(\mathbb{R})$ , positive, even, and such that  $\lim_{\infty} v_{\text{rad}} = 0$ .*

*Proof.* We proceed as in [13, Lemma 2.1] to establish that the global minimizer exists and is a smooth solution of (2.3) converging to 0 as  $|x| \rightarrow \infty$ . Next, we notice that  $v \not\equiv 0$  for  $\epsilon \ll 1$ . Indeed, by choosing a test function  $\psi \not\equiv 0$  supported in  $D(0; \rho) \cap \{x_2 > 0\}$ , and such that  $\psi^2 < 2\mu$ , one can see that

$$E(\psi) = \frac{\epsilon}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 + \frac{1}{4\epsilon} \int_{\mathbb{R}^2} \psi^2 (\psi^2 - 2\mu) < 0, \quad \epsilon \ll 1.$$

Let  $x_0 \in \mathbb{R}^2$  be such that  $v(x_0) \neq 0$ . Without loss of generality we may assume that  $v(x_0) > 0$ . Next, consider  $\tilde{v} = |v|$  which is another global minimizer and thus another solution. Clearly, in a neighborhood of  $x_0$  we have  $v = |\tilde{v}|$ , and as a consequence of the unique continuation principle (cf. [26]) we deduce that  $v \equiv \tilde{v} \geq 0$  on  $\mathbb{R}^2$ . Furthermore, the maximum principle implies that  $v > 0$ , since  $v \not\equiv 0$ . To prove that  $v$  is radial we consider the reflection with respect to the line  $x_1 = 0$ . We can check that  $E(v, \{x_1 > 0\}) = E(v, \{x_1 < 0\})$ , since otherwise by even reflection we can construct a map in  $H^1$  with energy smaller than  $v$ . Thus, the map  $\tilde{v}(x) = v(|x_1|, x_2)$  is also a minimizer, and since  $\tilde{v} = v$  on  $\{x_1 > 0\}$ , it follows by unique continuation that  $\tilde{v} \equiv v$  on  $\mathbb{R}^2$ . Repeating the same argument for any line

of reflection, we deduce that  $v$  is radial. To complete the proof, it remains to show the uniqueness of  $v$  up to change of  $v$  by  $-v$ . Let  $\tilde{v}$  be another global minimizer such that  $\tilde{v} > 0$ , and  $\tilde{v} \not\equiv v$ . Choosing  $\psi = u$  in (2.4), we find for any solution  $u \in H^1(\mathbb{R}^2)$  of (2.3) the following alternative expression of the energy:

$$E(u) = - \int_{\mathbb{R}^2} \frac{u^4}{4\epsilon}. \quad (2.7)$$

Formula (2.7) implies that  $v$  and  $\tilde{v}$  intersect for  $|x| = r > 0$ . However, setting

$$w(x) = \begin{cases} v(x) & \text{for } |x| \leq r \\ \tilde{v}(x) & \text{for } |x| \geq r, \end{cases}$$

we can see that  $w$  is another global minimizer, and again by the unique continuation principle we have  $w \equiv v \equiv \tilde{v}$ . This completes the proof of Lemma 2.2.  $\square$

On the other hand, in the class  $H_{\text{odd}}^1(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : u(x_1, x_2) = -u(x_1, -x_2)\}$  of odd functions with respect to  $x_2$ , there exists an odd minimizer with the following properties:

**Lemma 2.3.** *For every  $\epsilon > 0$  there exists  $u \in H_{\text{odd}}^1(\mathbb{R}^2)$  such that  $E(u) = \min_{H_{\text{odd}}^1(\mathbb{R}^2)} E$ . As a consequence,  $u$  is a  $C^\infty$  classical solution of (2.3). Moreover*

- (i)  $u(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ ;
- (ii)  $u(x_1, x_2) = u(-x_1, x_2)$ ;
- (iii) up to transformation  $u \mapsto -u$  we have  $u(x_1, x_2) > 0, \forall (x_1, x_2) \in \mathbb{R} \times (0, \infty)$ , provided that  $\epsilon \ll 1$ .

*Proof.* The existence of  $u \in H_{\text{odd}}^1(\mathbb{R}^2)$  such that  $E(u) = \min_{H_{\text{odd}}^1(\mathbb{R}^2)} E$ , follows as in [13, Lemma 2.1], and clearly  $u$  is a critical point of  $E$  in the subspace  $H_{\text{odd}}^1(\mathbb{R}^2)$ . In view of the radial symmetry of  $\mu$  it is easy to see that the Euler-Lagrange equation (2.4) holds also for every  $\phi \in H^1(\mathbb{R}^2)$ , such that  $\phi(x_1, x_2) = \phi(x_1, -x_2)$ . As a consequence,  $u$  is a  $C^\infty$  classical solution of (2.3).

For the proof of (i) we refer to [13, Lemma 2.1]. To show that  $u(x_1, x_2) = u(-x_1, x_2)$ , we first note that  $E(u, [0, \infty) \times \mathbb{R}) = E(u, (-\infty, 0] \times \mathbb{R})$ . Indeed, if we assume without loss of generality that  $E(u, [0, \infty) \times \mathbb{R}) < E(u, (-\infty, 0] \times \mathbb{R})$ , the function

$$\tilde{u}(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{when } x_1 \geq 0 \\ u(-x_1, x_2) & \text{when } x_1 \leq 0, \end{cases} \quad (2.8)$$

has strictly less energy than  $u$ , which is a contradiction. Thus,  $E(u, [0, \infty) \times \mathbb{R}) = E(u, (-\infty, 0] \times \mathbb{R})$ , and as a consequence the function  $\tilde{u}$  is also an odd minimizer and a solution. It follows by unique continuation [26] that  $\tilde{u} \equiv u$ , that is,  $u(x_1, x_2) = u(-x_1, x_2)$ .

Now, it remains to establish the uniqueness of the odd minimizer  $u$ , when  $\epsilon \ll 1$ . Proceeding as in Lemma 2.2, we can see that  $u \not\equiv 0$  for  $\epsilon \ll 1$ , and that

either  $u > 0$  or  $u < 0$  on  $\mathbb{R} \times (0, \infty)$ . Assume that  $u_1$  and  $u_2$  are two minimizers of  $E$  in  $H_{\text{odd}}^1(\mathbb{R}^2)$  such that  $u_1 > 0$  and  $u_2 > 0$  on  $\mathbb{R} \times (0, \infty)$ . Next, define the maps

$$u_*(x_1, x_2) = \begin{cases} \min(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \geq 0 \\ \max(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \leq 0, \end{cases} \quad (2.9)$$

$$u^*(x_1, x_2) = \begin{cases} \max(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \geq 0 \\ \min(u_1(x_1, x_2), u_2(x_1, x_2)) & \text{when } x_2 \leq 0, \end{cases} \quad (2.10)$$

and the set  $A := \{(x_1, x_2) \in \mathbb{R} \times (0, \infty) : u_1(x_1, x_2) < u_2(x_1, x_2)\}$ . We can see that  $E(u_1, A) = E(u_2, A)$  since otherwise we have either  $E(u_*) < E(u_2)$  or  $E(u^*) < E(u_1)$ , which contradicts the minimality of  $u_1$  and  $u_2$ . As a consequence,  $E(u_*) = E(u_2) = E(u_1) = E(u^*)$ , and it follows that  $u_*$  and  $u^*$  are also minimizers and solutions. Next, by unique continuation [26], we obtain that either  $u_1 \equiv u_*$  or  $u_1 \equiv u^*$ , *i.e.* we have either  $0 \leq u_1 \leq u_2$  or  $u_1 \geq u_2 \geq 0$  on  $\mathbb{R} \times [0, \infty)$ . Finally, applying (2.7) to  $E(u_1) = E(u_2)$ , we conclude in view of the ordering of  $u_1$  and  $u_2$  that  $u_1 \equiv u_2$ . This completes the proof.  $\square$

To study the limit of solutions as  $\epsilon \rightarrow 0$ , we need uniform bounds. Modifying slightly the arguments in [13, Section 2], we obtain:

**Lemma 2.4.** *We have  $\|u_\epsilon\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{\mu(0)}$ , for all solutions  $u_\epsilon$  of (2.3) converging to 0 as  $|x| \rightarrow \infty$ , and all  $\epsilon > 0$ .*

*Proof.* We drop the index and write  $u := u_\epsilon$ . Since  $\mu$  is bounded, the roots of the cubic equation  $u^3 - \mu(x)u = 0$  belong to a bounded interval, for all values of  $x$ . If  $u$  takes positive values, then it attains its maximum  $0 \leq \max_{\mathbb{R}^2} u = u(x_0)$ , at a point  $x_0 \in \mathbb{R}^2$ . In view of (2.3):

$$0 \geq \epsilon^2 \Delta u(x_0) = u^3(x_0) - \mu(x_0)u(x_0),$$

thus it follows that  $u(x_0)$  is uniformly bounded above by  $\sqrt{\mu(0)}$ . In the same way, we prove the uniform lower bound for  $u$ .  $\square$

**Lemma 2.5.** *For  $\epsilon \ll 1$  let  $u_\epsilon$  be a solution of (2.3) converging to 0 as  $|x| \rightarrow \infty$ . Then, there exist a constant  $K > 0$  such that*

$$|u_\epsilon(x)| \leq K(\sqrt{\max(\mu(x), 0)} + \epsilon^{1/3}), \quad \forall x \in \mathbb{R}^2. \quad (2.11)$$

*As a consequence, if for every  $\xi = \rho e^{i\theta}$  we consider the local coordinates  $s = (s_1, s_2)$  in the basis  $(e^{i\theta}, ie^{i\theta})$ , then the rescaled functions  $w_\epsilon(s)$  defined in (2.5) are uniformly bounded on the half-planes  $[s_0, \infty) \times \mathbb{R}$ ,  $\forall s_0 \in \mathbb{R}$ .*

*Proof.* As above we write  $u := u_\epsilon$ . Let us define the following constants:

- $M > 0$  is the uniform bound of  $|u_\epsilon|$  (*cf.* Lemma 2.4);
- $\lambda > 0$  is such that  $3\mu_{\text{rad}}(\rho - h) \leq 2\lambda h$ ,  $\forall h \in [0, \rho]$ ;
- $\kappa > 0$  is such that  $\kappa^4 \geq 6\lambda$ .

Next, we construct the following comparison function

$$\chi(x) = \begin{cases} \lambda \left( \rho - |x| + \frac{\epsilon^{2/3}}{2} \right) & \text{for } |x| \leq \rho \\ \frac{\lambda}{2\epsilon^{2/3}} (|x| - \rho - \epsilon^{2/3})^2 & \text{for } \rho \leq |x| \leq \rho + \epsilon^{2/3} \\ 0 & \text{for } |x| \geq \rho + \epsilon^{2/3}. \end{cases} \quad (2.12)$$

One can check that  $\chi \in C^1(\mathbb{R}^2 \setminus \{0\}) \cap H^1(\mathbb{R}^2)$  satisfies  $\Delta \chi \leq \frac{2\lambda}{\epsilon^{2/3}}$  in  $H^1(\mathbb{R}^2)$ .

Finally, we define the function  $\psi := \frac{|u|^2}{2} - \chi - \kappa^2 \epsilon^{2/3}$ , and compute:

$$\begin{aligned} \epsilon^2 \Delta \psi &= \epsilon^2 (|\nabla u|^2 + u \Delta u - \Delta \chi) \\ &\geq -\mu |u|^2 + |u|^4 - \epsilon^2 \Delta \chi \\ &\geq -\mu |u|^2 + |u|^4 - 2\epsilon^{4/3} \lambda. \end{aligned} \quad (2.13)$$

Now, one can see that when  $x \in \omega := \{x \in \mathbb{R}^2 : \psi(x) > 0\}$ , we have  $\frac{|u|^4}{3} - \mu |u|^2 \geq 0$ , since

$$x \in \omega \cap \overline{D(0; \rho)} \Rightarrow \frac{|u|^4}{3} \geq \frac{2\lambda}{3} \left( \rho - |x| + \frac{\epsilon^{2/3}}{2} \right) |u|^2 \geq \mu |u|^2.$$

In the open set  $\omega$  we also have:  $\frac{|u|^4}{3} \geq \frac{\kappa^4}{3} \epsilon^{4/3} \geq 2\epsilon^{4/3} \lambda$ , thus  $\Delta \psi \geq 0$  in  $\omega$  in the  $H^1$  sense. To conclude, we apply Kato's inequality that gives:  $\Delta \psi^+ \geq 0$  on  $\mathbb{R}^2$  in the  $H^1$  sense. Since  $\psi^+$  is subharmonic with compact support, we obtain by the maximum principle that  $\psi^+ \equiv 0$  or equivalently  $\psi \leq 0$  in  $\mathbb{R}^2$ . The statement of the lemma follows by adjusting the constant  $K$ .  $\square$

After this preparation we are ready to prove the main result of this section.

*Proof Theorem 2.1.* For every  $\xi = \rho e^{i\theta}$  we consider the local coordinates  $s = (s_1, s_2)$  in the basis  $(e^{i\theta}, ie^{i\theta})$ , and we rescale the solutions by setting  $\tilde{u}(s) = \frac{u_\epsilon(\xi + s\epsilon^{2/3})}{\epsilon^{1/3}}$ . Clearly  $\Delta \tilde{u}(s) = \epsilon \Delta u(\xi + s\epsilon^{2/3})$ , thus,

$$\Delta \tilde{u}(s) + \frac{\mu(\xi + s\epsilon^{2/3})}{\epsilon^{2/3}} \tilde{u}(s) - \tilde{u}^3(s) = 0, \quad \forall s \in \mathbb{R}^2.$$

Writing  $\mu(\xi + h) = \mu_1 h_1 + h \cdot A(h)$ , with  $\mu_1 := \mu'_{\text{rad}}(\rho) < 0$ ,  $A \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ , and  $A(0) = 0$ , we obtain

$$\Delta \tilde{u}(s) + (\mu_1 s_1 + A(s\epsilon^{2/3}) \cdot s) \tilde{u}(s) - \tilde{u}^3(s) = 0, \quad \forall s \in \mathbb{R}^2. \quad (2.14)$$

Next, we define the rescaled energy by

$$\tilde{E}(\tilde{u}) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla \tilde{u}(s)|^2 - \frac{\mu(\xi + s\epsilon^{2/3})}{2\epsilon^{2/3}} \tilde{u}^2(s) + \frac{1}{4} \tilde{u}^4(s) \right) ds. \quad (2.15)$$

With this definition  $\tilde{E}(\tilde{u}) = \frac{1}{\epsilon^{5/3}} E(u)$ . From Lemma 2.5 and (2.14), it follows that  $\Delta \tilde{u}$ , and also  $\nabla \tilde{u}$ , are uniformly bounded on compact sets. Moreover, by differentiating (2.14) we also obtain the boundedness of the second derivatives of  $\tilde{u}$ . Thanks to these uniform bounds, we can apply the theorem of Ascoli via a diagonal argument to obtain the convergence of  $\tilde{u}$  in  $C_{\text{loc}}^2(\mathbb{R}^2)$  (up to a subsequence) to a solution  $\tilde{z}$  of

$$\Delta \tilde{z}(s) + \mu_1 s_1 \tilde{z}(s) - \tilde{z}^3(s) = 0, \quad \forall s \in \mathbb{R}^2, \quad (2.16)$$

which is associated to the functional

$$\tilde{E}_0(\phi, J) = \int_J \left( \frac{1}{2} |\nabla \phi(s)|^2 - \frac{\mu_1}{2} s_1 \phi^2(s) + \frac{1}{4} \phi^4(s) \right) ds. \quad (2.17)$$

Given  $\tilde{\psi}(s)$  a test function supported in the compact set  $K$ , let

$$\psi(x) := \epsilon^{1/3} \tilde{\psi} \left( \frac{x - \xi}{\epsilon^{2/3}} \right) \Leftrightarrow \tilde{\psi}(s) = \frac{\psi(\xi + s \epsilon^{2/3})}{\epsilon^{1/3}}.$$

In the case where we take  $u$  to be the global minimizer  $v$ , since  $E(v_\epsilon + \psi, \text{supp } \psi) \geq E(v_\epsilon, \text{supp } \psi)$ , we have  $\tilde{E}(\tilde{v}_\epsilon + \tilde{\psi}, K) \geq \tilde{E}(\tilde{v}_\epsilon, K)$ , and at the limit  $\tilde{E}_0(\tilde{z} + \tilde{\psi}, K) \geq \tilde{E}_0(\tilde{z}, K)$ . Thus,  $\tilde{z}$  is a minimal solution of (2.16). In addition, the radial symmetry of  $v$ , implies that  $\tilde{z}$  depends only on the variable  $s_1$ . Indeed, noticing that  $\lim_{\epsilon \rightarrow 0} \frac{|\xi + \epsilon^{\frac{2}{3}}(s_1, s_2)| - \rho}{\epsilon^{\frac{2}{3}}} = s_1$ , it follows that  $\tilde{v}_\epsilon(s_1, s_2) = \tilde{v}_\epsilon(s_1 + o(1), 0)$ , and  $\tilde{z}(s_1, s_2) = \tilde{z}(s_1, 0)$ . Similarly, in the case where we take  $u$  to be the odd minimizer and  $\xi = (\pm \rho, 0)$ , we can see that  $\tilde{z}$  is a minimal solution of (2.16) for perturbations such that  $\tilde{\psi}(s_1, s_2) = -\tilde{\psi}(s_1, -s_2)$ . Finally, setting  $y(s) := \frac{1}{\sqrt{2}(-\mu_1)^{1/3}} \tilde{z} \left( \frac{s}{(-\mu_1)^{1/3}} \right)$ , (2.16) reduces to (2.6), that is,  $y$  solves (2.6). In the case where we take  $u$  to be the global minimizer  $v$ , we can see that either  $y(s_1, s_2) = h(s_1)$  or  $y(s_1, s_2) = -h(s_1)$ , since  $\pm h$  are the only minimal solutions of (1.4) (cf. [12, Theorem 1.3]). On the other hand, in the case where we take  $u$  to be the odd minimizer and  $\xi = (\pm \rho, 0)$ , it is clear that  $y$  is odd with respect to  $s_2$ , and minimal for perturbations such that  $\tilde{\psi}(s_1, s_2) = -\tilde{\psi}(s_1, -s_2)$ .  $\square$

### 3. Proof of Theorem 1.2

We will proceed in few steps. The proof of (i), (ii) and (iii) follows from Theorem 2.1, Lemma 2.5, and the fact that a minimal solution of 1.9 cannot be identically zero. To establish (v) we proceed as in Theorem 2.1. After rescaling appropriately  $y$  as  $x_1 \rightarrow -\infty$ , we compute uniform bounds of the rescaled functions. Then, by the theorem of Ascoli, we obtain at the limit a minimal solution of the Allen-Cahn equation (1.2). The proof of (vi) and (vii) is based on the moving plane method applied in a sector contained in the upper half-plane. The main difficulty is due to the unboundedness of the domain and to the availability of boundary conditions

only on the  $x_1$  axis where  $y(x_1, 0) = 0$ . We also utilize the asymptotic behaviour of  $y$ , as  $x_1 \rightarrow \pm\infty$ , provided respectively by (v) and Lemma 3.2. Our main tool is a version of the maximum principle in unbounded domains (*cf.* Lemma 3.1), that allows us to compute bounds for  $y_{x_1}$  and  $y_{x_2}$  when  $x_1$  is large enough and  $x_2 > 0$  (*cf.* Lemmas 3.3 and 3.4). Next, these bounds are extended to the whole half-plane  $x_2 > 0$  by applying the sliding method (*cf.* Lemma 3.5).

*Proof of (i), (ii) and (iii).* By applying Theorem 2.1 in a neighborhood of the point  $\xi = (\rho, 0)$  to the odd minimizer  $u$ , such that  $u > 0$  on  $\mathbb{R} \times (0, \infty)$ , it is clear that we obtain a solution  $y$  of (2.6) which is odd with respect to the second variable  $s_2$ , and such that  $y \geq 0$ , on  $\mathbb{R} \times (0, \infty)$ . For the sake of convenience in what follows we substitute the variables  $(s_1, s_2)$  by  $(x_1, x_2)$ . The properties (ii) and (iii) are also straightforward by Theorem 2.1 and Lemma 2.5. Thus, it remains to show that  $y(x_1, x_2) > 0, \forall x \in \mathbb{R} \times (0, \infty)$ . Assume by contradiction that  $y(x_1, x_2) = 0$ , for some  $x \in \mathbb{R} \times (0, \infty)$ , then it follows from the maximum principle that  $y \equiv 0$ . To conclude we are going to show that a solution  $y$  of (1.9) which is minimal for odd perturbations, cannot be identically zero. Indeed, the minimality of  $y$  implies that the second variation of the energy  $E_{P_{II}}$  is nonnegative:

$$\int_{\mathbb{R}^2} (|\nabla \phi(x)|^2 + (6y^2(x) + x_1)\phi^2(x))dx \geq 0, \quad \forall \phi \in C_0^1(\mathbb{R}^2), \quad (3.1)$$

such that  $\phi(x_1, x_2) = -\phi(x_1, -x_2)$ .

Clearly (3.1) does not hold when  $y \equiv 0$ , if we take  $\phi(x) = \phi_0(x_1 + l, x_2)$ , with  $l \rightarrow \infty$ , and  $\phi_0 \in C_0^1(\mathbb{R}^2)$  fixed, such that  $\phi_0(x_1, x_2) = -\phi_0(x_1, -x_2)$ , and  $\phi_0 \not\equiv 0$ .  $\square$

Next we recall a useful version of the maximum principle in unbounded domains [9, Lemma 2.1].

**Lemma 3.1.** *Let  $D$  be a domain (open connected set) in  $\mathbb{R}^n$ , possibly unbounded. Assume that  $\overline{D}$  is disjoint from the closure of an infinite open connected cone  $\Sigma$ . Suppose there is a function  $z$  in  $C(\overline{D})$  that is bounded above and satisfies for some continuous function  $c(x)$*

$$\begin{aligned} \Delta z - c(x)z &\geq 0 \text{ in } D \text{ with } c(x) \geq 0 \\ z &\leq 0 \text{ on } \partial D. \end{aligned}$$

*Then  $z \leq 0$  in  $D$ .*

As a first application of Lemma 3.1 we prove the exponential convergence of  $y$  to 0, as  $x_1 \rightarrow \infty$ .

**Lemma 3.2.**  $|y(x_1, x_2)| = O(e^{-\frac{2}{3}x_1^{3/2}})$ , as  $x_1 \rightarrow \infty$  (uniformly in  $x_2$ ).

*Proof.* We define  $\psi(x_1, x_2) := Me^{-\frac{2}{3}x_1^{3/2}}$ , in the domain  $D := \{(x_1, x_2) : x_1 > 1, x_2 > 0\}$ , where  $M \geq e^{\frac{2}{3}} \sup_{x_2 \geq 0} y(1, x_2)$  is a constant. It is easy to see that  $\Delta \psi \leq x_1 \psi$  in  $D$ , and  $\Delta(y - \psi) \geq x_1(y - \psi)$  in  $D$ . Since  $y - \psi \leq 0$  on  $\partial D$ , it follows from Lemma 3.1 that  $y \leq \psi$  in  $D$ .  $\square$

*Proof of (v).* We set  $(t_1, t_2) := \left(-\frac{2}{3}(-x_1)^{\frac{3}{2}}, (-x_1)^{\frac{1}{2}}r\right)$ , where  $x_1 \leq -1$  and  $r \in \mathbb{R}$ . Equivalently we have  $(x_1, r) = \left(-\left(-\frac{3}{2}t_1\right)^{\frac{2}{3}}, t_2(-\frac{3}{2}t_1)^{-\frac{1}{3}}\right)$ . Next we define  $\tilde{y}(t_1, t_2) := \frac{\sqrt{2}}{(-\frac{3}{2}t_1)^{\frac{1}{3}}} y(x_1, r + x_2)$ , for every  $x_2 \in \mathbb{R}$  *fixed*, or equivalently

$$y(x_1, r + x_2) = \frac{(-x_1)^{\frac{1}{2}}}{\sqrt{2}} \tilde{y}(t_1, t_2). \quad (3.2)$$

We are going to show that  $\tilde{y}(t_1, t_2)$  is uniformly bounded up to the second derivatives, when  $t_2$  belongs to a compact interval and  $t_1 \rightarrow -\infty$ . By differentiating (3.2) with respect to  $x_1$  and  $r$  we obtain

$$\sqrt{2}y_{x_2}(x_1, r + x_2) = (-x_1)\tilde{y}_{t_2}(t_1, t_2), \quad (3.3a)$$

$$\sqrt{2}y_{x_2x_2}(x_1, r + x_2) = (-x_1)^{\frac{3}{2}}\tilde{y}_{t_2t_2}(t_1, t_2), \quad (3.3b)$$

$$\begin{aligned} \sqrt{2}y_{x_1}(x_1, r + x_2) &= -\frac{1}{2}(-x_1)^{-\frac{1}{2}}\tilde{y}(t_1, t_2) + (-x_1)\tilde{y}_{t_1}(t_1, t_2) \\ &\quad - \frac{r}{2}\tilde{y}_{t_2}(t_1, t_2), \end{aligned} \quad (3.3c)$$

$$\sqrt{2}y_{x_1x_2} = -\tilde{y}_{t_2} + (-x_1)^{\frac{3}{2}}\tilde{y}_{t_1t_2} - \frac{r}{2}(-x_1)^{\frac{1}{2}}\tilde{y}_{t_2,t_2}, \quad (3.3d)$$

$$\begin{aligned} \sqrt{2}y_{x_1x_1} &= -\frac{1}{4}(-x_1)^{-\frac{3}{2}}\tilde{y} - \frac{3}{2}\tilde{y}_{t_1} + \frac{r}{4}(-x_1)^{-1}\tilde{y}_{t_2} \\ &\quad + (-x_1)^{\frac{3}{2}}\tilde{y}_{t_1t_1} - r(-x_1)^{\frac{1}{2}}\tilde{y}_{t_1t_2} + \frac{r^2}{4}(-x_1)^{-\frac{1}{2}}\tilde{y}_{t_2,t_2}. \end{aligned} \quad (3.3e)$$

Since by construction (*cf.* (2.11) in Lemma 2.5)  $y$  satisfies  $|y(x_1, x_2)| = O(|-x_1|^{\frac{1}{2}})$  as  $x_1 \rightarrow -\infty$  (*i.e.*  $\tilde{y}$  is bounded), we obtain by (1.9) and standard elliptic estimates [21, Section 3.4 page 37] that

$$|\nabla y(x_1, x_2)| = O(|-x_1|^{\frac{3}{2}}) \text{ and } |D^2 y(x_1, x_2)| = O(|-x_1|^{\frac{5}{2}}), \text{ as } x_1 \rightarrow -\infty. \quad (3.4)$$

From (3.4) and (3.3) it follows that

$$|\nabla \tilde{y}(t_1, t_2)| = O(|-x_1|^{\frac{1}{2}}) \text{ and } |D^2 \tilde{y}(t_1, t_2)| = O(|-x_1|), \text{ as } x_1 \rightarrow -\infty, \quad (3.5)$$

provided that  $(t_1, t_2) \in \Sigma_{t_0, r_0} := \{(t_1, t_2) : t_1 \leq t_0, |t_2| \leq r_0(-\frac{3}{2}t_1)^{\frac{1}{3}}\}$ , where  $t_0 < 0$  and  $r_0 > 0$  are arbitrary constants. In particular, we have  $\sqrt{2}\Delta y(x_1, x_2) = (-x_1)^{\frac{3}{2}}\Delta \tilde{y}(t_1, t_2) + O(|-x_1|^{\frac{3}{2}})$ , for  $(t_1, t_2) \in \Sigma_{t_0, r_0}$ . On the other hand it is clear by (1.9) that  $\sqrt{2}\Delta y(x_1, x_2) = (-x_1)^{\frac{3}{2}}(\tilde{y}^3(t_1, t_2) - \tilde{y}(t_1, t_2))$ , thus

$$|\Delta \tilde{y}(t_1, t_2)| \text{ and } |\nabla \tilde{y}(t_1, t_2)| \text{ are bounded, } \forall (t_1, t_2) \in \Sigma_{t_0, r_0}. \quad (3.6)$$

Similarly, by differentiating once more equations (3.3) with respect to  $x_1$  and  $r$ , one can show that

$$|D^2\tilde{y}(t_1, t_2)| \text{ is bounded, } \forall (t_1, t_2) \in \Sigma_{t_0, r_0}. \quad (3.7)$$

Next, we apply the theorem of Ascoli to the sequence  $\tilde{y}(t_1 + l, t_2)$  as  $l \rightarrow -\infty$ . Up to a subsequence  $l_n \rightarrow -\infty$ , we obtain via a diagonal argument, the convergence in  $C_{\text{loc}}^1(\mathbb{R}^2)$  of  $\tilde{y}_n(t_1, t_2) := \tilde{y}(t_1 + l_n, t_2)$  to a bounded function  $\tilde{z}(t_1, t_2)$  that we are going to determine. Our claim is that the limit  $\tilde{z}$  is a minimal solution of the Allen-Cahn equation (1.2), which is independent of the subsequence  $l_n$ . The proof of this property is based on the following energy considerations. Let  $(e_1, e_2)$  be the canonical basis of  $\mathbb{R}^2$ . The energy functional

$$\begin{aligned} & E_{\text{P}_\text{II}}(y, A) \\ &= \int_{A - x_2 e_2} \left[ \frac{1}{2} |\nabla y(x_1, r + x_2)|^2 + \frac{1}{2} x_1 y^2(x_1, r + x_2) + \frac{1}{2} y^4(x_1, r + x_2) \right] dx_1 dr, \end{aligned} \quad (3.8)$$

associated to (1.9), becomes after changing variables as in (3.2)

$$E_{\text{P}_\text{II}}(y, A) = \tilde{E}_{\text{P}_\text{II}}(\tilde{y}, \tilde{A}) = \tilde{F}(\tilde{y}, \tilde{A}) + \tilde{R}(\tilde{y}, \tilde{A}), \quad (3.9)$$

where

$$\tilde{A} := \{(t_1(x_1), t_2(x_1, r)) : (x_1, r) \in A - x_2 e_2\}, \quad (3.10)$$

$$\tilde{F}(\tilde{y}, \tilde{A}) := \int_{\tilde{A}} \frac{1}{2} \left( -\frac{3}{2} t_1 \right)^{\frac{2}{3}} \left[ \frac{1}{2} |\nabla \tilde{y}(t_1, t_2)|^2 - \frac{\tilde{y}^2(t_1, t_2)}{2} + \frac{\tilde{y}^4(t_1, t_2)}{4} \right] dt_1 dt_2, \quad (3.11)$$

and

$$\tilde{R}(\tilde{y}, \tilde{A}) := \int_{\tilde{A}} \left[ \frac{(\tilde{y} + t_2 \tilde{y}_{t_2})^2}{16(-\frac{3}{2} t_1)^{\frac{4}{3}}} - \frac{(\tilde{y} + t_2 \tilde{y}_{t_2}) \tilde{y}_{t_1}}{4(-\frac{3}{2} t_1)^{\frac{1}{3}}} \right] dt_1 dt_2. \quad (3.12)$$

Let  $\tilde{\phi}(t_1, t_2) \in C_0^\infty(\mathbb{R}^2)$  be a test function such that  $\tilde{B} := \text{supp } \tilde{\phi} \subset \{(t_1, t_2) : c - d \leq t_1 \leq c\}$ , for some constants  $c \in \mathbb{R}$  and  $d > 0$ . Given  $l \in \mathbb{R}$ , we consider the translated functions  $\tilde{\phi}^{-l}(t_1, t_2) := \tilde{\phi}(t_1 - l, t_2)$ , and  $\tilde{y}^l(t_1, t_2) = \tilde{y}(t_1 + l, t_2)$ . Note that  $\tilde{B}^l := \text{supp } \tilde{\phi}^{-l} = \tilde{B} + l e_1$ , and  $\text{supp } \tilde{\phi}^{-l} \subset \{(t_1, t_2) : t_1 < -1\}$  when  $l < 1 - c$ . Thus, for  $l < 1 - c$ , we can define  $\phi^{-l} \in C_0^\infty(\mathbb{R}^2)$  by  $\phi^{-l}(x_1, r + x_2) = \frac{(-x_1)^{\frac{1}{2}}}{\sqrt{2}} \tilde{\phi}^{-l}(t_1, t_2)$  as in (3.2). Let  $B^l := \{(x_1(t_1), r(t_1, t_2) + x_2) : (t_1, t_2) \in \tilde{B}^l\}$ .

We first examine the case where  $x_2 = 0$ , and assume that  $\tilde{\phi}(t_1, t_2) = -\tilde{\phi}(t_1, -t_2)$ . In view of the minimality of  $y$  and (3.9), we have

$$\tilde{E}_{\text{P}_\text{II}}(\tilde{y} + \tilde{\phi}^{-l}, \tilde{B}^l) = E_{\text{P}_\text{II}}(y + \phi^{-l}, B^l) \geq E_{\text{P}_\text{II}}(y, B^l) = \tilde{E}_{\text{P}_\text{II}}(\tilde{y}, \tilde{B}^l). \quad (3.13)$$

On the one hand, it is clear that the boundedness of  $\tilde{y}$  and (3.6) imply that  $\lim_{l \rightarrow -\infty} \tilde{R}(\tilde{y} + \tilde{\phi}^{-l}, \tilde{B}^l) = 0$  and  $\lim_{l \rightarrow -\infty} \tilde{R}(\tilde{y}, \tilde{B}^l) = 0$ . Next, setting  $t_0 := c + l$ , we have

$$\left( -\frac{3}{2} t_1 \right)^{\frac{2}{3}} \leq \left( -\frac{3}{2} t_0 \right)^{\frac{2}{3}} + d \left( -\frac{3}{2} t_0 \right)^{-\frac{1}{3}}, \quad \forall t_1 \in [t_0 - d, t_0]. \quad (3.14)$$

Thus, we obtain

$$\begin{aligned}\tilde{F}(\tilde{y}, \tilde{B}^l) &= \frac{1}{2} \left( -\frac{3}{2} t_0 \right)^{\frac{2}{3}} \tilde{G}(\tilde{y}, \tilde{B}^l) + O(|t_0|^{-\frac{1}{3}}) \\ &= \frac{1}{2} \left( -\frac{3}{2} t_0 \right)^{\frac{2}{3}} \tilde{G}(\tilde{y}^l, \tilde{B}) + O(|t_0|^{-\frac{1}{3}}),\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}\tilde{F}(\tilde{y} + \tilde{\phi}^{-l}, \tilde{B}^l) &= \frac{1}{2} \left( -\frac{3}{2} t_0 \right)^{\frac{2}{3}} \tilde{G}(\tilde{y} + \tilde{\phi}^{-l}, \tilde{B}^l) + O(|t_0|^{-\frac{1}{3}}) \\ &= \frac{1}{2} \left( -\frac{3}{2} t_0 \right)^{\frac{2}{3}} \tilde{G}(\tilde{y}^l + \tilde{\phi}, \tilde{B}) + O(|t_0|^{-\frac{1}{3}}),\end{aligned}\tag{3.16}$$

where we have set  $\tilde{G}(\tilde{z}, \tilde{B}) := \int_{\tilde{B}} (\frac{1}{2} |\nabla \tilde{z}|^2 - \frac{\tilde{z}^2}{2} + \frac{\tilde{z}^4}{4}) dt$ . Finally, since  $\tilde{y}^{l_n}(t_1, t_2) \rightarrow \tilde{z}(t_1, t_2)$  in  $C_{\text{loc}}^1(\mathbb{R}^2)$ , as  $n \rightarrow \infty$ , we conclude that

$$\begin{aligned}\tilde{G}(\tilde{z} + \tilde{\phi}, \tilde{B}) &= \lim_{n \rightarrow \infty} \frac{2}{(-\frac{3}{2}(c + l_n))^{\frac{2}{3}}} \tilde{E}_{\text{P}_\text{II}}(\tilde{y}^{l_n} + \tilde{\phi}, \tilde{B}) \\ &\geq \lim_{n \rightarrow \infty} \frac{2}{(-\frac{3}{2}(c + l_n))^{\frac{2}{3}}} \tilde{E}_{\text{P}_\text{II}}(\tilde{y}^{l_n} + \tilde{\phi}, \tilde{B}) = \tilde{G}(\tilde{z}, \tilde{B}),\end{aligned}\tag{3.17}$$

or equivalently  $E_{\text{AC}}(\tilde{z} + \tilde{\phi}, \tilde{B}) \geq E_{\text{AC}}(\tilde{z}, \tilde{B})$ . This means that  $\tilde{z}$  is a minimal solution of the Allen-Cahn equation (1.2) for odd perturbations  $\tilde{\phi}$ . In particular  $\tilde{z} \not\equiv 0$ , and as a consequence of the maximum principle,  $\tilde{z}(t_1, 0) = 0$ ,  $\forall t_1 \in \mathbb{R}$ , and  $\tilde{z}(t_1, t_2) \geq 0$ ,  $\forall (t_1, t_2) \in \mathbb{R} \times (0, \infty)$ , imply that  $\tilde{z}(t_1, t_2) > 0$ ,  $\forall (t_1, t_2) \in \mathbb{R} \times (0, \infty)$ . Thus, from [10, Theorem 1.5], it follows that  $\tilde{z}$  is a function of only  $t_2$ , which is the heteroclinic connection  $\tilde{z}(t_1, t_2) = \eta(t_2) = \tanh(t_2/\sqrt{2})$ . Furthermore, since the limit  $\tilde{z}$  is uniquely determined, the convergence  $\tilde{y}^l(t_1, t_2) \rightarrow \tilde{z}(t_1, t_2)$  holds as  $l \rightarrow -\infty$ .

It remains to examine the case where  $x_2 \neq 0$ . Without loss of generality we assume that  $x_2 > 0$ . Now (3.13) holds for arbitrary test functions  $\tilde{\phi}(t_1, t_2) \in C_0^\infty(\mathbb{R}^2)$ , since  $B^l \subset \{(x_1, x_2) : x_2 > 0\}$  as  $l \rightarrow -\infty$ . Repeating the previous arguments we find that  $\tilde{z}$  is a nonnegative minimal solution of (1.2). Applying [6, Corollary 5.2], we deduce that  $\tilde{z} \equiv 1$ . This completes the proof of (v).  $\square$

*Proof of (vi) and (vii).* The proofs of (vi) and (vii) which are based on the moving plane method, follow from the next lemmas.

**Lemma 3.3.** *We have  $y_{x_1}(x_1, x_2) < 0$ ,  $\forall x_1 \geq 0$ ,  $\forall x_2 > 0$ . In addition, for every  $d > 0$ , there holds  $\sup_{x_2 \geq d} y_{x_1}(1, x_2) < 0$ , and  $\inf_{x_2 \geq d} y(1, x_2) > 0$ .*

*Proof.* Given  $\lambda \geq 0$ , we define the function  $\psi_\lambda(x_1, x_2) := y(x_1, x_2) - y(-x_1 + 2\lambda, x_2)$  for  $(x_1, x_2) \in D_\lambda := \{(x_1, x_2) : x_1 > \lambda, x_2 > 0\}$ . One can check that  $\psi_\lambda = 0$  on  $\partial D_\lambda$ , and

$$\Delta \psi_\lambda - c(x_1, x_2) \psi_\lambda = 2(x_1 - \lambda) y(-x_1 + 2\lambda, x_2) \geq 0,$$

with  $c(x_1, x_2) = x_1 + 2(y^2(x_1, x_2) + y(x_1, x_2)y(-x_1 + 2\lambda, x_2) + y^2(-x_1 + 2\lambda, x_2)) \geq 0$ . Furthermore,  $\psi_\lambda$  is bounded above by Theorem 1.2 (ii), and not identically zero by Theorem 1.2 (v). As a consequence of Lemma 3.1, it follows that  $\psi_\lambda(x_1, x_2) < 0$ ,  $\forall x_1 > \lambda$ ,  $\forall x_2 > 0$ , and thus by Hopf's Lemma  $\frac{\partial \psi_\lambda}{\partial x_1}(\lambda, x_2) = 2y_{x_1}(\lambda, x_2) < 0$ ,  $\forall x_2 > 0$ . To establish that  $\sup_{x_2 \geq d} y_{x_1}(1, x_2) < 0$ , we proceed by contradiction and assume the existence of a sequence  $\{l_n\}$  such that  $\lim_{n \rightarrow \infty} l_n = \infty$  and  $\lim_{n \rightarrow \infty} y_{x_1}(1, l_n) = 0$ . Next, we set  $\tilde{y}_n(x_1, x_2) = y(x_1, x_2 + l_n)$ . In view of the bounds provided in Theorem 1.2 (ii), we obtain by the theorem of Ascoli that (up to subsequence)  $\tilde{y}_n$  converges in  $C_{\text{loc}}^1$  to a nonnegative minimal solution  $\tilde{y}$  of (1.9). Since  $\tilde{y}_{x_1}(1, 0) = \lim_{n \rightarrow \infty} y_{x_1}(1, l_n) = 0$ , and  $\tilde{y}_{x_1}(x_1, x_2) \leq 0$ ,  $\forall x_1 \geq 0$ ,  $\forall x_2 \in \mathbb{R}$ , the maximum principle applied to

$$\Delta \tilde{y}_{x_1} = \tilde{y} + (x_1 + 6\tilde{y}^2)\tilde{y}_{x_1} \geq (x_1 + 6\tilde{y}^2)\tilde{y}_{x_1}, \quad (3.18)$$

implies that  $\tilde{y}_{x_1}(x_1, x_2) = 0$ ,  $\forall x_1 \geq 0$ ,  $\forall x_2 \in \mathbb{R}$ . Then, since  $\lim_{x_1 \rightarrow \infty} \tilde{y}(x_1, x_2) = 0$ ,  $\forall x_2 \in \mathbb{R}$ , it follows that  $\tilde{y} \equiv 0$  in the half-plane  $x_1 \geq 0$ . Finally, we deduce by unique continuation that  $\tilde{y} \equiv 0$  in  $\mathbb{R}^2$ , which is a contradiction since  $\tilde{y}$  is minimal. Thus we have established that  $\sup_{x_2 \geq d} y_{x_1}(1, x_2) < 0$ . The proof that  $\inf_{x_2 \geq d} y(1, x_2) > 0$  is similar.  $\square$

**Lemma 3.4.** *For every vector  $\mathbf{n} = e^{i(\theta + \frac{\pi}{2})} \in \mathbb{C} \sim \mathbb{R}^2$ , with  $\theta \in (0, \frac{\pi}{2})$ , there exists  $s_{\mathbf{n}} > 0$  such that  $\nabla y(x_1, x_2) \cdot \mathbf{n} > 0$ ,  $\forall x_1 > s_{\mathbf{n}}$ ,  $\forall x_2 > 0$ .*

*Proof.* Our first claim is that there is a constant  $k_1 > 0$ , such that  $k_1 y_{x_1}(x_1, x_2) \leq -\sqrt{x_1} y(x_1, x_2)$ ,  $\forall x_1 \geq 1$ ,  $\forall x_2 \geq 0$ . Indeed, let  $\psi(x_1, x_2) = k_1 y_{x_1}(x_1, x_2) + \sqrt{x_1} y(x_1, x_2)$  for  $(x_1, x_2) \in D := \{x_1 > 1, x_2 > 0\}$ , where the constant  $k_1$  will be adjusted later. It is clear that  $\psi(x_1, 0) = 0$ ,  $\forall x_1 \geq 1$ . We also note that  $y_{x_1 x_2}(1, 0) < 0$  by Hopf's Lemma, since the function  $y_{x_1}$  vanishes at  $(1, 0)$ , is negative in  $\{x_1 > 0, x_2 > 0\}$ , and satisfies (3.18). This and  $\sup_{x_2 \geq d} y_{x_1}(1, x_2) < 0$ ,  $\forall d > 0$ , imply that when  $k_1$  is large enough, we have  $\psi(1, x_2) \leq 0$ ,  $\forall x_2 \geq 0$ . Next, we compute

$$\begin{aligned} \Delta \psi &= \left( x_1 + 6y^2 + \frac{1}{k_1 \sqrt{x_1}} \right) k_1 y_{x_1} + \left( x_1 + 2y^2 + \frac{k_1}{\sqrt{x_1}} - \frac{1}{4x_1^2} \right) \sqrt{x_1} y \\ &= \left( x_1 + 2y^2 + \frac{k_1}{\sqrt{x_1}} - \frac{1}{4x_1^2} \right) \psi + \left( 4y^2 + \frac{1}{k_1 \sqrt{x_1}} - \frac{k_1}{\sqrt{x_1}} + \frac{1}{4x_1^2} \right) k_1 y_{x_1}. \end{aligned}$$

By choosing  $k_1$  large enough we can ensure that  $(x_1 + 2y^2 + \frac{k_1}{\sqrt{x_1}} - \frac{1}{4x_1^2}) \geq 0$  and  $(4y^2 + \frac{1}{k_1 \sqrt{x_1}} - \frac{k_1}{\sqrt{x_1}} + \frac{1}{4x_1^2}) \leq 0$ , when  $x_1 \geq 1$  and  $x_2 \geq 0$ . Thus, by applying Lemma 3.1, our claim follows.

Similarly, we are going to establish that there is a constant  $k_2 > 0$ , such that  $y_{x_2}(x_1, x_2) \geq -k_2 y(x_1, x_2)$ ,  $\forall x_1 \geq 1$ ,  $\forall x_2 \geq 0$ . To do this we let  $\psi(x_1, x_2) = -y_{x_2}(x_1, x_2) - k_2 y(x_1, x_2)$  for  $(x_1, x_2) \in D$ , where the constant  $k_2$  will again be adjusted later. We first note that  $y_{x_2}(x_1, 0) > 0$ ,  $\forall x_1 \in \mathbb{R}$ , since the function  $y$  vanishes at  $(x_1, 0)$ , is positive in  $\{x_2 > 0\}$ , and satisfies (1.9). This and

$\inf_{x_2 \geq d} y(1, x_2) > 0, \forall d > 0$ , imply that when  $k_2$  is large enough, we have  $\psi(1, x_2) \leq 0, \forall x_2 \geq 0$ . On the other hand, it is clear that  $\psi(x_1, 0) < 0, \forall x_1 \geq 1$ . Next, we compute

$$\Delta\psi = (x_1 + 6y^2)(-y_{x_2}) + (x_1 + 2y^2)(-k_2 y) \geq (x_1 + 6y^2)\psi.$$

Thus, by applying Lemma 3.1, it follows that  $\psi \leq 0$  in  $D$ . Finally, given  $\theta \in (0, \pi/2)$ , we have

$$\nabla y \cdot \mathbf{n} = -y_{x_1} \sin \theta + y_{x_2} \cos \theta \geq \frac{\sqrt{x_1}}{k_1} y \sin \theta - k_2 y \cos \theta, \quad \forall x_1 \geq 1, \forall x_2 \geq 0,$$

and therefore  $\nabla y \cdot \mathbf{n} > 0$  for  $x_1 > s_n := \left(\frac{k_1 k_2}{\tan \theta}\right)^2$ , and  $x_2 > 0$ .  $\square$

**Lemma 3.5.** *Let  $\theta \in (0, \frac{\pi}{2})$  be fixed, and consider for every  $\lambda \in \mathbb{R}$  the reflection  $\sigma_\lambda$  with respect to the line  $\Gamma_\lambda := \{(x_1, x_2) : x_2 = \tan \theta(x_1 - \lambda)\}$ , and the domain  $D_\lambda := \{(x_1, x_2) : 0 < x_2 < \tan \theta(x_1 - \lambda)\}$ . Then, the function  $\psi_\lambda(x_1, x_2) := y(x_1, x_2) - y(\sigma_\lambda(x_1, x_2))$  is negative in  $D_\lambda$ , for every  $\lambda \in \mathbb{R}$ .*

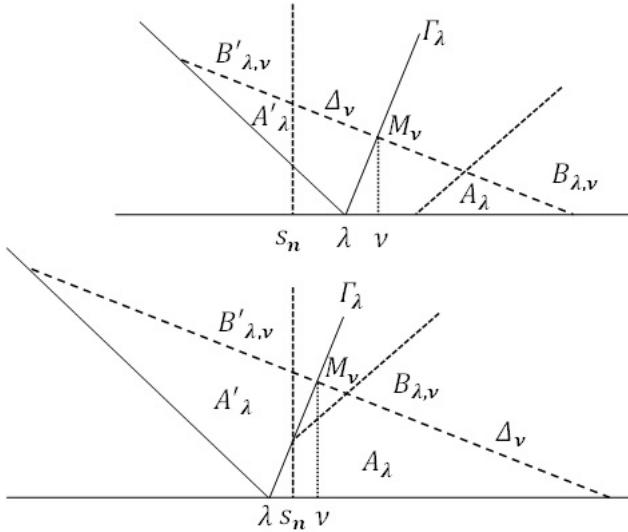
*Proof.* We set  $\mathbf{n} = e^{i(\theta + \frac{\pi}{2})}$  as in Lemma 3.4, and denote by  $(p', q')$  the image by  $\sigma_\lambda$  of a point  $(p, q) \in D_\lambda$ , and by  $D'_\lambda$  the set  $\sigma_\lambda(D_\lambda)$ . It is obvious that  $\psi_\lambda(x_1, 0) < 0, \forall x_1 > \lambda$ , and that  $\psi_\lambda(x_1, x_2) = 0, \forall (x_1, x_2) \in \Gamma_\lambda$ . Moreover,  $\psi_\lambda$  satisfies

$$\Delta\psi_\lambda(p, q) - c(p, q)\psi_\lambda = (p - p')y(p', q') \geq 0, \quad \forall (p, q) \in D_\lambda,$$

with  $c(p, q) = p + 2(y^2(p, q) + y(p, q)y(p', q') + y^2(p', q'))$ . For each  $\lambda \in \mathbb{R}$  we consider the statement

$$\psi_\lambda(p, q) < 0, \quad \forall (p, q) \in D_\lambda. \quad (3.19)$$

We shall first establish Lemma 3.5 in the case where  $\theta \in (0, \frac{\pi}{4})$ . According to Lemma 3.4, (3.19) is valid for each  $\lambda \geq s_n$ . Set  $\lambda_0 = \inf\{\lambda \in \mathbb{R} : \psi_\mu < 0 \text{ holds in } D_\mu, \text{ for each } \mu \geq \lambda\}$ . We will prove  $\lambda_0 = -\infty$ . Assume instead  $\lambda_0 \in \mathbb{R}$ . Then, there exist a sequence  $\lambda_k < \lambda_0$  such that  $\lim_{k \rightarrow \infty} \lambda_k = \lambda_0$ , and a sequence  $(p_k, q_k) \in D_{\lambda_k}$ , such that  $y(p_k, q_k) \geq y(p'_k, q'_k)$ . According to Lemma 3.4, we have  $p'_k \leq s_n$ , thus the sequence  $(p_k, q_k)$  is bounded, since by assumption  $\theta \in (0, \pi/4)$ . Up to subsequence we may assume that  $\lim_{k \rightarrow \infty} (p_k, q_k) = (p_0, q_0) \in \overline{D_{\lambda_0}}$ , with  $p'_0 \leq s_n$ . By definition of  $\lambda_0$ , we have  $\psi_{\lambda_0} \leq 0$  in  $D_{\lambda_0}$ , and  $\psi_{\lambda_0}(p_0, q_0) = 0$  i.e.  $y(p_0, q_0) = y(p'_0, q'_0)$ . Now we distinguish the following cases. If  $(p_0, q_0) \in D_{\lambda_0}$ , the maximum principle implies that  $\psi_{\lambda_0} \equiv 0$  in  $D_{\lambda_0}$ . Clearly, this situation is excluded, since  $y$  is positive in the half-plane  $\{x_2 > 0\}$ . On the other hand, the maximum principle also implies that  $\frac{\partial \psi_{\lambda_0}}{\partial \mathbf{n}}(p, q) = 2 \frac{\partial y}{\partial \mathbf{n}}(p, q) > 0$ , provided that  $(p, q) \in \Gamma_{\lambda_0}$  and  $q > 0$ . Furthermore, the previous inequality still holds at the vertex  $(p, q) = (\lambda_0, 0)$ , since  $y_{x_2}(x_1, 0) > 0$  and  $y_{x_1}(x_1, 0) = 0, \forall x_1 \in \mathbb{R}$  (cf. the proof of Lemma 3.4). As a consequence, in a neighborhood of the line segment  $\{(x_1, x_2) : x_2 = \tan \theta(x_1 - \lambda_0), 0 \leq x_1 \leq s_n\}$ , we have that  $\frac{\partial y}{\partial \mathbf{n}} > 0$ , and it



**Figure 3.1.** The sets  $A_\lambda$ ,  $A'_\lambda$ ,  $B_{\lambda,v}$ ,  $B'_{\lambda,v}$ , and the lines  $\Gamma_\lambda$ ,  $\Delta_v$ , in the case where  $\lambda > s_n$  and  $\lambda < s_n$ .

follows that  $(p_0, q_0)$  cannot belong to  $\Gamma_{\lambda_0}$ . Finally, since the case where  $p_0 > \lambda_0$  and  $q_0 = 0$  is ruled out (because  $y$  is positive in the half-plane  $\{x_2 > 0\}$ ), we have reached a contradiction.

Next, we establish Lemma 3.5 in the case where  $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$ , which is a little bit more involved. When  $\theta = \frac{\pi}{4}$ , it is clear that (3.19) is valid for each  $\lambda \geq s_n$ . Otherwise, when  $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ , let  $A'_\lambda := \{(p', q') \in D'_\lambda : p' \leq s_n\}$ , and let  $A_\lambda = \sigma_\lambda(A'_\lambda)$ . Our first claim is that  $m := \inf_{A'_{s_n+1}} y > 0$ . Indeed, proceeding as in the proof of Theorem 1.2 (v), one can see that

$$\lim_{(x_1, x_2) \in A'_{s_n+1}, x_1 \rightarrow -\infty} \frac{\sqrt{2}}{\sqrt{-x_1}} y(x_1, x_2) = 1.$$

In addition, proceeding as in the proof of Lemma 3.4, we obtain that  $\inf\{y(x_1, x_2) : (x_1, x_2) \in A'_{s_n+1}, s_n - l \leq x_1 \leq s_n\} > 0$ , for every constant  $l > 0$ . Thus,  $m > 0$ . On the other hand, we have  $\lim_{\lambda \rightarrow \infty} \sup\{y(x_1, x_2) : (x_1, x_2) \in A_\lambda\} = 0$ , since  $\lim_{\lambda \rightarrow \infty} \inf\{x_1 : (x_1, x_2) \in A_\lambda\} = 0$  (cf. Lemma 3.2). As a consequence when  $\lambda \geq s_n + 1$  is large enough, we have  $y(p', q') \geq m > y(p, q)$ ,  $\forall (p, q) \in A_\lambda$ , and also  $y(p', q') > y(p, q)$ ,  $\forall (p, q) \in D_\lambda \setminus A_\lambda$ , by definition of  $s_n$ . This establishes that (3.19) holds for  $\lambda$  large enough. Then, defining  $\lambda_0$  as previously, we assume by contradiction that  $\lambda_0 \in \mathbb{R}$ , and deduce in a similar way the existence of the sequences  $\lambda_k$  and  $(p_k, q_k) \in D_{\lambda_k}$ . We need to show that  $(p_k, q_k)$  is bounded. For  $v > \lambda$ , let  $M_v := (v, \tan \theta(v - \lambda)) \in \Gamma_\lambda$ , and let  $\Delta_v := \{(x_1, x_2) : x_2 = \tan(\theta + \frac{\pi}{2})(x_1 - v) + \tan \theta(v - \lambda)\}$  be the line parallel to  $\mathbf{n}$  and passing through  $M_v$ . Let also  $B'_{\lambda,v} := \{(p', q') \in A'_\lambda : q' \geq \tan(\theta + \frac{\pi}{2})(p' - v) + \tan \theta(v - \lambda)\}$  be the

subset of  $A'_\lambda$  which is above  $\Delta_\nu$ , and  $B_{\lambda,\nu} := \sigma_\lambda(B'_{\lambda,\nu})$ . Proceeding as previously, we can see that  $\forall \nu > \lambda_0 + 2, \forall \lambda > \lambda_0 - 1$ , we have  $\inf_{B'_{\lambda,\nu}} y > m$  for some constant  $m > 0$ , while  $\lim_{\nu \rightarrow \infty} \sup\{y(x_1, x_2) : (x_1, x_2) \in B_{\lambda,\nu}\} = 0$ . As a consequence, for  $\nu$  large enough and  $\lambda > \lambda_0 - 1$ , we have  $y(p', q') \geq m > y(p, q), \forall (p, q) \in B_{\lambda,\nu}$ , and thus  $(p_k, q_k) \notin B_{\lambda,\nu}$ . Furthermore, since  $p'_k \leq s_n$  by Lemma 3.4, we have established the boundedness of  $(p_k, q_k)$ . To complete the proof we utilize the same arguments detailed in the case where  $\theta \in (0, \frac{\pi}{4})$ .  $\square$

Lemma 3.5 implies that  $\forall \theta \in (0, \frac{\pi}{2}), \forall \lambda \in \mathbb{R}$ , and  $(p, q) \in \Gamma_\lambda$  with  $q > 0$ , we have  $\frac{\partial \psi_\lambda}{\partial n}(p, q) = 2 \frac{\partial y}{\partial n}(p, q) > 0$ , where  $n = e^{i(\theta + \frac{\pi}{2})}$ . It follows that  $y_{x_1}(x_1, x_2) \leq 0$ , and  $y_{x_2}(x_1, x_2) \geq 0, \forall x_1 \in \mathbb{R}, \forall x_2 \geq 0$ . Moreover, in the half-plane  $x_2 \geq 0$ ,  $y_{x_1}$  and  $y_{x_2}$  satisfy respectively  $\Delta y_{x_1} \geq (x_1 + 6y^2)y_{x_1}$ , and  $\Delta y_{x_2} = (x_1 + 6y^2)y_{x_2}$ , thus  $y_{x_1}$  (resp.  $y_{x_2}$ ) cannot vanish in the open half-plane  $x_2 > 0$ , since otherwise we would obtain by the maximum principle  $y_{x_1} \equiv 0$  (resp.  $y_{x_2} \equiv 0$ ). These situations are excluded by the fact that  $y > 0$  in the open half-plane  $x_2 > 0$ , and  $y_{x_2}(x_1, 0) > 0, \forall x_1 \in \mathbb{R}$ . Therefore we have proved that  $y_{x_1}(x_1, x_2) < 0, \forall x_1 \in \mathbb{R}, \forall x_2 > 0$ , and  $y_{x_2}(x_1, x_2) > 0, \forall x_1, x_2 \in \mathbb{R}$ . Finally, setting  $\tilde{y}_l(x_1, x_2) = y(x_1, x_2 + l)$ , we obtain by the Theorem of Ascoli, that up to a subsequence  $l_k \rightarrow \infty$ ,  $\tilde{y}_{l_k}$  converges in  $C_{\text{loc}}^2$  to a nonnegative minimal solution  $\tilde{y}_\infty$  of (1.9). Furthermore, the monotonicity of  $y$  along the  $x_2$  direction implies that  $\tilde{y}_\infty$  is independent of  $x_2$ . Thus, since  $h$  is the only nonnegative minimal solution of (1.9) (cf. [12, Theorem 1.3]), we deduce that  $\tilde{y}_\infty(x_1, x_2) = h(x_1)$ , and that  $\lim_{l \rightarrow \infty} y(x_1, x_2 + l) = h(x_1)$  is independent of the subsequence  $l_k$ . We also note that  $|y(x_1, x_2)| < h(x_1), \forall (x_1, x_2) \in \mathbb{R}^2$ , from which Theorem 1.2 (iv) follows. This completes the proof of Theorem 1.2.  $\square$

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