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Atomic decomposition and interpolation via the complex method for mixed norm Bergman spaces on tube domains over symmetric cones

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Abstract. Starting from an adapted Whitney decomposition of tube domains in $\mathbb{C}^n$ over irreducible symmetric cones of $\mathbb{R}^n$, we prove an atomic decomposition theorem in mixed norm weighted Bergman spaces on these domains. We also characterize the interpolation space via the complex method between two mixed norm weighted Bergman spaces.

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1. Introduction

The context and the notation are those of [20]. Let Ω be an irreducible symmetric cone of rank r in a vector space V of dimension n, endowed with an inner product $(.|.)$ for which Ω is self-dual.

We recall that Ω induces in V a structure of Euclidean Jordan algebra with identity $\mathbf{e}$ such that

$$\overline{\Omega} = \left\{x^2 : x \in V\right\}.$$

Let $\{c_1, \cdots, c_r\}$ be a fixed Jordan frame in V and

$$V = \bigoplus_{1 \le i \le j \le r} V_{i,j}$$

be its associated Peirce decomposition of V. We denote by

$$\Delta_1(x), \cdots, \Delta_r(x)$$

the principal minors of $x \in V$ with respect to the fixed Jordan frame $\{c_1, \cdots, c_r\}$. More precisely, $\Delta_k(x)$, $k = 1, \cdots, r$ is the determinant of the projection $P_k x$ of

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x, in the Jordan subalgebra $V^{(k)} = \bigoplus_{1\leq i\leq j\leq k} V_{i,j}$. We have $\Delta_k(x) > 0$, $k = 1, \cdots, r$, when $x \in \Omega$, and the determinant Δ of the Jordan algebra is given by $\Delta = \Delta_r$. The generalized power function on Ω is defined as

$$\Delta_{\mathbf{s}}(x) = \Delta_1^{s_1-s_2}(x)\Delta_2^{s_2-s_3}(x)\cdots\Delta_r^{s_r}(x), \quad x \in \Omega, \quad \mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r.$$

A typical example of an irreducible symmetric cone is the Lorentz cone Λ_n, $n \geq 3$, of $\mathbb{R}^n$, *i.e.* the set defined by

$$\Lambda_n = \left\{(y_1, \cdots, y_n) \in \mathbb{R}^n : y_1 + y_2 > 0 \text{ and } y_1^2 - \cdots - y_n^2 > 0\right\},$$

which is a symmetric cone of rank $r = 2$ with $\Delta_1(y) = y_1 + y_2$ and its determinant function is given by the Lorentz form

$$\Delta(y) = y_1^2 - \cdots - y_n^2.$$

We adopt the following standard notation:

$$n_k = 2(k-1)\frac{\frac{n}{r}-1}{r-1} \quad \text{and} \quad m_k = 2(r-k)\frac{\frac{n}{r}-1}{r-1}.$$

For $\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{R}^r$ and ρ real, the notation $\mathbf{s} + \rho$ will stand for the vector whose coordinates are $s_k + \rho$, $k = 1, \cdots, r$. For $1 \leq p \leq \infty$ and $1 \leq q < \infty$, let $L^{p,\,q}_{\mathbf{s}}$ denote the mixed norm Lebesgue space constisting of measurable functions F on T_Ω such that

$$\|F\|_{L^{p,\,q}_{\mathbf{s}}} = \left(\int_\Omega \|F(\cdot + iy)\|_p^q \Delta_{\mathbf{s}-\frac{n}{r}}(y)dy\right)^{\frac{1}{q}} < \infty$$

where

$$\|F(\cdot + iy)\|_p = \left(\int_V |F(x+iy)|^p dx\right)^{\frac{1}{p}}$$

(with the obvious modification if $p = \infty$). The *mixed norm weighted Bergman space* $A^{p,q}_{\mathbf{s}}$ is the (closed) subspace of $L^{p,\,q}_{\mathbf{s}}$ consisting of holomorphic functions. Following [18], $A^{p,q}_{\mathbf{s}}$ is non-trivial if and only if $s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. When $p = q$, we write $L^{p,\,q}_{\mathbf{s}} = L^p_{\mathbf{s}}$ and $A^{p,\,q}_{\mathbf{s}} = A^p_{\mathbf{s}}$ which are respectively the usual weighted Lebesgue space and the usual weighted Bergman space. Moreover, when $p = q = 2$ the orthogonal projector $P_{\mathbf{s}}$ from the Hilbert space $L^2_{\mathbf{s}}$ onto its closed subspace $A^2_{\mathbf{s}}$ is called *weighted Bergman projector*. It is well known that $P_{\mathbf{s}}$ is the integral operator on $L^2_{\mathbf{s}}$ given by the formula

$$P_{\mathbf{s}}F(z) = \int_{T_\Omega} B_{\mathbf{s}}(z, u+iv)F(u+iv)\Delta_{\mathbf{s}-\frac{n}{r}}(v)dudv,$$

where

$$B_{\mathbf{s}}(z, u+iv) = d_{\mathbf{s}} \Delta_{-\mathbf{s}-\frac{n}{r}} \left(\frac{z-u+iv}{i} \right)$$

is the reproducing kernel on $A^2_{\mathbf{s}}$, called *weighted Bergman kernel* of T_Ω. Precisely, $\Delta_{-\mathbf{s}-\frac{n}{r}}(\frac{x+iy}{i})$ is the holomorphic determination of the $(-\mathbf{s} - \frac{n}{r})$-power which reduces to the function $\Delta_{-\mathbf{s}-\frac{n}{r}}(y)$ when $x = 0$.

The atomic decomposition problem for Bergman spaces in tube domains over irreducible symmetric cones had been studied in the eighties by Coifman and Rochberg [17] by using the L^p-continuity properties of the Bergman projectors. But they assumed in their work that the Bergman projection is L^p bounded in these domains for all $p \in]1, \infty[$, $p \neq 2$, which happens not be true. In fact, the question of whether $P_{\mathbf{s}}$ extends or not as a bounded operator on $L^p_{\mathbf{s}}(T_\Omega)$ for $p \neq 2$ has attracted a lot of attention in recent years (see [1–6, 11, 12, 18] and the references therein). So far, only partial answers are known. For a brief review of these results, we suppose first that $\mathbf{s} = (s, \cdots, s)$, $s > \frac{n}{r} - 1$. In this case, we write P_s instead of $P_{\mathbf{s}}$ and L^p_s instead of $L^p_{\mathbf{s}}$. The following conjecture has been stated in [5] for these domains.

Conjecture. The Bergman projector P_s admits a bounded extension to $L^p_s(T_\Omega)$ if and only if

$$p'_s < p < p_s := \frac{s + \frac{2n}{r} - 1}{\frac{n}{r} - 1} - \frac{(1-s)_+}{\frac{n}{r} - 1};$$

where p'_s is the conjugate exponent of p_s.

The conjecture concerns the "if" part. A weaker result was proved in general in [4] and [3], namely the boundedness for

$$1 + \frac{s + \frac{n}{r} - 1}{s} < p < 1 + \frac{s + \frac{n}{r} - 1}{\frac{n}{r} - 1}.$$

Still for $\mathbf{s} = (s, \cdots, s)$, $s > \frac{n}{r} - 1$, we suppose next that p may be different from q. A more general conjecture was stated in [4] for tube domains over Lorentz cones, which proposed a necessary and sufficient condition on the couples (p, q) for P_s be bounded on $L^{p,q}_s(T_{\Lambda_n})$. Finally, the case of the tube domains over Lorentz cones has been completely settled after the works [4, 11, 22] and the recent proof of the l^2-decoupling conjecture by Bourgain and Demeter [13] in [12] and [7]. More precisely, the proofs of the two conjectures were announced in [12] and given in detail in [7].

We suppose finally that $\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{R}^r$, with $s_j > \frac{n_j}{2}$, $j = 1, \cdots, r$. Weighted $L^p_{\mathbf{s}}$ estimates were also considered for the tube domains over general symmetric cones. In this case, Debertol [19] and Nana and Trojan [24] established necessary conditions and sufficient conditions for the $L^p_{\mathbf{s}}$-boundedness of the weighted Bergman projector $P_{\mathbf{s}}$. To state their sufficient conditions, according to [19], we denote

$$p_{\mathbf{s}} = 1 + \min_{j=1,\dots,r} \frac{s_j + \frac{n}{r}}{\left((r-j)\frac{d}{2} - s_j\right)_+};$$
$$q_{\mathbf{s}}(p) = \min\{p, p'\} \min_{j=1,\dots,r} \left(1 + \frac{s_j - (j-1)\frac{d}{2}}{(r-j)\frac{d}{2}}\right).$$

Theorem 1.1 ([19] and [24]). *The weighted Bergman projector $P_{\mathbf{s}}$ extends to a bounded operator on $L^{p,q}_{\mathbf{s}}$ whenever $\frac{1}{q_{\mathbf{s}}(p)} < \frac{1}{q} < 1 - \frac{1}{q_{\mathbf{s}}(p)}$ in the following two cases:*

(i) $s_j > \frac{n_j}{2}$, $j = 1, \cdots r$ *and* $1 \le p < p_{\mathbf{s}}$ [19]*;*
(ii) $s_j > \frac{n}{r} - 1$, $j = 1, \cdots r$ *and* $1 \le p \le \infty$ [24].

Restricting again to tube domains over Lorentz cones ($r = 2$), Theorem 1.1 was also extended in [7] to other values of p and q again using the l^2-decoupling inequality of Bourgain and Demeter [13].

Our aim here is to develop atomic decomposition and interpolation of mixed norm weighted Bergman spaces by using L^p-continuity properties of the Bergman projectors. We shall then be assuming the boundedness of the Bergman projectors in the statement of our results.

Our first result is an atomic decomposition theorem for functions in mixed norm weighted Bergman spaces on tube domains over symmetric cones. It generalizes the result of [8] for usual weighted Bergman spaces on tube domains over symmetric cones and the result of [25] for mixed norm weighted Bergman spaces on the upper half-plane (the case $n = r = 1$).

Theorem A. *Let $\mathbf{s}$ be a vector of $\mathbb{R}^r$ such that $s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. Assume that $P_{\mathbf{s}}$ extends to a bounded operator on $L^{p,q}_{\mathbf{s}}$. Then there is a sequence of points $\{z_{l,j} = x_{l,j} + iy_j\}_{l\in\mathbb{Z},\, j\in\mathbb{N}}$ in T_Ω and a positive constant C such that the following assertions hold:*

(i) *For every sequence $\{\lambda_{l,j}\}_{l\in\mathbb{Z},\, j\in\mathbb{N}}$ such that*

$$\sum_j \left(\sum_l |\lambda_{l,j}|^p\right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) < \infty,$$

the series

$$\sum_{l,j} \lambda_{l,j} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) B_{\mathbf{s}}(z, z_{l,j})$$

is convergent in $A^{p,q}_{\mathbf{s}}$. Moreover, its sum F satisfies the inequality

$$\|F\|^q_{A^{p,q}_{\mathbf{s}}} \le C \sum_j \left(\sum_l |\lambda_{l,j}|^p\right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j);$$

(ii) *Every function $F \in A_{\boldsymbol{s}}^{p,q}$ may be written as*

$$F(z) = \sum_{l,j} \lambda_{l,j} \Delta_{\boldsymbol{s}+\frac{nq}{rp}}(y_j) B_{\boldsymbol{s}}\big(z, z_{l,j}\big),$$

with

$$\sum_j \left(\sum_l |\lambda_{l,j}|^p \right)^{\frac{q}{p}} \Delta_{\boldsymbol{s}+\frac{nq}{rp}}\big(y_j\big) \leq C \|F\|^q_{A_{\boldsymbol{s}}^{p,q}}.$$

Our second result is an interpolation theorem between mixed norm weighted Bergman spaces. It generalizes the result of [8] for usual weighted Bergman spaces.

For $\mathbf{s} \in \mathbb{R}^r$, we adopt the notation:

$$q_{\mathbf{s}} = \min_{1 \leq k \leq r} \left(1 + \frac{s_k - \frac{n_k}{2}}{\frac{m_k}{2}} \right).$$

Theorem B.

(1) *Let $\boldsymbol{s}_0, \boldsymbol{s}_1 \in \mathbb{R}^r$ be such that $(s_0)_k, (s_1)_k > \frac{n}{r} - 1, \quad k = 1, \cdots, r$ and let $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 < \infty$. Assume that there exists $\mathbf{t} \in \mathbb{R}^r$, $t_k > \frac{n}{r} - 1, \quad k = 1, \cdots, r$ such that $P_{\mathbf{t}}$ is bounded on $L_{\mathbf{s}_i}^{p_i,q_i}$, $i = 0, 1$. Then for every $\theta \in (0, 1)$, we have*

$$\left[A_{\boldsymbol{s}_0}^{p_0,q_0}, \;\; A_{\boldsymbol{s}_1}^{p_1,q_1} \right]_\theta = A_{\boldsymbol{s}}^{p,q}$$

with equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $\frac{\boldsymbol{s}}{q} = \frac{(1-\theta)\boldsymbol{s}_0}{q_0} + \frac{\theta \boldsymbol{s}_1}{q_1}$;

(2) *Let $\mathbf{s} \in \mathbb{R}^r$ be such that $s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. Assume that $P_{\boldsymbol{s}}$ extends to a bounded operator on $L_{\boldsymbol{s}}^{p_i,q_i}$, $i = 0, 1$ for $1 \leq p_0, p_1 \leq \infty$ and $1 < q_0, q_1 < \infty$. Then for every $\theta \in (0, 1)$, we have*

$$\left[A_{\boldsymbol{s}}^{p_0,q_0}, \;\; A_{\boldsymbol{s}}^{p_1,q_1} \right]_\theta = A_{\boldsymbol{s}}^{p,q}$$

with equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$;

(3) *Let $\mathbf{s} \in \mathbb{R}^r$ be such that $(s)_k > \frac{n}{r} - 1$, $k = 1, \cdots, r$, let $1 \leq p_0 < p_1 \leq \infty$ and let q_0, q_1 be such that $1 \leq q_0 < q_{\boldsymbol{s}} \leq q_1$. We assume that $P_{\boldsymbol{s}}$ extends to a bounded operator on $L_{\boldsymbol{s}}^{p_1,q_1}$. Then for some values of $\theta \in (0, 1)$, we have*

$$\left[A_{\boldsymbol{s}}^{p_0,q_0}, \;\; A_{\boldsymbol{s}}^{p_1,q_1} \right]_\theta = A_{\boldsymbol{s}}^{p,q}$$

with equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

We then mention that the assumption on $P_{\mathbf{t}}$ in assertion (1) of Theorem B is true whenever $1 \le q < q_{\mathbf{s}_i}$ for every $i = 0, 1$; in fact, it holds when all the components t_j, $j = 1, \cdots, r$ are sufficiently large. For $q > 1$, this is just the case $\alpha = 0$ in [7, Theorem 3.8], while the case $q = 1$ is an easy exercise (*cf. e.g.*, [9, Theorem II.7]).

The plan of this paper is as follows. In Section 2, we overview some preliminaries and useful results about symmetric cones and tube domains over symmetric cones. In Section 3, we study atomic decomposition of mixed norm Bergman spaces and we prove a more precise statement of Theorem A. In Section 4, we study interpolation via the complex method between mixed norm weighted Bergman spaces and we prove Theorem B. In particular we give a more precise statement of assertion (3) of this theorem (Theorem 4.6) and we ask an open question. A final remark will point out a connection between the two main theorems of the paper (Theorem A and Theorem B).

For $\mathbf{s} = (s, \cdots, s)$ real, Theorem A and Theorem B were presented in the PhD dissertation of the second author [21].

As usual, given two positive quantities A and B, the notation $A \lesssim B$ (respectively $A \gtrsim B$) means that there is an absolute positive constant C such that $A \le CB$ (respectively $A \ge CB$). When $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$ and say A and B are equivalent.

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2. Preliminaries

Materials of this section are essentially from [20]. We give some definitions and useful results.

Let Ω be an irreducible symmetric cone of rank r in a real vector space V of dimension n endowed with the structure of Euclidean Jordan algebra with identity $\mathbf{e}$. In particular, Ω is self-dual with respect to the inner product

$$(x|y) = \mathbf{tr}(xy)$$

on V.

2.1. Group action

Let $G(\Omega)$ be the group of linear transformations of the cone Ω and G its identity component. By definition, the subgroup G of $G(\Omega)$ is a semi-simple Lie group which acts transitively on Ω. This gives the identification $\Omega \sim G/K$, where $K := \{g \in G : \quad g \cdot \mathbf{e} = \mathbf{e}\}$ is a maximal compact subgroup of G. More precisely,

$$K = G \cap O(V),$$

where $O(V)$ is the orthogonal group in V. Furthermore, there is a solvable subgroup T of G acting simply transitively on Ω. That is, every $y \in \Omega$ can be written uniquely as $y = t \cdot \mathbf{e}$, for some $t \in T$.

Let $\{c_1, \cdots, c_r\}$ in $\mathbb{R}^n$ be a fixed Jordan frame in V (that is, a complete system of idempotents) and

$$V = \bigoplus_{1\le i\le j\le r} V_{i,j}$$

be its associated Peirce decomposition of V where

$$\begin{cases} V_{i,i} = \mathbb{R}c_i \\ V_{i,j} = \left\{x \in V : \ c_i x = c_j x = \frac{1}{2}x\right\} \text{ if } i < j. \end{cases}$$

We have $\mathbf{e} = \sum_{1\le i\le r} c_i$. Then the solvable Lie group T factors as the semidirect product $T = NA = AN$ of a nilpotent subgroup N consisting of lower triangular matrices, and an abelian subgroup A consisting of diagonal matrices. The latter takes the explicit form

$$A = \left\{ P(a) : \quad a = \sum_{i=1}^{r} a_i c_i, \ \ a_i > 0 \right\},$$

where P is the quadratic representation of $\mathbb{R}^n$. This also leads to the Iwasawa and Cartan decompositions of the semisimple Lie group G:

$$G = NAK \quad \text{and} \quad G = KAK.$$

Still following [20], we shall denote by $\Delta_1(x), \cdots, \Delta_r(x)$ the principal minors of $x \in V$, with respect to the fixed Jordan frame $\{c_1, \cdots, c_r\}$. These are invariant functions under the group N,

$$\Delta_k(nx) = \Delta_k(x),$$

where $n \in N$, $x \in V$, $k = 1, \cdots, r$, and satisfy a homogeneity relation under A,

$$\Delta_k(P(a)x) = a_1^2 \cdots a_k^2 \Delta_k(x),$$

if $a = a_1 c_1 + \cdots + a_r c_r$.

The determinant function $\Delta(y) = \Delta_r(y)$ is also invariant under K, and moreover, satisfies the formula

$$\Delta(gy) = \Delta(ge)\Delta(y) = \mathrm{Det}^{\frac{r}{n}}(g)\Delta(y), \ \forall g \in G, \ \forall y \in \Omega \tag{2.1}$$

where Det is the usual determinant of linear mappings. It follows from this formula that the measure $\frac{d\xi}{\Delta^{\frac{n}{r}}(\xi)}$ is G-invariant in Ω.

Finally, we recall the following version of Sylvester's theorem.

$$\Omega = \left\{x \in \mathbb{R}^n : \quad \Delta_k(x) > 0, \qquad k = 1, \cdots, r\right\}.$$

2.2. Geometric properties

With the identification $\Omega \sim G/K$, the cone can be regarded as a Riemannian manifold with the G-invariant metric defined by

$$\langle \xi, \eta \rangle_y := \left(t^{-1}\xi | t^{-1}\eta \right)$$

if $y = t \cdot \mathbf{e}$ with $t \in T$ and ξ and η are tangent vectors at $y \in \Omega$. We shall denote by d_Ω the corresponding invariant distance, and by $B_\delta(\xi)$ the associated ball centered at ξ with radius δ. Note that for each $g \in G$, the invariance of d_Ω implies that $B_\delta(g\xi) = gB_\delta(\xi)$. We also note that:

- On compact sets of $\mathbb{R}^n$ contained in Ω, the invariant distance d_Ω is equivalent to the Euclidean distance in $\mathbb{R}^n$;
- The associated balls B_δ in Ω are relatively compact in Ω.

We also need the following crucial invariance properties of d_Ω and Δ_k, obtained in [3,4].

Lemma 2.1. *Let $\delta_0 > 0$. Then there is a constant $\gamma 0$ depending only on δ_0 and Ω such that for every $0 < \delta \le \delta_0$ and for $\xi, \xi' \in \Omega$ satisfying $d_\Omega(\xi, \xi') \le \delta$ we have*

$$\frac{1}{\gamma} \le \frac{\Delta_k(\xi)}{\Delta_k(\xi')} \le \gamma, \ \ \forall k = 1, \cdots, r.$$

Lemma 2.2. *Let $\delta_0 > 0$ be fixed. Then there exist two constants $\eta_1 > \eta_2 > 0$, depending only on δ_0 and Ω, such that for every $0 < \delta \le \delta_0$ we have*

$$\{|\xi - e| < \eta_2\delta\} \subset B_\delta(e) \subset \{|\xi - e| < \eta_1\delta\}.$$

The next corollary is an easy consequence of the previous lemma for $\delta_0 = 1$.

Corollary 2.3. *There is a positive constant γ such that for every $\delta \in (0, 1)$ such that $\eta_1\delta < 1$, we have*

$$B_\delta(\xi) \subset \{y \in \Omega : \quad y - \gamma\xi \in \Omega\}$$

for all $\xi \in \Omega$.

2.3. Gamma function in Ω

The generalized gamma function in Ω is defined in terms of the generalized power functions by

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-(\xi|e)} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta^{\frac{n}{r}}(\xi)} \qquad \big(\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{R}^r\big).$$

This integral is known to converge absolutely if and only if $\Re e\, s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. In this case,

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{n-r}{2}} \prod_{k=1}^{r} \Gamma\left(s_k - \frac{n_k}{2}\right),$$

where Γ is the classical gamma function. We shall denote $\Gamma_\Omega(\mathbf{s}) = \Gamma_\Omega(s)$ when $\mathbf{s} = (s, \cdots, s)$. In view of [20], the Laplace transform of a generalized power function is given for all $y \in \Omega$ by

$$\int_\Omega e^{-(\xi|y)} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta^{\frac{n}{r}}(\xi)} = \Gamma_\Omega(\mathbf{s}) \Delta_{\mathbf{s}}\left(y^{-1}\right)$$

for each $\mathbf{s} \in \mathbb{C}^r$ such that $\Re e\, s_k > \frac{n_k}{2}$ for all $k = 1, \cdots, r$. We recall that $y^{-1} = t^{*-1} \cdot \mathbf{e}$ whenever $y = t \cdot \mathbf{e}$ with $t \in T$. Here $t^\star$ denotes the adjoint of the transformation $t \in T$ with respect to the inner product $(\cdot|\cdot)$.

The power function $\Delta_{\mathbf{s}}(y^{-1})$ can be expressed in terms of the rotated Jordan frame $\{c_r, \cdots, c_1\}$. Indeed if we denote by $\Delta_k^*, k = 1, \cdots, r$, the principal minors with respect to the rotated Jordan frame $\{c_r, \cdots, c_1\}$ then

$$\Delta_{\mathbf{s}}\left(y^{-1}\right) = \left[\Delta^*_{\mathbf{s}^*}(y)\right]^{-1}, \ \forall \mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r.$$

Here $\mathbf{s}^* = (s_r, \cdots, s_1)$.

2.4. Bergman distance on the tube domain T_Ω

Following [3], we define a matrix function $\{g_{j,k}\}_{1\le j,k\le n}$ on T_Ω by

$$g_{j,k}(z) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log B(z, z)$$

where B is the unweighted Bergman kernel of T_Ω,, *i.e.*, $B = B_{\mathbf{s}}$ with $\mathbf{s} = (\frac{n}{r}, \cdots \frac{n}{r})$. The map

$$T_\Omega \ni z \mapsto \mathcal{H}_z$$

with

$$\mathcal{H}_z(u, v) = \sum_{j,k=1}^{n} g_{j,k}(z) u_k \bar{v}_k, \ \ u = (u_1, \cdots, u_n), \ v = (v_1, \cdots, v_n) \in \mathbb{C}^n$$

defines a Hermitian metric on $\mathbb{C}^n$, called the Bergman metric. The Bergman length of a smooth path $\gamma : [0, 1] \to T_\Omega$ is given by

$$l(\gamma) = \int_0^1 \left\{\mathcal{H}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\right\}^{\frac{1}{2}} dt$$

and the Bergman distance $d(z_1, z_2)$ between two points z_1, z_2 of T_Ω is

$$d(z_1, z_2) = \inf_{\gamma} l(\gamma)$$

where the infimum is taken over all smooth paths $\gamma : [0, 1] \to T_\Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. It is well known that the Bergman distance d is equivalent to the Euclidean distance on the compact sets of $\mathbb{C}^n$ contained in T_Ω and the Bergman balls in T_Ω are relatively compact in T_Ω. Next, we again denote by $\mathbb{R}^n$ the group of translations by vectors in $\mathbb{R}^n$. Then the group $\mathbb{R}^n \times T$ acts simply transitively on T_Ω and the Bergman distance d is invariant under the automorphisms of $\mathbb{R}^n \times T$.

2.5. A Whitney decomposition of the tube domain T_Ω

In the sequel, the Bergman ball in T_Ω with centre at z and radius η will be denoted $\mathbf{B}_\eta(z)$.

Lemma 2.4. *There exists a constant $R > 1$ such that for all $\eta \in (0, 1)$ and $z_0 = x_0 + iy_0 \in T_\Omega$, the following inclusions hold:*

$$\left\{x + iy \in T_\Omega : \ \left\|g^{-1}(x - x_0)\right\| < \frac{\eta}{R} \quad \text{and } y \in B_{\frac{\eta}{R}}(y_0)\right\} \subset \mathbf{B}_\eta(z_0),$$
$$\mathbf{B}_\eta(z_0) \subset \left\{x + iy \in T_\Omega : \ \left\|g^{-1}(x - x_0)\right\| < R\eta \text{ and } y \in B_{R\eta}(y_0)\right\},$$

where g is the element of T satisfying $g \cdot e = y_0$.

Proof. From the invariance under translations and automorphisms of T we have that

$$g^{-1}(x - x_0) + ig^{-1}y \in \mathbf{B}_\eta(ie)$$

for all $x + iy \in \mathbf{B}_\eta(z_0)$. We recall that the Bergman distance d and the Euclidean distance d_{Eucl} are equivalent on compact sets of $\mathbb{C}^n$ contained in T_Ω. So there exists a constant $R > 1$ such that

$$\frac{1}{R} d(X + iY, ie) < d_{\text{Eucl}}(X + iY, ie) < R d(X + iY, ie)$$

for all $X + iY \in \overline{\mathbf{B}_1(ie)}$. The proof of the lemma follows from the following equivalence

$$d_{\text{Eucl}}(X_1 + iY_1, X_2 + iY_2) \approx \max(\|X_1 - X_2\|, \|Y_1 - Y_2\|)$$

and the equivalence given in Lemma 2.2 between d_Ω and the Euclidean distance $|| \cdot ||$ in $\mathbb{R}^n$ on compact sets of $\mathbb{R}^n$ contained in Ω. □

The starting point of our analysis is the following Whitney decomposition of the cone Ω which was obtained, *e.g.*, in [3,4].

Lemma 2.5. *There is a positive integer N such that given $\delta \in (0,1)$, one can find a sequence of points $\{y_j\}_{j=1,2,\dots}$ in Ω with the following properties:*

(i) *The balls $B_{\frac{\delta}{2}}(y_j)$ are pairwise disjoint;*
(ii) *The balls $B_\delta(y_j)$ cover Ω;*
(iii) *Each point of Ω belongs to at most N of the balls $B_\delta(y_j)$.*

Definition 2.6. The sequence $\{y_j\}$ is called a *δ-lattice of* Ω.

Our goal is to obtain an atomic decomposition theorem for holomorphic functions in $A_{\mathbf{s}}^{p,q}$ spaces. To this end, we need to derive a suitable version of the classical Whitney decomposition of $\mathbb{R}^n$. Let $\{y_j\}$ be a *δ-lattice of* Ω and let $g_j \in T$ be such that $g_j \cdot \mathbf{e} = y_j$. Let $R > 1$ be a constant like in Lemma 2.4. We adopt the following notation:

$$I_{l,j} = \left\{ x \in \mathbb{R}^n : \ \left\| g_j^{-1}(x - x_{l,j}) \right\| < \frac{\delta}{R} \right\}$$

$$I'_{l,j} = \left\{ x \in \mathbb{R}^n : \ \left\| g_j^{-1}(x - x_{l,j}) \right\| < \frac{\delta}{2R} \right\}$$

where $\{x_{l,j}\}$ is a sequence in $\mathbb{R}^n$ to be determined.

From Lemma 2.4 we have immediately the following.

Remark 2.7. For the constant $R > 1$ of Lemma 2.4, the following inclusion holds

$$I_{l,j} + iB_{\frac{\delta}{R}}(y_j) \subset \mathbf{B}_\delta(x_{lj} + iy_j).$$

Lemma 2.8. *Let $\delta \in (0,1)$. There exist a positive constant $R > 1$, a positive integer N and a sequence of points $\{x_{l,j}\}_{l\in\mathbb{Z},\, j\in\mathbb{N}}$ in $\mathbb{R}^n$ such that the following hold:*

(i) *$\{I_{l,j}\}_l$ form a cover of $\mathbb{R}^n$;*
(ii) *$\{I'_{l,j}\}_l$ are pairwise disjoint;*
(iii) *For each j, every point of $\mathbb{R}^n$ belongs to at most N balls $I_{l,j}$.*

Proof. Fix j in $\mathbb{N}$ and define the collection $\mathcal{A}_j$ of sets in $\mathbb{R}^n$ by

$$\mathcal{A}_j = \left\{ A \subset \mathbb{R}^n : \ \forall x,\, y \in A, x \neq y, \ \left\| g_j^{-1}(x-y) \right\| \geq \frac{\delta}{R} \right\}.$$

Clearly the collection $\mathcal{A}_j$ is non empty. Indeed the sets $\{\frac{\delta}{R} y_j, 0_{\mathbb{R}^n}\}$ are members of $\mathcal{A}_j$. Furthermore, the collection $\mathcal{A}_j$ is partially ordered with respect to inclusion.

Let $\mathcal{C}$ be a totally ordered subcollection of $\mathcal{A}_j$. We set $F = \cup_{A\in\mathcal{C}} A$. Given two distinct elements x, y of F, there are two members A_1 and A_2 of $\mathcal{C}$ such that $x \in A_1$ and $y \in A_2$. But either $A_1 \subset A_2$ or $A_2 \subset A_1$. So we have either $x, y \in A_1$ or $x, y \in A_2$. Hence $||g_j^{-1}(x-y)|| \geq \frac{\delta}{R}$. This shows that F is a member of $\mathcal{A}_j$. In other words, the collection $\mathcal{A}_j$ is inductive. An application of Zorn's lemma then gives that the collection $\mathcal{A}_j$ has a maximal member E_j. We write $E_j = \{x_{l,j}\}_{l\in L_j}$.

To prove assertion (ii), consider l and k such that $l \neq k$ and assume that $I'_{l,j} \cap I'_{k,j}$ contain at least an element x. Then

$$\left\| g_j^{-1}(x_{l,j} - x_{k,j}) \right\| \leq \left\| g_j^{-1}(x_{l,j} - x) \right\| + \left\| g_j^{-1}(x - x_{k,j}) \right\| < \frac{\delta}{2R} + \frac{\delta}{2R} = \frac{\delta}{R}.$$

This would contradict the property that $\{x_{l,j},\ x_{k,j}\}$ is a subset of E_j, which is a member of $\mathcal{A}_j$.

For assertion (i), let us suppose $\cup_{l \in L_j} I_{l,j} \neq \mathbb{R}^n$. Then there exists $\xi_j \in \mathbb{R}^n$ such that $\xi_j \notin \cup_{l \in L_j} I_{l,j}$. Clearly the set $E_j \cup \{\xi_j\}$ is a member of $\mathcal{A}_j$. This would contradict the maximality of E_j in $\mathcal{A}_j$. This completes the proof of assertion (i).

To prove assertion (iii), we fix j. Given $x \in \mathbb{R}^n$, it follows from assertion (i) that there exists a subset $L_j(x)$ of L_j such that

$$x \in \bigcap_{l \in L_j(x)} I_{l,j}.$$

We will show that there is a positive integer N independent of δ such that Card $L_j(x) \leq N$ for all $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. It follows from Lemma 2.4 and Lemma 2.5 that for every $l \in L_j$,

$$\mathbf{B}_{\frac{\delta}{2R^2}}(x_{l,j} + iy_j) \subset I'_{l,j} \times B_{\frac{\delta}{2R}}(y_j).$$

So the balls $\mathbf{B}_{\frac{\delta}{2R^2}}(x_{l,j} + iy_j), l \in L_j$ are pairwise disjoint since $R > 1$. Moreover, for every $l \in L_j(x)$, we have

$$\begin{aligned}\mathbf{B}_{\frac{\delta}{2R^2}}(x_{l,j} + iy_j) &\subset \left\{ \xi + i\sigma \in T_\Omega : \left\| g_j^{-1}(\xi - x) \right\| < \frac{3\delta}{2R} \quad \text{and} \quad \sigma \in B_{\frac{\delta}{2R}}(y_j) \right\} \\ &\subset \mathbf{B}_{\frac{3\delta}{2}}(x + iy_j).\end{aligned}$$

For the first inclusion, we applied the triangle inequality. We obtain

$$\bigcup_{l \in L_j(x)} \mathbf{B}_{\frac{\delta}{2R}}(x_{l,j} + iy_j) \subset \mathbf{B}_{\frac{3\delta R}{2}}(x + iy_j).$$

We call m the invariant measure on T_Ω given by

$$dm(\xi + i\sigma) = \Delta^{-\frac{2n}{r}}(\sigma) d\xi d\sigma.$$

We conclude that

$$\text{Card}\, L_j(x) \leq \frac{m\left(\mathbf{B}_{\frac{3\delta R}{2}}(i\mathbf{e})\right)}{m\left(\mathbf{B}_{\frac{\delta}{2R}}(i\mathbf{e})\right)},$$

because

$$m\left(\bigcup_{l \in L_j(x)} \mathbf{B}_{\frac{\delta}{2R}}(x_{l,j} + iy_j) \right) = \text{Card}\, L_j(x) \times m\left(\mathbf{B}_{\frac{\delta}{2R}}(i\mathbf{e})\right) \leq m\left(\mathbf{B}_{\frac{3\delta R}{2}}(i\mathbf{e})\right).$$

We finally prove that the collection $\{I_{l,j}\}_{l,j}$ is countable. It suffices to show that for each j, the collection $\{I_{l,j}\}_{l\in L_j}$ is countable. We fix j. To every set $I'_{l,j}$, we assign a point of $\mathbb{Q}^n$ belonging to $I'_{l,j}$. Since $\bigcap_l I'_{l,j} = \emptyset$, this defines a one-to-one correspondence from the collection $\{I'_{l,j}\}_{l\in L_j}$ to a subset of $\mathbb{Q}^n$. This shows that the collection $\{I'_{l,j}\}_{l\in L_j}$ is at most countable. Moreover the collection $\{I_{l,j}\}_{l\in L_j}$ which has the same cardinal as the collection $\{I'_{l,j}\}_{l\in L_j}$ is infinite: the proof is elementary since $\bigcup_{l\in L_j} I_{l,j} = \mathbb{R}^n$ is unbounded. The proof of the lemma is complete. □

Remark 2.9. We just proved in Lemma 2.8 that for each $j = 1, 2, \ldots,$ the index set L_j is countable. In analogy with the one-dimensional case [25], we took $L_j = \mathbb{Z}$ in the statement of Lemma 2.8 and in the statement of Theorem A.

2.6. A δ-lattice in T_Ω

Definition 2.10. The sequence $\{z_{lj} = x_{lj} + iy_j\}_{l\in\mathbb{Z}, j\in\mathbb{N}}$ defined in Lemma 2.8 will be called a *δ-lattice in T_Ω*.

We have the following lemma.

Lemma 2.11. *Let $\{z_{l,j} = x_{l,j} + iy_j\}_{l\in\mathbb{Z}, j\in\mathbb{N}}$ be a δ-lattice in T_Ω. There exists a positive constant $C = C(\delta, R)$ such that for all $l \in \mathbb{Z}$, $j \in \mathbb{N}$, the following hold.*

(a) $\displaystyle\int_{I_{l,j}} dx \simeq \Delta^{\frac{n}{r}}(y_j)$;

(b) $\displaystyle\int_{\mathbb{R}^n} \sum_{l\in L_j} \chi_{\{x\in I_{l,j}:\ \boldsymbol{d}(x+iy,w)<1\}}(x)dx \leq C\Delta^{\frac{n}{r}}(y_j),\ \ \forall y \in B_\delta(y_j), \forall w \in T_\Omega.$

Proof. We denote Det the usual determinant of an endomorphism of $\mathbb{R}^n$.

(a) We set $u = g_j^{-1}(x - x_{lj})$. Then

$$\int_{I_{l,j}} dx = \int_{\|u\|<\frac{\delta}{R}} \operatorname{Det}\left(g_j\right)du$$
$$= \Delta^{\frac{n}{r}}\left(y_j\right)\int_{\|u\|<\frac{\delta}{R}} du = C\Delta^{\frac{n}{r}}\left(y_j\right).$$

This proves assertion (a).

(b) By assertion (iii) of Lemma 2.8, we have

$$\sum_{l\in L_j} \chi_{I_{l,j}}(x) \leq N$$

for every j. Then

$$\int_{\mathbb{R}^n} \sum_{l\in L_j} \chi_{\{x\in I_{l,j}:\ \mathbf{d}(x+iy,w)<1\}}(x)dx \leq N\int_{\{x\in\mathbb{R}^n:\ x+iy\in\mathbf{B}_1(w)\}} dx.$$

We set $w = u + iv$. By Lemma 2.4, we have the implication

$$x + iy \in \mathbf{B}_1(w) \Rightarrow \left|\left|g^{-1}(x-u)\right|\right| < R \quad \text{and} \quad y \in B_R(v)$$

with $g \cdot \mathbf{e} = v$. So

$$\int_{\{x\in\mathbb{R}^n:\ x+iy\in\mathbf{B}_1(w)\}} dx \le \int_{\{x\in\mathbb{R}^n:\ ||g^{-1}(x-u)||<R\}} dx = C\,\mathrm{Det}\,(g) = C\Delta^{\frac{n}{r}}(v).$$

But $d_\Omega(y, y_j) < \delta$ and $d_\Omega(y, v) < R$. This implies that $d(v, y_j) < \delta + R$. Henceforth $\Delta^{\frac{n}{r}}(v) \le C\Delta^{\frac{n}{r}}(y_j)$ by Lemma 2.1. This gives assertion (b). □

3. Atomic decomposition

3.1. The sampling theorem

We first record the following lemma (see, *e.g.*, [3]).

Lemma 3.1. *Let* $1 \le p < \infty$. *Given* $\delta \in (0, 1)$, *there exists a positive constant* C *such that, for each holomorphic function* F *in* T_Ω *we have:*

(i) $|F(z)|^p \le C\delta^{-2n} \int_{\boldsymbol{B}_\delta(z)} |F(u+iv)|^p \frac{du\,dv}{\Delta^{\frac{2n}{r}}(v)}$;

(ii) *If* $d(z, \zeta) < \delta$ *then*

$$|F(z) - F(\zeta)|^p \le C\delta^p \int_{\boldsymbol{B}_1(z)} |F(u+iv)|^p \frac{du\,dv}{\Delta^{\frac{2n}{r}}(v)}.$$

For the second lemma, the reader should refer to [4, Lemma 4.5].

Lemma 3.2. *Suppose* $\delta \in (0, 1)$ *and* $1 \le p, q < \infty$. *There exists a positive constant* C *such that*

$$\|F(\cdot + iy)\|_p^q \le C \int_{B_\delta(y)} \|F(\cdot + iv)\|_p^q \frac{dv}{\Delta^{\frac{n}{r}}(v)} \tag{3.1}$$

for every holomorphic function F *on* T_Ω *and every* $y \in \Omega$.

The following is our sampling theorem.

Theorem 3.3. *Let* $\delta \in (0, 1)$ *satisfy the assumption of Corollary* 2.3 *and let* $\{z_{l,j} = x_{l,j} + iy_j\}_{l\in\mathbb{Z},\, j\in\mathbb{N}}$ *be a* δ*-lattice in* T_Ω. *Let* $1 \le p, q < \infty$ *and let* $\mathbf{s} \in \mathbb{R}^r$ *be such that* $s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. *There exists a positive constant* $C_\delta = C_\delta(\mathbf{s}, p, q)$ *such that for every* $F \in A_{\mathbf{s}}^{p,q}$, *we have*

$$\sum_j \left(\sum_l \left|F\left(z_{l,j}\right)\right|^p \right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}\left(y_j\right) \le C_\delta \|F\|_{A_{\mathbf{s}}^{p,q}}^q. \tag{3.2}$$

Moreover, if δ is small enough, the converse inequality

$$\|F\|^q_{A^{p,q}_{\mathbf{s}}} \leq C_\delta \sum_j \left(\sum_l \left|F(z_{l,j})\right|^p \right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) \tag{3.3}$$

is also valid.

Proof. From Lemma 3.1 we have

$$\left|F(z_{l,j})\right|^p \leq C\delta^{-2n} \int_{\mathbf{B}_{\frac{\delta}{2R^2}}(z_{l,j})} |F(u+iv)|^p \frac{du\,dv}{\Delta^{\frac{2n}{r}}(v)}. \tag{3.4}$$

It follows from the inclusion $\mathbf{B}_{\frac{\delta}{2R^2}}(z_{l,j}) \subset \{u+iv : u \in I'_{l,j},\ v \in B_{\frac{\delta}{2R}}(y_j)\}$ that

$$\left|F(z_{l,j})\right|^p \leq C\delta^{-2n} \int_{I'_{l,j}} du \int_{B_{\frac{\delta}{2R}}(y_j)} |F(u+iv)|^p \frac{dv}{\Delta^{\frac{2n}{r}}(v)}. \tag{3.5}$$

From the equivalence of $\Delta(v)$ and $\Delta(y_j)$ whenever $v \in B_{\frac{\delta}{2R}}(y_j)$, we obtain that

$$\left|F(z_{l,j})\right|^p \leq \frac{C\delta^{-2n}}{\Delta^{\frac{2n}{r}}(y_j)} \int_{I'_{l,j}} du \int_{B_{\frac{\delta}{2R}}(y_j)} |F(u+iv)|^p\, dv. \tag{3.6}$$

Next, a successive application of Lemma 2.8, Corollary 2.3 and the non-increasing property of the function $\Omega \ni v \mapsto \|F(\cdot + iv)\|^p_p$ gives the existence of a positive constant γ such that

$$\begin{aligned}
\sum_{l \in L_j} \left|F(z_{l,j})\right|^p &\leq \frac{C\delta^{-2n}}{\Delta^{\frac{2n}{r}}(y_j)} \int_{\mathbb{R}^n} du \int_{B_{\frac{\delta}{2R}}(y_j)} |F(u+iv)|^p\, dv \\
&= \frac{C\delta^{-2n}}{\Delta^{\frac{2n}{r}}(y_j)} \int_{B_{\frac{\delta}{2R}}(y_j)} \|F(\cdot + iv)\|^p_p\, dv \\
&\leq \frac{C\delta^{-2n}}{\Delta^{\frac{2n}{r}}(y_j)} \int_{B_{\frac{\delta}{2R}}(y_j)} \left\|F\left(\cdot + i\gamma y_j\right)\right\|^p_p\, dv \\
&\leq \frac{C\delta^{-2n}}{\Delta^{\frac{n}{r}}(y_j)} \left\|F\left(\cdot + i\gamma y_j\right)\right\|^p_p.
\end{aligned}$$

Finally, we obtain

$$\sum_j \left(\sum_l \left|F(z_{l,j})\right|^p \right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) \leq C_\delta^{\frac{q}{p}} \sum_j \left\|F\left(\cdot + i\gamma y_j\right)\right\|^q_p \Delta_{\mathbf{s}}(y_j). \tag{3.7}$$

We define the holomorphic function F_γ by

$$F_\gamma(x+iy) = F(\gamma(x+iy)).$$

By Lemma 3.2, we get

$$\begin{aligned}\|F(\cdot+i\gamma y_j)\|_p^q &= \gamma^{\frac{nq}{p}}\|F_\gamma(\cdot+iy_j)\|_p^q \\ &\leq C\gamma^{\frac{nq}{p}}\int_{B_{\frac{\delta}{2R^2}}(y_j)}\|F_\gamma(\cdot+iy)\|_p^q\frac{dy}{\Delta^{\frac{n}{r}}(y)}.\end{aligned}\tag{3.8}$$

It follows from (3.8), Lemma 2.5 and the equivalence of $\Delta(y)$ and $\Delta(y_j)$ whenever $y\in B_{\frac{\delta}{2R^2}}(y_j)$ that

$$\begin{aligned}\sum_j\|F(\cdot+i\gamma y_j)\|_p^q\Delta_{\mathbf{s}}(y_j) &\leq C\gamma^{\frac{nq}{p}}\int_\Omega\|F_\gamma(\cdot+iy)\|_p^q\Delta_{\mathbf{s}-\frac{n}{r}}(y)\,dy \\ &= C\int_\Omega\|F(\cdot+i\gamma y)\|_p^q\Delta_{\mathbf{s}-\frac{n}{r}}(y)dy.\end{aligned}$$

Moreover, taking $v=\gamma y$ we obtain

$$\sum_j\|F(\cdot+i\gamma y_j)\|_p^q\Delta_{\mathbf{s}}(y_j)\leq C(\gamma,\mathbf{s},p,q)\int_\Omega\|F(\cdot+iv)\|_p^q\Delta_{\mathbf{s}-\frac{n}{r}}(v)\,dv.\tag{3.9}$$

So the estimate (3.2) is a direct consequence of (3.7) and (3.9).

Conversely, a successive application of Lemma 2.8, the triangle inequality and assertion a) of Lemma 2.11 gives

$$\begin{aligned}\|F(\cdot+iy)\|_p^p &\leq C_p\left\{\sum_{l\in L_j}\int_{I_{l,j}}|F(x+iy)-F(z_{l,j})|^p\,dx+\sum_{l\in L_j}|F(z_{l,j})|^p\int_{I_{l,j}}dx\right\} \\ &\leq C_p\left\{\sum_{l\in L_j}\int_{I_{l,j}}|F(x+iy)-F(z_{l,j})|^p\,dx+\sum_{l\in L_j}|F(z_{l,j})|^p\Delta^{\frac{n}{r}}(y_j)\right\},\end{aligned}$$

for all $y\in\Omega$. In the sequel, for fixed $y\in\Omega$, we set

$$K_j(w) = \int_{\mathbb{R}^n}\sum_{l\in L_j}\chi_{\{x\in I_{l,j}:\, d(x+iy,w)<1\}}(x)dx$$

and we write

$$\begin{aligned}N_{p,q}(F) = &\int_{y\in\Omega}\sum_{j\in\mathbb{N}}\chi_{B_\delta(y_j)}(y) \\ &\times\left(\int_{v\in\Omega}\int_{\mathbb{R}^n}K_j(u+iv)|F(u+iv)|^p\chi_{d_\Omega(y,v)}(v)\frac{du\,dv}{\Delta(v)^{\frac{2n}{r}}}\right)^{\frac{q}{p}}\Delta_{\mathbf{s}-\frac{n}{r}}(y)dy.\end{aligned}$$

Using assertion (ii) of Lemma 3.1, we obtain easily that

$$\|F\|^q_{A^{p,q}_{\mathbf{s}}} \leq \int_{\bigcup_j B_\delta(y_j)} \|F(\cdot + iy)\|^q_p \Delta_{\mathbf{s}-\frac{n}{r}}(y)dy$$
$$\leq C_{p,q}\delta^q N_{p,q}(F) + C_{p,q}\sum_j \left(\sum_l |F(z_{l,j})|^p\right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}\big(y_j\big).$$

To prove (3.3) it suffices to establish the following inequality:

$$N_{p,q}(F) \leq C\|F\|^q_{A^{p,q}_{\mathbf{s}}}.$$

To this end, first observe that by assertion (b) of Lemma 2.11, we have

$$K_j(w) \leq C\Delta^{\frac{n}{r}}\big(y_j\big), \quad \forall y \in B_\delta\big(y_j\big), \quad \forall w \in \Omega.$$

Now by Lemma 2.1, we have the equivalence $\Delta(v) \sim \Delta(y_j) \sim \Delta(y)$ whenever $v \in B_R(y)$ and $y \in B_\delta(y_j)$ with equivalence constants independent of δ. This combined with an application of assertion (iii) of Lemma 2.5 gives that

$$N_{p,q}(F) \leq CN \int_\Omega \left(\int_{d(v,y)<R} \|F(\cdot + iv)\|^p_p \frac{dv}{\Delta^{\frac{n}{r}}(v)}\right)^{\frac{q}{p}} \Delta_{\mathbf{s}-\frac{n}{r}}(y)dy.$$

Next, from the non-increasing property of the mapping $v \in \Omega \mapsto \|F(\cdot + iv)\|_p$, Corollary 2.3 and the G-invariance of the measure $\frac{dv}{\Delta^{\frac{n}{r}}(v)}$ on Ω, there exists a positive constant γ independent of δ such that

$$N_{p,q}(F) \leq CN \int_\Omega \|F(\cdot + i\gamma y)\|^q_p \Delta_{\mathbf{s}-\frac{n}{r}}(y)dy.$$

Finally, taking $t = \gamma y$ on the right hand side of the previous inequality, we obtain that

$$N_{p,q}(F) \leq C(\gamma)\|F\|^q_{A^{p,q}_{\mathbf{s}}}.$$

□

3.2. Proof of Theorem A

We can now prove the atomic decomposition theorem (Theorem A). Here is its more precise statement.

Theorem 3.4. *Let $\delta \in (0,1)$ and let $\{z_{l,j} = x_{l,j} + iy_j\}_{l\in\mathbb{Z},\, j\in\mathbb{N}}$ be a δ-lattice in T_Ω. Let $\mathbf{s}$ be a vector of $\mathbb{R}^r$ such that $s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. Assume that $P_{\mathbf{s}}$ extends to a bounded operator on $L^{p,q}_{\mathbf{s}}$. Then there exists a positive constant C such that the following two assertions hold:*

(i) *For every sequence $\{\lambda_{l,j}\}_{l\in\mathbb{Z},\, j\in\mathbb{N}}$ such that*

$$\sum_j \left(\sum_l |\lambda_{l,j}|^p\right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) < \infty,$$

the series

$$\sum_{l,j} \lambda_{l,j} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) B_{\mathbf{s}}(z, z_{l,j})$$

is convergent in $A^{p,q}_{\mathbf{s}}$. Moreover, its sum F satisfies the inequality

$$\|F\|^q_{A^{p,q}_{\mathbf{s}}} \le C_\delta \sum_j \left(\sum_l \left|\lambda_{l,j}\right|^p\right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j);$$

(ii) *For δ small enough, every function $F \in A^{p,q}_{\mathbf{s}}$ may be written as*

$$F(z) = \sum_{l,j} \lambda_{l,j} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) B_{\mathbf{s}}(z, z_{l,j}),$$

with

$$\sum_j \left(\sum_l \left|\lambda_{l,j}\right|^p\right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j) \le C_\delta \|F\|^q_{A^{p,q}_{\mathbf{s}}}.$$

Proof. Let $p \in [1,\infty]$, $q \in (1,\infty)$, and call p' and q' their conjugate exponents, *i.e.*, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let $\mathbf{s} \in \mathbb{R}^r$ such that $s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. Recall that (*cf.* [18]) if $P_{\mathbf{s}} : L^{p',q'}_{\mathbf{s}} \to A^{p',q'}_{\mathbf{s}}$ is bounded, then the dual space of $A^{p',q'}_{\mathbf{s}}$ identifies with $A^{p,q}_{\mathbf{s}}$ with respect to the pairing

$$\langle F, G\rangle_{\mathbf{s}} = \int_{T_\Omega} F(x+iy)\overline{G(x+iy)}\Delta_{\mathbf{s}-\frac{n}{r}}(y)\,dx\,dy.$$

Denote by $l^{p,q}_{\mathbf{s}}$ the space of complex sequences $\{\lambda_{l,j}\}_{l\in\mathbb{Z},\, j\in\mathbb{N}}$ such that

$$\|\{\lambda_{l,j}\}\|_{l^{p,q}_{\mathbf{s}}} = \left(\sum_j \left(\sum_l \left|\lambda_{l,j}\right|^p\right)^{\frac{q}{p}} \Delta_{\mathbf{s}+\frac{nq}{rp}}(y_j)\right)^{\frac{1}{q}} < \infty.$$

We have the duality $l_{\mathbf{s}}^{p,q} = (l_{\mathbf{s}}^{p',q'})'$ with respect to the pairing

$$\langle \lambda,\ \mu \rangle_{l_{\mathbf{s}}^{p',q'},\ l_{\mathbf{s}}^{p,q}} = \sum_{l,j} \lambda_{l,j} \overline{\mu}_{l,j} \Delta_{\mathbf{s}+\frac{n}{r}}\big(y_j\big).$$

Then from the first part of the sampling theorem, the operator

$$\begin{aligned} R :\ A_{\mathbf{s}}^{p',q'} &\to l_{\mathbf{s}}^{p',q'} \\ F &\mapsto RF = \big\{F\big(z_{l,j}\big)\big\}_{l\in\mathbb{Z},\ j\in\mathbb{N}} \end{aligned}$$

is bounded. So the adjoint operator R^* of R is also a bounded operator from $l_{\mathbf{s}}^{p,q}$ to $A_{\mathbf{s}}^{p,q}$. Its explicit formula is

$$R^*\big(\{\lambda_{l,j}\}\big)(z) = \sum_{l,j} \lambda_{l,j} \Delta_{\mathbf{s}+\frac{n}{r}}\big(y_j\big) B_{\mathbf{s}}\big(z, z_{l,j}\big).$$

This completes the proof of assertion (i).

From the second part of the sampling theorem, if δ is small enough, the adjoint operator $R^* : l_\nu^{p,q} \to A_\nu^{p,q}$ of R is onto. Moreover, we call $\mathcal{N}$ the subspace of $l_\nu^{p,q}$ consisting of all sequences $\{\lambda_{l,j}\}_{l\in\mathbb{Z},\ j\in\mathbb{N}}$ such that the mapping

$$z \mapsto \sum_{l,j} \lambda_{l,j} \Delta_{\mathbf{s}+\frac{n}{r}}\big(y_j\big) B_{\mathbf{s}}\big(z, z_{l,j}\big)$$

vanishes identically. Then the linear operator

$$\begin{aligned} \varphi : l_{\mathbf{s}}^{p,q}/\mathcal{N} &\to A_{\mathbf{s}}^{p,q} \\ \big\{\lambda_{l,j}\big\} &\mapsto \sum_{l,j} \lambda_{l,j} B_\nu\big(z, z_{l,j}\big) \Delta_{\mathbf{s}+\frac{n}{r}}\big(y_j\big) \end{aligned}$$

is a bounded isomorphism from the Banach quotient space $l_\nu^{p,q}/\mathcal{N}$ to $A_{\mathbf{s}}^{p,q}$. The inverse operator φ^{-1} of φ is continuous. This gives assertion (ii). □

4. Interpolation

In this section we determine the interpolation space via the complex method between two mixed norm weighted Bergman spaces.

4.1. Interpolation via the complex method between Banach spaces

Throughout this section we denote by S the open strip in the complex plane defined by

$$S = \{z = x + iy \in \mathbb{C} :\ \ 0 < x < 1\}.$$

Its closure $\overline{S}$ is

$$\overline{S} = \{z = x + iy \in \mathbb{C} : \ 0 \le x \le 1\}.$$

Let X_0 and X_1 be two compatible Banach spaces, *i.e.* they are continuously embedded in a Hausdorff topological space. Then $X_0 + X_1$ becomes a Banach space with the norm

$$\|f\|_{X_0+X_1} = \inf\left\{\|f_0\|_{X_0} + \|f_1\|_{X_1} : \quad f = f_0 + f_1, \ f_0 \in X_0, \ f \in X_1\right\}.$$

We will denote by $\mathcal{F}(X_0, X_1)$ the space of analytic mappings

$$\begin{aligned} f : \overline{S} &\to X_0 + X_1 \\ \zeta &\mapsto f_\zeta \end{aligned}$$

with the following properties:

(1) f is bounded and continuous on $\overline{S}$;
(2) f is analytic in S;
(3) For $k = 0, 1$ the function $y \mapsto f_{k+iy}$ is bounded and continuous from the real line into X_k.

The space $\mathcal{F}(X_0, X_1)$ is a Banach space with the following norm:

$$\|f\|_{\mathcal{F}} = \max\left(\sup_{\Re e\, \zeta=0} \|f_\zeta\|_{X_0}, \sup_{\Re e\, \zeta=1} \|f_\zeta\|_{X_1}\right).$$

If $\theta \in (0,1)$, the complex interpolation space $[X_0, X_1]_\theta$ is the subspace of $\mathcal{F}(X_0, X_1)$ consisting of holomorphic functions g on T_Ω such that $f_\theta = g$ for some $f \in \mathcal{F}(X_0, X_1)$. The space $[X_0, X_1]_\theta$ is a Banach space with the following norm:

$$\|g\|_\theta = \inf\left\{||f||_{\mathcal{F}(X_0,X_1)} : g = f_\theta\right\}.$$

Referring to [10] and [26] (*cf.* also [29]), the complex method of interpolation spaces is functorial in the following sense: if Y_0 and Y_1 denote two other compatible Banach spaces of measurable functions on T_Ω, then if

$$T : \ X_0 + X_1 \to Y_0 + Y_1$$

is a linear operator with the property that T maps X_0 boundedly into Y_0 and T maps X_1 boundedly into Y_1, then T maps $[X_0, X_1]_\theta$ boundedly into $[Y_0, Y_1]_\theta$, for each $\theta \in (0, 1)$. See [10] for more information about complex interpolation.

A classical example of interpolation via the complex method concerns $L^{p,q}$ spaces with a change of measures. We state it in our setting of a tube domain T_Ω over a symmetric cone Ω.

Theorem 4.1 ([16,27]). *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$. *Given two positive measurable functions* (*weights*) ω_0, ω_1 *on* Ω, *then for every* $\theta \in (0,1)$, *we have*

$$\big[L^{q_0}\big((\Omega, \omega_0(y)dy); L^{p_0}(\mathbb{R}^n, dx)\big), L^{q_1}\big((\Omega, \omega_1(y)dy); L^{p_1}(\mathbb{R}^n, dx)\big)\big]_\theta$$
$$= L^q\big((\Omega, \omega(y)dy); L^p(\mathbb{R}^n, dx)\big)$$

with equal norms, provided that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \qquad \omega^{\frac{1}{q}} = \omega_0^{\frac{1-\theta}{q_0}} \omega_1^{\frac{\theta}{q_1}}.$$

We finally record the Wolff reiteration theorem [23,28].

Theorem 4.2. *Let* A_1, A_2, A_3, A_4 *be compatible Banach spaces. Suppose* $[A_1, A_3]_\theta = A_2$ *and* $[A_2, A_4]_\varphi = A_3$. *Then*

$$[A_1, A_4]_\xi = A_2, \quad [A_1, A_4]_\psi = A_3$$

with $\xi = \frac{\theta\varphi}{1-\theta+\theta\varphi}$, $\psi = \frac{\varphi}{1-\theta+\theta\varphi}$.

4.2. A preliminary property of weighted Bergman projectors on tube domains over symmetric cones

We recall the following notation given in the introduction:

$$n_k = \frac{2(\frac{n}{r}-1)(k-1)}{r-1}, \qquad m_k = \frac{2(\frac{n}{r}-1)(r-k)}{r-1}$$

for every $k = 1, \cdots, r$. We recall the following result [7,24].

Proposition 4.3. *Let* $\mathbf{s} \in \mathbb{R}^n$ *be such that* $s_k > \frac{n_k}{2}$, $k = 1, \cdots, r$. *Assume that* $\mathbf{t} \in \mathbb{R}^n$ *and* $1 \le p, q < \infty$ *are such that* $P_\mathbf{t}$ *extends to a bounded operator on* $L^{p,q}_\mathbf{s}$. *Then* $P_\mathbf{t}$ *is the identity on* $A^{p,q}_\mathbf{s}$; *in particular* $P_\mathbf{t}(L^{p,q}_\mathbf{s}) = A^{p,q}_\mathbf{s}$.

4.3. Proof of Theorem B

(1) We adopt the following notation:

$$||g||_\theta = ||g||_{\left[L^{p_0,q_0}_{\mathbf{s}_0},\, L^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta}, \qquad ||g||^{\text{anal}}_\theta = ||g||_{\left[A^{p_0,q_0}_{\mathbf{s}_0},\, A^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta}.$$

It suffices to show the existence of a positive constant C such that the following two estimates are valid:

$$||g||^{\text{anal}}_\theta \le C||g||_{A^{p,q}_\mathbf{s}} \qquad \forall g \in A^{p,q}_\mathbf{s}; \tag{4.1}$$

$$||g||_{A^{p,q}_\mathbf{s}} \le ||g||^{\text{anal}}_\theta \qquad \forall g \in \left[A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta. \tag{4.2}$$

We first the estimate (4.1). By Theorem 4.1, we have

$$\left[L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta = L^{p,q}_{\mathbf{s}}$$

with equivalent norms, provided that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \qquad \frac{\mathbf{s}}{q} = \frac{(1-\theta)\mathbf{s}_0}{q_0} + \frac{\theta \mathbf{s}_1}{q_1}.$$

In particular, for every $g \in L^{p,q}_{\mathbf{s}}$, we have

$$||g||_{L^{p,q}_{\mathbf{s}}} \simeq ||g||_\theta = \inf\left\{||f||_{\mathcal{F}\left(L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1}\right)} : g = f_\theta\right\}. \tag{4.3}$$

We assume that there exists $\mathbf{t} \in \mathbb{R}^r$, $t_k > \frac{n}{r} - 1$, $\quad k = 1, \cdots, r$ such that $P_{\mathbf{t}}$ is bounded on $L^{p_i,q_i}_{\mathbf{s}_i}$, $i = 0, 1$, and hence from $L^{p,q}_{\mathbf{s}}$ onto $A^{p,q}_{\mathbf{s}}$. Then by Proposition 4.3, for every $g \in A^{p_i,q_i}_{\mathbf{s}_i}$, $i = 0, 1$ and for every $g \in A^{p,q}_{\mathbf{s}}$, we have $P_{\mathbf{t}} g = g$.

Now let $g \in A^{p,q}_{\mathbf{s}}$. For $f \in \mathcal{F}(L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1})$, we define the mapping

$$P_{\mathbf{t}} \circ f : \quad \overline{S} \to A^{p_0,q_0}_{\mathbf{s}_0} + A^{p_1,q_1}_{\mathbf{s}_1}$$

by $(P_{\mathbf{t}} \circ f)_\zeta = P_{\mathbf{t}} \circ f_\zeta$. Then $P_{\mathbf{t}} \circ f \in \mathcal{F}(A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1})$ and if $f_\theta = g$, we have $(P_{\mathbf{t}} \circ f)_\theta = P_{\mathbf{t}} \circ f_\theta = P_{\mathbf{t}} g = g$. So

$$\begin{aligned} ||g||^{\text{anal}}_\theta &:= \inf\{||\varphi||_{\mathcal{F}\left(A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1}\right)} : g = \varphi_\theta\} \\ &\leq \|P_{\mathbf{t}} \circ f\|_{\mathcal{F}\left(A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1}\right)} \\ &:= \max\left\{\sup_{\Re e\, \zeta = 0} ||(P_{\mathbf{t}} \circ f)_\zeta||_{A^{p_0,q_0}_{\mathbf{s}_0}}, \sup_{\Re e\, \zeta = 1} ||(P_{\mathbf{t}} \circ f)_\zeta||_{A^{p_1,q_1}_{\mathbf{s}_1}}\right\} \end{aligned}$$

for every $f \in \mathcal{F}(L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1})$ such that $f_\theta = g$. By the boundedness of $P_{\mathbf{s}}$ on $L^{p_i,q_i}_{s_i}$, $i = 0, 1$, we get

$$||g||^{\text{anal}}_\theta \leq C_{\mathbf{t}} \inf\left\{||f||_{\mathcal{F}(L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1})} : f_\theta = g\right\} \sim C_{\mathbf{t}} ||g||_{L^{p,q}_{\mathbf{s}}}.$$

This proves the estimate (4.1).

We next prove the estimate (4.2). Let $g \in [A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1}]_\theta$. We first suppose that $||g||^{\text{anal}}_\theta = 0$, *i.e.* $g = 0$ in the Banach space $\left[A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta$. We notice that

$$||\varphi||_{\mathcal{F}(A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1})} = ||\varphi||_{\mathcal{F}\left(L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1}\right)} \tag{4.4}$$

for all $\varphi \in \mathcal{F}\left(A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1}\right)$. This implies that

$$||g||_{\left[L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta} \leq ||g||_{\left[A^{p_0,q_0}_{\mathbf{s}_0}, A^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta}$$

and hence $||g||_{\left[L^{p_0,q_0}_{\mathbf{s}_0}, L^{p_1,q_1}_{\mathbf{s}_1}\right]_\theta} = 0$. By the estimate (4.3), we obtain $||g||_{L^{p,q}_{\mathbf{s}}} = 0$.

We next suppose that $0 < ||g||_\theta^{\text{anal}} < \infty$. There exists $\varphi \in \mathcal{F}(A_{\mathbf{s}_0}^{p_0,q_0}, A_{\mathbf{s}_1}^{p_1,q_1})$ such that $g = f_\theta$ and $||\varphi||_{\mathcal{F}\left(A_{\mathbf{s}_0}^{p_0,q_0}, A_{\mathbf{s}_1}^{p_1,q_1}\right)} \leq 2||g||_\theta^{\text{anal}}$. By (4.3) and (4.4), we obtain:

$$||g||_{A_{\mathbf{s}}^{p,q}} = ||g||_{L_{\mathbf{s}}^{p,q}} \lesssim |g||_{\left[L_{\mathbf{s}_0}^{p_0,q_0}, L_{\mathbf{s}_1}^{p_1,q_1}\right]_\theta} \lesssim ||\varphi||_{\mathcal{F}\left(A_{\mathbf{s}_0}^{p_0,q_0}, A_{\mathbf{s}_1}^{p_1,q_1}\right)} \leq 2||g||_\theta^{\text{anal}}.$$

This proves the estimate (4.2).

(2) In this assertion, we have $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}$. The weighted Bergman projector $P_{\mathbf{s}}$ extends to a bounded operator from $L_{\mathbf{s}}^{p_i,q_i}$ onto $A_{\mathbf{s}}^{p_i,q_i}$, $i = 0, 1$ and hence from $L_{\mathbf{s}}^{p,q}$ onto $A_{\mathbf{s}}^{p,q}$. Then by Proposition 4.4, for every $g \in A_{\mathbf{s}}^{p,q}$, we have $P_{\mathbf{s}}g = g$. The proof of assertion (2) is the same as the proof of assertion (1) with $\mathbf{t} = \mathbf{s}$ in the present case. More precisely, for the proof of the estimate (4.1), we replace the mapping

$$P_{\mathbf{t}} \circ f : \quad \overline{S} \to A_{\mathbf{s}_0}^{p_0,q_0} + A_{\mathbf{s}_1}^{p_1,q_1}$$

with $f \in \mathcal{F}(L_{\mathbf{s}_0}^{p_0,q_0}, L_{\mathbf{s}_1}^{p_1,q_1})$, by the mapping

$$P_{\mathbf{s}} \circ f : \quad \overline{S} \to A_{\mathbf{s}}^{p_0,q_0} + A_{\mathbf{s}}^{p_1,q_1}$$

with $f \in \mathcal{F}(L_{\mathbf{s}}^{p_0,q_0}, L_{\mathbf{s}}^{p_1,q_1})$. The proof of the estimate (4.2) remains the same.

(3) We are going to prove the following more precise statement.

Recall that

$$q_s = \min_{1\leq k\leq r} \left(1 + \frac{s_k - \frac{n_k}{2}}{\frac{m_k}{2}}\right).$$

Theorem 4.4. *Let $\mathbf{s} \in \mathbb{R}^r$ be such that $s_k > \frac{n}{r} - 1, k = 1, \cdots, r$. Let $1 \leq p_0, p_1 \leq \infty$ and let q_0, q_1 be such that $1 \leq q_0 < q_{\mathbf{s}} \leq q_1 < \infty$. Assume that $P_{\mathbf{s}}$ extends to a bounded operator on $L_{\mathbf{s}}^{p_1,q_1}$. Let $\theta, \varphi \in (0, 1)$ be related by the equation*

$$\frac{1}{2} = \frac{1-\theta}{q_0} + \theta\left(\frac{1-\varphi}{2} + \frac{\varphi}{q_1}\right) \tag{$\star$}$$

and assume that

$$\varphi < \frac{\frac{1}{2} - \frac{1}{q_s}}{\frac{1}{2} - \frac{1}{q_1}}. \tag{$\star\star$}$$

Then for $\xi = \frac{\theta\varphi}{1-\theta+\theta\varphi}$, $\psi = \frac{\varphi}{1-\theta+\theta\varphi}$, we have

$$\left[A_{\mathbf{s}}^{p_0,q_0}, A_{\mathbf{s}}^{p_1,q_1}\right]_\xi = A_{\mathbf{s}}^{p_2,2} \quad \text{and} \quad \left[A_{\mathbf{s}}^{p_0,q_0}, A_{\mathbf{s}}^{p_1,q_1}\right]_\psi = A_{\mathbf{s}}^{p_3,q_3}$$

with equivalent norms, with

$$\frac{1}{p_2} = \frac{1-\xi}{p_0} + \frac{\xi}{p_1} \tag{$\star\star\star$}$$

$$\begin{cases} \dfrac{1}{p_3} = \dfrac{1-\varphi}{p_2} + \frac{\varphi}{p_1} \\ \dfrac{1}{q_3} = \dfrac{1-\varphi}{2} + \frac{\varphi}{q_1}. \end{cases} \tag{$\star\star\star\star$}$$

Proof. We apply the Wolff reiteration theorem (Theorem 4.2) with $A_1 = A_{\mathbf{s}}^{p_0,q_0}$, $A_2 = A_{\mathbf{s}}^{p_2,2}$, $A_3 = A_{\mathbf{s}}^{p_3,q_3}$ and $A_4 = A_{\mathbf{s}}^{p_1,q_1}$. On the one hand, we observe that $q_{\mathbf{s}} > 2$ and hence the couple $(p_2, 2)$ satisfies the condition

$$\frac{1}{q_{\mathbf{s}}(p_2)} < \frac{1}{2} < 1 - \frac{1}{q_{\mathbf{s}}(p_2)}$$

of Theorem 1.1. So $P_{\mathbf{s}}$ extends to a bounded operator on $L_{\mathbf{s}}^{p_2,2}$ as well as we assumed that $P_{\mathbf{s}}$ extends to a bounded operator on $L_{\mathbf{s}}^{p_1,q_1}$. We next apply assertion (2) of Theorem B to get the identity $[A_2, A_4]_\varphi = A_3$ with p_3 and q_3 defined by the system $(\star\star\star\star)$.

On the other hand, the condition $(\star\star)$ and the definition of q_3 given by the second equality of $(\star\star\star\star)$ imply that $1 < q_3 < q_{\mathbf{s}}$. We recall that $1 < q_0 < q_{\mathbf{s}}$. Then by assertion (1) of Theorem B and the remark immediately following the statement of this theorem, we obtain the identity $[A_1, A_3]_\theta = A_2$ with

$$\begin{cases} \dfrac{1}{p_2} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_3} \\ \dfrac{1}{2} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_3}. \end{cases}$$

The latter identity and the second identity of $(\star\star\star\star)$ give the relation $(\star)$. The former identity and the first identity of $(\star\star\star\star)$ give the relation $(\star\star\star)$. □

Question. Is $L^{p,q}$-boundedness of the Bergman projector necessary to conclude for both Theorem A and Theorem B? Or are there other methods that give a wider range of exponents?

Further notice. After the completion of this work, we became aware of the Arxiv preprint [14] where J.G. Christensen provides with a different method an atomic decomposition for functions in $A_s^{p,q}$ (s real) for the same range of exponents. His atoms are different from ours. More precisely, he uses the characterization of (Shilov) boundary values of functions in $A_s^{p,q}$ obtained in [4] as distributions in the Besov space $B_s^{p,q}$; he next applies atomic decompositions of the previous spaces established in [15]. We point out that the Laplace transforms of his atoms are compactly supported in the cone Ω; so by the Paley-Wiener theorem, his atoms are not "samples" of the Bergman kernel.

Final remark. We recall that $g \in [A_{\mathbf{s}_0}^{p_0,q_0}, A_{\mathbf{s}_1}^{p_1,q_1}]_\theta$ if there exists a mapping $f \in \mathcal{F}(A_{\mathbf{s}_0}^{p_0,q_0}, A_{\mathbf{s}_1}^{p_1,q_1})$ such that $f_\theta = g$. For $\mathbf{s}_0 = \mathbf{s}_1$ real and $p_i = q_i$, $i = 1, \dots, r$, an explicit construction was presented in [8] for such a mapping f in terms of an analytic family of operators and the atomic decomposition of the relevant (usual) Bergman spaces and this construction was generalized in [21] to mixed norm Bergman spaces associated to the same scalar parameter $\mathbf{s} = (\nu, \cdots, \nu)$. It may be interesting to extend this construction to mixed norm Bergman spaces $A_{\mathbf{s}_0}^{p_0,q_0}$, $A_{\mathbf{s}_1}^{p_1,q_1}$ associated to more general vectors $\mathbf{s}_0, \mathbf{s}_1 \in \mathbb{R}^r$.

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