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Spinorial classification of Spin(7) structures

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Abstract. We describe the different classes of Spin(7) structures in terms of spinorial equations. We relate them to the spinorial description of G_2 structures in some geometrical situations. Our approach enables us to analyze invariant Spin(7) structures on quasi Abelian Lie algebras.

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1. Introduction

Berger's list [3] (1955) of possible holonomy groups of simply connected, irreducible and non-symmetric Riemannian manifolds contains the so-called exceptional holonomy groups, G_2 and Spin(7), which occur in dimensions 7 and 8 respectively. Non-complete metrics with exceptional holonomy were given by Bryant in [4], complete metrics were obtained by Bryant and Salamon in [5], but compact examples were not constructed until 1996, when Joyce published [13, 14] and [15].

The remaining groups of Berger's list different from SO(n), called special holonomy groups, are U(n), SU(n), Sp(n) and Sp(n) · Sp(1). If the holonomy of a Riemannian manifold is contained in a group G, the manifold admits a G structure, that is, a reduction to G of its frame bundle. Therefore, holonomy is homotopically obstructed by the presence of G structures. Examples of manifolds endowed with G structures for some of the holonomy groups in the Berger list are not only easier to obtain than manifolds with holonomy in G, but also relevant in M-theory, especially if they admit a characteristic connection [11], that is, a metric connection with totally skew-symmetric torsion whose holonomy is contained in G. It is worth mentioning that Ivanov proved in [12] that each manifold with a Spin(7) structure admits a unique characteristic connection. Moreover, Friedrich proved in [10] that Spin(7) is the unique compact simple Lie group G such that every G structure admit a unique characteristic connection.

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The Lie group G_2 is compact, simple and simply connected. It consists of the endomorpisms of $\mathbb{R}^7$ which preserve the cross product from the imaginary part of the octonions [22]. Hence, a G_2 structure on a manifold Q determines a 3-form Ψ, a metric and an orientation. In [8], Fernández and Gray classify G_2 structures into 16 different classes in terms of the G_2 irreducible components of $\nabla\Psi$. Related to this, the analysis of the intrinsic torsion in [6] allowed to obtain equations involving $d\Psi$ and $d(*\Psi)$ for each of the 16 classes, determined by the G_2 irreducible components of $\Lambda^4 T^*Q$ and $\Lambda^5 T^*Q$. In particular, one obtains that the holonomy of Q is contained in G_2 if and only if $d\Psi = 0$ and $d(*\Psi) = 0$. The Lie group Spin(7) is also compact, simple and simply connected. It is the group of endomorphisms of $\mathbb{R}^8$ which preserve the triple cross product from the octonions [22]. Thus, a Spin(7) structure on a manifold M determines a 4-form Ω, a metric and an orientation. In [7], Fernández classifies Spin(7) structures into 4 classes in terms of differential equations for $d\Omega$, which are determined by the Spin(7)-irreducible components of $\Lambda^5 T^*M$. Parallel structures verify $d\Omega = 0$, locally conformally parallel structures satisfy $d\Omega = \theta \wedge \Omega$ for a closed 1-form θ and balanced structures verify $*(d\Omega) \wedge \Omega = 0$. A generic Spin(7) structure, which does not satisfy any of the previous conditions, is called mixed.

The relationship between G_2 and Spin(7) structures was firstly explored by Martín-Cabrera in [18]. Each oriented hypersurface of a manifold equipped with a Spin(7) structure naturally inherits a G_2 structure whose type is determined by the Spin(7) structure of the ambient manifold and some extrinsic information of the submanifold, such as the Weingarten operator. Following the same viewpoint, Martín-Cabrera constructed Spin(7) structures on S^1-principal bundles over G_2 manifolds in [19]. Both approaches allowed to construct manifolds with G_2 and Spin(7) structures of different pure types.

It turns out that manifolds admitting SU(3), G_2 and Spin(7) structures are spin and their spinor bundle has a unit section η which determines the structure. In [1], spinorial formalism was used to deal with the distinct aspects of SU(3) and G_2 structures, such as the classification of both types of structures, SU(3) structures on hypersurfaces of G_2 manifolds and different types of Killing spinors. A clear advantage of this viewpoint is that a unique object, the spinor, encodes the whole geometry of the structure. For instance, a G_2 structure on a Riemannian manifold (Q, g) with associated 3-form Ψ is determined by a suitable spinor η according to the formula $\Psi(X, Y, Z) = (X\eta, YZ\eta)$ where $(\cdot, \cdot)$ denotes the scalar product in the spinor bundle and juxtaposition of vectors indicates the Clifford product. Any oriented hypersurface Q' with normal vector field N inherits an SU(3) structure implicitly defined by $\Psi = N^* \wedge \omega + \mathrm{Re}(\Theta)$, where $N^*(X) = g(N, X)$ for $X \in TQ$. But both the 2-form ω and the (3, 0)-form $\mathrm{Re}(\Theta)$ turn out to be determined by the same spinor η.

In this paper we follow the ideas of [1] to describe the geometry of Spin(7) structures from a spinorial viewpoint, starting from the classification of these structures, continuing to analyze the relationship between G_2 and Spin(7) structures and finishing with the study of invariant Spin(7) structures on quasi Abelian Lie algebras.

Our first result, Theorem 4.8 in section 3, describes each type of Spin(7) structure in terms of differential equations involving the spinor η that determines the structure (see section 2 for details). Parallel Spin(7) structures have already been studied from a spinorial point of view and correspond to the equation $\nabla\eta = 0$. In order to state the spinorial equations for the remaining classes let D denote Dirac operator on the spinor bundle.

Theorem 1.1. *A* Spin(7) *structure determined by* η *is:*

1. *Balanced if* $D\eta = 0$*;*
2. *Locally Conformally Parallel if there exists* $V \in \mathfrak{X}(M)$ *such that* $\nabla_X \eta = \frac{2}{7}(X^* \wedge V^*)\eta$*. In this case,* $D\eta = V\eta$*.*

Moreover, in Proposition 5.2 we determine the torsion forms of the structure and we obtain that the Lee form is $\theta = \frac{7}{8}V^*$ where $D\eta = V\eta$.

Our techniques also allow us to identify the intrinsic torsion of the structure and to obtain the formula for the unique characteristic connection of each Spin(7) structure, given by Ivanov in [12, Theorem 1.1]. In Section 6 we also show that the spinorial equation for balanced structures can be obtained using [12, Theorem 9.1].

We also introduce the concept of G_2 distributions, a general setting to relate G_2 and Spin(7) structures.

Definition 1.2. Let (M, g) be an oriented 8-dimensional Riemannian manifold and let $\mathcal{D}$ be a cooriented distribution of codimension 1. We say that $\mathcal{D}$ has a G_2 structure if the principal SO(7) bundle $P(\mathcal{D})$ is spin and the spinor bundle $\Sigma(\mathcal{D})$ admits a unitary section.

This construction allows us to obtain the results which appear in [18] and [19] about G_2 structures on hypersurfaces of Spin(7) manifolds and S^1-principal bundles over G_2 manifolds. Related to this, we also study warped products of manifolds admitting a G_2 structure with $\mathbb{R}$.

The formalism of G_2 distributions enables us to study invariant Spin(7) structures on quasi-Abelian Lie algebras, that is, Lie algebras with a codimension 1 Abelian ideal. To state the result, which is Theorem 8.7, suppose that the Lie algebra is $\mathfrak{g} = \langle e_0, \dots, e_7\rangle$ with Abelian ideal $\mathbb{R}^7 = \langle e_1, \dots, e_7\rangle$ and it is endowed with the canonical metric and volume form.

Theorem 1.3. *Denote by* $\mathcal{E} = ad(e_0)|_{\mathbb{R}^7}$ *and let* $\mathcal{E}_{13}$ *and* $\mathcal{E}_{24}$ *be the symmetric and skew-symmetric parts of the endomorphism. Then,* $\mathfrak{g}$ *admits a* Spin(7) *structure of type:*

1. *Parallel, if and only if* $\mathcal{E}_{13} = 0$ *and the eigenvalues of* $\mathcal{E}_{24}$ *are* $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1 + \lambda_2)i$*, for some* $0 \leq \lambda_1 \leq \lambda_2$*;*
2. *Locally conformally parallel and non-parallel if and only if* $\mathcal{E}_{13} = h\,\mathrm{Id}$ *with* $h \neq 0$ *and the eigenvalues of* $\mathcal{E}_{24}$ *are* $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1 + \lambda_2)i$*, for some* $0 \leq \lambda_1 \leq \lambda_2$*;*
3. *Balanced if and only if* $\mathfrak{g}$ *is unimodular and the eigenvalues of* $\mathcal{E}_{24}$ *are* $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1 + \lambda_2)i$*, for some* $0 \leq \lambda_1 \leq \lambda_2$*.*

Moreover, if $\mathcal{E}_{24} \neq 0$ *then it admits a* Spin(7) *structure of mixed type.*

It follows from this (Corollary 8.8) that there are no quasi Abelian solvmanifolds which admit a locally conformally parallel Spin(7) structure. In addition, this result allows us to give an example of a nilmanifold admitting both an invariant balanced structure and an invariant mixed structure. We also compute an example of an invariant strict locally conformally balanced structure, that is a mixed structure whose Lee form is closed and non-exact.

A compact manifold admitting a parallel structure is also obtained as a quotient of a simply connected solvable Lie group whose Lie algebra is quasi Abelian. Despite not being diffeomorphic to a torus, it is flat. Indeed, we prove that quasi Abelian Lie algebras which admit an invariant Spin(7) parallel structure are flat (Corollary 8.9).

In addition, we also characterize which nilpotent quasi Abelian Lie algebras admitting invariant balanced and locally conformally balanced structures:

Theorem 1.4. *Let L_3 be the Lie algebra of the 3-dimensional Heisenberg group, L_4 the unique irreducible 4-dimensional nilpotent Lie algebra and A_j the j-dimensional Abelian Lie algebra.*

1. *Every invariant* Spin(7) *structure on the Abelian Lie algebra A_8 is parallel;*
2. *The Lie algebras $\mathfrak{g} = A_5 \oplus L_3$ or $\mathfrak{g} = A_3 \oplus L_4$ admit strict locally conformally balanced invariant structures. However, they do not admit invariant balanced structures;*
3. *The rest of quasi Abelian nilpotent Lie algebras admit a balanced structure and a strict locally conformally balanced structure.*

This paper is organized as follows. Section 2 contains a review of algebraic aspects of Spin(7) geometry. Section 3 identifies the instrinsic torsion of the Levi-Civita connection with a spinor, Section 4 contains the spinorial classification of Spin(7) structures, Section 5 is devoted to obtain the torsion forms of Spin(7) structures in terms of spinors and Section 6 provides an alternative proof of the existence of the characteristic connection. Section 7 provides a complete analysis of G_2 structures on distributions and then focuses on the particular cases described above. Section 8 deals with invariant structures on quasi Abelian Lie algebras and provides compact examples. Finally Section 9 is devoted to the study of quasi Abelian nilpotent Lie algebras and its Spin(7) structures.

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2. Preliminaries

In this section we introduce some aspects of Clifford algebras, 8-dimensional spin manifolds and Spin(7) representations, which can be found in [9, 16] and [22], as well as the notation that we will use in the sequel.

2.1. The irreducible representation of Cl_8

The Clifford algebra Cl_8 is isomorphic to the algebra of endomorphisms of $\mathbb{R}^{16}$. Such an isomorphism is denoted by $\rho\colon \mathrm{Cl}_8 \to \mathrm{End}(\mathbb{R}^{16})$ and is indeed the unique irreducible repesentation of Cl_8 up to equivalence [16, Chapter 1, Theorem 4.3]. There is also an inner product on $\mathbb{R}^{16}$, which we denote by $(\cdot,\cdot)$, such that the Clifford multiplication with a vector of $\mathbb{R}^8$ is a skew-symmetric transformation [16, Chapter 1, Theorem 5.3].

Fix an orientation of $\mathbb{R}^8$ and let ν_8 be the volume form of $\mathbb{R}^8$ that has length one and is positively oriented. Consider the Spin(8) equivariant endomorphism:

$$\nu_8\cdot\colon \mathbb{R}^{16} \to \mathbb{R}^{16}, \quad \phi \longmapsto \nu_8\phi.$$

Since $\nu_8^2 = 1$, there is a splitting $\mathbb{R}^{16} = \Delta^+ \oplus \Delta^-$ where $\Delta^{\pm}$ is the eigenspace associated to ± 1. In addition, this endomorphism anticommutes with the Clifford multiplication by a vector.

It is well known that Spin(8) contains three distinct conjugacy classes of the group Spin(7) [16, Chapter 4, Proposition 10.4]. The first one is obtained from the adjoint action $\mathrm{Ad}\colon \mathrm{Spin}(8) \to \mathrm{SO}(8)$ as the stabilizer of any non-zero $v \in \mathbb{R}^8$. The remaining ones, that we denote by $\mathrm{Spin}(7)^{\pm}$, are constructed from ρ as the stabilizer of a non-zero spinor $\phi_{\pm} \in \Delta^{\pm}$. The adjoint action embeds $\mathrm{Spin}(7)^{\pm}$ into SO(8) because $-1 \notin \mathrm{Stab}(\phi_{\pm})$. Note also that the conjugacy classes $\mathrm{Spin}(7)^{\pm}$ depend on the choice of an orientation of $\mathbb{R}^8$ and these are conjugated in Pin(8).

Remark 2.1. We can obtain ρ from the representation of the complex Clifford algebra and the real structure constructed in [9, Chapter 1]. The construction that allows to obtain an irreducible representation of Cl_6 is similar but there are is a difference that we outline. Let $\mathbb{C}\mathrm{l}_{2k}$ be the Clifford algebra of $(\mathbb{C}^{2k}, \sum_{i=1}^{2k} z_i^2)$, according to [9, page 13] there are 2^k-dimensional complex vector spaces Δ_{2k} and isomorphisms $\kappa_{2k}\colon \mathbb{C}\mathrm{l}_{2k} \to \mathrm{End}(\Delta_{2k})$. The multiplication by the complex volume form $\nu_{2k}^{\mathbb{C}} = i^k \nu_{2k}$ splits Δ_{2k} into two eigenspaces $\Delta_{2k}^{\pm}$ associated to the eigenvalue ± 1 which are irreducible under the action of $\mathrm{Spin}(2k)$.

1. There is a Spin(8) equivariant real structure φ_8 on Δ_8 which commutes with $\nu_8^{\mathbb{C}}$ (see [9, page 32]). Thus, a real representation is $(\Delta_8^+)_+ \oplus (\Delta_8^-)_-$, where $(\Delta_8^+)_\pm$ and $(\Delta_8^-)_\pm$ are the eigenspaces associated to the eigenvalue ± 1 of φ_8 on Δ_8^+ and Δ_8^-;
2. There is a Spin(6) equivariant real structure φ_6 on Δ_6 that anticommutes with $\nu_6^{\mathbb{C}}$. Thus the real representation of Cl_6 is $(\Delta_6)_+ = \{\phi + \varphi_6(\phi)\colon \phi \in \Delta_6^+\}$, the

eigenspace associated to $+1$ of φ_6. In addition, for a real spinor $\eta = \phi + \varphi_6(\phi) \neq 0$ we have that $\mathrm{Stab}_{\mathrm{Spin}(6)}(\eta) = \mathrm{Stab}_{\mathrm{Spin}(6)}(\phi) = \mathrm{Stab}_{\mathrm{Spin}(6)}(\varphi_6(\phi)) = \mathrm{SU}(3)$.

The Hermitian metric h on Δ_8 constructed in [9, page 24] makes the Clifford multiplication a skew-symmetric transformation. In particular, h is Spin(8) invariant. The fact that $\Delta_8^{\pm}$ are irreducible Spin(8) modules guarantees that $b(\phi, \eta) = h(\varphi_8(\phi), \eta)$ is a symmetric bilinear form on $\Delta_8^{\pm}$ and therefore the restrictions of h to the real and the imaginary part of $\Delta_8^{\pm}$ are real-valued. The subspaces Δ_8^{+} and Δ_8^{-} are orthogonal with respect to h because the multiplication by $\nu_{\mathbb{C}}$ preserves h. Therefore the real part of h is a scalar product on $(\Delta_8^{+})_+ \oplus (\Delta_8^{-})_-$ with the same properties as $(\cdot, \cdot)$.

2.2. Spin(7) structures

Let (M, g) be an oriented Riemannian 8-manifold and let $P(M)$ be the associated frame bundle. Provided that M is spin, that is $w_2(M) = 0$, we can take a Spin(8) principal bundle $\tilde{P}(M)$ over M which is a double covering $\pi\colon \tilde{P}(M) \to P(M)$ equivariant under the adjoint action $\mathrm{Ad}\colon \mathrm{Spin}(8) \to \mathrm{SO}(8)$. The associated spinor bundle is $\Sigma(M) = \tilde{P}(M) \times_\rho \mathbb{R}^{16}$ and it is endowed with a metric induced by $(\cdot, \cdot)$ which we denote by the same name. Moreover there is a splitting $\Sigma(M) = \Sigma(M)^+ \oplus \Sigma(M)^-$, where $\Sigma(M)^{\pm} = \tilde{P}(M) \times_\rho \Delta^{\pm}$.

Also note that $X(\Sigma(M)^{\pm}) \subset \Sigma(M)^{\mp}$ if $X \in \mathfrak{X}(M)$ and that for each nowhere vanishing spinor $\phi\colon M \to \Sigma(M)^{\pm}$ the map:

$$TM \to \Sigma(M)^{\mp}, \quad X \longmapsto X\phi, \tag{2.1}$$

is an isomorphism.

The Clifford multiplication with a vector field is extended to an action of $\Lambda^k T^*M$ defined as follows:

1. The product with a covector is defined by $X^*\phi = X\phi$, where we used the canonical identification between the tangent and the cotangent bundle: $X^* = g(X, \cdot)$;
2. If the product is defined on $\Lambda^\ell T^*M$ when $\ell \leq k$, we define

$$(X^* \wedge \beta)\phi = X(\beta\phi) + (i(X)\beta)\phi,$$

where $i(X)\beta$ denotes the contraction, $\beta \in \Lambda^k T^*M$ and $X \in TM$. This product is extended linearly to $\Lambda^{k+1} T^*M$.

For instance, we have:

$$(X^* \wedge Y^*)\phi = (XY + g(X, Y))\phi, \tag{2.2}$$

$$(X^* \wedge Y^* \wedge Z^*)\phi = (XYZ + g(X, Y)Z - g(X, Z)Y + g(Y, Z)X)\phi. \tag{2.3}$$

Observe also that $\Sigma^{\pm}(M) = \{\phi_p\colon \nu_p\phi_p = \pm\phi_p\}$ where ν is the positively oriented unit-length volume form of (M, g).

The action $\mathrm{Spin}(8) \times \mathbb{R}^{16} \to \mathbb{R}^{16}$ lifts to an action $\tilde{P}(M) \times \Sigma(M) \to \Sigma(M)$, so that the existence of a unit spinor $\eta \in \Gamma(\Sigma(M)^{\pm})$ determines an identification between $\mathrm{Spin}(7)^{\pm}$ and the stabilizer of η_p at each $p \in M$. This defines a Spin(7) principal subbundle $\mathrm{Stab}(\eta) \subset \tilde{P}(M)$ and therefore $\mathrm{Ad}(\mathrm{Stab}(\eta))$ is a Spin(7) reduction of $P(M)$. In this paper we focus on Spin(7) structures determined by positive spinors. This condition is not restrictive due to the following result which is not difficult to prove.

Lemma 2.2. *Let (M, g) be a connected oriented spin manifold and let $\Sigma(M)$ be its spinor bundle. Let $\overline{\Sigma(M)}$ be the spinor bundle associated to the opposite orientation on M. There is an isomorphism of* $\mathrm{Cl}(M)$ *modules $\mathcal{R}\colon \Sigma(M) \to \overline{\Sigma(M)}$. Therefore, $\mathcal{R}(\Sigma(M)^{\pm}) = \overline{\Sigma(M)}^{\mp}$.*

For the convenience of the reader, we shall relate this spinorial approach with the point of view of positive triple cross products [22, Definitions 6.1, 6.12]. That was the approach that M. Fernández followed in [7] to obtain the classification of Spin(7) structures.

Lemma 2.3. *Let (M, g) be a Riemannian oriented spin manifold that admits a unit spinor $\eta\colon M \to \Sigma(M)^{\pm}$. Then there is a well defined map:*

$$TM \times TM \times TM \to TM, \ (X,Y,Z) \longmapsto X \times Y \times Z \ s.t, \ (X \times Y \times Z)\eta = (X^* \wedge Y^* \wedge Z^*)\eta,$$

which is in turn a positive triple cross product.
The associated 4*-form $\Omega(W, X, Y, Z) = g(W, X \times Y \times Z)$ verifies that $*\Omega = \pm\Omega$.*

Moreover [22, Theorem 10.3] states that there is a 1 to 1 correspondence between 4-forms Ω that define a positive triple cross product and such that $\Omega \wedge \Omega > 0$ and sections of the projectivization of $\Sigma(M)^{+}$.

According to the previous discussion we summarize our basic assumptions in the following Proposition. In the sequel given a frame $(e_0 \dots, e_7)$ and a spinor ϕ we use short-hand notation e^i for $g(e_i, \cdot)$, e^{ijkl} for $e^i \wedge e^j \wedge e^k \wedge e^l$ and $e_{ijk}\phi$ for $e_i e_j e_k \phi$.

Proposition 2.4. *Let (M, g) be an oriented spin manifold and suppose that there exists a positive unit spinor. Consider the triple cross product on M defined as in Lemma* 2.3.

1. *The associated* 4*-form is self-dual and is determined by*

$$\Omega(W, X, Y, Z) = \frac{1}{2}((-WXYZ + WZYX)\eta, \eta);$$

2. *Given orthonormal vector local fields e_0, e_1, e_2, e_4 such that e_4 is perpendicular to $e_3 = e_0 \times e_1 \times e_2$ there exists a positive oriented orthonormal frame $(e_0, \dots, e_7)$ such that:*

$$\begin{aligned} \Omega = \ & e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} \\ & + e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256} - e^{1247}. \end{aligned} \tag{2.4}$$

A frame with this property is called Cayley frame.

Proof. Taking into account Lemma 2.3 and equation (2.3) the associated 4-form of the triple cross product, which is self-dual, is:

$$\begin{aligned}\Omega(W,X,Y,Z) &= ((X\times Y\times Z)\eta, W\eta)\\ &= ((XYZ+g(X,Y)Z-g(X,Z)Y+g(X,Y)Z)\eta, W\eta)\\ &= \frac{1}{2}((-WXYZ+WZYX)\eta,\eta).\end{aligned}$$

The third statement can be found in [22, Theorem 7.12]. Since Cayley frames verify $(e_0\cdots e_7)\eta=\eta$, they are positively oriented. □

2.3. Spin(7) representations

Let us denote the standard basis of $\mathbb{R}^8$ by $(e_0,\dots,e_7)$, and the standard Spin(7) structure of $\mathbb{R}^8$ by Ω_0, given by (2.4). We also denote $\Lambda^k=\Lambda^k(\mathbb{R}^8)^*$.

The representation of $\mathrm{Spin}(7)=\mathrm{Stab}(\Omega_0)\subset\mathrm{SO}(8)$ on Λ^k induces an orthogonal decomposition of this space into irreducible Spin(7) invariant subspaces. The expression Λ^k_ℓ denotes such an ℓ-dimensional subspace of Λ^k. The Hodge star operator $*$ gives isomorphisms between Λ^k and Λ^{8-k} determining that $\Lambda^k_\ell=*\Lambda^{8-k}_\ell$ if $k\leq 4$. We are going to describe briefly the splitting; a complete proof can be found in [7] and [22, Theorem 9.8]. The decomposition goes as follows:

$$\begin{aligned}\Lambda^2 &=\Lambda^2_7\oplus\Lambda^2_{21},\\ \Lambda^3 &=\Lambda^3_8\oplus\Lambda^3_{48},\\ \Lambda^4 &=\Lambda^4_1\oplus\Lambda^4_7\oplus\Lambda^4_{27}\oplus\Lambda^4_{35}.\end{aligned}$$

The first one comes from the orthogonal splitting $\Lambda^2=\mathfrak{so}(8)=\mathfrak{spin}(7)\oplus\mathfrak{m}$, where $\mathfrak{m}=\mathfrak{spin}(7)^\perp$. An alternative description is obtained from the map:

$$\Lambda^2\to\Lambda^2,\quad \beta\longmapsto *(\beta\wedge\Omega_0),$$

which is Spin(7)-equivariant, symmetric and traceless. Therefore, Λ^2 splits into eigenspaces which must coincide with the previous ones due to the irreducibility. It can be checked that the eigenvalues are 3 on Λ^2_7 and -1 on Λ^2_{21}. Moreover, the set $\{\alpha_j=\frac{1}{2}(e^{0j}+i(e_j)i(e_0)\Omega_0)\}_{j=1}^7$ is an orthonormal basis of Λ^2_7 and the projection $p^2_7\colon\Lambda^2\to\Lambda^2_7$ is consequently determined by the equation:

$$p^2_7(u^*\wedge v^*)=\frac{1}{4}(u^*\wedge v^*+i(v)i(u)\Omega_0). \tag{2.5}$$

The subspaces involved in the splitting of Λ^3 are:

$$\Lambda^3_8=i(\mathbb{R}^8)\Omega_0,\quad \Lambda^3_{48}=\ker(\cdot\wedge\Omega_0\colon\Lambda^3\to\Lambda^7).$$

In order to describe the last one observe that Hodge star operator splits Λ^4 into two 35-dimensional spaces: anti self-dual and self-dual forms. The space of anti self-dual forms is Λ^4_{35} and the space of self-dual forms is $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$. Obviously, $\Lambda^4_1 = \langle \Omega_0 \rangle$ and the space Λ^4_7 is the image of the map,

$$j\colon \mathfrak{m} \to \Lambda^4, \quad j(\beta) = \rho_*(\beta)\Omega_0,$$

with $\rho\colon\ \mathrm{SO}(8) \to \Lambda^4 T^*M$, $\rho(g) = (g^{-1})^*\Omega_0$. That is, j is the restriction to $\mathfrak{m}$ of the map determined by $j(u^* \wedge v^*) = u^* \wedge i(v)\Omega_0 - v^* \wedge i(u)\Omega_0$ and therefore, $\Lambda^7_4 = \{u^* \wedge i(v)\Omega_0 - v^* \wedge i(u)\Omega_0,\ u, v \in \mathbb{R}^8\}$. The subspace Λ^4_{27} is the orthogonal complement of $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$.

We now describe the irreducible decomposition of $\Lambda^1 \otimes \mathfrak{m}$ which is related with the intrinsc torsion of the Levi-Civita connection (see Section 3).

Proposition 2.5. *Let* $(e_0, \dots, e_7)$ *be a Cayley basis and let* $p^2_7\colon \Lambda^2 \to \mathfrak{m}$ *be the orthogonal projection. Consider the* Spin(7)*-equivariant maps:*

$$\Theta\colon \Lambda^3 \to \Lambda^1 \otimes \mathfrak{m}, \quad \beta \longmapsto \Theta(\beta) = \sum_{j=0}^{7} e_j \otimes p^2_7(i(e_j)\beta),$$
$$\Xi\colon \Lambda^1 \otimes \mathfrak{m} \to \Lambda^3, \quad \alpha \otimes \beta \longmapsto \alpha \wedge \beta = 3\,\mathrm{alt}(\alpha \otimes \beta),$$

where $\mathrm{alt}(\mathrm{T})(v_1, \dots, v_n) = \frac{1}{n!}\sum_{\sigma \in S_n}(-1)^{\mathrm{sgn}\,\sigma}\mathrm{T}(v_{\sigma(1)}, \dots, v_{\sigma(n)})$. *The eigenvalues of* $\Xi \circ \Theta$ *are* $\frac{9}{4}$ *and* $\frac{1}{2}$. *They are associated to the eigenspaces* Λ^3_8 *and* Λ^3_{48} *respectively.*

Proof. The map $\Xi \circ \Theta$ is symmetric and Spin(7)-equivariant, so that its eigenspaces must be Λ^3_8 and Λ^3_{48}. Taking $i(e_0)\Omega_0 \in \Lambda^3_8$ and $e^{123} + e^{145} \in \Lambda^3_{48}$ one can show that the eigenvalues are $\frac{9}{4}$ on Λ^3_8 and $\frac{1}{2}$ on Λ^3_{48}. □

We formulate an alternative description of $\Lambda^1 \otimes \mathfrak{m}$ which can be proved in the same manner.

Proposition 2.6. *Let* $(e_0, \dots, e_7)$ *be an orthonormal frame. Consider the* O(8) *equivariant maps,*

$$\iota\colon \mathbb{R}^8 \to \Lambda^1 \otimes \mathfrak{m}, \quad \iota(v) = \sum_{i=0}^{7} e^i \otimes (e^i \wedge v^*),$$

$$\kappa\colon \Lambda^1 \otimes \mathfrak{m} \to \mathbb{R}^8, \quad \kappa(\Gamma) = \sum_{i=0}^{7} (i(e_i)\Gamma(e_i))^\sharp,$$

which do not depend on the orthonormal basis chosen. Then $\iota(\mathbb{R}^8) = \Theta(\Lambda^3_8)$ *and* $\ker(\kappa) = \Theta(\Lambda^3_{48})$. *Moreover,* $\kappa \circ \iota(v) = \frac{7}{4}v$ *for any* $v \in \mathbb{R}^8$.

In the same manner one can study the space $\Lambda^1 \otimes \Lambda^4_7$ which is isomorphic to $\Lambda^1 \otimes \mathfrak{m}$. For instance, it is not difficult to check that the map alt: $\Lambda^1 \otimes \Lambda^4_7 \to \Lambda^5$ is a Spin(7) equivariant isomorphism.

A Spin(7) structure on the Riemannian manifold (M, g) determines a splitting of $\Lambda^k T^*M$ into subbundles $\Lambda^k_\ell T^*M = R \times_{\mathrm{Spin}(7)} \Lambda^k_\ell$ where R is the Spin(7) reduction R of the SO(8) principal bundle given by the Cayley frames. We also denote by $\Omega^k_\ell(M)$ the space of smooth sections of $\Lambda^k_\ell T^*M$. In addition, the maps $j, \Theta, \Xi, \iota, \kappa$ induce bundle homomorphisms that we call by the same name. We will also consider the subbundles of $T^*M \otimes \Lambda^7_2 T^*M$ defined by $\chi_1 = \Theta(\Lambda^3_{48} T^*M)$ and $\chi_2 = \Theta(\Lambda^3_8 T^*M)$.

3. The intrinsic torsion

We are going to compute the intrinsic torsion Γ of the Levi-Civita connection which is a section of the bundle $TM \otimes \Lambda^2_7 T^*M$. Recall that the Levi-Civita connection ∇ on TM induces a connection ω on $P(M)$. Then a connection on the Spin(7) reduction R is defined by $\omega' = p(\omega)|_{TR}$, where p denotes the orthogonal projection to $\mathfrak{spin}(7)$. The connection that ω' induces on TM is denoted by ∇' and determines the intrinsic torsion by means of the expression:

$$\nabla_X Y = \nabla'_X Y + \Gamma(X)Y.$$

The skew-symmetric endomorphism $\Gamma(X)$ can be identified with a 2-form which lies in $\Omega^2_7(M)$ for each $X \in TM$. To compute it, define H as the subspace of Δ_+ which is orthogonal to η with respect to the scalar product $(\cdot, \cdot)$ defined in Section 2.1. Of course, H depends on the choice of the spinor η. We first prove that the vector bundles $\Lambda^2_7 T^*M$ and H are isomorphic.

Lemma 3.1. *There is a well defined* Spin(7)*-equivariant map*

$$\Lambda^2 T^*M \to H, \quad \alpha \longmapsto \alpha\eta,$$

whose kernel is $\Lambda^2_{21} T^*M$. Indeed, its restriction $c\colon \Lambda^2_7 T^*M \to H$ is an isomorphism whose inverse is given by $(c^{-1}\phi)(X, Y) = \frac{1}{4}(\phi, (XY + g(X, Y))\eta)$.

Proof. The spinor $\beta\eta$ is perpendicular to η if $\beta \in \Lambda^2 T^*M$. Therefore, the map is well-defined and it is Spin(7)-equivariant since $\mathrm{Spin}(7) = \mathrm{Stab}(\eta_p)$.

To prove that c is an isomorphism, we first claim that if $(e_0, \ldots, e_7)$ is a Cayley frame then $\alpha_j\eta = 4e^{0j}\eta$. Observe that we only need to check this formula for $j = 1$ since c is Spin(7)-equivariant and $\mathrm{G}_2 = \mathrm{Spin}(7) \cap \mathrm{Stab}(e_0)$ acts transitively on the 6-sphere generated by $(e_1, \ldots, e_7)$. In this case, $\alpha_1 = e^{01} + e^{23} - e^{45} - e^{67}$ and if $(i, j) \in \{(2, 3), (5, 4), (7, 6)\}$ we have that $\Omega(e_0, e_1, e_i, e_j) = 1$. The previous equality means that $e_0\eta = e_{1ij}\eta$, so that $e^{01}\eta = e^{ij}\eta$.

Moreover, since $\{e^{0i}\eta\}_{i=1}^{7}$ is an orthonormal basis of H we have that

$$c^{-1}(\phi) = \frac{1}{4}\sum_{i=1}^{7}(\phi, e^{0i}\eta)\alpha_i.$$

If $X = e_0, Y = e_1$ are orthonormal vectors then $\alpha_j(e_0, e_1) = (e^{0j} - i(e_0)i(e_j)\Omega)(e_0, e_1) = \delta_{j1}$. Hence, $c^{-1}\phi(e_0, e_1) = \frac{1}{4}(\phi, e_0e_1\eta)$.

Finally, by dimensional reasons the Clifford product with η must vanish on $\Lambda^2_{21}T^*M$. □

Remark 3.2. These computations and others that we do in the sequel in terms of Cayley frames may be computed alternatively from a representation of Cl_8.

The previous result enables us to find a formula for the intrinsic torsion:

Proposition 3.3. *The intrinsic torsion is given by* $\Gamma(X) = 2c^{-1}\nabla_X\eta$.

Proof. We also denote by ∇ and ∇' the induced connections on the spinor bundle. According to [9, page 60] we have that:

$$\nabla_X\phi = \nabla'_X\phi + \frac{1}{2}\Gamma(X)\phi,$$

where $\Gamma(X)$ acts on ϕ as a 2-form. Since the holonomy of the connection ∇' is contained in Spin(7) and $\mathrm{Stab}(\eta_p) = \mathrm{Spin}(7)$ we have that $\nabla'\eta = 0$. Finally, if $X \in TM$ then $\nabla_X\eta \in H$ and $\Gamma(X) \in \Lambda^2_7T^*M$ thus, Lemma 3.1 shows that $\Gamma(X) = 2c^{-1}\nabla_X\eta$. □

4. Classification of Spin(7) structures

The classification of Spin(7) structures was obtained in [7, Theorem 5.3]. There it is proved that $\nabla\Omega \in \Gamma(TM^* \otimes \Lambda^4_7T^*M)$ and that $\Lambda^1 \otimes \Lambda^4_7$ splits into two irreducible Spin(7) subspaces that can be described via the isomorphism $\mathrm{Id} \otimes j\colon \Lambda^1 \otimes \mathfrak{m} \to \Lambda^1 \otimes \Lambda^4_7$ (see Section 2.3 for the definition of j). Those are of course $(\mathrm{Id} \otimes j) \circ \Theta(\Lambda^3_{48})$ and $(\mathrm{Id} \otimes j) \circ \Theta(\Lambda^3_8)$.

We also denote by $\mathrm{Id} \otimes j$ the induced map from $T^*M \otimes \Lambda^2_7T^*M$ to $T^*M \otimes \Lambda^4_7T^*M$ and we define $\mathcal{W}_1 = (\mathrm{Id} \otimes j)(\chi_1)$ and $\mathcal{W}_2 = (\mathrm{Id} \otimes j)(\chi_2)$, where χ_j are defined as in Section 2.3.

Moreover, it is straightforward to check that $\mathrm{Id}\otimes j(\Gamma)=\nabla\Omega$ and that $\mathrm{alt}(\nabla\Omega)= d\Omega$. These considerations allow us to describe the classification of Spin(7) structures in three different ways.

Definition 4.1. Let Γ be the intrinsic torsion of the Spin(7) structure determined by Ω. The type of the structure is given by the equivalent conditions:

	Γ	$\nabla\Omega$	$d\Omega$
Parallel	0	0	0
Balanced	χ_1	$\mathcal{W}_1$	$*(d\Omega)\wedge\Omega=0$
Locally conformally parallel	χ_2	$\mathcal{W}_2$	$\theta\wedge\Omega, \quad \theta\in\Omega^1(M)$

In other case, the structure is said to be mixed.

Definition 4.2. The Lee form of Ω is the unique $\theta \in \Omega^1(M)$ such that the orthogonal projection of $d\Omega$ to $\Omega^5_8(M)$ is $\theta\wedge\Omega$.

Remark 4.3. According to Proposition 2.6 locally conformally parallel Spin(7) structures are the class of Spin(7) structures with vectorial torsion in the sense of [2]. In [2, Proposition 2.2] the reader can find a characterization of compact manifolds with vectorial torsion and formulas for the Ricci tensor.

Remark 4.4. If the structure is locally conformally parallel then $d\theta = 0$. Let O be a contractible open set, take a primitive f of $-\frac{1}{4}\theta|_O$ and define the metric $g' = e^{2f}g|_O$. The associated Spin(7) structure is $\Omega' = e^{4f}\Omega|_O$ and it verifies $d\Omega' = 0$. Therefore, $\Omega|_O$ is conformal to a parallel structure. This justifies the name.

We now focus in obtaining an alternative description in terms of spinors. For that purpose, decompose $\Gamma = \Gamma_1 + \Gamma_2$ according to the splitting $\chi_1 \oplus \chi_2$ and write $\Gamma_2(X) = \frac{4}{7}p^2_7(X^*\wedge V^*)$. Taking into account Proposition 2.6 and equation (2.5) we obtain:

1. $\kappa(\Gamma_2) = V^*$;
2. $\Theta(\Gamma_2) = \frac{4}{7}\sum_{i=0}^7 e^i\wedge p^2_7(e^i\wedge V^*) = \frac{1}{7}\sum_{i=0}^7 e^i\wedge i(e_i)i(V)\Omega = \frac{3}{7}i(V)\Omega$.

Remark 4.5. Define $Z(V) = \{p\in M \text{ s.t } V(p)=0\}$ and let R denote the Spin(7) reduction of the SO(8) principal bundle of M. The frame bundle of $(M - Z(V), g)$ admits a G_2 reduction that consists of the orthonormal oriented frames on $R|_{M-Z(V)}$ that have the form $(V/\|V\|, e_1, \ldots, e_7)$.

Remark 4.6. We added a factor $\frac{4}{7}$ in order to avoid a constant on Theorem 4.7.

We compute the Dirac operator D of the spinor η that determines the Spin(7) structure.

Proposition 4.7. *Let Ω be a* Spin(7) *structure determined by a spinor η. Let $\Gamma = \Gamma_1 + \Gamma_2$ be its intrinsic torsion with $\Gamma_2(X) = \frac{4}{7}p^2_7(X^*\wedge V^*)$. Then,*

1. *The map $\Lambda^3T^*M \to \Sigma(M)^-$, $\alpha \longmapsto \alpha\eta$ is* Spin(7) *equivariant and its kernel is $\Lambda^3_{48}T^*M$.*
   Moreover, $(i(X)\Omega)\eta = 7X\eta$.
2. *The Dirac operator is determined by $D\eta = V\eta$.*

Proof. The first statement is a consequence of Schur's Lemma. To check that $i(X)\Omega\eta = 7X\eta$, one can suppose that X is unitary and use a Cayley frame such that $X = e_0$.

For the second we compute in terms of a Cayley local frame $(e_0, \dots, e_7)$,

$$\begin{aligned}2D\eta &= \sum_{i=0}^{7} e_i \Gamma(e_i)\eta = \sum_{i=0}^{7} (e^i \wedge \Gamma(e_i) - i(e_i)\Gamma(e_i))\eta \\ &= \Theta(\Gamma)\eta - \kappa(\Gamma)\eta = 2V\eta.\end{aligned}$$

□

Theorem 4.8. *The* Spin(7) *structure determined by a spinor* η *is,*

1. *Parallel if* $\nabla\eta = 0$*;*
2. *Balanced if* $D\eta = 0$*;*
3. *Locally Conformally Parallel if there exists* $V \in \mathfrak{X}(M)$ *such that* $\nabla_X \eta = \frac{2}{7}(X^* \wedge V^*)\eta$*. In this case,* $D\eta = V\eta$*.*

Proof. The equation for balanced structures follows from Proposition 4.7 and the equation for locally conformally balanced structures follows from Lemma 3.1. □

5. Torsion forms of a Spin(7) structure

In this section we describe the torsion forms of a Spin(7) structures by means of the spinor defining the structure. That is, we determine the projections of $*d\Omega$ to the spaces $\Omega^3_8(M)$ and $\Omega^3_{48}(M)$. Note that the projection is given by $p^3_8 \colon \Omega^3(M) \to \Omega^3_8(M)$, $p^3_8(\beta) = -\frac{1}{7} * ((*\beta \wedge \Omega) \wedge \Omega)$.

For that purpose, denote by D the Dirac operator on $\Sigma(M)$. The isomorphism (2.1) ensures the existence of a unique vector field V such that

$$D\eta = V\eta. \tag{5.1}$$

Then, the 3-form $\gamma_8(X, Y, Z) = (D\eta, (X \times Y \times Z)\eta) = (i(V)\Omega)(X, Y, Z)$ obviously lies in $\Omega^3_8(M)$.

Proposition 5.1. *Using the previous notation, we have:*

$$*d\Omega = 2(\gamma_8 - 12\,\mathrm{alt}(c^{-1}\nabla\eta)).$$

Proof. Since ∇ is a metric connection on the spinor bundle and acts as a derivation for the Clifford product, we get:

$$\begin{aligned}&(\nabla_T \Omega)(W, X, Y, Z) \\ &= \frac{1}{2}\Big(((-WXYZ + WZYX)\nabla_T\eta, \eta) + ((-WXYZ + WZYX)\eta, \nabla_T\eta)\Big) \\ &= \frac{1}{2}((-ZYXW + XYZW - WXYZ + WZYX)\eta, \nabla_T\eta).\end{aligned}$$

Take orthonormal vectors X, Y, Z and an orthonormal oriented basis $(X_0, \ldots X_7)$ such that $X_0 = X, X_1 = Y$ and $X_2 = Z$. Then,

$$\begin{aligned}
&\delta\Omega(X, Y, Z) \\
&= -\sum_{i=3}^{7} \nabla_{X_i}\Omega(X_i, X, Y, Z) = -2\sum_{i=3}^{7}(XYZ\eta, X_i\nabla_{X_i}\eta) \\
&= -2(D\eta, (X \times Y \times Z)\eta) + 2(XYZ\eta, X\nabla_X\eta + Y\nabla_Y\eta + Z\nabla_Z\eta) \\
&= -2((D\eta, (X \times Y \times Z)\eta) - (YZ\eta, \nabla_X\eta) + (XZ\eta, \nabla_Y\eta) - (XY\eta, \nabla_Z\eta) \\
&= -2((D\eta, (X \times Y \times Z)\eta) - 12\,\mathrm{alt}(c^{-1}\nabla\eta)(X, Y, Z)).
\end{aligned}$$

The third equality follows from $\sum_{i=3}^{7} X_i\nabla_{X_i}\eta = D\eta - \sum_{i=1}^{3} X_i\nabla_{X_i}\eta$. Note that the coefficient 12 comes from the normalization of alt and the expression $c^{-1}(\nabla_X\eta)(X, Y) = \frac{1}{4}((XY + g(X, Y))\eta, \nabla_X\eta)$. ☐

We are going to decompose $*d\Omega$ according to the previous splitting.

Proposition 5.2. *The* 3*-form* $\gamma_{48} = 3\gamma_8 - 84\,\mathrm{alt}(c^{-1}\nabla\eta)$ *lies in* $\Omega^3_{48}(M)$ *and*

$$*d\Omega = \frac{2}{7}\gamma_{48} + \frac{8}{7}\gamma_8.$$

Moreover, the Lee form is given by $\theta = \frac{8}{7}V^*$*, where* V *is defined as in the equation* (5.1).

Proof. Take a unitary vector X and a Cayley frame $(e_0, e_1, \ldots, e_7)$ such that $X = e_0$. Then:

$$\begin{aligned}
&(\gamma_8 \wedge \Omega)(e_1, \ldots, e_7) \\
=&(D\eta, (e_{123} - e_{145} - e_{167} - e_{246} + e_{257} - e_{347} - e_{356})\eta) \\
=&7(D\eta, e_0\eta) = 7V^*(X),
\end{aligned}$$

$$\begin{aligned}
&(12\,\mathrm{alt}(c^{-1}\nabla\eta) \wedge \Omega)(e_1, \ldots, e_7) \\
=&\mathfrak{S}(\nabla_{e_1}\eta, e_{23}\eta) - \mathfrak{S}(\nabla_{e_1}\eta, e_{45}\eta) - \mathfrak{S}(\nabla_{e_1}\eta, e_{67}\eta) \\
&- \mathfrak{S}(\nabla_{e_2}\eta, e_{46}\eta) + \mathfrak{S}(\nabla_{e_2}\eta, e_{57}\eta) - \mathfrak{S}(\nabla_{e_3}\eta, e_{47}\eta) \\
&- \mathfrak{S}(\nabla_{e_3}\eta, e_{56}\eta) = 3(D\eta, e_0\eta) = 3V^*(X).
\end{aligned}$$

We denoted by $\mathfrak{S}$ the cyclic sums in the indices involved. To arrange the last term observe that each index appears 3 times and:

$$\begin{aligned}
\mathfrak{S}(\nabla_{e_1}\eta, e_{23}\eta) =&(e_1\nabla_{e_1}\eta + e_2\nabla_{e_2}\eta + e_3\nabla_{e_3}\eta, e_{123}\eta) \\
=&(e_1\nabla_{e_1}\eta + e_2\nabla_{e_2}\eta + e_3\nabla_{e_3}\eta, e_0\eta),
\end{aligned}$$

$$\begin{aligned}
-\mathfrak{S}(\nabla_{e_1}\eta, e_{45}\eta) =&(e_1\nabla_{e_1}\eta + e_4\nabla_{e_4}\eta + e_5\nabla_{e_5}\eta, -e_{145}\eta) \\
=&(e_1\nabla_{e_1}\eta + e_4\nabla_{e_4}\eta + e_5\nabla_{e_5}\eta, e_0\eta),
\end{aligned}$$

and so on. Note that we have used, as in the proof of Lemma 3.1, that $e_{123}\eta = e_0\eta = -e_{145}\eta$.

Since Cayley bases are positively oriented, we get $*(V^*) = \frac{1}{7}(\gamma_8 \wedge \Omega) = 4\,\mathrm{alt}(c^{-1}\nabla\eta)$, so that γ_{48} as defined above lies in $\Omega^3_{48}(M)$. Finally, taking into account the formula for $*d\Omega$ in Proposition 5.1, we get $*d\Omega = \frac{2}{7}\gamma_{48} + \frac{8}{7}\gamma_8$.

To compute the Lee form we have used that the projection of $d\Omega$ to $\Omega^3_8(M)$ is $-\frac{8}{7}*\gamma_8$ and the formula $i(X)\Omega = *(X^*\wedge\Omega)$, which shall be checked by considering a Cayley frame and $X = e_0$. □

6. The characteristic connection

The *characteristic connection* of a Spin(7) structure is a connection ∇^c with totally skew-symmetric torsion such that $\nabla^c\Omega = 0$. The computations above allow us to prove the existence and uniqueness of the characteristic connection for manifolds with a Spin(7) structure. This is a well known result which appears in [12, Theorem 1.1]. Our proof is based on the argument of Theorem 3.1 in [10] and uses the notation of Section 2.3.

Proposition 6.1. *Given a* Spin(7) *structure, there exists a unique characteristic connection whose torsion* $\mathrm{T} \in \Omega^3(M)$ *is given by:*

$$\mathrm{T} = -\delta\Omega - \frac{7}{6} * (\theta \wedge \Omega).$$

Proof. A connection with skew-symetric torsion $\mathrm{T} \in \Omega^3(M)$ is given by $\nabla_X Y + \frac{1}{2}\mathrm{T}(X,Y,\cdot)^\sharp$, where $\mathrm{T}(X,Y,\cdot)^\sharp$ is the vector field such that $(\mathrm{T}(X,Y,\cdot)^\sharp)^* = \mathrm{T}(X,Y,\cdot)$. Thus, the lift to the spinor bundle is $\nabla_X\phi + \frac{1}{4}i(X)\mathrm{T}\phi$.

Since the condition $\nabla^c\Omega = 0$ is equivalent to $\nabla^c\eta = 0$ and the kernel of the Clifford product by η on Λ^2T^*M is $\Lambda^2_{21}T^*M$, the set of characteristic connections is isomorphic to the set of 3-forms $\mathrm{T} \in \Omega^3(M)$ such that

$$-4c^{-1}\nabla_X\eta = i(X)\mathrm{T}\eta = p^2_7(i(X)\mathrm{T}), \quad \forall X \in \mathfrak{X}(M).$$

The last equality may be rewritten as $-4c^{-1}\nabla\eta = \Theta(\mathrm{T})$. From the definition of γ_{48} given in Proposition 5.2 we have: $-4\Xi(c^{-1}\nabla\eta) = -12\,\mathrm{alt}(c^{-1}\nabla\eta) = \frac{1}{7}(\gamma_{48} - 3\gamma_8)$. Finally, taking into account the eigenvalues of $\Xi \circ \Theta$, we get:

$$\mathrm{T} = \frac{1}{7}(2\gamma_{48} - \frac{4}{3}\gamma_8) = *d\Omega - \frac{4}{3}\gamma_8 = -\delta\Omega - \frac{7}{6} * (\theta \wedge \Omega).$$

To obtain the second equality we have used the formula for $d\Omega$ from Lemma 5.2. To check the last one, note that $\gamma_8 = i(V)\Omega = *(V^* \wedge \Omega) = \frac{7}{8} * \theta \wedge \Omega$. □

Remark 6.2. The Spin(7) structure is balanced if and only if $\mathrm{T} \in \Omega^3_{48}(M)$ and locally conformally parallel if and only if $\mathrm{T} \in \Omega^3_8(M)$.

Remark 6.3. The equation for balanced structures given in Theorem 4.8 can be deduced from [12, Theorem 9.1], which states that the Spin(7) structure determined by η on a Riemannian manifold (M, g) is balanced for the metric $e^{\frac{6}{7}f} g$ if and only if it verifies the following equations:

$$\nabla^{\mathrm{T}}\eta = 0, \tag{6.1}$$

$$(df - \frac{1}{2}\mathrm{T})\eta = 0, \tag{6.2}$$

where ∇^{T} is the g-metric connection with totally skew-symmetric torsion T. That is, $\nabla^{\mathrm{T}}\phi = \nabla_X\phi + \frac{1}{4}i(X)\mathrm{T}\phi$ for $\phi \in \Sigma(M)$. This connection has an associated Dirac operator, which is related to D:

$$D^{\mathrm{T}}\phi = \sum_{i=0}^{7} e_i \nabla^T_{e_i}\phi = D\phi + \frac{1}{4}\sum_{i=0}^{7} e_i \wedge (i(e_i)\mathrm{T})\phi = D\phi + \frac{3}{4}\mathrm{T}\phi.$$

Assuming [12, Theorem 9.1], if we suppose that the structure is balanced for the metric g, equations (6.1) and (6.2) imply that $0 = D^{\mathrm{T}}\eta = D\eta + \frac{3}{4}\mathrm{T}\eta = D\eta$. Conversely if we suppose that $D\eta = 0$ and we choose T the torsion of the characteristic connection, we have obviously that $\nabla^{\mathrm{T}}\eta = 0$ and that $0 = D^{\mathrm{T}}\eta = D\eta + \frac{3}{4}\mathrm{T}\eta$, so that $\mathrm{T}\eta = 0$. According to Proposition 4.7, $\mathrm{T} \in \Omega^3_{48}(M)$ so that structure is balanced.

7. G_2 distributions

In this section we define the notion of a G_2 distribution on a Spin(7) manifold in terms of spinors and we study the torsion of the structure with respect to a suitable connection on the distribution. Then we relate the Spin(7) structure of the ambient manifold with the G_2 structure of the distribution. This approach enables us to study G_2 structures on submanifolds of Spin(7) manifolds, S^1-principal fibre bundles over G_2 manifolds and warped products of manifolds admitting a G_2 structure with $\mathbb{R}$. Our analysis is very similar to the description of G_2 structures from a spinorial viewpoint done in [1], which we briefly recall.

A 7-dimensional Riemannian manifold (Q, g) can be equipped with a G_2 structure if it is spin and its spinor bundle $\Sigma(Q)$ admits a unit section η. A cross product is then constructed from the spinor and is determined by a 3-form Ψ. Denote by ∇^Q both the Levi-Civita connection of the manifold and its lift to the spinor bundle; an endomorphism $\mathcal{S}$ of TQ is defined by the condition:

$$\nabla^Q_X\eta = \mathcal{S}(X)\eta.$$

The intrinsic torsion is $-\frac{2}{3}i(\mathcal{S})\Psi$ [1, Proposition 4.4], so that pure types of G_2 structures are given by the G_2 irreducible components of $\mathrm{End}(TQ)$. It is known that $\mathrm{End}(\mathbb{R}^7) = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$, where χ_i are irreducible G_2 representations, defined by:

$$\begin{aligned}
\chi_1 &= \langle \mathrm{Id} \rangle, \\
\chi_2 &= \mathfrak{g}_2, \\
\chi_3 &= \mathrm{Sym}_0^2(\mathbb{R}^7), \\
\chi_4 &= \{A\colon \mathbb{R}^7 \to \mathbb{R}^7 \colon A(X) = X \times S, \quad S \in \mathbb{R}^7\},
\end{aligned}$$

where $\mathrm{Sym}_0^2(\mathbb{R}^7)$ denotes the set of symmetric and traceless endomorphisms. The dimensions of the previous spaces are $1, 14, 27$ and 7 respectively.

If we denote by R_Q a G_2 reduction of the SO(7) principal bundle $P(Q)$ and define $\chi_i(Q) = R_Q \times_{\mathrm{G}_2} \chi_i$, then the pure classes of G_2 structures are determined by the condition $\mathcal{S} \in \chi_i(Q)$. For instance, nearly parallel G_2 structures verify $\mathcal{S} \in \chi_1(Q)$, almost parallel or calibrated are those with $\mathcal{S} \in \chi_2(Q)$, and locally conformally calibrated are such that $\mathcal{S} \in \chi_4(Q)$. Indeed in the nearly parallel case it holds that $\mathcal{S}(X) = \lambda_0 X$ for some $\lambda_0 \in \mathbb{R}$. Moreover mixed classes are also relevant, for instance cocalibrated structures verify $\mathcal{S} \in \chi_1(Q) \oplus \chi_3(Q)$.

Taking this into account, we define G_2 structures on distributions and characterise the existence of such structures.

Definition 7.1. Let (M, g) be an oriented 8-dimensional Riemannian manifold and let $\mathcal{D}$ be a cooriented distribution of codimension 1. We say that $\mathcal{D}$ has a G_2 structure if the principal SO(7) bundle $P(\mathcal{D})$ is spin and the spinor bundle $\Sigma(\mathcal{D})$ admits a unit section.

Lemma 7.2. *Consider an oriented* 8*-dimensional Riemannian manifold* (M,g) *and a cooriented distribution* $\mathcal{D}$ *of codimension* 1*. Take a unit vector field* N *perpendicular to* $\mathcal{D}$ *such that* $TM = \langle N \rangle \oplus \mathcal{D}$ *as oriented bundles. The manifold* M *is spin if and only if the bundle* $P(\mathcal{D})$ *is spin. In this case, the spinorial bundles are related by* $\Sigma(\mathcal{D}) = \Sigma^+(M)$ *and it holds*

$$X \cdot_{\mathcal{D}} \phi = N X \phi, \text{ if } X \in \mathcal{D}, \quad \phi \in \Sigma(\mathcal{D}), \tag{7.1}$$

where we have suppressed the symbol $\cdot_M$ *to denote the Clifford product on* M*.*

Therefore M *has a* Spin(7) *structure if and only if* $\mathcal{D}$ *has a* G_2 *structure.*

Proof. The bundle $P(\mathcal{D})$ is a reduction of $P(M)$ because of the following inclusion:

$$i\colon P(\mathcal{D}) \to P(M), \quad (X_1, \dots, X_7) \to (N, X_1, \dots, X_7).$$

Suppose that $P(\mathcal{D})$ is spin and denote the spin bundle by $\pi_{\mathcal{D}}\colon \tilde{P}(\mathcal{D}) \to P(\mathcal{D})$. Then, we can define the principal Spin(8) bundle $\tilde{P}(M) = \tilde{P}(\mathcal{D}) \times_{\mathrm{Spin}(7)} \mathrm{Spin}(8)$ and the map:

$$\pi_M\colon \tilde{P}(M) \to P(M), \quad [\tilde{F}, \tilde{\varphi}] \to \mathrm{Ad}(\tilde{\varphi})(i(\pi_{\mathcal{D}}(\tilde{F}))),$$

which is a double covering and Ad-equivariant. Therefore, M is spin. Conversely, if M is spin then the pullback $i^*(\tilde{P}(M))$ is the spin bundle of $P(\mathcal{D})$.

Moreover, the irreducible 8-dimensional representation of Cl_7 which maps the volume form to the identity can be constructed from the composition

$$\mathrm{Cl}_7 \to \mathrm{Cl}_8^0 \xrightarrow{\rho} \mathrm{GL}(\Delta^+),$$

where the first map is induced by the embedding $\mathbb{R}^7 \to \mathrm{Cl}_8^0$, $v \to e_0 v$, denoting by $(e_0, \ldots, e_7)$ the canonical basis of $\mathbb{R}^8$.

Therefore, the spinor bundle $\Sigma(\mathcal{D})$ coincides with $\Sigma(M)^+$ and Clifford products of vectors and spinors are related by the formula (7.1). □

From now on we assume that the manifold (M, g) has a Spin(7) structure Ω, constructed from a unit section η of the spinor bundle $\Sigma(M)^+$, as in Proposition 2.4. We equip M with a distribution $\mathcal{D}$ as in Lemma 7.2. We denote by $\Omega^k(\mathcal{D})$ the space of smooth sections of $\Lambda^k \mathcal{D}^*$.

Remarks 7.3. In this situation, we have the following:

1. If $\beta \in \Omega^{2k}(\mathcal{D})$ and $\phi \in \Sigma(\mathcal{D})$ then $\beta \cdot_{\mathcal{D}} \phi = \beta\phi$;
2. There is an orthogonal decomposition $\Sigma(\mathcal{D}) = \langle \eta \rangle \oplus (\mathcal{D} \cdot_{\mathcal{D}} \eta)$;
3. The section η defines a cross product on $\mathcal{D}$ by means of:

   $$(X \times Y)\eta = (X^* \wedge Y^*)\eta = (XY + g(X, Y))\eta,$$

   which is determined by $\Psi_{\mathcal{D}}(X, Y, Z) = (X\eta, (Y \times Z)\eta) = -(\eta, XYZ\eta)$;
4. The cross product is determined by $\Psi_{\mathcal{D}} = i(N)\Omega$. Therefore, using that $*\Omega = \Omega$ we get $\Omega = N^* \wedge \Psi_{\mathcal{D}} + *_{\mathcal{D}}\Psi_{\mathcal{D}}$.

We equip $\mathcal{D}$ with a suitable connection which is determined by the covariant derivative of the ambient manifold.

Definition 7.4. The covariant derivative of $\mathcal{D}$ induced by M, $\nabla^{\mathcal{D}}$, is given by the expression:

$$\nabla^M_X Y = \nabla^{\mathcal{D}}_X Y + g(\mathcal{T}(X), Y)N, \quad X, Y \in \mathcal{D},$$

where $\mathcal{T} \in \mathrm{End}(\mathcal{D})$ is given by: $2g(\mathcal{T}(X), Y) = -N(g(X, Y)) - g([X, N], Y) - g([Y, N], X) + g([X, Y], N)$.

We will decompose $\mathcal{T}$ into its symmetric and skew-symmetric parts, which we call $\mathcal{W}$ and $\mathcal{L}$ respectively. The connection $\nabla^{\mathcal{D}}$ is a metric connection and the tensor $\mathcal{L} = -\frac{1}{2} dN^*$ measures the lack of integrability of the distribution.

We will also denote by $\nabla^{\mathcal{D}}$ the lift of this connection to the spinor bundle $\Sigma(\mathcal{D})$. This connection is metric with respect to $(\cdot, \cdot)$ and behaves as a derivation with respect to the Clifford product. Hence, $\nabla^{\mathcal{D}}\eta \in \langle \eta \rangle^{\perp}$, and there is an endomorphism of $\mathcal{D}$ that we denote by $\mathcal{S}_{\mathcal{D}}$ such that $\nabla^{\mathcal{D}}_X \eta = \mathcal{S}_{\mathcal{D}}(X)\eta$. Let us define

$\chi_i(\mathcal{D}) = R_{\mathcal{D}} \times \chi_i$, where $R_{\mathcal{D}}$ is the G_2 reduction of $P(\mathcal{D})$ determined by $\Psi_{\mathcal{D}}$, we have a splitting of $\mathrm{End}(\mathcal{D})$ and we can decompose $\mathcal{S}$ according to it:

$$\mathcal{S}_{\mathcal{D}}(X) = \lambda \mathrm{Id} + S_2 + S_3 + S_4,$$

where $\lambda \in C^\infty(M)$, $S_2 \in \chi_2(\mathcal{D})$, $S_3 \in \chi_3(\mathcal{D})$, $S_4 \in \chi_4(\mathcal{D})$, and let $S \in \mathfrak{X}(\mathcal{D})$ be such that $S_4(X) = X \times S$.

We can relate these components with the Spin(7) structure defined on M. First of all, since $g(\nabla_X N, Y) = -g(\nabla_X Y, N)$ we get that the connection ∇^M at $\Sigma(M)^+$ in the direction of $\mathcal{D}$ is given by:

$$\nabla^M_X \eta = \nabla^{\mathcal{D}}_X \eta - \frac{1}{2} N \mathcal{T}(X)\eta = N\mathcal{A}(X)\eta,$$

where $\mathcal{A} = \mathcal{S}_{\mathcal{D}} - \frac{1}{2}\mathcal{T}$. We can decompose $\mathcal{L}$ and $\mathcal{W}$ according to the splitting of $\mathrm{End}(\mathcal{D})$ into irreducible parts and then decompose $\mathcal{A}$:

1. $\mathcal{L} = L_2 + L_4$, where $L_2 \in \chi_2(\mathcal{D})$, $L_4 \in \chi_4(\mathcal{D})$ and let $L \in \mathfrak{X}(\mathcal{D})$ such that $L_4(X) = X \times L$;
2. $\mathcal{W} = h\mathrm{Id} + W_3$, where $h \in C^\infty(M)$, $W_3 \in \chi_3(\mathcal{D})$;
3. $\mathcal{A} = \mu\mathrm{Id} + A_2 + A_3 + A_4$, where $\mu = \lambda - \frac{h}{2}$, $A_2 = S_2 - \frac{1}{2}L_2$, $A_3 = S_3 - \frac{1}{2}W_3$, $A_4 = S_4 - \frac{1}{2}L_4$. We will also denote $A = S - \frac{1}{2}L$.

We are going to compute $*d\Omega$ in terms of the previous endomorphisms and $\nabla^{\mathcal{D}}_N \eta$. Our first lemma is deduced from [1, Theorems 4.6,4.8].

Lemma 7.5. *Let $(X_1, \dots, X_7)$ be an orthonormal local frame of $\mathcal{D}$. Then*

$$\sum_{i=1}^{7} X_i \mathcal{A}(X_i)\eta = -7\mu\eta - 6NA\eta.$$

Proof. We will split the endomorphism $\mathcal{A}$ into its G_2 irreducible components and then compute each term separately. It is obvious that $\sum_{i=1}^7 X_i \mu X_i \eta = -7\mu\eta$. Moreover,

$$\sum_{i=1}^{7} X_i (X_i \times A)\eta = \sum_{i=1}^{7} X_i (X_i N A - g(X_i, A)N)\eta = -6NA.$$

Finally consider the G_2-equivariant map, $m\colon \mathcal{D} \otimes \mathcal{D} \to \Sigma(\mathcal{D})$, $m(X, Y) = XY\eta$. By dimensional reasons, its kernel must be $\chi_2(\mathcal{D}) \oplus \chi_3(\mathcal{D})$. Therefore, if $k \in \{2, 3\}$ we have that:

$$\sum_{i=1}^{7} X_i A_k(X_i)\eta = m\left(\sum_{i=1}^{7} (A_k)_{ij} X_i X_j\right) = 0,$$

where we have denoted $(A_k)_{ij}$ the entries of the matrix A_k with respect to the basis $(X_1, \dots, X_7)$. □

Remarks 7.6.

1. Since $\nabla^M_N \eta$ is perpendicular to η we can take $U \in \mathfrak{X}(\mathcal{D})$ such that $\nabla^M_N \eta = -NU\eta$.
   In order to compute $\nabla^M_N \eta$ we may take $F = (X_0, X_1, \ldots, X_7)$ a local orthonormal frame of M such that $N = X_0$, a lifting $\tilde{F} \in \tilde{P}(M)$ and write $\eta(p) = [\tilde{F}, s(p)]$. With this notation we have:

$$\nabla^M_{X_0}\eta = [\tilde{F}, ds(X_0)] + \frac{1}{2}\sum_{0\le i<j\le 7} g(\nabla_{X_0}X_i, X_j)X_iX_j\eta \tag{7.2}$$

$$= [\tilde{F}, ds(X_0)] + \frac{1}{2}\left(X_0\nabla_{X_0}X_0 + \sum_{1\le i<j\le 7} g(\nabla_{X_0}X_i, X_j)X_iX_j\right)\eta. \tag{7.3}$$

   Then, U depends on the local information of the section and $\nabla_{X_0}X_i$;
2. The Dirac operator of M is

$$D^M\eta = U\eta + \sum_{i=1}^{7} X_iN\mathcal{A}(X_i)\eta = (U - 6A + 7\mu N)\eta.$$

Lemma 7.7. *Define the forms $\beta_2 \in \Omega^2(\mathcal{D})$ and $\beta_3 \in \Omega^3(\mathcal{D})$ by:*

$$\beta_2(X, Y) = g(A_2(X), Y), \quad \beta_3(X, Y, Z) = \operatorname{alt}(i(A_3)(\cdot)\Psi_{\mathcal{D}})(X, Y, Z).$$

Then

1. $N^* \wedge i(N)(12\operatorname{alt}(c^{-1}\nabla\eta)) = i(U - 2A)(N^* \wedge \Psi_{\mathcal{D}}) - 2N^* \wedge \beta_2$;
2. $12\operatorname{alt}(c^{-1}\nabla\eta)|_Q = 3i(\mu N - A)\Omega|_Q + 3\beta_3$.

Proof. The first equality is a consequence of the symmetric or skew-symmetric properties of each factor:

$$\begin{aligned}12\operatorname{alt}(c^{-1}\nabla\eta)(N,X,Y) &= -(XY\eta, NU\eta) - (NY\eta, N\mathcal{A}(X)\eta) + (NY\eta, N\mathcal{A}(X)\eta)\\ &= -i(U)\Psi_{\mathcal{D}}(X, Y) - 2(Y\eta, (A_2(X) + X \times A)\eta)\\ &= \big(i(U - 2A)(N^* \wedge \Psi_{\mathcal{D}}) - 2N^* \wedge \beta_2\big)(N, X, Y).\end{aligned}$$

To check the second one, note that $12\operatorname{alt}(c^{-1}\nabla\eta)|_Q = 3\operatorname{alt}(i(\mathcal{A}(\cdot))\Psi_{\mathcal{D}})$. We compute separately each term in the decomposition of $\mathcal{A}$. It is evident that $3\operatorname{alt}(i(\mu\mathrm{Id})\Psi_{\mathcal{D}})(X,Y,Z) = 3\mu\Psi_{\mathcal{D}}(X,Y,Z)$ and $3\operatorname{alt}(i(A_3(\cdot))\Psi_{\mathcal{D}}) = 3\beta_3$. Moreover, $\operatorname{alt}(i(A_2(\cdot))\Psi_{\mathcal{D}}) = 0$ because $A_2 \in \chi_2(Q)$. Finally, if X, Y and Z are orthonormal vectors in TQ, then:

$$i(A_4(X))\Psi_{\mathcal{D}}(Y, Z) = (X \times A\eta, Y \times Z\eta) = (XA\eta, YZ\eta) = -(A\eta, (X \times Y \times Z)\eta).$$

Therefore, $3\operatorname{alt}(i(A_4(\cdot))\Psi_{\mathcal{D}})(X, Y, Z) = -3(A\eta, X \times Y \times Z\eta)$. ☐

From Lemmas 7.5 and 7.7 and the decomposition of $*d\Omega$ obtained in Proposition 5.2 we conclude:

Proposition 7.8. *Let* $U \in \mathfrak{X}(\mathcal{D})$ *such that* $\nabla^M_N \eta = -NU\eta$ *and define the forms* $\beta_2 \in \Omega^2(\mathcal{D})$ *and* $\beta_3 \in \Omega^3(\mathcal{D})$ *by:*

$$\beta_2(X,Y) = g(A_2(X),Y), \quad \beta_3(X,Y,Z) = \mathrm{alt}(i(A_3)(\cdot)\Psi_{\mathcal{D}})(X,Y,Z).$$

Then, the pure components of $*d\Omega$ *in terms of the* G_2 *structure are:*

$$(*d\Omega)_{48} = \frac{2}{7}\left(-4i(A+U)N^* \wedge \Psi + 3i(A+U) *_{\mathcal{D}} \Psi_{\mathcal{D}}\right) + 4N^* \wedge \beta_2 - 6\beta_3,$$

$$(*d\Omega)_8 = \frac{8}{7} i(U - 6A + 7\mu N)(N^* \wedge \Psi_{\mathcal{D}} + *_{\mathcal{D}}\Psi_{\mathcal{D}}).$$

7.1. Hypersurfaces

Consider an 8-dimensional Spin(7) manifold (M,g), whose Spin(7) form is constructed from a unit section η of the spinor bundle $\Sigma(M)^+$, as in Definition (2.4). Let Q be an oriented hypersurface and take a unit vector field N such that $TM = \langle N \rangle \oplus TQ$ as oriented vector bundles.

The tubular neighbourhood theorem guarantees the existence of a cooriented distribution $\mathcal{D}$ defined on a neighbourhood O of Q such that $\mathcal{D}|_Q = TQ$. The coorientation is determined by a unit extension of the normal vector field that we also denote by N. Both $\mathcal{D}$ and Q have G_2 structures determined by the spinor η; we are going to relate them using Proposition 7.8 in the manifold O.

Note that the Levi-Civita connection of the hypersurface Q is $\nabla^{\mathcal{D}}|_Q$. Moreover, $\mathcal{L}|_Q = 0$ and $\mathcal{W}|_Q$ is the Weingarten operator. Therefore, the restriction of $\mathcal{S}_{\mathcal{D}}$ at Q is the endomorphism $\mathcal{S}$ of the submanifold Q. Decompose $\mathcal{S}|_Q$ and $\mathcal{W}|_Q$ with respect to the G_2 splitting of $\mathrm{End}(TQ)$:

1. $\mathcal{S} = \lambda \mathrm{Id} + S_2 + S_3 + S_4$;
2. $\mathcal{W}|_Q = H\mathrm{Id} + W_3$,

where $\lambda \in C^\infty(M)$, $S_2 \in \chi_2(Q)$, $S_3, W_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$ and $H \in C^\infty(Q)$ is the mean curvature. We will also denote by S the vector field on Q such that $S_4(X) = X \times S$.

Corollary 7.9. *Let* $U \in \mathfrak{X}(Q)$ *such that* $\nabla^M_N \eta|_Q = -NU\eta$ *and* $\Psi_Q = i(N)\Omega$. *Define the forms* $\beta_2 \in \Omega^2(Q)$ *and* $\beta_3 \in \Omega^3(Q)$ *by:*

$$\beta_2(X,Y) = g(S_2(X),Y), \quad \beta_3(X,Y,Z) = \mathrm{alt}(i((S_3 - \frac{1}{2}W_3)(\cdot))\Psi_{\mathcal{D}})(X,Y,Z).$$

Then, the pure components of $*d\Omega$ *in terms of the* G_2 *structure are:*

$$(*d\Omega)_{48} = \frac{2}{7}\left(-4i(S+U)N^* \wedge \Psi_Q + 3i(S+U) *_Q \Psi_Q\right) + 4N^* \wedge i^*\beta_2 - 6\beta_3,$$

$$(*d\Omega)_8 = \frac{8}{7} i\left(U - 6S + 7(\lambda - \frac{1}{2}H)N\right)(N^* \wedge \Psi_Q + *_Q\Psi_Q).$$

Remark 7.10. Note that the condition $\nabla_N \eta|_Q = -NU\eta$ does not depend on the extension of the vectors. Moreover, we can compute U taking into account equation (7.2). The terms involved are extrinsic and not encoded in $\mathcal{S}$ and $\mathcal{W}$.

Therefore, the Spin(7) type of the ambient manifold provides relations between the G_2 type of the hypersurface, the vector U and the Weingarten operator. Before stating the result, we recall that a hypersurface is said to be totally geodesic if $\mathcal{W} = 0$, totally umbilic if $W_3 = 0$ and minimal if $H = 0$.

Theorem 7.11. *Let (M, g) be a Riemannian manifold endowed with a* Spin(7) *structure determined by a spinor η. Let Q be an oriented hypersurface with normal vector N and let $U \in \mathfrak{X}(Q)$ be such that $\nabla_N \eta|_Q = -NU\eta$.*

1. *If M has a parallel $Spin(7)$ structure, then Q has a cocalibrated* G_2 *structure. Moreover:*
   - 1.1 $\mathcal{S} = 0$ *if and only if Q is totally geodesic;*
   - 1.2 $\mathcal{S} \in \chi_1(Q)$ *if and only if Q is totally umbilic;*
   - 1.3 $\mathcal{S} \in \chi_3(Q)$ *if and only if Q is a minimal hypersurface;*
2. *If M has a locally conformally parallel $Spin(7)$ structure, then $\mathcal{S} \in \chi_1(Q) \oplus \oplus \chi_3(Q) \oplus \chi_4(Q)$. Indeed:*
   - 2.1 $\mathcal{S} \in \chi_1(Q)$ *if and only if $U = 0$ and Q is totally umbilic;*
   - 2.2 $\mathcal{S} \in \chi_1(Q) \oplus \chi_4(Q)$ *if and only if Q is totally umbilic;*
3. *If M has a balanced $Spin(7)$ structure, then:*
   - 3.1 $\mathcal{S} \in \chi_2(Q) \oplus \chi_3(Q)$ *if and only if $U = 0$ and Q is a minimal hypersurface;*
   - 3.2 $\mathcal{S} \in \chi_1(Q) \oplus \chi_2(Q) \oplus \chi_3(Q)$ *if and only if $U = 0$;*
   - 3.3 $\mathcal{S} \in \chi_2(Q) \oplus \chi_3(Q) \oplus \chi_4(Q)$ *if and only if Q is a minimal hypersurface.*

Proof. The parallel case follows from the equalities $U = S = 0$, $S_2 = 0$, $2\lambda = H$ and $2S_3 = W_3$. The locally conformally parallel case follows from the equalities $U = -S$, $S_2 = 0$ and $2S_3 = W_3$, which imply that $S \in \chi_1(Q) \oplus \chi_2(Q) \oplus \chi_3(Q)$. Finally the balanced case follows from $U = 6S$ and $2\lambda = 7H$. □

7.2. Principal bundles over a G_2 manifold

Let Q be a G_2 manifold and let $\pi\colon M \to Q$ be a $G = \mathbb{R}$ or $G = S^1$ principal bundle over Q; identify its Lie algebra $\mathfrak{g}$ with $\mathbb{R}$.

Define the vertical field $N(p) = \frac{d}{dt}\Big|_{t=0}(p\exp(t))$. A connection $\omega\colon TM \to \mathfrak{g}$ defines a horizontal distribution $\mathcal{H}$. Consider the metric on M such that:

1. The map $d\pi : \mathcal{H}_p \to T_{\pi(p)}Q$ is an isometry;
2. The vector $N(p)$ is unitary and perpendicular to $\mathcal{H}_p$.

The projection $d\pi$ induces a map $p\colon P(\mathcal{H}) \to P(Q)$ so that the pullback to $\tilde{P}(Q)$ defines a spin structure $\tilde{P}(\mathcal{H})$ on $P(\mathcal{H})$. The map $\tilde{p}\colon \tilde{P}(\mathcal{H}) \to \tilde{P}(Q)$, which is canonically defined, has the property that $\tilde{p}(\tilde{\varphi}\tilde{F}) = \tilde{\varphi}\tilde{p}(\tilde{F})$ if $\tilde{\varphi} \in \mathrm{Spin}(8)$, inducing a map between the spinorial bundles, that we denote by $\bar{p}$. Note that this map yields isomorphisms $\Sigma(\mathcal{H})_p \to \Sigma(Q)_{\pi(p)}$. Moreover, let $X \in TQ$ and denote by X^h its horizontal lift, then $\bar{p}(X^h \cdot_{\mathcal{H}} \phi) = X\bar{p}(\phi)$. Therefore, a section $\bar{\eta}\colon Q \to \Sigma(Q)$ allows us to define a section $\eta\colon M \to \Sigma(\mathcal{H})$ by means of the expression $\bar{p}(\eta) = \bar{\eta}$. Denote by Ψ_Q the G_2 form on Q, then $\Psi_{\mathcal{D}} = \pi^*\Psi_Q$.

Furthermore, one can check that $\nabla^{\mathcal{H}}_{X^h} Y^h = (\nabla^Q_X Y)^h$. Hence, if we take $\mathcal{S} \in \mathrm{End}(Q)$ such that $\nabla^Q_X \bar{\eta} = \mathcal{S}(X)\eta$, we get that the endomorphism of the distribution $\mathcal{S}_{\mathcal{D}}$ is the lifting of $\mathcal{S}$, that is:

$$\nabla^{\mathcal{H}}_{X^h}\eta = \mathcal{S}(X)^h\eta.$$

Therefore the distribution $\mathcal{H}$ and the manifold Q have the same type of G_2 structure. In order to classify the Spin(7) structure on M, denote the curvature of the connection ω by:

$$\mathfrak{L}(X, Y) = [X^h, Y^h] - [X, Y]^h \in \langle N\rangle, \qquad X, Y \in TQ.$$

Since $\mathfrak{L}(X, Y) \in \langle N\rangle$ we also denote by $\mathfrak{L}$ the 2-form that we obtain contracting the tensor with the metric. As a skew-symmetric endomorphism, we can decompose $\mathfrak{L} = \bar{L}_2 + \bar{L}_4$ where $\bar{L}_4(X) = X \times \bar{L}$ for some vector field $\bar{L} \in \mathfrak{X}(Q)$.

Corollary 7.12. *Suppose that* $\nabla^Q_X \bar{\eta} = \mathcal{S}(X)\cdot_Q \bar{\eta}$ *with* $\mathcal{S}(X) = \lambda\mathrm{Id} + S_2 + S_3 + S_4$ *where* $\lambda \in C^\infty(Q)$, $S_2 \in \chi_2(Q)$, $S_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$ *and let* $S \in \mathfrak{X}(Q)$ *be such that* $S_4(X) = X \times S$. *Define* $\beta_2 \in \Omega^2(Q)$ *and* $\beta_3 \in \Omega^3(Q)$ *by:*

$$\beta_2(X, Y) = g\left(S_2(X) - \frac{1}{4}\bar{L}_2(X), Y\right), \quad \beta_3(X, Y, Z) = \mathrm{alt}(i(S_3(\cdot))\Psi_Q)(X, Y, Z).$$

The pure components of $*d\Omega$ *in terms of the* G_2 *structure are:*

$$\begin{aligned}(*d\Omega)_{48} &= \frac{2}{7}\left(-4i(S^h + \frac{1}{2}\bar{L}^h)N^* \wedge \pi^*\Psi_Q + 3i(S^h + \frac{1}{2}\bar{L}^h)\pi^*(*_Q\Psi_Q)\right)\\ &\quad - 4N^* \wedge \pi^*\beta_2 + 6\pi^*\beta_3,\\ (*d\Omega)_8 &= \frac{8}{7}i\left(\frac{15}{4}\bar{L}^h - 6S^h + 7\lambda N\right)(N^* \wedge \pi^*\Psi_Q + \pi^*(*_Q\Psi_Q)).\end{aligned}$$

Proof. The result follows immediately from Proposition 7.8 once we check that $\mathcal{W} = 0$, $g(\mathcal{L}(X), Y) = \frac{1}{2}\pi^*\mathfrak{L}(X, Y)$, and $U = \frac{3}{4}\bar{L}^h$.

First of all, since the connection ω is left-invariant we have that $[X^h, N] = 0$ if $X \in \mathfrak{X}(Q)$. Thus, $\mathcal{W} = 0$. Moreover, $\mathcal{L}(X^h)(Y^h) = \frac{1}{2}\mathfrak{L}(X, Y)$. Furthermore, let $F = (X_1, \ldots, X_7)$ be a local frame of $\mathcal{H}$ which lifts some local frame of TQ. Take a lift $\tilde{F} \in \tilde{P}(\mathcal{H})$ and write $\eta(p) = [\tilde{F}, s(p)]$. We denote $X_0 = N$ and compute U using the formula (7.2).

By definition, if $\bar{\eta}(\pi(p)) = [\tilde{p}(\tilde{F}(p)), \bar{s}(\pi(p))]$ then $s(p) = \bar{s}(\pi(p))$ so that $ds_p(N) = 0$. Besides, according to Koszul formulas we have:

$$\begin{aligned} \nabla_N N &= 0, \\ g(\nabla_N X_i, X_j) &= -\frac{1}{2} g([X_i, X_j], N) = -\frac{1}{2} g(N, \mathfrak{L}(d\pi(X_i), d\pi(X_j))). \end{aligned}$$

Therefore, if we define $\gamma_i(X, Y) = g(\bar{L}_i(X), Y)$, for $i \in \{2, 4\}$, then:

$$\nabla_N \eta = -\frac{1}{4}\pi^* \mathfrak{L}\eta = -\frac{1}{4}\pi^* \gamma_4 \eta = -\frac{3}{4} N \bar{L}^h \eta,$$

where we have used that $\pi^*\gamma_2\eta = 0$ because $\mathfrak{g}_2 \subset \mathrm{Spin}(7) = \Lambda^2_{21}$ and $\pi^*\gamma_4 = -i(N)i(\bar{L}^h)\Omega$ so that $\pi^*\gamma_4\eta = 3N\bar{L}^h\eta$, as we noted in the proof of Lemma 3.1. □

7.3. Warped products

We analyze Spin(7) structures on warped products of a G_2 manifold with $\mathbb{R}$. Recall that a warped product of two Riemannian manifolds (X_1, g_1) and (X_2, g_2) is $(X_1 \times X_2, g_1 + f_1 g_2)$ where $f_1 \colon X_1 \to \mathbb{R}$ is a smooth function. Therefore, we have to distinguish two cases.

7.3.1.

Consider a G_2 manifold (Q, g) and a smooth function $f \colon \mathbb{R} \to \mathbb{R}$. Define the Riemannian manifold $(M = Q \times \mathbb{R}, e^{2f} g + dt^2)$. This is the so-called spin cone.

The distribution $\mathcal{D} = TQ$ obviously admits a G_2 structure. The spinor bundle is given by $\Sigma(M)^+ = \Sigma(TQ \times \mathbb{R}) = \Sigma(Q) \times \mathbb{R}$ and Clifford products are related by $(X \cdot_Q \phi, t) = e^{-f} X \cdot_{\mathcal{D}} (\phi, t) = e^{-f} \frac{\partial}{\partial t} X(\phi, t)$ if $X \in TQ$. In the last expression, we have suppressed the symbol $\cdot$ to denote the Clifford product on M.

A unit section η is constructed from a section $\bar{\eta} \colon Q \to \Sigma(Q)$ by defining $\eta \colon M \to \Sigma(\mathcal{D})$, $\eta(x, t) = (\bar{\eta}(x), t)$. If we denote by Ψ_Q the G_2 form on Q, then $\Psi_{\mathcal{D}} = e^{3f}\pi^*\Psi_Q$ and $*_{\mathcal{D}}(\Psi_{\mathcal{D}}) = e^{4f} *_Q (\Psi_Q)$. In addition, since $\nabla^{\mathcal{D}}_X Y = \nabla^Q_X Y$ when $X, Y \in \mathfrak{X}(Q)$, we have that $\nabla^{\mathcal{D}}_X \eta = e^{-f} \mathcal{S}(X) \cdot_{\mathcal{D}} \eta$ and $\nabla^Q_X \bar{\eta} = \mathcal{S}(X)\bar{\eta}$. That is, $\mathcal{S}_{\mathcal{D}} = e^{-f}\mathcal{S}$.

Corollary 7.13. *Suppose that $\nabla^Q_X \bar{\eta} = \mathcal{S}(X) \cdot_Q \bar{\eta}$ with $\mathcal{S}(X) = \lambda \mathrm{Id} + S_2 + S_3 + S_4$ where $\lambda \in C^\infty(Q)$, $S_2 \in \chi_2(Q)$, $S_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$. Let $S \in \mathfrak{X}(Q)$ be such that $S_4(X) = X \times S$. Denote by Ψ_Q the G_2-form on Q and define $\beta_2 \in \Omega^2(Q)$ and $\beta_3 \in \Omega^3(Q)$ by:*

$$\beta_2(X, Y) = g\left(S_2(X), Y\right), \quad \beta_3(X, Y, Z) = \mathrm{alt}(i(S_3(\cdot))\Psi_Q)(X, Y, Z).$$

The pure components of $*d\Omega$ *in terms of the* G_2 *structure are:*

$$(*d\Omega)_{48} = \frac{2}{7}\left(-4e^{2f}i(S)dt\wedge\pi^*\Psi_Q + 3e^{3f}i(S)\pi^*(*_Q\Psi_Q)\right) + 4e^f dt^*\wedge\pi^*\beta_2 - 6e^{2f}\pi^*\beta_3,$$

$$(*d\Omega)_8 = \frac{8}{7}i\left(-6e^{-f}S+7\left(\lambda e^{-f}+\frac{1}{2}f'\right)\frac{\partial}{\partial t}\right)(e^{3f}dt\wedge\pi^*\Psi_Q+e^{4f}\pi^*(*_Q\Psi_Q)).$$

Proof. The result follows immediately from Proposition 7.8 once we check that $\mathcal{W} = -f'Id, \mathcal{L} = 0$ and $U = 0$.

Since the distribution $\mathcal{D}$ is integrable, we have that $\mathcal{L} = 0$. Take an orthonormal frame of TQ, $(X_1,\dots,X_7)$ and note that $\mathcal{W}(X_i,X_j) = -f'e^{2f}\delta_{ij}$ so that $\mathcal{W} = -f'$. Moreover, using the Koszul formulas we get:

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = 0 = \nabla_{\frac{\partial}{\partial t}}(e^{-f}X_i).$$

Therefore, using formula (7.2) we conclude that $\nabla_{\frac{\partial}{\partial t}}\eta = 0$. □

7.3.2.

Consider a G_2 manifold (Q,g) and a smooth function $f\colon Q\to\mathbb{R}$. Define the Riemannian manifold $(M = Q\times\mathbb{R}, g+e^{2f}dt^2)$.

The distribution $\mathcal{D} = TQ$ obviously admits a G_2 structure. The spinor bundle is given by $\Sigma(M)^+ = \Sigma(TQ\times\mathbb{R}) = \Sigma(Q)\times\mathbb{R}$ and the Clifford products are related by $(X\cdot_Q\phi,t) = X\cdot_{\mathcal{D}}(\phi,t) = e^{-f}\frac{\partial}{\partial t}X(\phi,t)$ if $X\in TQ$. We have suppressed again the symbol $\cdot$ to denote the Clifford product on M.

A unit section η is constructed from a section $\bar\eta\colon Q\to\Sigma(Q)$ by defining $\eta\colon M\to\Sigma(\mathcal{D})$, $\eta(x,t) = (\bar\eta(x),t)$. If we denote by Ψ_Q the G_2 form on Q, then $\Psi_{\mathcal{D}} = \pi^*\Psi_Q$ and $*_{\mathcal{D}}(\Psi_{\mathcal{D}}) = *_Q(\Psi_Q)$. In addition, since $\nabla^{\mathcal{D}}_X Y = \nabla^Q_X Y$ when $X,Y\in\mathfrak{X}(Q)$, if we take $\mathcal{S}\in\mathrm{End}(TQ)$ with $\nabla^Q_X\bar\eta = \mathcal{S}(X)\bar\eta$, then $\mathcal{S}_{\mathcal{D}} = \mathcal{S}$.

Corollary 7.14. *Suppose that* $\nabla^Q_X\bar\eta = \mathcal{S}(X)\cdot_Q\bar\eta$ *with* $\mathcal{S}(X) = \lambda\mathrm{Id}+S_2+S_3+S_4$ *where* $\lambda\in C^\infty(Q)$, $S_2\in\chi_2(Q)$, $S_3\in\chi_3(Q)$, $S_4\in\chi_4(Q)$. *Let* $S\in\mathfrak{X}(Q)$ *be such that* $S_4(X) = X\times S$. *Denote by* Ψ_Q *the* G_2*-form on* Q *and define* $\beta_2\in\Omega^2(Q)$ *and* $\beta_3\in\Omega^3(Q)$ *by:*

$$\beta_2(X,Y) = g(S_2(X),Y),\quad \beta_3(X,Y,Z) = \mathrm{alt}(i(S_3(\cdot))\Psi_Q)(X,Y,Z).$$

The pure components of $*d\Omega$ *in terms of the* G_2 *structure are:*

$$(*d\Omega)_{48} = \frac{2}{7}\left(-4i\left(S+\frac{1}{2}\mathrm{grad}(f)\right)e^f dt\wedge\pi^*\Psi_Q + 3i\,(S+\mathrm{grad}(f))\,\pi^*(*_Q\Psi_Q)\right) + 4e^f dt\wedge\pi^*\beta_2 - 6\pi^*\beta_3,$$

$$(*d\Omega)_8 = \frac{8}{7}i\left(\frac{1}{2}\mathrm{grad}(f) - 6S + 7\lambda e^{-f}\frac{\partial}{\partial t}\right)(e^f dt\wedge\pi^*\Psi_Q + \pi^*(*_Q\Psi_Q)).$$

Proof. The result follows immediatly from Proposition 7.8 once we check that $\mathcal{W} = 0, \mathcal{L} = 0$ and $U = \frac{1}{2}\operatorname{grad}(f)$.

Since the distribution $\mathcal{D}$ is integrable, we have that $\mathcal{L} = 0$. Take an orthonormal frame of TQ, $(X_1, \ldots, X_7)$ and note that $\mathcal{W}(X_i, X_j) = 0$. Moreover, using the Koszul formulas we get:

$$g(\nabla_{e^{-f}\frac{\partial}{\partial t}} X_i, X_j) = 0,$$
$$g\left(\nabla_{e^{-f}\frac{\partial}{\partial t}} e^{-f}\frac{\partial}{\partial t}, X_i\right) = -X_i(f).$$

Therefore, using formula (7.2) we conclude that $\nabla_N \eta = -\frac{1}{2}e^{-f}\left(\frac{\partial}{\partial t}\right)\operatorname{grad}(f)\eta$. □

8. Spin(7) structures on quasi Abelian Lie algebras

As an application of the previous section, we are going to study Spin(7) structures on quasi Abelian Lie algebras. The geometric setting will be that of a simply connected Lie group with an invariant Spin(7) structure, endowed with an integrable distribution which inherits a G_2 structure. The integral submanifolds of the distribution are actually flat, so that the G_2 distribution is parallel and these submanifolds have non-trivial Weingarten operators. In some cases, finding a lattice in the Lie group will allow us to give compact examples.

First of all, let us recall the following definition:

Definition 8.1. A Lie algebra $\mathfrak{g}$ is called quasi Abelian if it contains a codimension 1 Abelian ideal $\mathfrak{h}$.

The information of $\mathfrak{g}$ is then encoded in $ad(x)$ for any vector x transversal to $\mathfrak{h}$. The following result shows that $\mathfrak{h}$ is unique in $\mathfrak{g}$ with exception of the Lie algebras $\mathbb{R}^n$ and $L_3 \oplus \mathbb{R}^{n-3}$, where L_3 is the Lie algebra of the 3-dimensional Heisenberg group, which is generated by x,y,z with relations $[x, y] = z$ and $[x, z] = [y, z] = 0$.

Lemma 8.2. *Let $\mathfrak{g}$ be a n-dimensional quasi Abelian Lie algebra with $n \geq 3$. If $\mathfrak{g}$ is not isomorphic to $\mathbb{R}^n$ or $L_3 \oplus \mathbb{R}^{n-3}$, then it has a unique codimension 1 Abelian ideal. Moreover, codimension 1 Abelian ideals on $L_3 \oplus \mathbb{R}^{n-3}$ are parametrized by $\mathbb{RP}^1$.*

Proof. Suppose that $\mathfrak{g}$ is not isomorphic to $\mathbb{R}^n$ and let $\mathfrak{h}$ be a codimension 1 Abelian ideal with a transversal vector x. Let $\mathfrak{h}'$ be a codimension 1 Abelian ideal different from $\mathfrak{h}$. If $u \in \mathfrak{h}$ is such that $x + u \in \mathfrak{h}'$ and $v \in \mathfrak{h} \cap \mathfrak{h}'$, then $0 = [x + u, v] = ad(x)(v)$. Since $\mathfrak{h} \cap \mathfrak{h}'$ is $(n-2)$-dimensional and $\mathfrak{g}$ is not Abelian we conclude that $\mathfrak{h} \cap \mathfrak{h}' = \ker(ad(x)|_{\mathfrak{h}})$ and $ad(x)(\mathfrak{h}) = \langle z \rangle$ for some $z \in \mathfrak{h}$. Take $y \in \mathfrak{h}$ with $[x, y] = z$ and observe that $z \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}'$, that is, $z \in \mathfrak{h} \cap \mathfrak{h}'$ and $[x, z] = 0$. Therefore, $\mathfrak{g}$ is isomorphic to $L_3 \oplus \mathbb{R}^{n-3}$.

Moreover, from the discussion above we get that $\mathfrak{h}' = \langle v, z \rangle \oplus \mathbb{R}^5$ for some $v \in \langle x, y \rangle$. Conversely, all the subspaces of the previous form are actually codimension 1 Abelian ideals. Therefore, they are parametrized by $\mathbb{RP}^1$. □

An invariant Spin(7) structure on a Lie group is determined by the choice of a Spin(7) form Ω, which is in turn determined by a direction of the spinorial space Δ^+.

Define the set $\mathcal{QA}$ with elements $(\mathfrak{g}, \mathfrak{h}, g, \nu_g, \Omega)$ where $\mathfrak{g}$ is a non-trivial quasi Abelian Lie algebra with a marked codimension 1 Abelian ideal $\mathfrak{h}$, g is a metric on $\mathfrak{g}$, ν_g is a volume form on $\mathfrak{g}$ and Ω is a Spin(7) structure on $(\mathfrak{g}, g, \nu_g)$. We will say that $\varphi' \colon (\mathfrak{g}, \mathfrak{h}, g, \nu_g, \Omega) \to (\mathfrak{g}', \mathfrak{h}', g', \nu_{g'}, \Omega')$ is an isomorphism if φ is an isomorphism of Lie algebras such that $\varphi'(\mathfrak{h}) = \mathfrak{h}'$, $(\varphi')^* g' = g$, $\varphi^* \nu_{g'} = \nu_g$ and $\varphi^* \Omega' = \Omega$.

Lemma 8.3. *The set $\overline{\mathcal{QA}}$ of isomorphisms classes of $\mathcal{QA}$ is given by:*

$$\overline{\mathcal{QA}} = \Big((\mathrm{End}(\mathbb{R}^7) - \{0\}) \times \mathbb{P}(\Delta^+) \Big) \,/\, \mathrm{O}(7),$$

where O(7) *acts via*

$$\varphi \cdot (\mathcal{E}, [\eta]) = (\det(\varphi)\varphi \circ \mathcal{E} \circ \varphi^{-1}, [\rho(\tilde{\varphi})\eta]), \tag{8.1}$$

where $\tilde{\varphi}$ is a lifting to Spin(8) *of the unique $\varphi' \in \mathrm{SO}(8)$ such that $\varphi'|_{\mathbb{R}^7} = \varphi$.*

Proof. A map $(\mathrm{End}(\mathbb{R}^7) - \{0\}) \times \mathbb{P}(\Delta^+) \to \mathcal{QA}$ can be defined as follows. Take $(\mathcal{E}, \bar{\eta})$ and define the Lie structure on $\mathbb{R}^8$ with oriented basis $(e_0, \dots, e_7)$ such that $\mathbb{R}^7 = \langle e_1, \dots, e_7 \rangle$ is a maximal Abelian ideal and $\mathcal{E} = ad(e_0)|_{\mathbb{R}^7}$. We will endow this algebra with the canonical metric, the standard volume form and the spin structure determined by η.

It is obvious that a representative of each element of $\overline{\mathcal{QA}}$ can be chosen to lie in the image of our map. Moreover, if two structures given by $(\mathcal{E}, \bar{\eta})$ and $(\mathcal{E}', \bar{\eta}')$ are isomorphic via φ', we have the following:

1. $\varphi'(e_0) = \pm e_0$ and $\varphi = \varphi'|_{\mathbb{R}^7} \in \mathrm{O}(7)$, since φ' preserves the metric and the orientation;
2. Denote by $\tilde{\varphi}$ any lifting of φ' to Spin(8). Since $(\varphi')^*\Omega' = \Omega$, we have that $\mathrm{Stab}(\Omega) = (\varphi')^{-1} \circ \mathrm{Stab}(\Omega') \circ (\varphi')$, thus $\mathrm{Stab}(\eta) = \tilde{\varphi}^{-1}\,\mathrm{Stab}(\eta')\tilde{\varphi}$. But $\mathrm{Stab}(\rho(\tilde{\varphi})^{-1}\eta') = \tilde{\varphi}^{-1}\,\mathrm{Stab}(\eta')\tilde{\varphi}$, so that $\eta = \pm\rho(\tilde{\varphi})^{-1}\eta'$;
3. $\varphi \circ \mathcal{E} = det(\varphi)\mathcal{E}' \circ \varphi$, since φ' is an isomorphism of Lie algebras. □

From now on we denote by $(\mathbb{R}^8, \mathcal{E}, [\eta])$ to $(\mathfrak{g}, \mathfrak{h}, g, \nu, \Omega) \in \mathcal{QA}$ where $\mathfrak{g}$ is the Lie algebra $\mathbb{R}^8$ with maximal Abelian ideal $\mathfrak{h} = \mathbb{R}^7$, $ad(e_0) = \mathcal{E}$, g is the canonical metric, ν is the canonical volume form and the Spin(7) form Ω is determined by $[\eta]$.

Remark 8.4. To obtain an analogue of Lemma 8.3, suppressing the condition $\varphi'(\mathfrak{h}) = \mathfrak{h}'$ in the definition of isomorphism, we have to treat separatedly the case of the Lie algebra $L_3 \oplus \mathbb{R}^5$. For this purpose, define $\mathcal{E}(x) = e_1^*(x)e_2$ and observe that Lemmas 8.2 and 8.3 allow us to suppose that any isomorphism of structures with underlying Lie algebra $L_3 \oplus \mathbb{R}^5$ is represented by $\varphi' \colon (\mathbb{R}^8, \lambda\mathcal{E}, [\eta]) \to (\mathbb{R}^8, \lambda'\mathcal{E}, [\eta'])$, for some $\lambda, \lambda' \neq 0$.

The set $\varphi'(\mathbb{R}^7)$ is a codimension 1 Abelian ideal, hence Lemma 8.2 guarantees that $\varphi'(e_0) = \cos(\theta)e_0 + \sin(\theta)e_1$. Denote $\mathbb{R}^6 = \langle e_2, \dots, e_7\rangle$ and let $v, v' \in \mathbb{R}^6$ be such that $\varphi'(v) = -\mu\sin(\theta)e_0 + \mu\cos(\theta)e_1 + v'$. Then, $0 = \varphi'[e_0, v] = [\cos(\theta)e_0 + \sin(\theta)e_1, -\mu\sin(\theta)e_0 + \mu\cos(\theta)e_1 + v'] = \mu\lambda' e_2$. Therefore $\mu = 0$, $\mathbb{R}^6$ is φ'-invariant and $\varphi'(e_1) = \mp\sin(\theta)e_0 \pm \cos(\theta)e_1$.

Denote by φ_1 the restriction of φ' to $\langle e_0, e_1\rangle$ and note that: $\lambda\varphi'(e_2) = \varphi'[e_0, e_1] = [\varphi'(e_0), \varphi'(e_1)] = \det(\varphi_1)\lambda' e_2$. Hence $\varphi'(e_2) = \det(\varphi_1)\frac{\lambda'}{\lambda}e_2$ and $|\lambda| = |\lambda'|$. Then, φ' is determined by φ_1 and $\varphi_2 = \varphi'|_{\mathbb{R}^5}$, where $\mathbb{R}^5 = \langle e_3, \dots, e_7\rangle$, under the conditions $\frac{\lambda'}{\lambda}\det(\varphi_2) = 1$ and $\varphi'(e_2) = \det(\varphi_1)\frac{\lambda'}{\lambda}e_2$.

The condition over the spinor is obviously $\eta' = \pm\rho(\tilde{\varphi})\eta$, where $\tilde{\varphi}$ is any lifting of φ' to Spin(8).

In the following result we describe the action which appears in Lemma 8.3.

Lemma 8.5. *Under the action of* O(7) *on* $\mathrm{End}(\mathbb{R}^7)$,

$$\varphi \cdot \mathcal{E} = det(\varphi)\varphi \circ \mathcal{E} \circ \varphi^{-1}, \tag{8.2}$$

the sets $\langle \mathrm{Id}\rangle$, $\mathrm{Sym}^2_0(\mathbb{R}^7)$ *and* $\Lambda^2\mathbb{R}^7$ *are parametrized respectively by:*

1. $[0, \infty)$;
2. $\{(\lambda_1, \dots, \lambda_7)\colon \lambda_i \le \lambda_{j+1}, \sum_{j=1}^{7}\lambda_i = 0\}/\sim$, *where* $(\lambda_1, \dots, \lambda_7) \sim (-\lambda_7, \dots, -\lambda_1)$;
3. $\{(\lambda_1, \lambda_2, \lambda_3)\colon 0 \le \lambda_1 \le \lambda_2 \le \lambda_3\}$.

Proof. The first claim is obvious and the second follows from the fact that each symmetric matrix has an oriented orthonormal basis of ordered eigenvectors. Note also that $-\mathrm{Id}\cdot \mathrm{diag}(\lambda_1, \dots, \lambda_7) = \mathrm{diag}(-\lambda_7, \dots, -\lambda_1)$, hence $(\lambda_1, \dots, \lambda_7)$ is related to $(-\lambda_7, \dots, -\lambda_1)$.

If $\mathcal{E}$ is a skew-symmetric endomorphism of $\mathbb{R}^7$ we can find a Hermitian basis in $\mathbb{C}^7$ of eigenvectors and the eigenvalues are of the form $(-\lambda_3 i, -\lambda_2 i, \lambda_1 i, 0, \lambda_1 i, \lambda_2 i, \lambda_3 i)$ with $0 \le \lambda_j \le \lambda_{j+1}$. Moreover, the real parts of the eigenspaces associated to $-\lambda_j i$ and $\lambda_j i$ coincide. Thus, we can find a positive oriented orthonormal basis $(v_1, w_1, v_2, w_2, v_3, w_3, u)$ of $\mathbb{R}^7$, such that $\mathcal{E}(v_j) = \lambda_j w_j$ and $\mathcal{E}(u) = 0$. Finally note that $(\lambda_1, \lambda_2, \lambda_3)$ are invariantly defined in the orbit. □

In Lemma 8.3, the second factor of the product of $\mathcal{QA}$ depends on $\mathrm{Stab}(\mathcal{E})$ under the action defined by (8.2) and it is determined by the number of equal eigenvalues. Now we compute the invariants that we defined for G_2 distributions on $\mathbb{R}^7$:

Proposition 8.6. *Consider* $(\mathbb{R}^8, \mathcal{E}, [\eta]) \in \mathcal{QA}$ *and decompose* $\mathcal{E}$ *according to the* G_2 *structure induced by* η, *that is* $\mathcal{E} = h\mathrm{Id} + E_2 + E_3 + E_4$, *where* $h \in \mathbb{R}$, $E_2 \in \chi_2$, $E_3 \in \chi_3$, $E_4 \in \chi_4$ *and* $E_4(X) = X \times E$ *for some* $E \in \mathbb{R}^7$. *Define* $\Psi, \beta_3 \in \Lambda^3 T^*\mathbb{R}^7$

by $\Psi = \Omega|_{\mathbb{R}^7}$ *and* $\beta_3(X, Y, Z) = \mathrm{alt}(i(E_3(\cdot))\Psi)]$. *We have:*

$$(*d\Omega)_{48} = \frac{2}{7}\left(6i(E)e^0 \wedge \Psi - \frac{9}{4}i(E) *_{\mathbb{R}^7} \Psi\right) + 6\beta_3,$$
$$(*d\Omega)_8 = -\left(\frac{12}{7}E + 4he_0\right)(e^0 \wedge \Psi + *_{\mathbb{R}^7}\Psi).$$

Proof. The result follows immediately from Proposition 7.8 once we check that: $\mu = -\frac{1}{2}h$, $A_2 = 0$, $A_3 = -\frac{1}{2}E_3$, $A = 0$ and $U = -\frac{3}{2}E$.

To obtain this, first observe that $\nabla^{\mathfrak{h}}\eta = 0$ and $\mathcal{L} = 0$ because $\mathfrak{h}$ is an Abelian ideal. From the formula of the Weingarten operator we get: $\mathcal{W} = h\mathrm{Id} + E_3$. To compute U we use again equation (7.2), obtaining that:

$$\nabla_N \eta = \frac{3}{2}e_0 E\eta,$$

since $\nabla_{e_0} e_0 = 0$ because $\mathfrak{h}$ is an ideal and $\nabla_{e_0} e_j = (E_2 + E_4)(e_j)$ if $j > 0$. □

In the next result we characterise in terms of Lemma 8.5 the type of Spin(7) structure on quasi Abelian Lie algebras. For this purpose, recall that a Lie algebra is called unimodular if the volume form is not exact. In the case of the Lie algebra $(\mathbb{R}^8, \mathcal{E})$, it is equivalent to say that $\mathcal{E}$ is traceless.

Theorem 8.7. *Consider the Lie algebra* $(\mathbb{R}^8, \mathcal{E})$ *endowed with the standard metric and volume form. Denote by* $\mathcal{E}_{13}$ *and* $\mathcal{E}_{24}$ *the symmetric and skew-symmetric parts of the endomorphism* $\mathcal{E} \neq 0$. *Then, the Lie algebra admits a* Spin(7) *structure of type:*

1. *parallel, if and only if* $\mathcal{E}_{13} = 0$ *and* $\mathcal{E}_{24}$ *is associated to* $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ *with* $0 \leq \lambda_1 \leq \lambda_2, \lambda_2 > 0$ *as in Lemma* 8.5*;*
2. *locally conformally parallel and non-parallel if and only if* $\mathcal{E}_{13} = h\mathrm{Id}$ *with* $h \neq 0$ *and* $\mathcal{E}_{24}$ *is associated to* $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ *with* $0 \leq \lambda_1 \leq \lambda_2$*, as in Lemma* 8.5*;*
3. *balanced if and only if it is unimodular and* $\mathcal{E}_{24}$ *is associated to* $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ *with* $0 \leq \lambda_1 \leq \lambda_2$*, as in Lemma* 8.5.

Moreover, if $\mathcal{E}_{24} \neq 0$ *then it admits a* Spin(7) *structure of mixed type.*

Proof. We identify $\mathcal{E}_{24}$ with a 2-form γ which can be written with respect to a positive oriented orthonormal basis $(X_1, \ldots, X_7)$ of $\mathbb{R}^7$ as $\gamma = \lambda_1 X^{23} + \lambda_2 X^{45} + \lambda_3 X^{67}$, where $0 \leq \lambda_j \leq \lambda_{j+1}$ and $X^{ij} = X_i^* \wedge X_j^*$.

Due to Proposition 8.6, to prove the first part we have to check that under the assumption $\mathcal{E}_{24} \neq 0$, the existence of a spinor η such that $\gamma\eta = 0$ is equivalent to the fact that $\mathcal{E}_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$. This spinor exists if and only if $\rho_7(\lambda_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7)$ is non-invertible for some 8-dimensional real irreducible representation $\rho_7\colon \mathrm{Cl}_7 \to \mathrm{End}(\mathbb{R}^8)$ which maps the volume form ν_7 to the identity, since they are all equivalent [16, Proposition 5.9].

It is known that the two distinct irreducible representations of Cl_7 can be constructed from the octonions $\mathbb{O}$ [16, page 51]. Specifically, those are the extension to Cl_7 of the maps $\rho_\theta\colon \mathbb{R}^7 \to \mathrm{End}(\mathbb{R}^8)$, $\rho_\theta(v)(x) = \theta v x$, where $\theta = \pm 1$ and $\mathbb{R}^7$ is viewed as the imaginary part of the octonions. Define the isometry φ of $\mathbb{R}^7$ which maps X_i to e_i and note that the volume form is fixed by the extension of φ to the Clifford algebra. The extensions of ρ_θ and φ to Cl_7 are denoted in the same way. We check the previous condition using the representation $\rho_7 = \rho_\theta \circ \varphi\colon \mathrm{Cl}_7 \to \mathrm{End}(\mathbb{R}^8)$, taking θ such that $\rho_\theta(\nu_7) = \mathrm{Id}$. The determinant of $\rho_7(\lambda_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7)$ is given by:

$$(\lambda_1 + \lambda_2 + \lambda_3)^2(\lambda_1 + \lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_2 + \lambda_3)^2.$$

Since $\lambda_1 \le \lambda_2 \le \lambda_3$, the endomorphism is non-invertible if and only if $\lambda_3 = \lambda_2 + \lambda_1$.

Finally, if $\mathcal{E}_{24} \neq 0$ then $\rho_7(\lambda_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7) \neq 0$ so that there is a spinor η such that $E \neq 0$; Proposition 8.6 guarantees that η induces a Spin(7) structure of mixed type. □

Recall that solvmanifolds are compact quotients G/Γ, where G is a simply connected solvable Lie group and Γ is a discrete lattice. This forces the Lie algebra $\mathfrak{g}$ of G to be unimodular [20, Lemma 6.2]. Thefore, using Proposition 8.6, we conclude the following:

Corollary 8.8. *There exists no quasi Abelian solvmanifold with an invariant locally conformally parallel and non-parallel* Spin(7) *structure.*

Of course, a torus is solvmanifold which admits a parallel Spin(7) structure.

Corollary 8.9. *If* $(\mathbb{R}^8, \mathcal{E})$ *is a quasi Abelian Lie algebra such that* $\mathcal{E}$ *is skew-symmetric, then it is flat. In particular, quasi Abelian Lie algebras which admit an invariant parallel* Spin(7) *structure are flat.*

Proof. Let $(\mathbb{R}^8, \mathcal{E})$ be a quasi Abelian Lie algebra and denote by $\mathcal{E}_{13}$ and $\mathcal{E}_{24}$ the symmetric and skew-symmetric parts of $\mathcal{E}$. It is straightforward to check that if $i, j > 0$ then:

$$\nabla_{e_0} e_0 = 0, \quad \nabla_{e_0} e_j = \mathcal{E}_{24}(e_j), \quad \nabla_{e_i} e_0 = -\mathcal{E}_{13}(e_i), \quad \nabla_{e_i} e_j = g(\mathcal{E}_{13}(e_i), e_j) e_0.$$

From this, one can deduce that if $i, j, k > 0$, then the curvature tensor is given by:

$$\begin{aligned} R(e_0, e_j)e_0 &= -(\mathcal{E}_{24} \circ \mathcal{E}_{13} + \mathcal{E}_{13} \circ \mathcal{E}_{24})(e_j), \\ R(e_0, e_j)e_k &= -g(\mathcal{E}_{13}(e_k), (\mathcal{E} + \mathcal{E}_{24})(e_j)) e_0, \\ R(e_i, e_j)e_0 &= 0, \\ R(e_i, e_j)e_k &= g(\mathcal{E}_{13}(e_j), e_k)\mathcal{E}_{13}(e_i) - g(\mathcal{E}_{13} e_i, e_k)\mathcal{E}_{13}(e_j). \end{aligned}$$

Therefore, if $\mathcal{E}$ is skew-symmetric then the Lie group is flat. □

Examples

Let $\mathfrak{g}$ be a quasi Abelian Lie algebra determined by an endomorphism $\mathcal{E}$. Consider the unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$. The split exact sequence of Lie algebras $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ lifts to a split exact sequence of Lie groups $0 \to (\mathbb{R}^7, +) \to G \to (G/\mathbb{R}^7 = \mathbb{R}, +) \to 0$. This splitting and the conjugation ϵ on G by the elements of $(\mathbb{R}, +)$, provide an isomorphism $(\mathbb{R}, +) \ltimes_\epsilon (\mathbb{R}^7, +)$. Therefore $\frac{d}{dt}\big|_{t=s} d(\epsilon(t)) = s\mathcal{E}$, so that $d(\epsilon(t)) = \exp(t\mathcal{E}) = \epsilon(t)$, using that the exponential of $\mathbb{R}^7$ is the identity.

A nilmanifold with a balanced and a locally conformal balanced Spin(7) structure.

Define the endomorphism of $\mathbb{R}^7$

$$\mathcal{E} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and consider the quasi Abelian Lie algebra $(\mathbb{R}^8, \mathcal{E})$. Note that this is a nilpotent Lie algebra with $(de^0, de^1, de^2, \ldots, de^7) = (0, e^{02}, 2e^{03}, e^{04}, e^{05}, e^{06}, e^{07}, 0)$, where $d\beta(X, Y) = -\beta([X, Y])$.

The symmetric part of $\mathcal{E}$ is traceless and the eigenvalues of its skew-symmetric part are of the form $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$. Therefore, Theorem 8.7 guarantees the existence of an invariant Spin(7) structure of type balanced and other invariant Spin(7) structure which is mixed. To avoid computing the eigenvalues, one can observe that if we take the standard form Ω_0 in $\mathbb{R}^8$, determined by a spinor η, it holds that $e_2 e_3 \eta = -e_4 e_5 \eta = -e_6 e_7 \eta$ and $e_1 e_2 \eta = -e_5 e_6 \eta$. Therefore, if we identify the skew-symmetric part of $\mathcal{E}$ with the 2-form $\gamma = e^{23} + \frac{1}{2}(e^{12} + e^{45} + e^{56} + e^{67})$, we get that $\gamma\eta = 0$. Therefore, the 4-form associated to the structure is the standard Ω_0.

On some nilpotent Lie algebras, the existence of a lattice is guaranteed by general theorems [17]. This case is really simple and we can compute it explicitly. The matrix of the endomorphism $\exp(t\mathcal{E})$ is:

$$\begin{pmatrix} 1 & -t & t^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & \frac{t^2}{2} & -\frac{t^3}{6} \\ 0 & 0 & 0 & 0 & 1 & -t & \frac{t^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we define $\Gamma = 6\mathbb{Z}e_0 \times_\epsilon (\mathbb{Z}e_1 \times \mathbb{Z}e_2 \times \cdots \times \mathbb{Z}e_7)$, then G/Γ is a compact manifold with $\pi_1(G/\Gamma) = \Gamma$ which inherits both a balanced and a mixed Spin(7) invariant structure.

Moreover, we claim that G/Γ is not diffeomorphic to $Q \times S^1$ for any 7-dimensional submanifold Q. Since $b_1(G/\Gamma) = 2$, it is sufficient to prove that if a nilmanifold G'/Γ' is diffeomorphic to $Q \times S^1$ then, $b_1(Q \times S^1) \geq 3$, or equivallently, $b_1(Q) \geq 2$. This assertion turns out to be true because we can check that Q is homotopically equivalent to a nilmanifold. On the one hand, Q is an Eilenberg-MacLance space $K(1, \pi_1(Q))$, because G' is contractible. On the other hand a group is isomorphic to a lattice of a nilpotent Lie group if and only if it is nilpotent, torsion-free and finitely generated [21, Theorem 2.18]. Since $\Gamma' = \pi_1(G'/\Gamma') = \pi_1(Q) \times \mathbb{Z}$, both $\pi_1(Q)$ and Γ' verify the conditions listed above. Thus, there is a nilmanifold Q' such that $\pi_1(Q') = \pi_1(Q)$, which is an Eilenberg-MacLane space $K(1, \pi_1(Q))$. Therefore, Q' and Q have the same homotopy type and $b_1(Q) = b_1(Q') \geq 2$, because Q' is a nilmanifold.

This nilmanifold has also a stict locally conformally balanced Spin(7) structure (see Definition 9.1), a structure of mixed type with closed and non-exact Lee form. According to Theorem 8.6, if we show that there exists a spinor η and $\lambda \neq 0$ such that $\gamma\eta = -\lambda e^7\eta$, then the Lee form of the Spin(7) structure determined by η is μe^7 for some $\mu \in \mathbb{R}$ and $d(\mu e^7) = 0$. Take the octonionic representation ρ, which extends to Cl_7 the map $\rho\colon \mathbb{R}^7 \to \mathrm{End}(\mathbb{R}^8)$, $\rho(v)(x) = vx$ where $\mathbb{R}^7$ is viewed as the imaginary part of the octonions.

The previous condition is then equivalent to $(\rho(e_7)\rho(\gamma) - \lambda \mathrm{Id})\eta = 0$ for some $\eta \in \mathbb{R}^8$, that is, $\lambda \neq 0$ is a real eigenvalue of $\rho(e_7)\rho(\gamma)$. Computing this condition we get:

1. The eigenvalue $\lambda_+ = \sqrt{3}$ has associated unit eigenvectors
   $\eta_+^1 = \frac{1}{\sqrt{15}}(0, -\sqrt{3}, 0, -\sqrt{3}, 0, 3, 0, 0)$ and
   $\eta_+^2 = \frac{1}{\sqrt{75}}(-\sqrt{3}, 0, 3\sqrt{3}, 0, -6, 0, 3, 0)$;
2. The eigenvalue $\lambda_- = -\sqrt{3}$ has associated unit eigenvectors
   $\eta_-^1 = \frac{1}{\sqrt{15}}(0, -\sqrt{3}, 0, \sqrt{3}, 0, 3, 0, 0)$ and
   $\eta_-^2 = \frac{1}{\sqrt{75}}(\sqrt{3}, 0, -3\sqrt{3}, 0, -6, 0, 3, 0)$.

The 4-form associated to η_+^1 is $\Omega_0 = e^0 \wedge \Psi + *\Psi$, where $*$ is the Hodge star of the canonical metric on $\mathbb{R}^7$ and:

$$\begin{aligned}\Psi = {} & e^{12} \wedge \left(-\frac{1}{5}e^3 - 2\frac{\sqrt{3}}{5}e^5 + 2\frac{\sqrt{3}}{5}e^7\right) - 2\frac{\sqrt{3}}{5}e^{13} \wedge (e^4 + e^6) \\ & - \frac{1}{5}e^{14} \wedge (3e^5 + 2e^7) - \frac{2}{5}e^{156} + \frac{3}{5}e^{167} \\ & - 2\frac{\sqrt{3}}{5}e^{23}(e^5 + e^7) + e^{246} + \frac{1}{5}e^{257} + \frac{1}{5}e^{34} \wedge (-2e^5 + 3e^7) \\ & - \frac{3}{5}e^{356} - \frac{2}{5}e^{367} - 2\frac{\sqrt{3}}{5}e^{457} + 2\frac{\sqrt{3}}{5}e^{567}.\end{aligned}$$

A compact manifold with a parallel and a mixed Spin(7) structure

Take the same spinor and basis of $\mathbb{R}^7$ as the previous example. Consider the skew-symmetric endomorphism such that $\mathcal{E}(e_2) = e_3$, $\mathcal{E}(e_4) = e_5$ and $\mathcal{E}(X) = 0$ on $\langle e_2, e_3, e_4, e_5\rangle^{\perp}$. The rank of this matrix is two and it is associated to $(0, 1, 1)$. Therefore, Theorem 8.7 guarantees the existence of a parallel invariant Spin(7) structure and other invariant Spin(7) structure which is mixed. The matrix of the endomorphism $\exp(t\mathcal{E}_2)$ in the previous basis is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & \sin(t) & 0 & 0 & 0 & 0 \\ 0 & -\sin(t) & \cos(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(t) & -\sin(t) & 0 & 0 \\ 0 & 0 & 0 & \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $t \in \pi\mathbb{Z}$, the previous matrix has integers coefficients so that $\gamma = \pi\mathbb{Z}e_0 \times_{\epsilon} (\mathbb{Z}e_1 \times \mathbb{Z}e_2 \times \cdots \times \mathbb{Z}e_7)$ is a subgroup. Moreover, G/Γ is a compact manifold with $\pi_1(G/\Gamma) = \Gamma$ and inherits from G both a parallel invariant Spin(7) structure and a mixed invariant one.

According to Remark 8.9, this manifold is flat. It is the mapping torus of $\exp(\pi\mathcal{E})\colon X \to X$, where X is a 7 torus. Indeed, since $\exp(\pi\mathcal{E})^2 = \mathrm{Id}$, the 8-torus is a 2-fold connected covering of G/Γ.

9. Balanced and locally conformally balanced structures on quasi Abelian Lie algebras

In this section we focus on invariant structures on quasi Abelian nilpotent Lie algebras. As Corollary 8.8 states, a locally conformally calibrated structure on a quasi Abelian nilpotent Lie algebra is automatically parallel. Indeed, if a quasi Abelian nilmanifold $(\mathbb{R}^8, \mathcal{E})$ admits an invariant parallel structure, then $\mathcal{E}$ is symmetric so that $(\mathbb{R}^8, \mathcal{E})$ is a torus. Therefore, we search for quasi Abelian nilpotent Lie algebras which admit a balanced structure. In addition, we introduce a special type of mixed structure, which we call locally conformally balanced and we analyze its existence on quasi Abelian nilpotent Lie algebras.

A Spin(7) structure on a Riemannian manifold is locally conformally balanced if at each contractible neighbourhood there is a conformal change of the metric whose associated Spin(7) structure is balanced, that is:

Definition 9.1. A Spin(7) structure is locally conformally balanced if its Lee form is closed. In addition, if the Lee form is not exact, we say that it is strict locally conformally balanced.

Of course, balanced and locally conformally calibrated structures are locally conformally balanced. The interesting case is when the structure is mixed and the Lee form is not exact.

Remark 9.2. Our spinorial approach enables us to characterise locally conformally balanced structures. Let $V \in TM$ such that $D\eta = V\eta$. We are going to compute the Dirac operator of V as an element of $\mathrm{Cl}(M)$, that is, $DV = \sum_{i=1}^{7} X_i \nabla_{X_i} V$ for an orthonormal local basis $(X_0, \ldots, X_7)$:

$$\begin{aligned} DV &= \sum_{i,j=0}^{7} g(\nabla_{X_i} V, X_j) X_i X_j \\ &= \sum_{i<j} \big(g(\nabla_{X_i} V, X_j) - g(\nabla_{X_j} V, X_i)\big) X_i X_j - \sum_{i=0}^{7} g(\nabla_{X_i} V, X_i) \\ &= 2\sum_{i<j} dV^*(X_i, X_j) X_i X_j + \mathrm{div}(V). \end{aligned}$$

Since the Lee form is $\frac{8}{7}V^*$, the structure is locally conformally balanced if and only if $DV = \mathrm{div}(V)$.

If we focus on invariant structures on unimodular quasi Abelian Lie algebras $(\mathbb{R}^8, \mathcal{E})$ the problem of determining whether or not the Lee form of a structure is homothetic to a unitary 1-form θ becomes an eigenvalue problem.

As Theorem 8.6 states, the Lee form of the Spin(7) structure defined by η is homothetic to a 1-form $E^* \in \mathfrak{h}^*$ determined by the equation $\gamma \cdot_{\mathfrak{h}} \eta = 3E \cdot_{\mathfrak{h}} \eta$, where γ is the 2-form associated to the skew-symmetric part of $\mathcal{E}$. For a unitary 1-form θ, the condition $\gamma \cdot_{\mathfrak{h}} \eta = -\lambda\theta \cdot_{\mathfrak{h}} \eta$ for some $\lambda \neq 0$ is equivalent to $(\theta\gamma - \lambda) \cdot_{\mathfrak{h}} \eta = 0$, that is, the endomorphism of Δ^+ given by $\phi \longmapsto \theta \cdot_{\mathfrak{h}} \gamma \cdot_{\mathfrak{h}} \phi$ has λ as an eigenvalue.

This argument enables us to prove that if a nilpotent quasi Abelian $\mathfrak{g}$ Lie algebra is decomposable that is, $\mathfrak{g} = \mathfrak{g}' \oplus \langle W \rangle$ as Lie algebras then W is homothetic to the Lee form of a Spin(7) structure.

Lemma 9.3. *Let* $(\mathbb{R}^8, \mathcal{E})$ *be a unimodular quasi Abelian Lie algebra. If* $\mathcal{E}_{24} \neq 0$ *and* $\mathcal{E}_{24}(W) = 0$ *for some non-zero vector* $W \in \mathbb{R}^7$*, then* $(\mathbb{R}^8, \mathcal{E})$ *admits a spinor* η *whose associated* Spin(7) *structure has Lee form homothetic to* W^*.

In particular, if a decomposable quasi Abelian Lie algebra $\mathfrak{g} = \mathfrak{g}' \oplus \langle W \rangle$ *is non-Abelian and nilpotent, it admits a* Spin(7) *structure whose Lee form is homothetic to* W^*.

Proof. First note that $\gamma \in \Lambda^2 \langle W^* \rangle^{\perp}$ so that $(W^*\gamma) \cdot_{\mathfrak{h}} \phi = (W^* \wedge \gamma) \cdot_{\mathfrak{h}} \phi$ for all $\phi \in \Delta^+$. But the product by an element of $\Lambda^3(\mathbb{R}^7)^*$ is a symmetric endomorphism of Δ^+. Therefore, the condition $\mathcal{E}_{24} \neq 0$ guarantees the existence of a non-zero eigenvalue of the product by $W^* \wedge \gamma$ and therefore, a spinor η whose associated Spin(7) structure has Lee form homothetic to W^*.

Suppose that a decomposable quasi Abelian Lie algebra $\mathfrak{g} = \mathfrak{g}' \oplus \langle W \rangle$ is non-Abelian and nilpotent. It is straightforward to check that W lies on the Abelian ideal $\mathfrak{h}$. Thus, if we take a metric g with e_0 perpendicular to $\mathfrak{h}$ and W perpendicular to $\mathfrak{h} \cap \mathfrak{g}'$ then $(\mathfrak{g}, g)$ is identified in terms of Lemma 8.3 with $(\mathbb{R}^8, \mathcal{E})$ such that $\mathcal{E}_{24}(W) = 0$. In addition, $\mathcal{E}_{24} \neq 0$ because the algebra is non-Abelian and nilpotent. □

A more detailed analysis of the eigenvalue problem provides the following result:

Lemma 9.4. *Let* $(\mathbb{R}^8, \mathcal{E})$ *be a unimodular quasi Abelian Lie algebra and suppose that* $\mathcal{E}_{24}$ *is associated to* $(\lambda_1, \lambda_2, \lambda_3)$ *as in Lemma 8.5 with* $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ *with* $\lambda_3 \leq \lambda_1 + \lambda_2$. *Then, each* $\theta \in (\mathbb{R}^7)^*$ *is homothetic to the Lee form of a* Spin(7) *structure.*

Proof. Let θ in $(\mathbb{R}^7)^*$ and take $(X_1, \dots, X_7)$ an orthonormal oriented basis of $\mathbb{R}^7$ such that:

$$\begin{aligned}\gamma =& \lambda_1 X_1^* \wedge X_2^* + \lambda_2 X_3^* \wedge X_4^* + \lambda_3 X_5^* \wedge X_6^*,\\ \theta^\sharp =& \mu_1 X_1 + \mu_3 X_3 + \mu_5 X_5 + \mu_7 X_7.\end{aligned}$$

Let ρ be the representation of Cl_7 constructed as in the proof of Theorem 8.7. The characteristic polynomial of the matrix $\rho(\theta^\sharp)\rho(\lambda_1 X_1 X_2 + \lambda_2 X_3 X_4 + \lambda_3 X_5 X_6)$ is $p(t) = (t^4 + a_2 t^2 + a_1 t + a_0)^2$, where:

$$\begin{aligned}a_0 =& -(\lambda_1 + \lambda_2 + \lambda_3)(-\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3),\\ a_1 =& 8\lambda_1\lambda_2\lambda_3\mu_7,\\ a_2 =& -2(\mu_1^2(-\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + \mu_3^2(\lambda_1^2 - \lambda_2^2 + \lambda_3^2) + \mu_5^2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2)\\ &+ \mu_7^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)).\end{aligned}$$

Therefore, we have the following:

1. If $\lambda_3 < \lambda_1 + \lambda_2$ then $a_0 < 0$ so that $p(t)$ has a non-zero eigenvalue.
2. If $\lambda_3 = \lambda_1 + \lambda_2$ then $p(t) = t^2(t^3 + a_2 t + a_1)^2$ with $a_2 < 0$. Therefore, p has a non-zero eigenvalue. □

9.1. Quasi Abelian nilpotent Lie algebras and Spin(7) structures

Quasi Abelian nilpotent Lie algebras are classified by the adjoint action a vector which is transverse to the Abelian ideal. Therefore, each isomorphism type is associated to a unique element of $\mathcal{N}_7/\mathrm{GL}(7)$, where $\mathcal{N}_7$ is the set of nilpotent matrices of $\mathbb{R}^7$ and GL(7) acts via conjugation. The orbits are matrices with the same Jordan normal form, and therefore, classified by the dimension those blocks. We have 15 types that we will denote by $(n_1, \dots, n_k)$ with $n_i \leq n_{i+1}$ and $\sum_{i=1}^k n_i = 7$.

We are going to determine those which admit an invariant balanced Spin(7) structure or an invariant Spin(7) structure with closed Lee form in the cohomology of the algebra. Note that the last type will induce strict locally conformally balanced structures on each nilmanifold associated to the algebra because the cohomology of the algebra is isomorphic to the cohomology of any associated nilmanifold. In this context we say that the Spin(7) structure of a nilpotent Lie algebra is strict locally conformally balanced.

First of all observe that the Abelian Lie algebra only admits parallel invariant structures. Next, we analyze the algebras $L_3 \oplus A_5$ and $L_4 \oplus A_4$, where L_3 denotes the Lie algebra of the 3-dimensional Heisenberg group, L_4 the unique irreducible 4-dimensional nilpotent Lie algebra and A_j the j-dimensional Abelian Lie algebra. In our previous notation, they are associated to $(2, 1, 1, 1, 1, 1)$ and $(3, 1, 1, 1, 1)$.

Proposition 9.5. *The Lie algebras $A_4 \oplus L_3$ and $A_3 \oplus L_4$, do not admit any balanced structure. However, both of them admit strict locally conformal balanced structures.*

Proof. Let $\mathfrak{h}$ be an Abelian ideal of $\mathfrak{g}$ and let g be a metric. Take a vector e_0 orthogonal to $\mathfrak{h}$ and denote $\mathcal{E} = ad(e_0)|_{\mathfrak{h}}$. We write in both cases the endomorphism $\mathcal{E}$ with respect to a suitable orthonormal basis $(e_1, \dots, e_7)$ of $\mathfrak{h}$:

1. If $\mathfrak{g} = A_4 \oplus L_3$ we can suppose that $\ker(\mathcal{E}) = \langle e_1, \dots, e_6 \rangle$ and $\mathcal{E}(e_7) = -\lambda e_6$ for some $\lambda \neq 0$. Thus, $\gamma = \lambda e^{67}$ so that $\gamma\eta \neq 0$ for all η;
2. If $\mathfrak{g} = A_3 \oplus L_4$ we can suppose that $\ker(\mathcal{E}) = \langle e_1, \dots, e_5 \rangle$, $\mathcal{E}(e_6) = -\lambda_1 e_5$ and $\mathcal{E}(e_7) = -\lambda_2 e_4 - \lambda_3 e_5 - \lambda_4 e_6$, where $\lambda_1\lambda_4 \neq 0$. Thefore, $\gamma = \lambda_1 e^{56} + (\lambda_2 e^4 + \lambda_3 e^5) \wedge e^7 + \lambda_4 e^{67}$. The spinor $\lambda_4 e^{67}\eta$ is non-zero and orthogonal to $(\lambda_1 e^{56} + (\lambda_2 e^4 + \lambda_3 e^5) \wedge e^7)\eta$. Therefore, $\gamma\eta \neq 0$ for all η.

The existence of strict locally conformally balanced structures is a consequence of Lemma 9.3. □

Now, we focus in types associated to matrices with two distinct Jordan blocks of dimension greater than 1, which are $(5, 2)$, $(4, 3)$, $(4, 2, 1)$, $(3, 3, 1)$, $(3, 2, 2)$, $(3, 2, 1, 1)$, $(2, 2, 2, 1)$ and $(2, 2, 1, 1, 1)$.

Proposition 9.6. *Nilpotent quasi Abelian algebras with two distinct Jordan blocks of dimension greater than* 1 *admit a metric with a both a balanced and a strict locally conformally balanced* Spin(7) *structure.*

Proof. Let e_0 be transversal to the Abelian ideal $\mathfrak{h}$ and observe that there is a splitting $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ with $\dim \mathfrak{h}_2 \in \{2, 3\}$, $\mathfrak{h}_3$ Abelian and $ad(e_0)(\mathfrak{h}_i) \subset \mathfrak{h}_i$. Observe that $\mathfrak{h}_3$ may be $\{0\}$. We are going to define a metric g which makes e_0 perpendicular to $\mathfrak{h}$ and $g|_{\mathfrak{h}} = g_1 + g_2 + g_3$ where g_i are metrics on $\mathfrak{h}_i$.

Therefore $\mathcal{E}$ is going to be a block matrix $\begin{pmatrix} \mathcal{E}_1 & 0 & 0 \\ 0 & \mathcal{E}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with respect to an orthonormal basis adapted to the splitting of $\mathfrak{h}$.

Obviously, for each $\lambda > 0$ there exists an upper triangular matrix of dimension 2 or 3, conjugated to a Jordan block of dimension 2 or 3, such that its skew-symmetric part has eigenvalues $\pm\lambda i$ or $0, \pm\lambda i$. Therefore, once obtained the eigenvalues of the skew-symmetric part of $\mathcal{E}_1$ with respect to any metric g_1 we can change g_2 so that g satisfies the balanced condition.

Except for $(2, 2, 1, 1, 1)$, $(3, 2, 1, 1)$, $(3, 3, 1)$ we can change g_1 so that the skew-symmetric part of $\mathcal{E}_1$ has two distinct eigenvalues. Lemma 9.4 ensures the existence of strict locally conformally balanced structures. Finally, the algebras considered except $(5, 2)$ and $(4, 3)$ are verify that $\mathcal{E}_{24}(W) = 0$ for some non-zero vector W so that Lemma 9.3 ensures the existence of a strict locally conformally balanced structure associated to the metric that we have previously defined. □

Remark 9.7. A similar construction ensures the existence of metrics without associated balanced structures which admit strict locally conformally balanced structures.

Finally we analyze the case of the algebras associated to (4, 1, 1, 1), (5, 1, 1), (6, 1), (7).

Proposition 9.8. *The quasi Abelian nilpotent Lie algebras associated to* (4, 1, 1, 1)*,* (5, 1, 1)*,* (6, 1)*,* (7) *have both a balanced and a strict locally conformally balanced* Spin(7) *structure.*

Proof. Lemma 9.3 guarantees the existence of strict locally conformally balanced structures in the algebras associated to (4, 1, 1, 1), (5, 1, 1), (6, 1). We are going to prove that all of them admit a balanced structure giving an explicit example of an structure of the type $(\mathbb{R}^8, \mathcal{E})$. In the case of (7), the range of $\mathcal{E}_{24}$ will be 6 so that the same metric also admits a strict locally conformally balanced Spin(7) structure as Lemma 9.4 states. Define:

$$\mathcal{E} = -\begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1+a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $a = b = c = 0$, the Lie algebra is associated to (4, 1, 1, 1), if $a = b = 0$ and $c \neq 0$ to (5, 1, 1), if $a = 0$, $b \geq 0$ and $c \neq 0$ to (6, 1) and if $a \geq 0$, $b \geq 0$ and $c \neq 0$, to (7). The skew-symmetric part of $\mathcal{E}$ is associated to the 2-form:

$$\gamma = ae^{12} + be^{23} + ce^{25} + ce^{34} - e^{45} - e^{47} + (1+a)e^{56} + (1+b)e^{67}.$$

Take the spinor η whose associated 4-form is the standard Spin(7) form Ω_0. We have that $\gamma\eta = 0$ as a consequence of the following equalities:

$$e^{67}\eta = e^{45}\eta,\ e^{56}\eta = e^{47}\eta,\ e^{34}\eta = -e^{25}\eta,\ e^{23}\eta = -e^{67}\eta,\ e^{12}\eta = -e^{56}\eta.$$ □

We have proven the following result:

Theorem 9.9.

1. *Every invariant* Spin(7) *structure on the Abelian Lie algebra* A_8 *is parallel;*
2. *The Lie algebras* $\mathfrak{g} = A_5 \oplus L_3$ *or* $\mathfrak{g} = A_3 \oplus L_4$ *admit strict locally conformally balanced invariant structures. They do not admit invariant balanced structures;*
3. *The rest of quasi Abelian nilpotent Lie algebras admit a balanced structure and a strict locally conformally balanced structure.*

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