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The motion of a fluid in an open channel

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Abstract. We consider a free boundary value problem for a viscous, incompressible fluid contained in an uncovered three-dimensional rectangular channel, with gravity and surface tension, governed by the Navier-Stokes equations. We obtain existence results for the linear and nonlinear time-dependent problem. We analyse the qualitative behavior of the flow using tools of bifurcation theory. The main result is a Hopf bifurcation theorem with $\mathbb{Z}_k$-symmetry.

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0. Introduction

In this work we investigate the motion of a viscous, incompressible fluid contained in an uncovered three-dimensional rectangular channel. The upper surface changes with the motion of the fluid, so we deal with a free boundary problem. We consider small perturbations of a uniform flow with a flat free surface. We include the effect of surface tension; the external forces are gravity, and the wind force which acts on the free boundary (in Section 5).

The motion of the fluid in the channel is governed by the Navier-Stokes equations. The variables are, as usual, the velocity and the pressure of the fluid in the interior of the domain and a function parameterizing the free boundary. The pressure can be expressed in terms of the other two variables, which are coupled as follows: the fluid velocity at the free boundary prescribes the speed of the boundary, and the mean curvature of the free surface creates a pressure jump via the surface tension.

We consider the system to be periodic in the direction of the length of the channel. Technically, we identify the inflow boundary with the outflow boundary of the channel and then we consider the second spatial variable belonging to the circle S^1. In order to obtain a well-posed model, we have to prescribe the value of the dynamic contact angle between the walls and the free boundary (see [11, 9]) and we choose it to be $\frac{\pi}{2}$. As boundary conditions, we consider that the walls are impenetrable together with a perfect slip condition, and a no slip condition for the bottom.

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The main aim of this paper is to analyse the qualitative behavior of the flow (oscillations of periodic solutions) using tools of bifurcation theory. In order to do this we need fundamental facts of existence and regularity of solutions, spectral analysis of the linear system connected with the free boundary value problem taking into account the underlying symmetries, and techniques of equivariant Hopf bifurcation theorem.

J. T. Beale studied the problem of the motion of a viscous incompressible fluid in a semi-infinite domain, bounded below by a solid floor and above by an atmosphere of constant pressure, either with ([2]) or without ([3]) surface tension. In [2] he used the Fourier transformation to prove resolvent estimates. These estimates combined with the Laplace transformation in time were used to prove the solvability of the time-dependent problem. He transformed the free boundary value problem to an initial boundary value problem on a fixed domain in a special way. This method is crucial in his existence proof and was also adapted and used by [9, 10, 11]. We will apply it also in this paper.

B. Schweizer treated in [10] the case of a liquid drop (with viscosity and surface tension) in a free space, so a full free boundary problem. With the help of semigroup methods, he studied linearized equations and get also existence results for the nonlinear problem. He computed the spectrum of the generator of the semigroup. Nonreal eigenvalues appeared for large values of the surface tension. An additional exterior linear force proportional to the normal velocity and acting on the free surface leaded to a Hopf bifurcation with $O(2)$-symmetry.

As soon as contact between a fixed boundary and a free boundary arises, the analytic investigations are getting more complicated. Already in case of a flow in a domain with non smooth fixed boundary, the regularity of the solutions is restricted (see e.g. [5]). The problem how to prescribe conditions for the contact is still in discussion. There exists a huge number of publications dealing with the solvability of free boundary problems with contact points and lines and therefore only some of the works and authors can be mentioned.

V. A. Solonnikov proved existence results for free boundary problems for the Navier-Stokes equations for both static or dynamic contact points and lines. He proved estimates for stationary problem for limiting values of contact angle 0 or π, in weighted Hölder spaces (see [13, 14] and the references presented there). For the solvability of stationary free boundary problems with a Navier type slip condition on the rigid walls see [8] and [12]. This condition can be applied in the case of a domain with rough boundaries by replacing the rough boundary with a smooth one where the Navier condition is fulfilled.

M. Renardy ([9]) proved existence and uniqueness results for a two dimensional free surface flow problem with open boundaries. Both steady and initial value problems are investigated. He considered the case where velocity boundary conditions are prescribed on both the inflow and the outflow boundary. The smoothness of the solution is limited by the singularity at the corner between the free surface and the inflow (or outflow) boundary.

In [11], B. Schweizer discussed conditions for the dynamic contact angle and well-posedness of the equations for a flow in a two dimensional domain. For the case of $\frac{\pi}{2}$ contact angle and slip boundary conditions he proved resolvent estimates which, using techniques developed in [9], yielded an existence result for the nonlinear initial boundary value problem.

The studies of the oscillatory behavior of a fluid in a channel is continuing the research of B. Schweizer who analyzed the oscillation of a liquid drop [10]. Due to the solid boundary in our problem, the techniques in this paper have to be changed due to difficulties arising from the additional boundary conditions. We are able to obtain results for the channel similar to those B. Schweizer obtained for the oscillating drop.

1. Formulation of the problem

We first collect the nonlinear equations describing the nonstationary motion of a viscous, incompressible fluid contained in an uncovered rectangular channel. The unknown functions are not only the velocity field u and the pressure $\bar{p}$, but also the domain Ω. We consider the channel of width b and length $l = 2\pi$ to be deep enough such that the fluid will never overflow it. We impose a periodicity condition in the direction of the length of the channel (for all unknown functions).

Let $(0, b) \times (0, 2\pi) \times (-h, +\infty)$, $b, h > 0$ be the channel and Ω the domain occupied by the fluid with the free boundary denoted by Γ and fixed boundary Σ composed from the walls Σ_1, Σ_2 and the bottom Σ_{-h}. Let C_1, C_2 be the intersection curves between the free boundary and the walls. The periodicity in x_2 is technically incorporated by considering the independent variable x_2 belonging to the circle S^1. So, we have identified (and actually eliminated as boundaries) the surfaces $(0, b) \times \{0\} \times (-h, +\infty)$ and $(0, b) \times \{2\pi\} \times (-h, +\infty)$. The channel $(0, b) \times S^1 \times (-h, +\infty)$ is now considered "without curvature in the x_2-direction", *i.e.* the equations will not be transformed (this is not a domain transformation, it is only an identification).

We take the domain of the fluid at equilibrium to be

$$\Omega_0 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < b,\ x_2 \in S^1, -h < x_3 < 0\right\},$$

with the upper boundary Γ_0

$$\Gamma_0 = (0, b) \times S^1 \times \{0\},$$

and the fixed boundary composed from the walls $\Sigma_{1,0}, \Sigma_{2,0}$ and the bottom Σ_{-h}. The contact curves between the free boundary and the walls are denoted by $C_{1,0}, C_{2,0}$. Where no confusion can appear, we will omit the index 0 from the notation for the walls and contact lines of the equilibrium domain. When we want to refer to the walls together, we will denote them by $\Sigma_{1,2}$ and the same for contact lines $C_{1,2}$.

To describe the free surface of the fluid, we assume small perturbations of the equilibrium surface Γ_0 and parametrize the free boundary of the liquid with a function $\eta(t,\cdot) : \Gamma_0 \longrightarrow \mathbb{R}$. Thus the height of the free surface is a function of horizontal coordinates: $x_3 = \eta(t, x_1, x_2)$, $(x_1, x_2) \in \Gamma_0$ and the graph of η gives the shape of Γ. The domain occupied by the fluid is

$$\Omega = \Omega(t) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < b, x_2 \in S^1, -h < x_3 < \eta(t, x_1, x_2)\right\}.$$

The velocity field is a function $u(t,\cdot) : \Omega(t) \longrightarrow \mathbb{R}^3$.

As usual, we introduce the deformation tensor S_u with the components

$$(S_u)_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

and the stress tensor σ with the components

$$\sigma_{ij} = -\bar{p}\delta_{ij} + 2\nu(S_u)_{ij}.$$

The motion of the fluid in the interior is governed by the Navier-Stokes equations for an incompressible fluid with viscosity ν:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla\bar{p} + g\nabla x_3 = 0 \tag{1.1}$$
$$\nabla \cdot u = 0 \tag{1.2}$$

where g is the acceleration of gravity. It is natural to substract the hydrostatic presure from $\bar{p}$, so we se

$$p := \bar{p} - P_0 + gx_3$$

where P_0 is the atmospheric pressure above the liquid. The density does not appear because of the nondimensionalization. After substitution, the gravity term in (1.1) is eliminated.

On the free surface we have the kinematic boundary condition which states that the fluid particles do not cross the free surface (which is equivalent with the geometric condition that η always parametrizes the free surface):

$$\partial_t\eta = u_3 - (\partial_1\eta)u_1 - (\partial_2\eta)u_2 \quad \text{on } \Gamma. \tag{1.3}$$

If we neglected the surface tension, the remaining boundary condition on Γ would be the continuity of the stress across the free surface, so $-\sum_{j=1}^3 \sigma_{ij}n_j = P_0\, n_i + f_i n_i$ for $i = 1, 2, 3$, where $n = (n_1, n_2, n_3)$ is the outward normal to Γ and $f = (f_1, f_2, f_3)$ is the exterior force (for example the wind force). The effect of surface tension is to introduce a discontinuity in the normal stress, proportional to the mean curvature $H(\eta)$ of the free surface Γ. Our boundary condition on Γ is therefore (using $p := \bar{p} - P_0 + gx_3$ and $x_3 = \eta$ on Γ)

$$p\,n_i - \nu\sum_{j=1}^{3}(\partial_i u_j + \partial_j u_i)n_j = (g\eta + \beta H(\eta) + f_i)n_i \qquad i = 1, 2, 3 \tag{1.4}$$

where $\beta > 0$ is the nondimensionalized coefficient of the surface tension and the mean curvature of the surface Γ is given by

$$H(\eta) = -\overline{\nabla} \cdot \frac{\overline{\nabla}\eta}{\sqrt{1 + |\overline{\nabla}\eta|^2}}. \tag{1.5}$$

We have denoted here by $\overline{\nabla}$ the gradient with respect to the first two variables x_1, x_2; then let $\underline{\Delta} := \overline{\nabla} \cdot \overline{\nabla}$.

If nothing else is specified, in the following, we denote by n the outward normal and by τ_i, $i = 1, 2$, the two tangential directions to the surface.

From a physical point of view, the usual boundary condition $u = 0$ on Σ can not be considered here because of the unknown contact between the free surface and the walls (we can not assume that it is not moving at all on the walls, so we can not "stick" the free surface on the fixed boundary); but it is natural to consider the no-slip condition on the bottom:

$$u\big|_{\Sigma_{-h}} = 0 \tag{1.6}$$

and the velocity vanishing in the normal direction of the walls

$$u \cdot n\big|_{\Sigma_1 \cup \Sigma_2} := u_n\big|_{\Sigma_{1,2}} = u_1\big|_{\Sigma_{1,2}} = 0 \tag{1.7}$$

together with a perfect slip condition

$$n \cdot S_u \cdot \tau_i\big|_{\Sigma_{1,2}} = 0. \tag{1.8}$$

We need also to prescribe the contact angle between the free surface and the fixed boundary. We shall choose it to be $\frac{\pi}{2}$. So, the free surface is moving on the walls, but the value of the contact angle should remain constant. This condition can be written as:

$$\overline{\nabla}\eta \cdot n^{\Sigma_1} = \overline{\nabla}\eta \cdot n^{\Sigma_2} = \partial_1\eta = 0 \quad \text{ on } C_1 \cup C_2. \tag{1.9}$$

For similar problems with contact angle 0 or π see [13, 14] and the references presented there.

The unknown functions u, p, η are periodic in the x_2 direction of the channel, so

$$(u, p, \eta)(t, x_1, x_2, x_3) = (u, p, \eta)(t, x_1, x_2 + 2\pi, x_3). \tag{1.10}$$

The initial condition is

$$(u, \eta)\big|_{t=0} = (u_0, \eta_0). \tag{1.11}$$

The Equations (1.1)–(1.11) are the evolutionary nonlinear equations describing the oscillations of a fluid in an uncovered channel.

2. The existence theory

The linear problem for which we derive estimates is the one obtained by linearizing Equations (1.1)–(1.11) about equilibrium, replacing the initial data by zero and introducing a right hand side. We note that the linearization of the mean curvature in Γ_0 is $-\underline{\Delta}\eta$, where $\underline{\Delta}$ is the Laplacian with respect to the "horizontal" variables x_1, x_2. Because $\Gamma_0 = \{x_3 = 0\}$, we have $n_i = \delta_{i3}$, $i = 1, 2, 3$, in the equation (1.4).

For the beginning we consider the exterior force to be zero. The influence of a nonzero exterior force (for example the wind force) will be considered for the study of the Hopf bifurcation in Section 5.

We observe that the equation (2.5) is equivalent to the condition on the vanishing of the tangential stress on Γ_0, so it can be written also in the form $n \cdot S_u \cdot \tau_i\big|_{\Gamma_0} = 0$ We also use the notations $S_u : S_v := \sum_{i,j=1}^{3} (S_u)_{ij}(S_v)_{ij}$, $S_u^n := n \cdot S_u \cdot n$, $S_u^{\tau_i} := n \cdot S_u \cdot \tau_i$. Our linear problem becomes: $u(t, \cdot) : \Omega_0 \longrightarrow \mathbb{R}^3$, $p(t, \cdot) : \Omega_0 \longrightarrow \mathbb{R}$, $\eta(t, \cdot) : \Gamma_0 \longrightarrow \mathbb{R}$,

$$\partial_t u - \nu \Delta u + \nabla p = 0 \tag{2.1}$$

$$\nabla \cdot u = 0 \tag{2.2}$$

$$\partial_t \eta = u_3\big|_{\Gamma_0} = u_n\big|_{\Gamma_0} \tag{2.3}$$

$$(p - 2\nu\partial_3 u_3)\big|_{\Gamma_0} = (p - 2\nu S_u^n)\big|_{\Gamma_0} = g\eta - \beta\underline{\Delta}\eta \tag{2.4}$$

$$(\partial_3 u_i + \partial_i u_3)\Big|_{\Gamma_0} = n \cdot S_u \cdot \tau_i\big|_{\Gamma_0} = 0 \quad (i = 1, 2) \tag{2.5}$$

$$u\big|_{\Sigma_{-h}} = 0 \tag{2.6}$$

$$u_1\big|_{\Sigma_{1,2}} = u_n\big|_{\Sigma_{1,2}} = 0 \tag{2.7}$$

$$\partial_1 u_i\big|_{\Sigma_{1,2}} \overset{(2.7)}{=} n \cdot S_u \cdot \tau_i\big|_{\Sigma_{1,2}} = 0 \quad (i = 2, 3) \tag{2.8}$$

$$\partial_1 \eta\big|_{x_1 \in \{0, b\}} = 0 \tag{2.9}$$

$$(u, p, \eta)(t, x_1, x_2, x_3) = (u, p, \eta)(t, x_1, x_2 + 2\pi, x_3) \tag{2.10}$$

$$(u, \eta)\big|_{t=0} = (0, 0)\,. \tag{2.11}$$

We want to write the linear equations in the form $\partial_t x + \mathcal{L}x = 0$ and to satisfy the boundary conditions by the choice of appropriate function spaces. To estimate solutions of this equation, we use the Laplace transform in time.

Following [2] and [10], we use a harmonic extension operator and replace the pressure term from the equation (2.1) by a gradient term which is determined by the other unknowns (u and η). The harmonic extension function is defined as the unique solution of the problem

$$\begin{aligned} \Delta p &= 0 \quad \text{in } \Omega_0 && (a) \\ p\big|_{\Gamma_0} &= 2\nu S_u^n\big|_{\Gamma_0} + g\eta - \beta\underline{\Delta}\eta && (b) \\ \partial_n p\big|_{\Sigma_{1,2}} &= 0 && (c) \\ \partial_n p\big|_{\Sigma_{-h}} &= \nu(\partial_n S_u^n)\big|_{\Sigma_{-h}} . && (d) \end{aligned} \tag{2.12}$$

So, define the linear operator

$$\tilde{\mathcal{H}} : H^{r-1/2}(\Gamma_0) \times H^{r-3/2}(\Sigma_{-h}) \longrightarrow H^r(\Omega_0)^3$$

which essentially maps a function defined on Γ_0 to its harmonic extension in Ω_0. The order r of the Sobolev space will be established later. We can consider p as a harmonic function defined on the whole domain,

$$\begin{aligned} p &= \tilde{\mathcal{H}}(2\nu S_u^n\big|_{\Gamma_0} + g\eta - \beta\underline{\Delta}\eta, \nu(\partial_n S_u^n)\big|_{\Sigma_{-h}}) \\ &= \tilde{\mathcal{H}}(2\nu S_u^n\big|_{\Gamma_0}, \nu\partial_n S_u^n\big|_{\Sigma_{-h}}) + \tilde{\mathcal{H}}(g\eta - \beta\underline{\Delta}\eta, 0) \\ &:= \mathcal{H}(2\nu S_u^n\big|_{\Gamma_0}) + \mathcal{H}(g\eta - \beta\underline{\Delta}\eta). \end{aligned} \tag{2.13}$$

In the last equality of (2.13), we have only simplified the notation for the operator $\tilde{\mathcal{H}}$ (*i.e.* we have not included the condition on the bottom Σ_{-h}), because generally we are more interested to solve the problem near the free surface. Anytime when we refer to $\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0})$ we have to understand the condition (2.12)(d) to be satisfied too, and when we refer to $\mathcal{H}(g\eta - \beta\underline{\Delta}\eta)$ we have to understand the condition (2.12)(d) with zero right hand side, *i.e.* $\partial_n p\big|_{\Sigma_{-h}} = 0$.

In the following we will consider complex valued functions and denote with $\bar{u}$ the complex conjugate of u. We use the following notations for the norms: $\forall r \in \mathbb{R}$ ($r = 0$ denotes the L^2-norm) $\|u\|_{H^r(\Omega_0)^3} := \|u\|_{r,\Omega_0}$, $\|\eta\|_{H^r(\Gamma_0)} := \|\eta\|_{r,\Gamma_0}$.

Definition 2.1. Define the Hilbert spaces (over $\mathbb{C}$):

$$X^r := \{(u,\eta) \in H^r(\Omega_0)^3 \times H^{r+1/2}(\Gamma_0) \;\big|\; \nabla\cdot u = 0,\; u_n\big|_{\Sigma_{1,2,-h}} = 0\}$$

$$\tilde{X}^r := \{(u,\eta) \in X^r \;\big|\; n\cdot S_u\cdot\tau_i\big|_{\Gamma_0\cup\Sigma_{1,2}} = 0,\; u_{\tau_i}\big|_{\Sigma_{-h}} = 0,\; \partial_1\eta\big|_{x_1\in\{0,b\}} = 0\}$$

with the natural norm inherited from the product space, and the operator

$$\mathcal{L} : \tilde{X}^{r+2} \longrightarrow X^r,$$

by

$$\mathcal{L}\begin{pmatrix} u \\ \eta \end{pmatrix} := \begin{pmatrix} -\nu\Delta u + \nabla\mathcal{H}(2\nu S_u^n\big|_{\Gamma_0}) + \nabla\mathcal{H}(g\eta - \beta\underline{\Delta}\eta) \\ -u_n\big|_{\Gamma_0} \end{pmatrix}.$$

The next lemma can be proved by simple calculations. We will use it especially in the particular case when u and v satisfy the same conditions.

Lemma 2.2. *For smooth functions $u, v : \Omega_0 \to \mathbb{C}^3$ with $\nabla \cdot u = 0$ there holds*

$$2\int_{\Omega_0} S_u : S_{\bar{v}} = -\int_{\Omega_0} \Delta u \cdot \bar{v} + 2\int_{\partial\Omega_0} n \cdot S_u \cdot \bar{v}.$$

In the case $\nabla \cdot v = 0$, $v\big|_{\Sigma_{-h}} = 0$, $v_n\big|_{\Sigma_{1,2}} = 0$, and $n \cdot S_u \cdot \tau_i\big|_{\Gamma_0 \cup \Sigma_{1,2}} = 0$ (where τ_i is any tangent vector and n the normal vector corresponding to Γ_0, Σ_1 or Σ_2 respectively), we obtain the identity

$$2\int_{\Omega_0} S_u : S_{\bar{v}} = \int_{\Omega_0} [-\Delta u + \nabla\mathcal{H}(2S_u^n\big|_{\Gamma_0})] \cdot \bar{v}\,.$$

Definition 2.3 (Energy-norms). For functions $u, v : \Omega_0 \to \mathbb{C}^3$, $\eta, \sigma : \Gamma_0 \to \mathbb{C}$ we define the scalar products:

$$\langle u, v\rangle_{E,\Omega_0} := \int_{\Omega_0} u \cdot \bar{v}$$

$$\langle \eta, \sigma\rangle_{E,\Gamma_0} := \int_{\Gamma_0} \eta \cdot (g\bar{\sigma} - \beta\underline{\Delta}\bar{\sigma})$$

$$\left\langle \begin{pmatrix} u \\ \eta \end{pmatrix}, \begin{pmatrix} u \\ \eta \end{pmatrix} \right\rangle_E := \langle u, v\rangle_{E,\Omega_0} + \langle \eta, \sigma\rangle_{E,\Gamma_0}\,.$$

The corresponding norms are denoted by $\|\cdot\|_{E,\Omega_0}$, $\|\cdot\|_{E,\Gamma_0}$ and $\|\cdot\|_E$.

The next two theorems prove properties of the spectrum of $\mathcal{L}$ in the complex plane. The proofs follow by simple calculations using partial integration; the first is similar with that of [10, Lemma 2.4]; see also [4, Theorem 1.2.5, Theorem 1.2.9].

Theorem 2.4 (Position of eigenvalues of $\mathcal{L}$ with respect to $\|\cdot\|_E$). *Let $(u,\eta) \in \tilde{X}^r$ be an eigenfunction (considered complex) of $\mathcal{L}$ with eigenvalue λ. Then*

$$\mathrm{Re}\lambda \left\| \begin{pmatrix} u \\ \eta \end{pmatrix} \right\|_E^2 = 2\nu \int_{\Omega_0} |S_u|^2 \tag{2.14}$$

$$\mathrm{Im}\lambda \left\| \begin{pmatrix} u \\ \eta \end{pmatrix} \right\|_E^2 = 2\mathrm{Im}\int_{\Gamma_0} (-u_n\big|_{\Gamma_0})(g\bar{\eta} - \beta\underline{\Delta}\bar{\eta}). \tag{2.15}$$

In the case of $\mathrm{Im}\lambda \neq 0$*, the energy equality holds:*

$$\|u\|^2_{E,\Omega_0} = \|\eta\|^2_{E,\Gamma_0} = \frac{1}{2}\left\|\begin{pmatrix} u \\ \eta \end{pmatrix}\right\|^2_E . \tag{2.16}$$

Theorem 2.5. *The spectrum of* $\mathcal{L}$ *consists only of eigenvalues and is contained in a sector*

$$S_C = \{\lambda \in \mathbb{C} \big| \, |\mathrm{Im}\lambda| \leq C\mathrm{Re}\lambda\}.$$

Using the properties of the Stokes operator, we can easily prove the following proposition; see [4, Proposition 1.2.6]:

Proposition 2.6.

(a) *The operator* $\mathcal{L}^{-1} : X^r \to \tilde{X}^{r+1}$, $r \geq 1$, *is bounded.*
(b) *The operator* $\mathcal{L}^{-1} : X^0 \to \tilde{X}^2$ *is not bounded.*

In order to prove the existence of a solution of the linear problem, for instance with the help of the Laplace transformation in time, we prove an estimate for the resolvent of $-\mathcal{L}$, first only on a subspace of the form $\{(f, 0)\big| f \in L^2(\Omega_0)^3\}$. We denote the transformed functions also by (u, η) and investigate the solutions of the equation

$$(\lambda + \mathcal{L})\begin{pmatrix} u \\ \eta \end{pmatrix} := \begin{pmatrix} \lambda u - \nu\Delta u + \nabla\mathcal{H}(2\nu S^n_u\big|_{\Gamma_0}) + \nabla\mathcal{H}(g\eta - \beta\underline{\Delta}\eta) \\ \lambda\eta - u_n\big|_{\Gamma_0} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{2.17}$$

We can prove the next two results:

Theorem 2.7 (The resolvent $(\lambda+\mathcal{L})^{-1}$ **in the case** $(f, 0) \in X^r$**).** *There exist constants* C_R *and* c *such that solutions* (u, η) *of* (2.17) *with* $\lambda \in \mathbb{C} \setminus (-S_C)$ *satisfy for* $(f, 0) \in X^r$, *with* $r \geq 0$, *the regularity*

$$\|(u, \eta)\|_{X^{r+2}} \leq c\|(f, 0)\|_{X^r} \tag{2.18}$$

and for $|\lambda|$ *large enough, the resolvent estimate*

$$\|(u, \eta)\|_{X^r} \leq \frac{C_R}{|\lambda|}\|(f, 0)\|_{X^r} . \tag{2.19}$$

Corollary 2.8 (The resolvent $(\lambda+\mathcal{L})^{-1}$ **for** $(f, h) \in X^r$ **with** $h \neq 0$**).** *Let* (u, η) *be a solution of the equation*

$$(\lambda + \mathcal{L})\begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}, \tag{2.20}$$

with $(f, h) \in X^{r+2}$, $r \geq 0$. *Then there exists a constant* $M > 0$ *such that for all* $\lambda \in \mathbb{C} \setminus (-S_C)$, $|\lambda|$ *large enough, there holds:*

$$\|(u, \eta)\|_{X^{r+2}} \leq \frac{M}{|\lambda|}\|(f, h)\|_{X^{r+2}} . \tag{2.21}$$

The proofs of these results for $r = 0$ are very similar with those of [10, Theorem 3.1, Corollary 3.2] and they will not be presented here. In order to increase the regularity order r and to avoid the difficulties with the corners of the domain, we can perform a reflection across the walls. Our boundary conditions on the walls allow us to define symmetric extensions of $(\vec{u}, \eta, p)$ across e.g. Σ_1, as follows: u_1 will be extended to be odd and u_2, u_3, η and p will be extended to be even with respect to x_1. The right hand sides can be extended consistently, such that the new functions satisfy the a problem similar to the old ones, but now in the extended domain $(-b, b) \times S^1 \times (-h, 0)$ and they will be periodic with respect to x_1. In a similar way like the case $r = 0$, we can prove now estimates for the symmetric extended function, using well-known techniques: we differentiate the equations with respect to the variables x_1 and x_2, then the corresponding derivatives of u satisfy the same equations with the differentiated right hand side. The estimates of the derivatives with respect to x_3 can be obtained directly from the Navier-Stokes equation in the extended domain. We can now restrict ourselves to the initial domain and obtain the desired estimates for the solution (u, η) of the initial problem (2.17). For more details see [4, Theorem 1.2.10, Corollary 1.2.14].

We can now apply the inverse of the Laplace transformation and formulate our existence result for the linear problem.

Theorem 2.9 (Linear existence result for $(f, 0)$). *We consider $\mathcal{L} : \tilde{X}^{r+2} \to X^r$, $r \geq 1$ and $(f, 0) \in L^2([0, T], X^r)$, $T > 0$. Then the problem*

$$(\partial_t + \mathcal{L}) \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

with initial conditions $(u, \eta)\big|_{t=0} = (u_0, \eta_0) \in \tilde{X}^{r+2}$ has a unique solution

$$(u, \eta) \in H^1([0, T], \tilde{X}^r) \cap L^2([0, T], \tilde{X}^{r+2}).$$

The particular linear existence result obtained for the special form of the right hand side $(f, 0) \in X^r$ will be not enough for the proof of the Hopf bifurcation theorem. We can formulate a result stronger then Theorem 2.7, *i.e.* for a nonzero second component of the right hand side, if this is more regular than the space X^r requires. This means we have to introduce a new space

$$X^r_{3/2} := \{(f, h) \in H^r(\Omega_0)^3 \times H^{r+3/2}(\Gamma_0) \,\big|\, \nabla \cdot f = 0,\, f_n\big|_{\Sigma_{1,2,-h}} = 0\} \tag{2.22}$$

with the natural norm inherited from the product space. Using this notation, our X^r spaces coincide with the $X^r_{1/2}$ spaces, but we will keep the old notation for X^r. The properties of $\mathcal{L} : \tilde{X}^{r+2} \to X^r_{3/2}$ are stated in the next theorem and follow in the same way like those one for the special right hand side.

Theorem 2.10 (Properties of $\mathcal{L}$: $\tilde{X}^{r+2} \to X^r_{3/2}$). *The operator $\mathcal{L} : \tilde{X}^{r+2} \to X^r_{3/2}$, $r \geq 0$, is invertible, the inverse is bounded and we have the regularity estimate*

$$\|(u, \eta)\|_{X^{r+2}} \leq c\|(f, h)\|_{X^r_{3/2}} . \tag{2.23}$$

The same result holds for the operator $\lambda + \mathcal{L}$, too, when λ is not an eigenvalue of $-\mathcal{L}$.

We can immediately formulate the analog of the linear existence Theorem 2.9:

Theorem 2.11 (Linear existence result for (f, h) with $h \neq 0$). *We consider $\mathcal{L} : \tilde{X}^{r+2} \to X^r_{3/2}$, $r \geq 1$, and $(f, h) \in L^2([0, T], X^r_{3/2})$, $T > 0$. Then the problem*

$$(\partial_t + \mathcal{L}) \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

with initial conditions $(u, \eta)\big|_{t=0} = (u_0, \eta_0) \in \tilde{X}^{r+2}$ has a unique solution

$$(u, \eta) \in H^1([0, T], \tilde{X}^r_{3/2}) \cap L^2([0, T], \tilde{X}^{r+2}).$$

Following [2, 9] and [10], we convert our (initial) nonlinear problem (1.1) - (1.11) defined on the unknown domain Ω to one on the equilibrium domain Ω_0 by stretching or compressing on the vertical line segments. We state here only the main existence results. For more details, see [4, Section 1.3].

Theorem 2.12 (Nonlinear existence result for $\tilde{X}$-spaces). *For $r \geq 1$, small enough $(f, 0) \in L^2([0, T], X^r)$ and small enough initial values $(u_0, \eta_0) \in \tilde{X}^{r+2}$, there exists a unique solution $(u, \eta) \in H^1([0, T], \tilde{X}^r) \cap L^2([0, T], \tilde{X}^{r+2})$ of the nonlinear problem. (f is a right hand side introduced in the Equation* (1.1)).

Theorem 2.13 (Nonlinear existence result for $\tilde{X}_{3/2}$-spaces). *For $r \geq 1$, small enough $(f, h) \in L^2([0, T], X^r_{3/2})$ and small enough initial values $(u_0, \eta - 0) \in \tilde{X}^{r+2}$, there exists a unique solution $(u, \eta) \in H^1([0, T], \tilde{X}^r_{3/2}) \cap L^2([0, T], \tilde{X}^{r+2})$ of the nonlinear problem. (f and h are right hand sides introduced in the Equations* (1.1) *and* (1.3)*, respectively*).

3. The $\mathcal{L}$-invariant decomposition

We want to split X^r and $\tilde{X}^r$ into a direct sum of $\mathcal{L}$-invariant subspaces $(X^r_i)_{i \in I}$. The normed eigenvectors of $-\underline{\Delta}$ on Γ_0, with Neumann boundary conditions in the x_1-direction of the channel, form an orthonormal basis for $L^2(\Gamma_0)$. In order to find this basis explicitly, we solve the eigenvalue problem

$$\begin{aligned} -\underline{\Delta}\eta(x_1, x_2) &= \lambda\eta(x_1, x_2) \\ \partial_1\eta\big|_{x_1 \in \{0,b\}} &= 0 \end{aligned}$$

using the method of separation of variables. It is well-known (see e.g. [6], Chapter VIII, Theorem 8 and the applications presented here) that this problem has a countable number of eigenvalues $\lambda^{n,k}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$ which are real, positive and simple. The eigenfunctions are

$$\eta^{n,k}(x_1, x_2) = c^{n,k} \cos\left(\frac{\pi}{b} n x_1\right) e^{ikx_2},$$

the constants $c^{n,k}$ being chosen in such a way that

$$\int_{\Gamma_0} |\eta^{n,k}|^2 dx_1\, dx_2 = 1.$$

So, $L^2(\Gamma_0)$ can be decomposed into a direct Hilbert sum

$$L^2(\Gamma_0) = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} l^2_{n,k}$$

where

$$l^2_{n,k}(\Gamma_0) = \mathrm{span}\{\eta^{n,k}(x_1, x_2)\}.$$

Using the basis we found for $L^2(\Gamma_0)$, we will construct a basis for $L^2(\Omega_0)^3$ in the next proposition. The proof is elementary and uses a special Helmholz decomposition (see [15]). For more details, see [4]. Let $\vec{e}_3 = (0, 0, 1)$ be the normal vector on Γ_0, $\overline{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2}$ and $\overline{\nabla}^{\perp} = \vec{e}_1 \frac{\partial}{\partial x_2} - \vec{e}_2 \frac{\partial}{\partial x_1}$, where $\vec{e}_1 = (1, 0, 0)$ and $\vec{e}_2 = (0, 1, 0)$ are two tangent vectors to Γ_0.

Proposition 3.1. *The set*

$$\mathcal{B} = \{\eta^{n,k}(x_1, x_2)\vec{e}_3,\ \overline{\nabla}\eta^{n,k}(x_1, x_2)\,,\ \overline{\nabla}^{\perp}\eta^{n,k}(x_1, x_2)\}$$

is a basis for $L^2(\Gamma_0)^3$.

Using the basis $\mathcal{B}$ for $L^2(\Gamma_0)^3$, we can decompose a function $u(x_1, x_2, x_3) \in L^2(\Omega_0)^3$:

$$\begin{aligned} u(x_1, x_2, x_3) &= \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} U_1^{n,k}(x_3)\overline{\nabla}\eta^{n,k}(x_1, x_2) + U_2^{n,k}(x_3)\overline{\nabla}^{\perp}\eta^{n,k}(x_1, x_2) \\ &\qquad + U_3^{n,k}(x_3)\eta^{n,k}(x_1, x_2)\vec{e}_3 \\ &=: \sum_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} u^{n,k}(x_1, x_2, x_3) \end{aligned}$$

where $U^{n,k}_{1,2,3}$ are arbitrary real functions depending only on x_3, not all of them identically zero. Then,

$$L^2(\Omega_0)^3 = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} L^2_{n,k},$$

where $L^2_{n,k}$ is the corresponding space in the decomposition of $L^2(\Omega_0)^3$, for n, k fixed.

In order to find a $\mathcal{L}$-invariant decomposition for X^r, we have to see how the divergence free condition and the boundary conditions are carried over. We fix $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ and find after simple calculations:

Proposition 3.2. *The $\mathcal{L}$-invariant decompositions of the spaces X^r and $\tilde{X}^r$ are:*

$$X^r = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} X^r_{n,k} \qquad \tilde{X}^r = \bigoplus_{\substack{n\in\mathbb{N}\\ k\in\mathbb{Z}}} \tilde{X}^r_{n,k}$$

with

$$\begin{aligned} X^r_{n,k} = \{(u^{n,k}, \eta^{n,k}) \in H^r(\Omega_0)^3 \times H^{r+1/2}(\Gamma_0) \ \Big| \\ \eta^{n,k}(x_1, x_2) &= c^{n,k} \cos\left(\tfrac{\pi}{b} n x_1\right) e^{ikx_2}, \\ \vec{u}^{n,k}(x_1, x_2, x_3) &= U_1^{n,k}(x_3)\overline{\nabla}\eta^{n,k}(x_1, x_2) + U_3^{n,k}(x_3)\eta^{n,k}(x_1, x_2)\vec{e}_3, \\ (U_3^{n,k})'(x_3) &= \lambda^{n,k} U_1^{n,k}(x_3), \ \ x_3 \in (-h, 0), \\ U_3^{n,k}(-h) &= 0\} \end{aligned}$$

$$\tilde{X}^r_{n,k} = \Big\{(u^{n,k}, \eta^{n,k}) \in X^r_{n,k} \ \Big| \ U_1^{n,k}(-h) = 0 \qquad U_3^{n,k}(0) + (U_1^{n,k})'(0) = 0\Big\}.$$

Since we study the eigenvalue problem for $\mathcal{L}$, we can restrict ourselves to such a space $X^r_{n,k}$ and make all considerations there. This is stated in the next proposition. For the proof, see [4].

Proposition 3.3. *Let λ be an arbitrary eigenvalue of $\mathcal{L}$. Then there exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that λ is an eigenvalue for $\mathcal{L}\big|_{\tilde{X}^r_{n,k}}$.*

4. A bifurcation picture with respect to α

Since the Navier-Stokes equations are invariant under the Euclidean group E_3 of all translations, rotations and reflections of space, the group of symmetries of a given model is a subgroup of E_3 determined by the shape of the domain and the boundary conditions. In our problem, we consider the symmetries obtained by translations along x_2 and reflections through a plane perpendicular to the x_2-axis. The assumption on periodic boundary conditions in the x_2-direction allows us to identify these translations with the action of a circle group. These lead to an $O(2)$ symmetry, so our problem provides an $O(2)$-equivariance.

Remark 4.1. A reflection through the plane $\{x_1 = \frac{b}{2}\}$ is also a symmetry for our model. We did not consider it because it does not increase the dimension of the kernel spaces in the bifurcation theorem. This will become clear from the form of the function $\eta^{n,k}$.

$O(2)$ is generated by $SO(2)$ together with the flip $\varkappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $SO(2)$ consists of planar rotations $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. We refer to elements of $O(2)$ as 3×3 matrices, adding the third line and the third column $(0, 0, 1)$. We define the action of an element $\gamma \in O(2)$ on X^r by

$$\left.\begin{aligned} \gamma * u &:= u \circ \gamma^{-1} \\ \gamma * \eta &:= \eta \circ \gamma^{-1} \\ \gamma * \begin{pmatrix} u \\ \eta \end{pmatrix} &:= \begin{pmatrix} \gamma * u \\ \gamma * \eta \end{pmatrix} . \end{aligned}\right\} \tag{4.1}$$

$SO(2)$ may be identified with the circle group S^1, the identification being $R_\theta \mapsto \theta$. Using this identification, we describe the action of $O(2) = \{se^{i\theta} : \theta \in \mathbb{R}, s \in \{id, \varkappa\}\}$ on X^r as follows: if $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$ is the velocity field,

$$\left.\begin{aligned} \theta * \vec{u}(x_1, x_2, x_3) &:= u_1(x_1, x_2 - \theta, x_3)\vec{e}_1 + u_2(x_1, x_2 - \theta, x_3)\vec{e}_2 \\ &\quad\; + u_3(x_1, x_2 - \theta, x_3)\vec{e}_3 \\ \varkappa * \vec{u}(x_1, x_2, x_3) &:= u_1(x_1, -x_2, x_3)\vec{e}_1 - u_2(x_1, -x_2, x_3)\vec{e}_2 \\ &\quad\; + u_3(x_1, -x_2, x_3)\vec{e}_3 \\ \theta * \eta(x_1, x_2) &:= \eta(x_1, x_2 - \theta) \\ \varkappa * \eta(x_1, x_2) &:= \eta(x_1, -x_2) \; . \end{aligned}\right\} \tag{4.2}$$

It is easy to see that $\mathcal{L}$ is $O(2)$-equivariant with respect to this action, *i.e.*

$$\gamma * \mathcal{L}\begin{pmatrix} u \\ \eta \end{pmatrix} = \mathcal{L}\left(\gamma * \begin{pmatrix} u \\ \eta \end{pmatrix}\right).$$

Lemma 4.2. *The function $\eta^{n,k}$ has an isotropy subgroup $\Sigma_{\eta^{n,k}}$ of $O(2)$ isomorphic to $\mathbb{Z}_k$.*

The proof follows by simple calculations.

We are now able to study the position of the eigenvalues of $\mathcal{L}$ depending on the gravity g and on the surface tension β. The position can be calculated explicitly for $g = \beta = 0$ and for $g, \beta \to +\infty$. It is not of interest to study the problem for g and β separately. Anyway, these parameters are physical measures and they are fixed for a given liquid, but the ”formal” analysis we are presenting here gives us useful ideas for the study of Hopf bifurcation in the next section. Then

$$(g - \beta\underline{\Delta})\eta^{n,k} = (g + \beta\lambda^{n,k})\eta^{n,k} =: \alpha\, \eta^{n,k},$$

with $\alpha := g + \beta\lambda^{n,k} \in [0, \infty)$.

Remark 4.3. In this section, n and k are fixed, so $\lambda^{n,k}$ is fixed, and varying α in the Theorem 4.8 means actually to vary g and β. This is also the reason for which we do not introduce n and k in the notation α for $g + \beta\lambda^{n,k}$.

Let $A : (u, p) \mapsto -\nu\Delta u + \nabla p$ together with the following conditions:

$$\left.\begin{aligned} \text{in } \Omega_0 : \quad \nabla \cdot u &= 0 \\ n \cdot S_u \cdot \tau_i\big|_{\Gamma_0 \cup \Sigma_{1,2}} &= 0 \\ u_n\big|_{\Sigma_{1,2}} &= 0 \\ u\big|_{\Sigma_{-h}} &= 0 \end{aligned}\right\} \tag{4.3}$$

be the Stokes operator. In order to study eigenvalue problems for A, we have to impose one boundary condition more, *i.e.* one for the normal velocity on the free boundary Γ_0. We have two possibilities, to prescribe the normal velocity on Γ_0 (and obtain than a "Dirichlet" problem for the Stokes operator) or to prescribe the normal stress on the free boundary (and obtain than a "Neumann" problem for the Stokes operator). As soon as we have imposed a condition for $u_n\big|_{\Gamma_0}$ or for $(p - 2\nu S_u^n)\big|_{\Gamma_0}$, we can calculate the value of the other one. Because we are in $X^r_{n,k}$, both of them should be multiple of $\eta^{n,k}$. Also, for fixed $\eta^{n,k}$, the pressure p is known as a function of u and $\eta^{n,k}$ (see (2.13)). Therefore, when we don't need to write the pressure explicitly, we will simplify the notation:

$$A(u, p) = -\nu\Delta u + \nabla p =: Au.$$

Definition 4.4 (The Stokes operators A_D and A_N). Denote by A_D the Stokes operator A on $\tilde{X}^r$ together with the boundary condition of a vanishing normal component of the velocity at the free boundary. It is known that its eigenvalues are countable, real, positive and simple; we denote them by $\{\kappa_j\}_{j\in\mathbb{N}}$. The corresponding eigenfunctions with symmetry $\mathbb{Z}_k$ are unique up to a multiplicative constant. Let $\{u_j\}_{j\in\mathbb{N}}$ be the normed eigenfunctions with symmetry $\mathbb{Z}_k$ and $\{p_j\}_{j\in\mathbb{N}}$ be the pressure functions such that $(p_j - 2\nu S^n_{u_j})\big|_{\Gamma_0} = \eta^{n,k}$.

Denote by A_N the Stokes operator A on $\tilde{X}^r$ together with the boundary condition of a vanishing normal stress on the free boundary. It is known that its eigenvalues are countable, real, positive and simple; we denote them by $\{\rho_j\}_{j\in\mathbb{N}}$.

The Stokes operators A_D and A_N are elliptic in the sense of Agmon, Douglis and Nirenberg (see [1], and also [2, 10]).

Following [10], we define for every $\mu \in \mathbb{C} \setminus \{\kappa_j \mid j \in \mathbb{N}\}$, $(\tilde{u}(\mu), \tilde{p}(\mu))$ to be the unique solution of the problem ($\tilde{p}(\mu)$ is here unique up to an additive constant):

$$(\mu - A)\tilde{u}(\mu) = 0 \tag{4.4}$$

$$\tilde{u}_n(\mu)\big|_{\Gamma_0} = -\mu\eta^{n,k}\,. \tag{4.5}$$

We know from the perturbation theory for linear operators (see [7], and also [10]) that $(\tilde{u}(\mu), \tilde{p}(\mu))$ is an analytic family of functions for $\mu \in \mathbb{C} \setminus \{\kappa_j \mid j \in \mathbb{N}\}$.

One verifies easily that $X^r_{n,k}$ are invariant subspaces also for A_D and A_N. Therefore the (unique) solution of (4.4)-(4.5) must be in $X^r_{n,k}$. In particular $(\tilde{p}(\mu) - 2\nu S^n_{\tilde{u}(\mu)})\big|_{\Gamma_0}$ is a multiple of $\eta^{n,k}$. We define $\tilde{r}(\mu) \in \mathbb{C}$ by

$$(\tilde{p}(\mu) - 2\nu S^n_{\tilde{u}(\mu)})\big|_{\Gamma_0} =: \tilde{r}(\mu)\eta^{n,k}. \tag{4.6}$$

Of course, every $\mu \neq \kappa_j$ eigenvalue of $\mathcal{L}$ together with the corresponding eigenfunction satisfy the problem (4.4)-(4.5). Reciprocally, a $\mu \in \mathbb{C}$ is an eigenvalue on $\mathcal{L}$ with eigenfunction $(\tilde{u}(\mu), \eta^{n,k})$, if and only if

$$\tilde{r}(\mu) = \alpha.$$

Lemma 4.5. *We have: $\mu \in \mathbb{R}$ implies $\tilde{r}(\mu) \in \mathbb{R}$.*

Proof. Testing the eigenvalue equation (4.4) with $\bar{\tilde{u}}$ we obtain:

$$\begin{aligned}
\mu \int_{\Omega_0} |\tilde{u}(\mu)|^2 &= \int_{\Omega_0} [-\nu\Delta\tilde{u}(\mu) + \nabla\mathcal{H}(2\nu S^n_{\tilde{u}(\mu)}\Big|_{\Gamma_0})]\,\bar{\tilde{u}}(\mu) \\
&\quad + \int_{\Omega_0} [\nabla\tilde{p}(\mu) - \nabla\mathcal{H}(2\nu S^n_{\tilde{u}(\mu)}\Big|_{\Gamma_0})]\,\bar{\tilde{u}}(\mu) \\
&= 2\nu \int_{\Omega_0} S_{\tilde{u}(\mu)} : S_{\bar{\tilde{u}}(\mu)} + \int_{\Gamma_0} (\tilde{p}(\mu) - 2\nu S^n_{\tilde{u}(\mu)})\,\bar{\tilde{u}}_n(\mu) \\
&= 2\nu \int_{\Omega_0} |S_{\tilde{u}(\mu)}|^2 + \int_{\Gamma_0} \tilde{r}(\mu)\eta^{n,k}(-\bar{\mu}\bar{\eta}^{n,k}) \\
&= 2\nu \int_{\Omega_0} |S_{\tilde{u}(\mu)}|^2 - \bar{\mu} \int_{\Gamma_0} \tilde{r}(\mu)\,|\eta^{n,k}|^2
\end{aligned}$$

and the lemma is proved. □

In the following we abbreviate by $\|\cdot\|$ (without indices) the $L^2(\Omega_0)^3$-norm or the $L^2(\Gamma_0)$-norm.

The next two proposition are very important for the bifurcation analysis. The proofs are similar to those presented in [10].

Proposition 4.6 (Properties of $\tilde{u}(\mu)$).

(a) *In κ_j there holds*

$$\|\tilde{u}(\mu)\| \to +\infty \quad for \quad \mu \to \kappa_j. \tag{4.7}$$

(b) *The rescaled functions approximate the eigenfunctions of A_D, so*

$$u_j := \lim_{\mathbb{R}\ni\mu\nearrow\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|} = -\lim_{\mathbb{R}\ni\mu\searrow\kappa_j} \frac{\tilde{u}(\mu)}{\|\tilde{u}(\mu)\|}. \tag{4.8}$$

(c)

$$\|\tilde{u}(\mu)\| \to +\infty \qquad for \qquad |\mu| \to +\infty. \tag{4.9}$$

Proposition 4.7 (Properties of $\tilde{r}(\mu)$). *The function $\tilde{r}(\mu)$ satisfies:*

(a)

$$\lim_{\mathbb{R}\ni\mu\searrow 0} \tilde{r}(\mu) = 0\,; \tag{4.10}$$

(b)

$$\lim_{\mathbb{R}\ni\mu\searrow\kappa_j} \tilde{r}(\mu) = -\lim_{\mathbb{R}\ni\mu\nearrow\kappa_j} \tilde{r}(\mu) = +\infty\,; \tag{4.11}$$

(c) *$\tilde{r}(\mu)$ is positive for small $\mu > 0$ and $\partial_\mu \tilde{r}(\mu)\big|_{\mu=0} > 0$;*
(d) *it has exactly one turning point on each interval (κ_j, κ_{j+1}), $j \in \mathbb{N}$;*
it does not have turning points on the interval $(-\infty, \kappa_0)$;
(e) *critical values of $\tilde{r}(\mu)$ are positive.*

We can draw now the graph of $\tilde{r}$ for $\mu \in \mathbb{R}$ (see Figure 1). On $(0, \kappa_0)$ we know exactly how it looks like, on (κ_j, κ_{j+1}) we have two possibilities: $\tilde{r}$ is monoton descending or has a local maximum and a local minimum, both positive. We have drawn the graph of $\tilde{r}$ also for negative μ (because we need it for the next section). We know that $\tilde{r}$ has no negative zeros, and no turning points on $(-\infty, \kappa_0)$, so it should looks like a “parabola” on this interval.

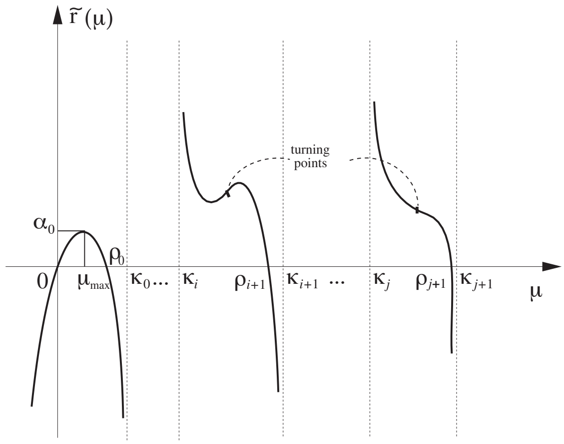

Figure 1. The Graph of $\tilde{r}(\mu)$.

The numbers $\rho_j > 0$ are zeros of the $\tilde{r}(\mu)$, so the shape of $\tilde{r}$ implies

$$\rho_j < \kappa_j < \rho_{j+1} \qquad \forall j \in \mathbb{N}.$$

Theorem 4.8 (The global bifurcation picture in α). *For $\alpha = 0$ all the eigenvalues of $\mathcal{L}_\alpha\big|_{\tilde{X}^r_{n,k}}$ are real. Denoting them by $\{\mu_j\}_{j\in\mathbb{N}}$, it holds*

$$\mu_0 = 0, \quad \mu_{j+1} = \rho_j \qquad \forall j \in \mathbb{N}.$$

For some $\alpha_0 > 0$ the first two eigenvalues merge and leave the real axis.
Given a number $\omega \in \mathbb{R}$, there exists $\alpha_\omega > 0$ such that for $\alpha > \alpha_\omega$ every interval (κ_j, κ_{j+1}) with $\kappa_{j+1} < \omega$ contains one and only one eigenvalue $\mu(\alpha)$ of $\mathcal{L}_\alpha$ (which is the unique real solution of the equation $\tilde{r}(\mu) = \alpha$ on this interval) and this real eigenvalue satisfies

$$\mu^{\mathbb{R}}(\alpha) \searrow \kappa_j \qquad for \qquad \alpha \to +\infty.$$

For the nonreal eigenvalues it holds

$$|\mu^{\mathbb{C}}(\alpha)| \to +\infty \qquad for \qquad \alpha \to +\infty.$$

Proof. The statements for the real eigenvalues of $\mathcal{L}_\alpha$ are clear from the graph of $\tilde{r}$.
For $\alpha = 0$ we can compute a complete set of eigenfunctions in $X^r_{n,k}$:

$$\begin{aligned} \mu_0 = 0 &\text{ with eigenfunction } (0, \eta^{n,k}) \\ \mu_{j+1} = \rho_j &\text{ with eigenfunction } (\tilde{u}(\rho_j), \eta^{n,k}) \,. \end{aligned}$$

Let $\mu_{\max}$ be the critical point of $\tilde{r}$ on $(0, \kappa_0)$. Then $\alpha_0 := \tilde{r}(\mu_{\max})$ and from the shape of $\tilde{r}$ we see that for $\alpha \geq \alpha_0$ the first two eigenvalues merge and leave the real axis.

Let $\omega \in \mathbb{R}$ be given, then there exists $i \in \mathbb{N}$ such that $0 < \kappa_0 < \ldots < \kappa_i < \omega$ and define α_ω to be the biggest local maximum of $\tilde{r}(\mu)$ on (κ_j, κ_{j+1}) for all j, $j < i$. The rest is clear from the shape of $\tilde{r}$.

It remains now to prove only the assertion on the nonreal eigenvalues. We suppose we have a sequence of nonreal eigenvalues $\mu(\alpha)$ of $\mathcal{L}_\alpha$ which are bounded independent of α, so suppose:

$$\mu(\alpha) \to \mu_\infty \in \mathbb{C} \qquad \text{for a sequence } \alpha \to +\infty.$$

Denoting the corresponding eigenfunctions of $\mathcal{L}_\alpha$ with $(\tilde{u}(\mu(\alpha)), \eta^{n,k})$, they satisfy the energy equality $\forall \alpha \in \mathbb{R}$:

$$\|\tilde{u}(\mu(\alpha))\|^2 = \alpha \|\eta^{n,k}\|^2$$

and the condition for the normal stress on the free boundary:

$$(\tilde{p}(\mu(\alpha)) - 2\nu S^n_{\tilde{u}(\mu(\alpha))})\big|_{\Gamma_0} = \alpha \eta^{n,k}$$

where $\tilde{p}(\mu(\alpha)$ is the corresponding pressure function. Because $\mu(\alpha)$ is nonreal $\forall \alpha$, it never meets κ_j and the pair $(\tilde{u}(\mu(\alpha)), \tilde{p}(\mu(\alpha)))$ is also a nonzero solution of the problem (4.4)–(4.5).

Then the pair $(v(\mu(\alpha)), q(\mu(\alpha)))$,

$$v(\mu(\alpha)) := \frac{\tilde{u}(\mu(\alpha))}{\alpha} \quad \text{and} \quad q(\mu(\alpha)) := \frac{\tilde{p}(\mu(\alpha))}{\alpha},$$

satisfies the equations:

$$\begin{aligned}(\mu(\alpha) - A)v(\mu(\alpha)) &= 0\\ v_n(\mu(\alpha))\big|_{\Gamma_0} &= -\frac{\mu(\alpha)}{\alpha}\eta^{n,k}\\ (q(\mu(\alpha)) - 2\nu S^n_{v(\mu(\alpha))})\big|_{\Gamma_0} &= \eta^{n,k}.\end{aligned}$$

Passing to the limit $\alpha \to +\infty$ in all these equations, using our hypothesis $\mu(\alpha) \to \mu_\infty \in \mathbb{C}$ and continuity with respect to μ of the functions v and q, the pair of the limit functions $(v(\mu_\infty), q(\mu_\infty))$

$$v(\mu_\infty) := \lim_{\alpha\to+\infty} v(\mu(\alpha)) \quad \text{and} \quad q(\mu_\infty) := \lim_{\alpha\to+\infty} q(\mu(\alpha))$$

satisfies the following equations:

$$\begin{aligned}(\mu_\infty - A)v(\mu_\infty) &= 0\\ v_n(\mu_\infty)\big|_{\Gamma_0} &= 0\\ (q(\mu_\infty) - 2\nu S^n_{v(\mu_\infty)})\big|_{\Gamma_0} &= \eta^{n,k}\end{aligned}$$

and because the normal stress on the free boundary is $\eta^{n,k}$, the solution $v(\mu_\infty) \not\equiv 0$.

On the other hand, using the energy equality we can calculate:

$$\begin{aligned}0 \neq \|v(\mu_\infty)\|^2 &= \lim_{\alpha\to+\infty} \left\| \frac{\tilde{u}(\mu(\alpha))}{\alpha} \right\|^2\\ &= \lim_{\alpha\to+\infty} \frac{\alpha\|\eta^{n,k}\|^2}{\alpha^2}\\ &= 0,\end{aligned}$$

a contradiction, so for nonreal eigenvalues, $|\mu(\alpha)| \to +\infty$ for $\alpha \to +\infty$. □

Proposition 4.9.

(a) *Eigenvalues of $\mathcal{L}_\alpha$ leave the real axis with an infinite speed (with respect to α).*
(b) *The qualitative shape of $\tilde{r}_\nu(\mu)$ is independent of the viscosity ν:*

$$\tilde{r}_{\epsilon\nu}(\epsilon\mu) = \epsilon^2 \tilde{r}_\nu(\mu).$$

The proof is similar to that presented in [10].

5. Hopf bifurcation with symmetry

The Hopf bifurcation refers to a phenomenon in which a steady state of an evolution equation evolves into a periodic orbit as a bifurcation parameter is varied. When the symmetry appears, the problem becomes more complicated because the symmetry can lead to multiple eigenvalues. In order to state an equivariant Hopf bifurcation theorem we have to prove the existence of a pair of purely imaginary eigenvalues of $\mathcal{L}$ which are $\mathbb{Z}_k$-simple together with the transversality condition that these eigenvalues cross the imaginary axis with a nonzero speed, when the bifurcation parameter is varied.

In this section we consider the influence of an exterior force (e.g. the wind force) acting on the free surface of the fluid. In general such a force will depend on the position and the velocity of the free surface and result in an increase or decrease of the pressure at the free boundary. With a parameter ξ for the strength we write

$$(p - 2\nu S_u^n)\big|_{\Gamma_0} = g\eta - \beta\underline{\Delta}\eta + \xi F(\eta, u_n\big|_{\Gamma_0}).$$

Linearizing F in 0 we notice that $D_1 F \cdot \eta$ acts like an additional surface tension, the effect of which we know in any subspace $X^r_{n,k}$ (Section 4). So we will concentrate on a linear force of the form

$$F(\eta, u_n\big|_{\Gamma_0}) = D_2 F \cdot u_n\big|_{\Gamma_0}.$$

This force can be written in terms of the representation $X^r = \oplus X^r_{n,k}$. We assume that the decomposition remains invariant and study the force $D_2 F = -id$ in $X^r_{n,k}$ which has the structure of a negative damping. We are interested in the position of eigenvalues and restrict all the calculations to $X^r_{n,k}$. The linearized equation are the same like that one in Chapter 1, except the Equation (5.4) where the term $\xi u_n\big|_{\Gamma_0}$ appears additionally:

$$\partial_t u - \nu\Delta u + \nabla p = 0 \tag{5.1}$$

$$\nabla \cdot u = 0 \tag{5.2}$$

$$\partial_t \eta^{n,k} = u_n\big|_{\Gamma_0} \tag{5.3}$$

$$(p - 2\nu S_u^n)\big|_{\Gamma_0} = g\eta^{n,k} - \beta\underline{\Delta}\eta^{n,k} - \xi u_n\big|_{\Gamma_0} \tag{5.4}$$

$$n \cdot S_u \cdot \tau_i\big|_{\Gamma_0} = 0, \quad i = 1, 2 \tag{5.5}$$

$$u\big|_{\Sigma_{-h}} = 0 \tag{5.6}$$

$$u_n\big|_{\Sigma_{1,2}} = 0 \tag{5.7}$$

$$n \cdot S_u \cdot \tau_i\big|_{\Sigma_{1,2}} = 0 \tag{5.8}$$

$$\partial_1 \eta^{n,k}\big|_{x_1 \in \{0,b\}} = 0 \tag{5.9}$$

$$(u, p, \eta^{n,k})(t, x_1, x_2, x_3) = (u, p, \eta^{n,k})(t, x_1, x_2 + 2\pi, x_3)\,. \tag{5.10}$$

Because we are working in the space $\tilde{X}^r_{n,k}$ or $X^r_{n,k}$, so we have a special form for $\eta^{n,k}$, some of the conditions (5.1)-(5.10) are automatically satisfied; however, for the seek of completeness we wrote the whole Stokes problem.

In analogy with the previous sections we define the operator

$$\mathcal{L}_\xi\begin{pmatrix}u\\ \eta^{n,k}\end{pmatrix}:=\begin{pmatrix}-\nu\Delta u+\nabla\mathcal{H}(2\nu S^n_u\big|_{\Gamma_0})+\nabla\mathcal{H}(g\bar{\eta}^{n,k}-\beta\underline{\Delta}\bar{\eta}^{n,k})-\nabla\mathcal{H}(\xi u_n\big|_{\Gamma_0})\\ -u_n\big|_{\Gamma_0}\end{pmatrix},\tag{5.11}$$

where $\mathcal{H}(\xi u_n\big|_{\Gamma_0}) := \tilde{\mathcal{H}}(\xi u_n\big|_{\Gamma_0}, 0)$. We denote by $\mathcal{L}_\xi u$ the first component in the definition (5.11).

We prove how the Theorem 2.4 carries over. We observe that the next Theorem is true also in the whole space $\tilde{X}^r$ (*i.e.* for an eigenvector $(u,\eta) \in \tilde{X}^r$ of $\mathcal{L}_\xi$).

Theorem 5.1 (Position of eigenvalues of $\mathcal{L}_\xi$ with respect to $\|\cdot\|_E$). *Let* $\begin{pmatrix}u\\ \eta^{n,k}\end{pmatrix} \in \tilde{X}^r_{n,k}$ *be an eigenfunction (considered complex) of* $\mathcal{L}_\xi$ *with eigenvalue* μ. *Then*

$$\mathrm{Re}\mu\left\|\begin{pmatrix}u\\ \eta^{n,k}\end{pmatrix}\right\|^2_E = 2\nu\int_{\Omega_0}|S_u|^2 - \xi|\mu|^2\|\eta^{n,k}\|^2_{0,\Gamma_0}\tag{5.12}$$

$$\mathrm{Im}\mu\left\|\begin{pmatrix}u\\ \eta^{n,k}\end{pmatrix}\right\|^2_E = 2\mathrm{Im}\int_{\Gamma_0}(-u_n\big|_{\Gamma_0})(g\bar{\eta}^{n,k}-\beta\underline{\Delta}\bar{\eta}^{n,k}).\tag{5.13}$$

In the case of $\mathrm{Im}\mu \neq 0$ *the energy equality holds:*

$$\|u\|^2_{0,\Omega_0} = \|u\|^2_{E,\Omega_0} = \|\eta^{n,k}\|^2_{E,\Gamma_0} = \frac{1}{2}\left\|\begin{pmatrix}u\\ \eta^{n,k}\end{pmatrix}\right\|^2_E = \alpha\|\eta^{n,k}\|^2_{0,\Gamma_0}\ .\tag{5.14}$$

The proof is similar to that of Theorem 2.4 and follows by simple calculations.

We abbreviate again by $\|\cdot\|$ without indices the $L^2(\Omega_0)^3$-norm (or $L^2(\Gamma_0)$-norm) and by $\langle\cdot,\cdot\rangle$ the L^2-scalar product.

We want to get a global picture of the position of eigenvalues as in the previous section, but now depending on the parameter ξ. Looking at the results of Theorem 5.1, we see that two important differences will appear:

(a) *The eigenvalues may have a negative real part;*
(b) *The energy equality for eigenvectors* $(u,\eta^{n,k})$ *remains unchanged and does not depend on the bifurcation parameter* ξ*; we will exploit this to prove that for* $|\xi| \to +\infty$*, the nonreal eigenvalues are bounded.*

Proposition 5.2.

(a) *The modulus of nonreal eigenvalues is bounded independent of* ξ.
(b) *For* $|\xi| \to +\infty$ *all eigenvalues of* $\mathcal{L}_\xi$ *are real.*

Proof. (a) We suppose that for $|\xi| \to +\infty$ we can find a sequence of nonreal eigenvalues $\mu(\xi) \in \mathbb{C} \backslash \mathbb{R}$ of $\mathcal{L}_\xi$ with $|\mu(\xi)| \to +\infty$. For every such complex eigenvalue with the eigenfunction $u(\mu(\xi))$, we know from the energy equality (5.14):

$$\|u(\mu(\xi))\|^2 = \alpha\|\eta^{n,k}\|^2 \text{ is bounded independent of } \xi.$$

The function $u(\mu(\xi))$ satisfies the problem (4.4)–(4.5) (together with the corresponding pressure function). We can use the result of Proposition 4.6(c) from the previous section, because its proof did not exploit $\xi = 0$, and we conclude:

$$\|u(\mu(\xi))\| \to +\infty, \qquad \text{for} \qquad |\mu(\xi)| \to +\infty,$$

a contradiction.

(b) For the second part we treat separately the cases $\xi \to -\infty$ and $\xi \to +\infty$.

(i) $\xi \to -\infty$

The Equation (5.12) implies

$$\mathrm{Re}\mu(\xi) \to +\infty \qquad \text{for} \qquad \xi \to -\infty$$

which implies $\mu(\xi) \in \mathbb{R}$ (because the nonreal eigenvalues are bounded).

(ii) $\xi \to +\infty$

We suppose that for any ξ arbitrary large, we can find a nonreal eigenvalue $\mu(\xi)$ of $\mathcal{L}_\xi$, so we can construct a sequence of nonreal eigenvalues (which are bounded) and consider $\mu(\xi) \to \mu_\infty$. Let $(\tilde{u}(\mu(\xi)), \eta^{n,k})$ be an eigenfunction of $\mathcal{L}_\xi$ corresponding to $\mu(\xi)$ and $\tilde{p}(\mu(\xi))$ be the corresponding pressure function. Because $\mu(\xi) \in \mathbb{C} \setminus \mathbb{R}$, it never meets κ_j, so $(\tilde{u}(\mu(\xi)), \tilde{p}(\mu(\xi)))$ is a nonzero solution of (4.4)–(4.5) (for $\mu(\xi)$).
We distinguish two cases:

(1) $\mu_\infty = 0$

Letting $\xi \to +\infty$, the limit function $(\tilde{u}(0), \tilde{p}(0))$ is a solution of the problem (4.4)–(4.5) for $\mu = 0$, so $\tilde{u}(0)$ is identically zero. On the other hand, every $\tilde{u}(\mu(\xi))$ satisfies the energy equality (5.14) and passing to the limit, we obtain

$$0 = \|\tilde{u}(0)\|^2 = \lim_{\xi \to +\infty} \|\tilde{u}(\mu(\xi))\|^2 = \alpha\|\eta^{n,k}\|^2 \neq 0$$

a contradiction.

(2) $\mu_\infty \neq 0$

Then the pair $(v(\mu(\xi)), q(\mu(\xi)))$,

$$v(\mu(\xi)) := \frac{\tilde{u}(\mu(\xi))}{\xi} \quad \text{and} \quad q(\mu(\xi)) := \frac{\tilde{p}(\mu(\xi))}{\xi}$$

satisfies the equations:

$$\begin{aligned}(\mu(\xi) - A)v(\mu(\xi)) &= 0 \\ v_n(\mu(\xi))\big|_{\Gamma_0} &= -\frac{\mu(\xi)}{\xi}\eta^{n,k} \\ (q(\mu(\xi)) - 2\nu S^n_{v(\mu(\xi))})\big|_{\Gamma_0} &= \frac{\alpha}{\xi}\eta^{n,k} + \mu(\xi)\eta^{n,k}.\end{aligned}$$

Passing to the limit $\xi \to +\infty$, using our hypothesis $\mu(\xi) \to \mu_\infty \in \mathbb{C}$ and continuity with respect to μ of the functions v and q, the pair of the limit functions $(v(\mu_\infty), q(\mu_\infty))$

$$v(\mu_\infty) := \lim_{\xi \to +\infty} v(\mu(\xi)) \quad \text{and} \quad q(\mu_\infty) := \lim_{\xi \to +\infty} q(\mu(\xi))$$

satisfies the following equations:

$$\begin{aligned}(\mu_\infty - A)v(\mu_\infty) &= 0 \\ v_n(\mu_\infty)\big|_{\Gamma_0} &= 0 \\ (q(\mu_\infty) - 2\nu S^n_{v(\mu_\infty)})\big|_{\Gamma_0} &= \mu_\infty\eta^{n,k}.\end{aligned}$$

and $v(\mu_\infty) \not\equiv 0$ because the normal stress on the free boundary is still nonzero.

On the other hand, using the energy equality (5.14) we have:

$$\begin{aligned}0 \neq \|v(\mu_\infty)\|^2 &= \lim_{\xi \to +\infty} \left\| \frac{\tilde{u}(\mu(\xi))}{\xi} \right\|^2 \\ &= \lim_{\xi \to +\infty} \frac{\alpha \|\eta^{n,k}\|^2}{\xi^2} \\ &= 0,\end{aligned}$$

a contradiction.

So, $\exists \xi_0 > 0$ such that for $|\xi| > \xi_0$ all eigenvalues of $\mathcal{L}_\xi$ are real. □

We resume now two useful results from the previous Section 4. First, for $\xi = 0$ we know:

the first two eigenvalues of $\mathcal{L}_{0,\alpha}$ become nonreal when α exceeds α_0

and for the analysis in this section we fixed such an α (and omit it from the notation of $\mathcal{L}_{\xi,\alpha}$). Second, for $\mu \in \mathbb{C} \setminus \{\kappa_j : j \in \mathbb{N}\}$ we have defined $\tilde{u}(\mu)$ as the unique solution of the problem (4.4)–(4.5), and $\tilde{r}(\mu)$. We know that $\mu(\alpha)$ is an eigenvalue of $\mathcal{L}_{0,\alpha} \Leftrightarrow \tilde{r}(\mu) = \alpha$. With the exterior force acting through ξ, we have:

$$\begin{aligned} \mu(\xi) \in \mathbb{C} \textit{ is an eigenvalue of } \mathcal{L}_\xi \iff \tilde{r}(\mu)\eta^{n,k} &= \alpha\eta^{n,k} - \xi\tilde{u}_n\big|_{\Gamma_0} \\ &= \alpha\eta^{n,k} + \xi\mu\eta^{n,k} \\ \iff \qquad \tilde{r}(\mu) &= \alpha + \xi\mu, \end{aligned}$$

so, we find the real eigenvalues of $\mathcal{L}_\xi$ at the intersection of the graph of the function $\tilde{r}(\mu) - \alpha$ (which is already known) with the line $y = \xi\mu$ (see Figure 2).

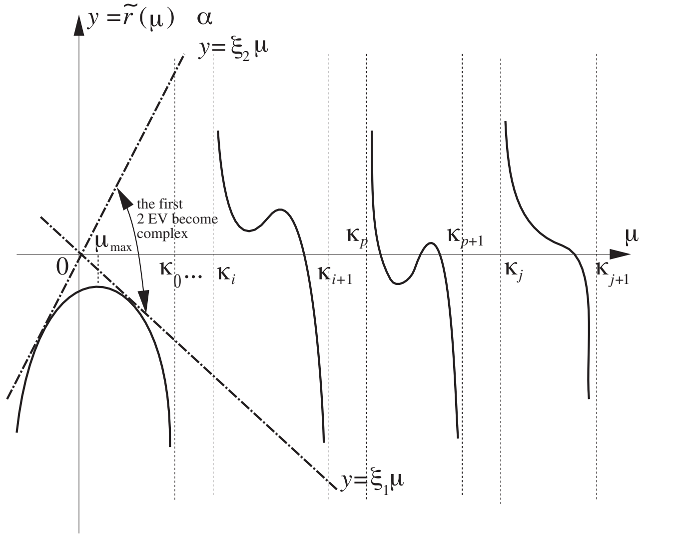

Figure 2. The intersection of the graph of $\tilde{r}(\mu) - \alpha$ with the line $y = \xi\mu$.

We observe:

- For any $\xi \in \mathbb{R}$, the line $y = \xi\mu$ intersects the graph of $\tilde{r}(\mu) - \alpha$ on each interval (κ_j, κ_{j+1}), $j \in \mathbb{N}$, at least once, so $\mathcal{L}_\xi$ has at least one real eigenvalue lying on each interval (κ_j, κ_{j+1}).
- There exists the values $\xi_1 < 0$ and $\xi_2 > 0$ such that the lines $y = \xi_1\mu$ and $y = \xi_2\mu$ are tangent to the graph of $\tilde{r}(\mu) - \alpha$ on the interval $(0, \kappa_0)$ and $(-\infty, 0)$ respectively. For $\xi \in (\xi_1, \xi_2)$ the line $y = \xi\mu$ does not intersect the graph of $\tilde{r}(\mu) - \alpha$ for $\mu \in (-\infty, \kappa_0)$. Because of the analyticity of $\tilde{r}$ (the number of zeros of $\tilde{r}$, each counted with its multiplicity, is locally constant), a pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ appears for $\xi = \xi_1 + \epsilon$ and $\xi = \xi_2 - \epsilon$ ($\epsilon > 0$ small). Denote them by $\mu_0(\xi)$ and $\mu_1(\xi)$ with $\mu_0(\xi) = \bar{\mu}_1(\xi)$.
- For $\xi \in (-\infty, \xi_1)$, the line $y = \xi\mu$ intersects the graph of $\tilde{r}(\mu) - \alpha$ twice for $\mu \in (0, \kappa_0)$, so the first two eigenvalues are real and positive.

- For $\xi \in (\xi_2, +\infty)$, the line $y = \xi\mu$ intersects the "first part" of the graph of $\tilde{r}(\mu) - \alpha$ twice, but for $\mu \in (-\infty, 0)$, so the first two eigenvalues are real and negative.

We denote the first two eigenvalues of $\mathcal{L}_\xi$ with $\mu_0(\xi)$, $\mu_1(\xi)$ and the ordered sequence of the (rest) real eigenvalues with $\{\mu_j(\xi)\}_{j\in\mathbb{N}, j\geq 2}$.

Theorem 5.3 (The global bifurcation picture in ξ). *For $\xi \in (-\infty, \xi_1)$ the first two eigenvalues of $\mathcal{L}_\xi$ are real and positive:*

$$0 < \mu_0(\xi) < \mu_1(\xi) < \kappa_0.$$

For $\xi \to -\infty$ all eigenvalues of $\mathcal{L}_\xi$ are real, every interval (κ_j, κ_{j+1}) contains one real eigenvalue μ_{j+2} of $\mathcal{L}_\xi$ and $\mu_0(\xi) \searrow 0$, $\mu_{j+2} \nearrow \kappa_{j+1}$, $j \in \mathbb{N} \cup \{-1\}$.

For $\xi \in (\xi_2, +\infty)$ the first two eigenvalues of $\mathcal{L}_\xi$ are real and negative:

$$\mu_0(\xi), \mu_1(\xi) < 0.$$

For $\xi \to +\infty$ all eigenvalues of $\mathcal{L}_\xi$ are real, every interval (κ_j, κ_{j+1}) contains one real eigenvalue μ_{j+2} of $\mathcal{L}_\xi$ and $\mu_{j+2} \searrow \kappa_{j+1}$, $j \in \mathbb{N}$.

There exists a point $\xi^* \in (\xi_1, \xi_2)$ where a pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ crosses the imaginary axis transversally. The imaginary axis can be crossed only with negative real part of the velocity.

Proof. During this proof we have to keep in mind that each of μ, u, p depends on ξ, but we will not write this explicitly.

The first two statements are clear from Proposition 5.2(b) and Figure 2, which also implies (because $\tilde{r}$ is an analytic function): for small $\epsilon > 0$,

- for $\xi = \xi_1 + \epsilon$ the pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ has a positive real part;
- for $\xi = \xi_2 - \epsilon$ the pair of complex conjugate eigenvalues of $\mathcal{L}_\xi$ has a negative real part.

The eigenvalues of $\mathcal{L}_\xi$ depend continuously on ξ and together with Proposition 5.2(a) we can conclude: there exists $\xi^* \in (\xi_1, \xi_2)$ such that $\mu(\xi^*)$ is purely imaginary, $\mathrm{Re}\mu(\xi^*) = 0$.

The eigenvalues $\mu(\xi) \neq \kappa_j$ of $\mathcal{L}_\xi$ are geometrically simple (in every $\tilde{X}^r_{n,k}$ and up to the $\mathbb{Z}_k$-symmetry) because for every eigenfunction $(u(\mu(\xi)), \eta^{n,k})$, $u(\mu(\xi))$ satisfies also the problem (4.4)–(4.5) which has unique solution. The eigenvalues have the same geometric and algebraic multiplicity for $|\xi| \to +\infty$; for the proof see [10].

We have to prove now the transversality (for $\xi = \xi^*$, $\partial_\xi(\mathrm{Re}\mu) \neq 0$) and the direction of crossing (for $\xi = \xi^*$, $\partial_\xi(\mathrm{Re}\mu) < 0$).

From the energy equality (5.14) we see that the norm of the eigenfunction u does not depend on ξ, and we can calculate:

$$0 = \partial_\xi \|u\|^2 = \partial_\xi \int_{\Omega_0} u \cdot \bar{u} = \int_{\Omega_0} u \cdot \partial_\xi \bar{u} + \overline{u \cdot \partial_\xi \bar{u}} = 2\mathrm{Re}\langle u, \partial_\xi u\rangle\,. \tag{5.15}$$

We make first some further calculations, (5.16) and (5.17), for $\partial_\xi u \neq 0$. Multiplying the first component of the eigenvalue equation for $\mathcal{L}_\xi$ with $\partial_\xi \bar{u}$, integrating over Ω_0 and using Theorem 5.1, we obtain:

$$\begin{aligned}
\langle \mu u, \partial_\xi u\rangle &= \langle \mathcal{L}_\xi u, \partial_\xi u\rangle \\
&= 2\nu \int_{\Omega_0} S_u : S_{\partial_\xi \bar{u}} + \int_{\Gamma_0} (\alpha \eta^{n,k} - \xi u_n\big|_{\Gamma_0}) \cdot \partial_\xi \bar{u}_n\big|_{\Gamma_0} \\
&= \frac{1}{2}\partial_\xi \left(2\nu \int_{\Omega_0} S_u : S_{\bar{u}}\right) + \int_{\Gamma_0} (\alpha + \xi\mu)\eta^{n,k}(-\partial_\xi \bar{\mu})\bar{\eta}^{n,k} \\
&= \frac{1}{2}\partial_\xi \left(\mathrm{Re}\mu \cdot 2\alpha \|\eta^{n,k}\|^2 + \xi |\mu|^2 \|\eta^{n,k}\|^2\right) \\
&\quad -\partial_\xi \bar{\mu} \cdot \alpha \|\eta^{n,k}\|^2 - \xi\mu \partial_\xi \bar{\mu} \|\eta^{n,k}\|^2 \\
&= \underbrace{\partial_\xi (\mathrm{Re}\mu)\alpha \|\eta^{n,k}\|^2 + \frac{1}{2}|\mu|^2 \|\eta^{n,k}\|^2 + \frac{1}{2}\xi(\partial_\xi |\mu|^2)\|\eta^{n,k}\|^2}_{\in \mathbb{R}} \\
&\quad -\partial_\xi \bar{\mu} \cdot \alpha \|\eta^{n,k}\|^2 - \xi\mu \partial_\xi \bar{\mu} \|\eta^{n,k}\|^2\,.
\end{aligned} \tag{5.16}$$

Differentiating the first component of the eigenvalue equation for $\mathcal{L}_\xi$ with respect to ξ, multiplying with $\partial_\xi \bar{u}$, integrating over Ω_0 and using Theorem 5.1, we obtain:

$$\begin{aligned}
0 &= \langle \partial_\xi(\mu u), \partial_\xi u\rangle - \langle \partial_\xi(\mathcal{L}_\xi u), \partial_\xi u\rangle \\
&= \partial_\xi \mu \langle u, \partial_\xi u\rangle + \mu \|\partial_\xi u\|^2 \\
&\quad + \langle \nu \Delta \partial_\xi u - \nabla \mathcal{H}(2\nu S^n_{\partial_\xi u}\big|_{\Gamma_0}), \partial_\xi u\rangle - \langle \nabla \mathcal{H}(\partial_\xi(-\xi u_n\big|_{\Gamma_0})), \partial_\xi u\rangle \\
&= \partial_\xi \mu \langle u, \partial_\xi u\rangle + \mu \|\partial_\xi u\|^2 - 2\nu \int_{\Omega_0} S_{\partial_\xi u} : S_{\partial_\xi \bar{u}} - \int_{\Gamma_0} \partial_\xi(\xi\mu)\eta^{n,k}(-\partial_\xi \bar{\mu})\bar{\eta}^{n,k} \\
&= \partial_\xi \mu \underbrace{\langle u, \partial_\xi u\rangle}_{\in \mathbb{C}\setminus\mathbb{R}} + \mu \underbrace{\|\partial_\xi u\|^2}_{\in \mathbb{R}} \underbrace{- 2\nu \|S_{\partial_\xi u}\|^2 + \xi |\partial_\xi \mu|^2 \|\eta^{n,k}\|^2}_{\in \mathbb{R}} \\
&\quad + \mu \partial_\xi \bar{\mu} \|\eta^{n,k}\|^2\,.
\end{aligned} \tag{5.17}$$

We prove now that the speed of nonreal eigenvalues never vanishes. Let μ be a nonreal eigenvalue of $\mathcal{L}_\xi$ and suppose $\partial_\xi \mu = 0$, so $\partial_\xi(\mathrm{Re}\mu) = \partial_\xi(\mathrm{Im}\mu) = 0$. We prove first that this implies also $\partial_\xi u = 0$. Suppose $\partial_\xi u \neq 0$, so $\partial_\xi \bar{u} \neq 0$, too. Introducing this in the equation (5.17) we obtain

$$\mu \|\partial_\xi u\|^2 = 2\nu \|S_{\partial_\xi u}\|^2$$

which implies $\mu \in \mathbb{R}$, a contradiction. So

$$\partial_\xi \mu = 0 \implies \partial_\xi u = \partial_\xi \bar{u} = 0 \implies \partial_\xi S_u = 0.$$

Differentiating the equation (5.12) with respect to ξ we obtain

$$\begin{aligned} 0 &= \partial_\xi (\mathrm{Re}\mu) 2\alpha \|\eta^{n,k}\|^2 \\ &= 2\nu \partial_\xi \left(\int_{\Omega_0} |S_u|^2 \right) - \xi \partial_\xi |\mu|^2 \|\eta^{n,k}\|^2 - |\mu|^2 \|\eta^{n,k}\|^2 \\ &= -|\mu|^2 \|\eta^{n,k}\|^2, \end{aligned}$$

a contradiction. Therefore we know for nonreal eigenvalues: $\partial_\xi \mu \neq 0, \forall \xi$.

In order to prove the transversality condition for $\xi = \xi^*$ and the direction of crossing of the imaginary axis, we take the real part of (5.16) together with (5.15) to obtain:

$$-\mathrm{Im}\mu \cdot \mathrm{Im}\langle u, \partial_\xi u\rangle \overset{(5.15)}{=} \mathrm{Re}\langle \mu u, \partial_\xi u\rangle \overset{(5.16)}{=} \frac{1}{2} |\mu|^2 \|\eta^{n,k}\|^2$$

and the imaginary part of (5.17) to obtain

$$0 = \partial_\xi (\mathrm{Re}\mu) \mathrm{Im}\langle u, \partial_\xi u\rangle + \mathrm{Im}\mu \|\partial_\xi u\|^2 + \mathrm{Im}(\mu \partial_\xi \bar{\mu}) \|\eta^{n,k}\|^2.$$

Multiplying the last equation with $2\mathrm{Im}\mu \neq 0$ and using the previous equation, we obtain:

$$\partial_\xi (\mathrm{Re}\mu) |\mu|^2 \|\eta^{n,k}\|^2 = 2\mathrm{Im}^2 \mu \|\partial_\xi u\|^2 + 2\mathrm{Im}\mu \cdot \mathrm{Im}(\mu \partial_\xi \bar{\mu}) \|\eta^{n,k}\|^2.$$

For $\xi = \xi^*$, we are on the imaginary axis, so we have

$$\begin{gathered} \mathrm{Re}\mu = 0 \Rightarrow \mathrm{Im}(\mu \partial_\xi \bar{\mu}) = \mathrm{Im}\mu \cdot \partial_\xi (\mathrm{Re}\mu) \\ |\mu|^2 = \mathrm{Im}^2 \mu \neq 0 \end{gathered}$$

and then

$$-\partial_\xi (\mathrm{Re}\mu) \|\eta^{n,k}\|^2 = 2\|\partial_\xi u\|^2 > 0.$$

□

We can formulate results similar to Proposition 2.6, Theorem 2.7 and Theorem 2.10 for $\mathcal{L}_\xi$ $(\forall \xi)$. The proofs follow immediately because only the value of $p\big|_{\Gamma_0}$ is modified with $\xi u_n\big|_{\Gamma_0}$ and we can estimate $\|\nabla \mathcal{H}(\xi u_n|_{\Gamma_0})\|_{r,\Omega_0} \leq c\|u\|_{r+1,\Omega_0}$:

Proposition 5.4 (Properties of $\mathcal{L}_\xi$).

(a) *The operator* $\mathcal{L}_\xi^{-1} : X^r \to \tilde{X}^{r+1}$, $r \geq 1$ *is bounded* $\forall \xi$.

(b) *The solution* (u, η) *of the equation* $\mathcal{L}_\xi(u, \eta) = (f, 0) \in X^r$ *satisfies the regularity:*

$$\|(u, \eta)\|_{X^{r+2}} \leq c\|(f, 0)\|_{X^r}.$$

(c) *The operator* $\mathcal{L}_\xi : \tilde{X}^{r+2} \to X^r_{3/2}$, $r \geq 0$, *is invertible and the inverse is bounded* $\forall \xi$. *The same result holds for* $\lambda + \mathcal{L}_\xi$, *too, when* $-\lambda$ *is not an eigenvalue of* $\mathcal{L}_\xi$.

(d) *Linear existence results, similar to Theorem* 2.9 *and Theorem* 2.11, *hold for* $\mathcal{L}_\xi$, $\forall \xi$, *too.*

Definition 5.5 (Generalized nonresonance condition). We say that the pair $\mu^\pm$ of pure imaginary eigenvalues of $\mathcal{L}_{\xi^*}$ satisfies the generalized nonresonance condition, when the following two requirements are fullfiled:

(a) the usual nonresonance condition: $\forall a \in \mathbb{Z} \setminus \{\pm 1\}$, $a\mu^+$ is not an eigenvalue of $\mathcal{L}_{\xi^*}$;

(b) a simplicity condition: for the fixed value ξ^* of the bifurcation parameter (for which we have proved the transversality condition), the eigenvalues $\mu^\pm$ of $\mathcal{L}_{\xi^*}$ are eigenvalues of $\mathcal{L}_{\xi^*}\big|_{\tilde{X}_{n,k}}$ only for one $n \in \mathbb{N}$ and for one $k \in \mathbb{Z}$.

We are now in position to formulate a Hopf bifurcation theorem for the full nonlinear problem. We can consider we have written it in the form (after similar transformations we have done in Section 2):

$$(\partial_t + \mathcal{L}_\xi)\begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} F(u, \eta) \\ 0 \end{pmatrix} \tag{5.18}$$

where F contains all the nonlinearities and correction terms. We recall that F has the following properties: for $r \geq 1$, $F : X^{r+2} \to H^r(\Omega_0)^3$, $F(0, 0) = 0$, DF exists and $DF(0, 0) = 0$.

Theorem 5.6 (Hopf bifurcation theorem). *For every space* $X_{n,k}$ *there exists a critical value* ξ^* *of the bifurcation parameter* ξ *such that* $\mathcal{L}_{\xi^*}$ *has a pair* $\mu^\pm$ *of purely imaginary eigenvalues and the transversality condition is fullfiled. We assume that this pair of eigenvalues satisfies the generalized nonresonance condition of Definition* 5.5.
Then a Hopf bifurcation occurs and there exists a branch of $\mathbb{Z}_k$*-symmetric, periodic solutions of the nonlinear equation.*

The proof is classical an will not be presented here. The only difficulty which appears is to verify the Fredholm-index-zero-property. For more details see [4].

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