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Smooth quotients of Abelian varieties by finite groups

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Abstract. We give a complete classification of smooth quotients of Abelian varieties by finite groups that fix the origin. In the particular case where the action of the group G on the tangent space at the origin of the Abelian variety A is irreducible, we prove that A is isomorphic to the self-product of an elliptic curve and $A/G \cong \mathbb{P}^n$. In the general case, assuming $\dim(A^G) = 0$, we prove that A/G is isomorphic to a direct product of projective spaces.

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1. Introduction

Quotients of Abelian varieties by finite groups have appeared in many different contexts and topics of research. For example, in [7] Kollár and Larsen study groups acting on simple Abelian varieties in dimension greater than or equal to 4, and prove that the quotient has canonical singularities and Kodaira dimension 0. This is done in the context of studying quotients of Calabi-Yau varieties by finite groups. In [6], Im and Larsen study the existence of rational curves lying on quotients of Abelian varieties by finite groups, and they find a condition on the group that implies that rational curves actually exist on the quotient.

Along another line, in [14] Yoshihara initiates the study of Galois embeddings of varieties, where he asks when a projective variety embedded into projective space admits a finite linear projection that is a Galois morphism. In particular, the existence of a Galois embedding implies that the variety has a finite group of automorphisms such that the quotient variety is isomorphic to projective space. Yoshihara finishes the paper by analyzing the case of Abelian surfaces. In [1], the first author generalizes Yoshihara's results to arbitrary dimension, and proves that if the quotient of an Abelian variety by a finite group is projective space, then the Abelian

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variety is isogenous to the self-product of an elliptic curve. As a matter of fact, when there is an action of an irreducible finite subgroup of $\mathrm{GL}(T_0(A))$ with Schur index 1 on an Abelian variety A, then A is isogenous to the self-product of an elliptic curve, as was proven in [11].

These examples show that quotients of Abelian varieties by finite groups have indeed garnered attention in varied contexts within algebraic geometry. On the other hand, group actions on Abelian varieties over $\mathbb{C}$ lead to the study of finite-dimensional complex representations via their universal covering space, and viceversa. In this sense, a classic article by Looijenga relates root systems and self-products of elliptic curves in [8]. There is also work by Popov [10] and Tokunaga-Yoshida [13] on complex crystallographic reflection groups, which are extensions Γ of a finite complex reflection group G by a G-stable lattice Λ in $\mathbb{C}^n$. In [13] the authors study the corresponding quotient $\mathbb{C}^n/\Gamma$ for $n = 2$ and in [4], Bernstein and Schwarzman do the same in arbitrary dimension for complex crystallographic groups of Coxeter type. Note that such quotients correspond to the quotient of the Abelian variety $A = \mathbb{C}^n/\Lambda$ by G. However, for a given finite complex reflection group G, *not every G-stable lattice comes from a complex crystallographic reflection group* and hence the study of smooth quotients of Abelian varieties remains an open question.

The purpose of this paper is to give a full classification of smooth quotients of Abelian varieties by finite groups in the particular case in which the group fixes the origin. Our main theorem states the following:

Theorem 1.1. *Let A be an Abelian variety of dimension $n \geq 3$, and let G be a (non trivial) finite group of automorphisms of A that fix the origin. Then the following conditions are equivalent:*

(1) *A/G is smooth and the action of G on T_0A is irreducible;*
(2) *A/G is smooth of Picard number 1;*
(3) *$A/G \cong \mathbb{P}^n$;*
(4) *There exists an elliptic curve E such that $A \cong E^n$ and (A, G) satisfies exactly one of the following:*
   (a) *$G \cong C^n \rtimes S_n$ where C is a non-trivial (cyclic) subgroup of automorphisms of E that fix the origin; here the action of C^n is coordinatewise and S_n permutes the coordinates;*
   (b) *$G \cong S_{n+1}$ and acts on*

$$A \cong \left\{(x_1, \ldots, x_{n+1}) \in E^{n+1} : x_1 + \cdots + x_{n+1} = 0\right\}$$

by permutations.

The two cases found in item (4) of the above theorem were studied in detail in [1], where it was proven that both examples give projective spaces as quotients. This gives the proof of (4) $\Rightarrow$ (3). Our theorem shows that these are the only cases that give smooth quotients in dimension $n \geq 3$. Throughout the paper we will refer to these two examples as Example (a) and Example (b), respectively.

Note that the case of dimension $n = 1$ is obvious: every pair (A, G) gives $\mathbb{P}^1$ as a quotient. For $n = 2$, according to Yoshihara (*cf.* [14]), this classification was already done by Tokunaga and Yoshida in [13]. This paper classifies 2-dimensional complex crystallographic reflection groups. However, as stated above, these do not cover all possible G-stable lattices and hence not all possible group actions on Abelian surfaces. The classification in this case was thus incomplete, but was recently achieved by P. Quezada and the authors in [2]. The outcome is that, in the irreducible case, there is only one example different from Examples (a) and (b) giving a smooth quotient: it is the pair (A, G) with $A = E^2$ for $E = \mathbb{C}/\mathbb{Z}[i]$ and G is the order 16 subgroup of $\mathrm{GL}_2(\mathbb{Z}[i])$ generated by:

$$\left\{\begin{pmatrix}-1 & 1+i\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-i & i-1\\ 0 & i\end{pmatrix}, \begin{pmatrix}-1 & 0\\ i-1 & 1\end{pmatrix}\right\},$$

acting on A in the obvious way.

An interesting corollary, which was a first motivation for writing this paper is the following:

Corollary 1.2. *If G is a finite group that acts on an Abelian variety A such that the elements of G fix the origin and $A/G \cong \mathbb{P}^n$, then A is isomorphic to the self-product of an elliptic curve.*

The general case is quickly reduced to the irreducible case.

Theorem 1.3 (*Cf.* Theorem 2.7). *Let G be a group that acts by algebraic homomorphisms on an Abelian variety A such that A/G is smooth. Assume that $\dim(A^G) = 0$. Then $G = \prod_{i=1}^r G_i$, $A = \prod_{i=1}^r A_i$ and each pair (A_i, G_i) satisfies the equivalent conditions from Theorem* 1.1 *above.*

When A^G has positive dimension, the situation does not necessarily split, but we can still describe the quotient A/G as a fibration over an Abelian variety with smooth fibers that are isomorphic to the quotients in Theorem 1.3. Actually, we prove in the general case that A/G is smooth if and only if P_G/G is smooth, where P_G is the complementary Abelian subvariety of the connected component of A^G that contains 0, *cf.* Proposition 2.9. The notation P_G comes from the fact that in the case that A is the Jacobian of a curve X and G is a group of automorphisms of X, P_G is the Prym variety associated to the morphism $X \to X/G$.

As an application of our main theorems, we expect to give in a subsequent paper a classification of quotients of principally polarized Abelian varieties by groups preserving the divisor class of the polarization. This will be applicable to the specific case of Jacobian varieties with group action coming from an action on the corresponding curve. As a final application, B. Lim pointed out to us that our classification would be a key ingredient in solving a conjecture by Polishchuk and Van den Bergh (*cf.* [9, Conjecture A]) on semiorthogonal decompositions of categories of equivariant coherent sheaves in the case of Abelian varieties.

The structure of this paper is as follows: In Section 2, we cover some basic properties of Abelian varieties with a finite group action and smooth quotient. In particular, we prove in Section 2.1 the implication (2) $\Rightarrow$ (1) from Theorem 1.1, while Section 2.2 is dedicated to the study of G-equivariant isogenies in this context, which are used in the sequel. In Section 2.3 we prove Theorem 1.3 and we briefly look at the ultimate general case in which A^G may have positive dimension. Section 3 is dedicated to the proof of (1) $\Rightarrow$ (4) (note that (3) $\Rightarrow$ (2) is evident and (4) $\Rightarrow$ (3) was established in [1], so this concludes the proof of Theorem 1.1). This is the heart of the article and therefore its longest and most technical part. Here we use Shephard-Todd's classification of irreducible complex reflection groups in order to study them case by case. The case of the symmetric group S_n is studied in Section 3.1 and the infinite family of groups $G(m, p, n)$ for $m \geq 2$ is studied in Section 3.2. Finally, Section 3.3 is dedicated to the remaining sporadic cases.

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2. Groups acting on Abelian varieties with smooth quotient

2.1. Generalities

Let A be an Abelian variety of dimension n and let G be a group of automorphisms of A that fix the origin, such that the quotient variety A/G is smooth. By the Chevalley-Shephard-Todd Theorem, the stabilizer in G of each point in A must be generated by pseudoreflections; that is, elements that fix a divisor pointwise, such that the divisor passes through the point. In particular, G is generated by pseudoreflections and G acts on the tangent space at the origin $T_0(A)$ (this is the analytic representation). Thus, an element in G is a pseudoreflection (at the origin) if and only if it fixes a hyperplane in $T_0(A)$ pointwise. We will often abuse notation and display G as either acting on A or $T_0(A)$; it will be clear from the context which action we are considering.

In what follows, let $\mathcal{L}$ be a fixed G-invariant polarization on A (take the pullback of an ample class on A/G, for example). For σ a pseudoreflection in G of order r, define

$$D_\sigma := \operatorname{im}\big(1 + \sigma + \cdots + \sigma^{r-1}\big),$$
$$E_\sigma := \operatorname{im}(1 - \sigma).$$

These are both Abelian subvarieties of A.

Lemma 2.1. *We have the following:*

1. *D_σ is the connected component of* $\operatorname{Fix}(\sigma) := \ker(1-\sigma)$ *that contains 0 and E_σ is the complementary Abelian subvariety of D_σ with respect to $\mathcal{L}$. In particular, D_σ is a divisor and E_σ is an elliptic curve;*

2. *$\langle\sigma\rangle$ acts faithfully on E_σ and hence $r \in \{2, 3, 4, 6\}$;*
3. *For $a \not\equiv 0 \pmod r$, $E_{\sigma^a} = E_\sigma$ and $D_{\sigma^a} = D_\sigma$;*
4. *$D_\sigma \cap E_\sigma$ consists of 2-torsion points for $r = 2, 4$, of 3-torsion points for $r = 3$ and $D_\sigma \cap E_\sigma = 0$ for $r = 6$.*

Proof. Since

$$\big(1 + \sigma + \cdots + \sigma^{r-1}\big)(1 - \sigma) = (1 - \sigma)\big(1 + \sigma + \cdots + \sigma^{r-1}\big) = 1 - \sigma^r = 0,$$

we see that $D_\sigma \subset \ker(1-\sigma)$ and $E_\sigma \subset \ker(1+\sigma+\cdots+\sigma^{r-1})$. If $x \in \ker(1-\sigma)$, then

$$rx = x + \sigma(x) + \cdots + \sigma^{r-1}(x) = \big(1 + \sigma + \cdots + \sigma^{r-1}\big)(x) \in D_\sigma,$$

and so after possibly adding an r-torsion point to x we obtain that it lies in D_σ. Therefore both spaces are of the same dimension and, since D_σ is irreducible, we get that it corresponds to the connected component containing 0.

To show that E_σ is the complementary Abelian subvariety of D_σ, let H be the first Chern class of $\mathcal{L}$, seen as a Hermitian form H on $T_0(A) = \mathbb{C}^n$. Then, since σ preserves the numerical class of $\mathcal{L}$, we have that $\sigma^t H = H\sigma^{-1}$. Hence

$$\left(\sum_{i=0}^{r-1} \sigma^i\right)^t H(I_n - \sigma) = H\left(\sum_{i=0}^{r-1} \sigma^{-i}\right)(I_n - \sigma) = 0.$$

This shows that the vector subspaces of $T_0(A)$ induced by D_σ and E_σ are orthogonal with respect to H; *i.e.* they are complementary Abelian subvarieties. This proves 1.

Since σ and $(1 - \sigma)$ clearly commute, we see that $\sigma(E_\sigma) = E_\sigma$ by definition. Moreover, if σ^i acts trivially on E_σ, then σ^i acts trivially on the whole variety A since $A = D_\sigma + E_\sigma$. This implies immediately that the action of $\langle\sigma\rangle$ is faithful and hence $r \in \{2, 3, 4, 6\}$. This proves 2. For the third assertion, we know that both D_σ and D_{σ^a} are irreducible divisors. But clearly $\ker(1 - \sigma^a) \supset \ker(1 - \sigma)$ and hence $D_\sigma = D_{\sigma^a}$. Complementarity implies then that $E_\sigma = E_{\sigma^a}$. Finally, note that since $D_\sigma \subset \ker(1 - \sigma)$ and $E_\sigma \subset \ker(1 + \sigma + \cdots + \sigma^{n-1})$, for every $x \in D_\sigma \cap E_\sigma$ we have

$$rx = x - \sigma(x) + x + \sigma(x) + \cdots + \sigma^{r-1}(x) = (1-\sigma)(x) + \big(1+\sigma+\cdots+\sigma^{r-1}\big)(x) = 0.$$

This proves that $D_\sigma \cap E_\sigma$ consists of r-torsion points. Using the third assertion for $a = 2, 3$ we prove 4. □

We are now in a position to prove that (2) $\Rightarrow$ (1) in Theorem 1.1; the proof goes along the lines of [1, Remark 2.1].

Proposition 2.2. *Let G be a finite group acting on an Abelian variety A via algebraic homomorphisms. Assume that A/G is smooth and the Picard number of A/G is 1. Then the analytic representation of G is irreducible.*

Proof. Assume that A/G is of Picard number 1. We will first show that G does not leave a non-trivial Abelian subvariety invariant. Indeed, let $X \subseteq A$ be an Abelian subvariety on which G acts, and let $N_X \in \mathrm{End}(A)$ be its norm endomorphism with respect to some fixed G-invariant polarization $\mathcal{L}$ (N_X on tangent spaces is just the orthogonal projection onto the linear subspace that defines X with respect to the first Chern class of $\mathcal{L}$). Now

$$N_X^*\mathcal{L} \in \mathrm{NS}(A)_{\mathbb{Q}}^G \cong \mathrm{NS}(A/G)_{\mathbb{Q}} \cong \mathbb{Q},$$

where the subscript $\mathbb{Q}$ indicates that we extended scalars to $\mathbb{Q}$. Since $\mathcal{L} \in \mathrm{NS}(A)_{\mathbb{Q}}^G$, we have that $N_X^*\mathcal{L}$ is a rational multiple of $\mathcal{L}$ and therefore the self-intersection number $(N_X^*\mathcal{L})^n$ is non zero. However, by [3, Proposition 3.1], if X is non-trivial then this number must be zero. Therefore X must be trivial.

Now let W be a G-stable linear subspace of $T_0(A)$, and let $\sigma \in G$ be a pseudoreflection. Since the image of $1-\sigma$ is an elliptic curve on A induced, say, by a linear subspace $\langle z_0 \rangle \leq T_0(A)$, we have that for every $z \in W$, $(1-\sigma)(z) = \lambda_z z_0$ for some $\lambda_z \in \mathbb{C}$.

If $\lambda_z \neq 0$ for some $z \in W$, then $z_0 \in W$. Now, since the translates of z_0 by G all lie in W and $\sum_{\tau \in G} \tau(E_\sigma) = A$ by the previous discussion, we have that $W = T_0(A)$.

Assume now that $\lambda_z = 0$ for every $z \in W$ and every pseudoreflection $\sigma \in G$. In particular, W is fixed by every σ, and since these generate the group, we have that W is fixed pointwise by G. Now, since G does not fix pointwise any non-trivial Abelian subvariety of A, we have that

$$\bigcap_{\tau \in G} \ker(1-\tau) \subseteq A$$

is finite and so its preimage in $T_0(A)$ is discrete. However W is contained in this preimage, and so it must be trivial. □

2.2. *G*-equivariant isogenies

We will consider now a new Abelian variety B equipped with a G-equivariant isogeny to A, which we will call a G-isogeny from now on. Let Λ_A denote the lattice in $\mathbb{C}^n$ such that $A = \mathbb{C}^n/\Lambda_A$. Let $\Lambda_B \subseteq \Lambda_A$ be a G-invariant sublattice, and let $B := \mathbb{C}^n/\Lambda_B$ be the induced Abelian variety, along with the G-isogeny

$$\pi : B \to A,$$

whose analytic representation is the identity. Note that this implies that $\sigma \in G$ is a pseudoreflection of B if and only if it is a pseudoreflection of A. We may then consider the subvarieties $E_\sigma, D_\sigma \subset A$ defined as above, which we will denote by $E_{\sigma,A}$ and $D_{\sigma,A}$. Now, we can do the same thing for B and hence we obtain subvarieties $E_{\sigma,B}, D_{\sigma,B} \subset B$. Note that, by definition, π sends $E_{\sigma,B}$ to $E_{\sigma,A}$ and $D_{\sigma,B}$ to $D_{\sigma,A}$.

Define $\Delta := \ker(\pi)$. Since π is G-equivariant, G acts on Δ and hence we may consider the group $\Delta \rtimes G$. This group acts on B in the obvious way: Δ acts by translations and G by automorphisms. In particular, we see that the quotient $B/(\Delta \rtimes G)$ is isomorphic to A/G.

Our goal is to reduce as much as we can the structure of B/G and Δ and to prove that the latter must be trivial in several cases. Fix then a pseudoreflection $\sigma \in G$ of order r and consider the subvarieties $E_{\sigma,A}$, $D_{\sigma,A} \subset A$ and $E_{\sigma,B}$, $D_{\sigma,B} \subset B$. Define moreover $F_{\sigma,A} = E_{\sigma,A} \cap D_{\sigma,A}$ and $F_{\sigma,B}$ similarly. Then the isogeny $\pi : B \to A$ sends $F_{\sigma,B}$ to $F_{\sigma,A}$.

Lemma 2.3. *Assume that the map $E_{\sigma,B} \to E_{\sigma,A}$ is injective and that the map $F_{\sigma,B} \to F_{\sigma,A}$ is surjective. Then $\Delta \subset D_{\sigma,B}$.*

Proof. Since $E_{\sigma,B}$ and $D_{\sigma,B}$ generate B, we have $z = x + y \in \Delta = \ker(\pi)$ with $x \in E_{\sigma,B}$ and $y \in D_{\sigma,B}$ for every $z \in \Delta$. Then $\pi(z) = 0$ implies $\pi(x) = -\pi(y) \in F_{\sigma,A}$. But since $F_{\sigma,B} \to F_{\sigma,A}$ is surjective and $E_{\sigma,B} \to E_{\sigma,A}$ is injective, we have that $x \in E_{\sigma,B} \cap \pi^{-1}(F_{\sigma,A}) = F_{\sigma,B}$. Thus $x \in D_{\sigma,B}$ and hence $\Delta \subset D_{\sigma,B}$. □

Since all conjugates of a pseudoreflection are pseudoreflections and everything is G-equivariant, we immediately get the following result.

Proposition 2.4. *Let $\sigma \in G$ be a pseudoreflection and assume that the map $E_{\sigma,B} \to E_{\sigma,A}$ is injective and that the map $F_{\sigma,B} \to F_{\sigma,A}$ is surjective. Then the subgroup $\Delta = \ker(\pi)$ is contained in $D_{\tau\sigma\tau^{-1},B}$ for* every $\tau \in G$.

We conclude this section by studying pseudoreflections in $\Delta \rtimes G$.

Lemma 2.5. *Let $\sigma \in \Delta \rtimes G$ be a pseudoreflection. Then $\sigma = (t, \tau)$ with $\tau \in G$ a pseudoreflection and $t \in \Delta \cap E_{\tau,B}$.*

Proof. Let $t \in \Delta$ and $\tau \in G$ be such that $\sigma = (t, \tau) \in \Delta \rtimes G$. This element acts on B sending x to $\tau(x) + t$. By definition, σ must fix a divisor pointwise, that is, there is a subvariety $C \subset B$ of codimension 1 such that $x = \tau(x) + t$ for all $x \in C$, or equivalently, $x \in (1 - \tau)^{-1}(t)$. But since $1 - \tau \in \mathrm{End}(B)$, we see that C is a translate of $\ker(1 - \tau)$, which is a divisor if and only if τ is a pseudoreflection and $t \in (1 - \tau)(B) = E_{\tau,B}$. □

2.3. Reduction to irreducible representations

Let G be a group that acts by algebraic homomorphisms on an Abelian variety A such that A/G is smooth. In particular the analytic representation of G on $T_0(A)$ is a finite complex reflection group. It is well-known (*cf.* for instance [12] or [10, Section 1.4]), that $G \cong G_1 \times \cdots \times G_r$ and $T_0(A) = W_0 \oplus W_1 \oplus \cdots \oplus W_r$ where:

- W_i is an irreducible complex representation of G_i that makes G_i an irreducible finite complex reflection group for $i > 0$;
- G_j acts trivially on W_i for $i \neq j$.

In particular, $W_0 = T_0(A)^G$.

Lemma 2.6. *The subspace W_i induces a G-stable Abelian subvariety A_i of A such that G_j acts trivially on A_i for $i \neq j$. Moreover, $A_i/G = A_i/G_i$ is smooth.*

Proof. Since $W_0 = T_0(A)^G$, then A_0 is the neutral connected component of A^G and $A_0/G = A_0$. Assume now $i > 0$, let $\sigma \in G_i$ be a pseudoreflection and let L be the linear subspace of $T_0(A)$ that induces E_σ. It is clear that $L \subseteq W_i$, since $L = (1-\sigma)(T_0(A))$. Since the representation of G_i on W_i is irreducible, we have that

$$W_i = \sum_{\tau \in G} (\tau(L)).$$

Therefore, W_i is the tangent space of the Abelian subvariety $A_i = \sum_{\tau \in G} \tau(E_\sigma)$. It is clear that A_i is G-stable and G_j acts trivially on A_i for $i \neq j$ so that $A_i/G = A_i/G_i$. Moreover, since $\mathrm{Stab}_{G_i}(x) = \mathrm{Stab}_G(x) \cap G_i$ for $x \in A_i$ and every pseudoreflection in G belongs to some G_j, it is easy to see that $\mathrm{Stab}_{G_i}(x)$ is generated by pseudoreflections in G_i whenever $\mathrm{Stab}_G(x)$ is generated by pseudoreflections in G. This is the case by the Chevalley-Shephard-Todd Theorem because A/G is smooth and therefore A_i/G_i is smooth. □

We can now prove that, whenever A_0 is trivial, it is enough to understand the case when the action of G on $T_0(A)$ is irreducible.

Theorem 2.7. *Let G be a group that acts by algebraic homomorphisms on an Abelian variety A such that A/G is smooth. Assume that* $\dim(A^G) = 0$. *Then A is the direct product of the A_i, defined as above. In particular,*

$$A/G \cong A_1/G_1 \times \cdots \times A_r/G_r.$$

We will need the following small result on irreducible finite complex reflection groups:

Lemma 2.8. *Let G be a finite complex reflection group acting irreducibly on $\mathbb{C}^n$. Then there exists $\tau \in G$ such that $(1-\tau)$ is surjective.*

Proof. This amounts to finding an element $\tau \in G$ such that 1 is not an eigenvalue of τ. Now this follows directly from [12, Theorem 5.4]. □

Proof of Theorem 2.7. Consider the subvarieties $A_i \subset A$ from Lemma 2.6 for $i \geq 1$ (A_0 is trivial by the hypothesis on A^G). Then there is a natural G-isogeny

$$B := A_1 \times \cdots \times A_r \to A,$$

given by the sum in A. In particular, the kernel of this isogeny is

$$\Delta := \left\{ (a_1, \ldots, a_r) \in A_1 \times \cdots \times A_r \mid \sum_{i=1}^{r} a_i = 0 \right\}.$$

We claim that Δ is fixed pointwise by G. Indeed, since $a_i \in A_i$, we know that G_j acts trivially on it for $j \neq i$; but since $a_i = -\sum_{j\neq i} a_j \in \sum_{j\neq i} A_j$, we also know that G_i acts trivially on it (since it acts trivially on every A_j for $j \neq i$). We see then that G acts trivially on every coordinate of every element of Δ, which proves the claim.

Thus, $\Delta \times G$ acts on B and hence A/G is isomorphic to $B/(\Delta \times G)$, *i.e.*

$$A/G \cong \big[(A_1/G_1) \times \cdots (A_r/G_r)\big]/\Delta.$$

All we need to prove now is that Δ has to be trivial. Assume then that this is not the case and note that the action of $(a_1, \dots, a_r) \in \Delta$ on $X := (A_1/G_1) \times \cdots (A_r/G_r)$ corresponds coordinatewise to the action of a_i on A_i/G_i (which is well defined since a_i is G_i-invariant and thus its action commutes with that of G_i). Now, the action of a_i on A_i/G_i always has a fixed point p_i. Indeed, by Lemma 2.8 we know that there exists $\tau \in G_i$ such that $(1 - \tau)$ is surjective. Thus, there exists $x_i \in A_i$ such that $x_i - \tau(x_i) = a_i$, which implies that the image p_i of x_i in A_i/G_i is fixed by a_i. We see then that $(p_1, \dots, p_r) \in X$ is a point that is fixed by $(a_1, \dots, a_r)$ and thus the action of Δ on X is not free. It is also a non-trivial action since the image of $0 \in B$ in X is clearly moved by Δ.

Since $A/G = X/\Delta$ is smooth, the Chevalley-Shephard-Todd Theorem tells us then that every stabilizer of this action has to be generated by pseudoreflections. Now this is impossible since, for every non-trivial $(a_1, \dots, a_r) \in \Delta$, its fixed locus in X corresponds to the product of the fixed loci in each A_i/G_i via a_i. We see then that if any element in Δ is a pseudoreflection, it must fix all but one A_i/G_i (otherwise the fixed locus would not be a divisor), which amounts to $a_i = 0$ for all but one i, and this is impossible since $\sum_{i=1}^r a_i = 0$. This proves that Δ is trivial. ☐

Let us consider now the "degenerate" case in which $\dim(A^G) > 0$.

Proposition 2.9. *Let G be a group that acts by algebraic homomorphisms on an Abelian variety A. Let A_0 be the connected component of A^G containing 0 and let P_G be its complementary Abelian subvariety with respect to a G-invariant polarization. Then there exists a fibration $A/G \to A_0/(A_0 \cap P_G)$ with fibers isomorphic to P_G/G. Moreover, A/G is smooth if and only if P_G/G is smooth.*

Proof. Consider as in the last proof the natural G-isogeny $A_0 \times P_G \to A$ and denote its kernel by Δ. This can be rewritten as

$$A \cong (A_0 \times P_G)/\Delta.$$

Now, the same argument from the proof above shows that Δ is fixed pointwise by G. In particular, the actions of G and Δ on P_G commute and it is easy to see then that

$$A/G \cong ((A_0 \times P_G)/G)/\Delta \cong (A_0 \times (P_G/G))/\Delta.$$

Recalling that $\Delta \cong A_0 \cap P_G$, we get a natural projection $A/G \to A_0/(A_0 \cap P_G)$ whose fibers are easily seen to be isomorphic to P_G/G.

Finally note that, since the action of Δ on A_0 is free, the quotient $A/G = (A_0 \times P_G/G)/\Delta$ is smooth whenever P_G/G is. On the other hand, by the same argument we used for A_i/G_i, P_G/G is smooth if A/G is. □

Note that this fibration is non-trivial in general, as shown by the following example: Let E be an elliptic curve and let $e \in E[2]$. Define $B = E \times E$ and let $G = \{\pm 1\}$ act on the second factor. Note in particular that (e, e) is G-invariant. Put then $A = B/\langle (e, e)\rangle$ and denote by $\pi : B \to A$ the projection. We have that $A_0 = \pi(E \times \{0\})$, $P_G = \pi(\{0\} \times E)$, $B = A_0 \times P_G$ and $\Delta = \langle (e, e)\rangle$. We see then that

$$A/G \cong B/(\Delta \times G) \cong (B/G)/\Delta \cong \big(E \times \mathbb{P}^1\big)/\Delta,$$

where, up to a base change in $\mathbb{P}^1$, Δ acts on $E \times \mathbb{P}^1$ by sending (x, y) to $(x+e, -y)$. Looking at the first coordinate, we see then that the action is free and thus defines by étale descent a non-trivial $\mathbb{P}^1$-bundle over the elliptic curve $E' = E/\langle e\rangle$. This bundle can be given explicitly as follows: consider the constant sections $E \to E \times \mathbb{P}^1$ given by 0 and ∞. Since the action of Δ fixes both points on $\mathbb{P}^1$, we see that this section passes to the quotient, defining two sections $E' \to A/G$ with trivial intersection. Then by [5, Ex. V.2.2], the bundle is decomposable. A direct computation tells us then that it corresponds to the projectivization of $\mathcal{O}_{E'} \oplus \mathcal{O}_{E'}(D)$, where $D = [e'] - [0]$ and e' generates the kernel of the isogeny $E' \to E$ dual to the natural projection $E \to E'$.

3. Quotients by irreducible finite complex reflection groups

Given the results from the last section, we will now concentrate on group actions on Abelian varieties that satisfy the following condition (which is condition (1) from Theorem 1.1):

$$A/G \text{ is smooth and the analytic representation of } G \text{ is irreducible.} \tag{$\star$}$$

If the pair (A, G) satisfies ($\star$), we see that the analytic representation makes G an irreducible finite complex reflection group, in the sense of Shephard-Todd [12]. These groups were completely classified by Shephard and Todd in [12], where they discovered that any finite irreducible complex reflection group is either a group $G(m, p, n)$ depending on $m, p, n \in \mathbb{Z}_{>0}$ where $p \mid m$ and $n \geq 1$, or is one of 34 sporadic cases. The group $G(m, p, n)$ consists of the semidirect product $H \rtimes S_n$ of the Abelian group

$$H = H(m, p, n) = \big\{\big(\zeta_m^{a_1}, \dots, \zeta_m^{a_n}\big) \mid a_1 + \cdots + a_n \equiv 0 \pmod{p}\big\} \subset \mu_m^n \tag{3.1}$$

with the symmetric group S_n, where ζ_m is a primitive m-th root of unity and S_n acts on each member by permuting the coordinates in the obvious way. For $m = p = 1$,

$G(1, 1, n)$ is just the symmetric group on n letters, and acts irreducibly on an $(n-1)$-dimensional complex vector space. For $m > 1$, $G(m, p, n)$ acts irreducibly on an n-dimensional complex vector space.

The purpose of this section is to describe which of these actions actually appear on Abelian varieties of dimension $n \geq 3$ such that ($\star$) is satisfied. In the following subsections we will analyze each case of the Shephard-Todd classification. In particular, in this section we prove (1) $\Rightarrow$ (4) of Theorem 1.1.

3.1. The case $m = p = 1$: the standard representation of S_{n+1}

Let $G(1, 1, n+1) = S_{n+1}$ act on an Abelian variety A of dimension $n \geq 2$ in such a way that its action on $T_0(A)$ is the standard one. Let $\sigma = (1\,2)$ and $E = E_\sigma$ be induced by a line $L_\sigma \subseteq T_0(A)$, and define the lattice

$$\Lambda_B := \sum_{\tau \in S_{n+1}} \tau(L_\sigma \cap \Lambda_A).$$

This gives us a G-invariant sublattice of Λ_A, and we therefore get a G-equivariant isogeny $\pi : B \to A$ with kernel Δ. Applying this construction to Example (b), we see that it gives the whole lattice and hence corresponds to Example (b) itself. We can thus see B as

$$B = \left\{(x_1, \ldots, x_{n+1}) \in E^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\right\}$$

and S_{n+1} acts coordinatewise in the natural way. Using the notations from Section 2.2, we see by inspection that $F_{\sigma,B} = E_{\sigma,B}[2] \cong E[2]$, hence the map $\pi : F_{\sigma,B} \to F_{\sigma,A}$ is surjective since by Lemma 2.1 we have $F_{\sigma,A} \subset E_{\sigma,A}[2] \cong E[2]$. Moreover, the induced map $E_{\sigma,B} \to E_{\sigma,A}$ is injective by construction. Thus, by Proposition 2.4, we have that Δ is contained in the fixed locus of all the conjugates of σ. In other words, Δ consists of elements of the form $(x, \ldots, x) \in E^{n+1}$ such that $(n+1)x = 0$. Note that this implies that the direct product $\Delta \times G$ acts on B.

Proposition 3.1. *Let $n \geq 2$. If S_{n+1} acts on A in such a way that its analytic representation is the standard representation and (A, S_{n+1}) satisfies ($\star$), then $A \cong E^n$ and S_{n+1} acts as in Example* (b).

Proof. Let $\pi : B \to A$ be the G-isogeny defined above. We have to prove then that $\Delta = \{0\}$. Let $\bar{t} = (t, \ldots, t) \in \Delta$ be a non-trivial element and let $\tau \in G$ be an element such that $(1 - \tau)$ is surjective (such an element exists by Lemma 2.8). Then there exists an element $z \in B$ such that $z - \tau(z) = \bar{t}$ and thus the stabilizer of z contains the element $(\bar{t}, \tau) \in \Delta \times G$.

Note now that $\Delta \cap E_{\sigma,B} = \{0\}$ for every pseudoreflection $\sigma \in G$. Thus, by Lemma 2.5, the only pseudoreflections in $\Delta \times G$ are the transpositions in $G = S_{n+1}$, and so $\mathrm{Stab}_G(z)$ cannot be generated by pseudoreflections. Therefore if $\Delta \neq 0$, A/G is not smooth by the Chevalley-Shephard-Todd Theorem, which contradicts condition ($\star$). $\square$

3.2. The case of $G(m, p, n)$, $m \geq 2$, $n \geq 3$

Now we will study when $G = G(m, p, n)$ acts on an Abelian variety A of dimension n for $m \geq 2$. *We assume here that* $n \geq 3$ (recall that the case of dimension 2 was already dealt with elsewhere). Recall that $G = H \rtimes S_n$, where $H \subset \mu_m^n$ is defined in (3.1) and it acts coordinatewise on $\mathbb{C}^n = T_0(A)$, while S_n permutes the variables in the obvious way.

Remark 3.2. In what follows, we will try as much as we can to prove results on G without splitting into subcases depending on the value of p. Hence, in the following arguments we will only consider elements in $G(m, m, n) \subset G(m, p, n)$, even if in some cases a simpler argument can be found for certain values of p. We will also keep all arguments (with one exception) depending on at most three dimensions, so that they are all valid for $n \geq 3$.

Let E_i be the image of $\mathbb{C}e_i$ in A via the exponential map. We claim that it corresponds to an elliptic curve. Indeed, consider the element $\tau = (1, \zeta_m, \zeta_m^{-1}, 1, \dots, 1) \in H$ and denote $\rho = 1 + \tau + \cdots + \tau^{m-1}$. Then a direct computation shows that, for $\sigma = (1\,2) \in S_n \subset G$, $\operatorname{im}(\rho(1-\sigma)) = \mathbb{C}e_1$. This tells us that $E_1 = \rho(1-\sigma)(A)$ and hence it corresponds to an elliptic curve. This allows us to prove the following.

Lemma 3.3. *Assume that G acts on A as above. Then $m \in \{2, 3, 4, 6\}$ and, if $m \geq 3$, then the curves $E_i \subset A$ have non-trivial automorphisms.*

Proof. Consider the curve $E_1 \subset A$ defined as above. We see then that the element $(\zeta_m, \zeta_m^{-1}, 1, \dots, 1) \in H$ induces an automorphism of order m of E_1. Therefore $m \in \{2, 3, 4, 6\}$ and, if $m \geq 3$, then E_1 has non-trivial automorphisms. The other E_i are obtained from E_1 via the action of S_n and hence are isomorphic to it. □

Now, let Λ_A be a lattice for A in $\mathbb{C}^n$. Then $\mathbb{C}e_i \cap \Lambda_A$ corresponds to the lattice of E_i in $\mathbb{C} = \mathbb{C}e_i$. We can thus define the G-stable sublattice of Λ_A

$$\Lambda_B := \bigoplus_{i=1}^{n} (\mathbb{C}e_i \cap \Lambda_A).$$

As in Section 2.2, this defines a G-isogeny $\pi : B \to A$. Moreover, we see that $B \cong E_1 \times \cdots \times E_n \cong E^n$ and that $\pi|_{E_i}$ is an injection. As in the previous section, let Δ be the kernel of π. We will study the different possible quotients A/G by studying the possible quotients $B/(\Delta \rtimes G)$ and thus by studying the possible Δ's. Let us start with the case of a trivial Δ:

Proposition 3.4. *Let $G = G(m, p, n)$ with $n \geq 2$ act on $B = E^n$ as above. Then the quotient B/G is smooth if and only if $p = 1$.*

Proof. By Lemma 3.3, we know that $m \in \{2, 3, 4, 6\}$. Thus, if $p = 1$, the action of G on B is by construction the same as in Example (a), which tells us that $B/G \cong \mathbb{P}^n$ and hence it is smooth.

Assume now that $p \geq 2$. By Lemma 3.3, we also know that if $m \neq 2$ then E has non-trivial automorphisms given by multiplication by ζ_m. In particular, E is a very specific curve in each of these cases and it is easy to see that:

- If $m = 3, 6$, then there exists a non-trivial $t \in E[3]$ such that $\zeta_6 t = -t$;
- If $m = 4$, then there exists a non-trivial $t \in E[2]$ such that $\zeta_4 t = t$.

Consider one such element $t \in E$ unless $(m, p) \in \{(2, 2), (6, 2)\}$, in which case take any non-trivial element $t \in E[2]$. Let $(x_3, \dots, x_n) \in E^{n-2}$ be a general element. Then, if $\bar{x} = (t, 0, x_3, \dots, x_n) \in B = E^n$, we immediately see that an element in $\mathrm{Stab}_G(\bar{x})$ must be in $H \subset G$ since the coordinates cannot be permuted, even after applying automorphisms on some coordinates via H. A direct computation tells us then that $\mathrm{Stab}_G(\bar{x})$ is equal to the (Abelian) subgroup of $H \subset G$ given in each case by the following table:

(m, p)	Generators of $\mathrm{Stab}_G(\bar{x})$
(2,2)	$(-1, -1, 1, \dots, 1)$
(3,3)	$(\zeta_3, \zeta_3^{-1}, 1, \dots, 1)$
(4,2)	$(\zeta_4, \zeta_4, 1, \dots, 1), (-1, 1, 1, \dots, 1), (1, -1, 1, \dots, 1)$
(4,4)	$(\zeta_4, \zeta_4^{-1}, 1, \dots, 1)$
(6,2)	$(-1, -1, 1, \dots, 1), (1, \zeta_3, 1, \dots, 1)$
(6,3)	$(\zeta_3, \zeta_3^{-1}, 1, \dots, 1), (1, -1, 1, \dots, 1)$
(6,6)	$(\zeta_3, \zeta_3^{-1}, 1, \dots, 1)$

However we observe that in all cases the first element is not a pseudoreflection, since its fixed locus is of codimension 2. Moreover, the only pseudoreflections in $\mathrm{Stab}_G(\bar{x})$ are the other given generators (and their powers) and hence they cannot generate the first one. Therefore, by the Chevalley-Shephard-Todd Theorem, the quotient B/G is not smooth. ☐

Let us consider now the case of a non-trivial kernel Δ. We start with an application of Proposition 2.4.

Lemma 3.5. *If Δ is non-trivial, then $m \neq 6$ and, if we define the following type of elements in Δ:*

- *Diagonal: $(t, \dots, t)$ with $t \in E$;*
- *Hyperplanar: $(t, -t, 0, \dots, 0)$ with $t \in E$;*

then Δ contains a non-trivial hyperplanar element unless *it consists purely of diagonal elements. Moreover, the coordinates of every hyperplanar element are invariant by ζ_m, so in particular these elements are 2-torsion if $m = 2, 4$ and 3-torsion if $m = 3$.*

Proof. Let $\sigma = (1\,2)$ and note that $(t, -t, 0, \dots, 0) \in E_{\sigma,B}$. Then there being no non-trivial hyperplanar element in Δ amounts to $E_{\sigma,B} \to E_{\sigma,A}$ being an isomorphism. By inspection, we see that $F_{\sigma,B} = E_{\sigma,B}[2]$ and we can thus apply Proposition 2.4, which tells us that elements in Δ are invariant by *every* transposition, hence diagonal.

Assume now that Δ contains a hyperplanar element $\bar{t}$. Then, since Δ is G-stable, we have that, for $\rho_1 = (\zeta_m, 1, \zeta_m^{-1}, 1, \dots, 1) \in H$,

$$(1 - \rho_1)(\bar{t}) = ((1 - \zeta_m)t, 0, \dots, 0) \in \Delta.$$

But, by construction, there are no elements of the form $(x, 0, \dots, 0)$ in Δ. We deduce then that t is ζ_m-invariant. The assertion on the torsion of its coordinates follows immediately.

Assume finally that $m = 6$ and let $(t_1, \dots, t_n) \in \Delta$. Define $\sigma_i = (1\,i) \in S_n \subset G$ and $\rho_2 = (\zeta_6^{-1}, \zeta_6, 1, \dots, 1) \in H \subset G$. Then

$$[(1 - \rho_2)(1 - \rho_1)\sigma_i](\bar{t}) = (t_i, 0, \dots, 0) \in \Delta,$$

which implies as above that $t_i = 0$ and thus $\Delta = 0$. □

Let us study now pseudoreflections in $\Delta \rtimes G$. Define the elements

$$\begin{aligned} \rho &:= \big(\zeta_m, \zeta_m^{-1}, 1, \dots, 1\big) \in H \subset G; \\ \sigma &:= (1\,2) \in S_n \subset G; \\ \tau &:= \big(\zeta_m^p, 1, \dots, 1\big) \in H \subset G. \end{aligned}$$

Then there are two types of pseudoreflections in G:

(I) Conjugates of $\rho^a\sigma$ for $0 \le a < p$;
(II) Conjugates of powers of τ (these do not exist if $m = p$);

and the corresponding elliptic curves in B are respectively:

$$\begin{aligned} E_{\rho^a\sigma} &= \big\{(x, -\zeta_m^a x, 0, \dots, 0) \mid x \in E\big\}; \\ E_\tau &= \{(x, 0, 0, \dots, 0) \mid x \in E\}. \end{aligned}$$

Note now that elements of the form $(x, 0, \dots, 0)$ are not in Δ by construction of the isogeny $\pi : B \to A$. Using Lemmas 2.5 and 3.5, we see then that pseudoreflections in $\Delta \rtimes G$ that are not in G must be of the form:

(III) Conjugates of $(\bar{t}, \rho^a\sigma) \in \Delta \rtimes G$ for $0 \le a < p$;

where $\bar{t} = (t, -t, 0, \dots, 0) \in \Delta$ and t is ζ_m-invariant.

With these considerations, we can restrict further the structure of Δ. For instance, diagonal elements in Δ are bound to bring problems since they do not belong to any elliptic curve E_υ for a pseudoreflection $\upsilon \in G$. Thus, they cannot bring up new pseudoreflections in $\Delta \rtimes G$ unless they are generated by hyperplanar elements. This is explained by the following proposition.

Proposition 3.6 (Δ is not diagonal). *Assume that there exists $s \in E$ such that $(s, \dots, s) \in \Delta$ but $(s, -s, 0, \dots, 0) \notin \Delta$. Then A/G is not smooth.*

In particular, we see that Δ has to contain at least one hyperplanar element.

Proof. Since $A/G \cong B/(\Delta \rtimes G)$, we will work with this last quotient using the Chevalley-Shephard-Todd theorem.

Let $\bar{s} \in \Delta$ denote the diagonal element in the statement of the proposition. We will prove first that an element of the form $(\bar{s}, \upsilon)$ cannot be generated by pseudoreflections in $\Delta \rtimes G$. Indeed, the only pseudoreflections that are not in G are those of type (III), so that if $(\bar{s}, \upsilon)$ was generated by pseudoreflections, we should be able to write

$$\bar{s} = \sum_{i=1}^{\ell} \upsilon_i(\bar{t}_i), \tag{3.2}$$

with $\bar{t}_i = (t_i, -t_i, 0 \dots, 0) \in \Delta$ a hyperplanar element and $\upsilon_i \in G$. In particular, $\bar{s}$ would be contained in the sub-G-module of Δ generated by the $\bar{t}_i$. But since t_i is ζ_m-invariant, the only way in which G acts on the $\bar{t}_i$ is by permuting their coordinates. Thus, by looking at the first coordinate in equation (3.2), we get that s is a linear combination of the t_i, which implies immediately that $(s, -s, 0, \dots, 0)$ is a linear combination of the $\bar{t}_i$ and hence is in Δ, contradicting our hypothesis.

Having proved this, it suffices then to exhibit an element $\bar{x} \in B$ such that its stabilizer in $\Delta \rtimes G$ has an element of the form $(\bar{s}, \upsilon)$. In other words, we need $\upsilon \in G$ and $\bar{x} \in B$ such that $\upsilon(\bar{x}) + \bar{s} = \bar{x}$, and this is a direct consequence of Lemma 2.8. □

Denote by E_0 the subgroup of ζ_m-invariant elements of E. Then E_0 is equal to $E[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ if $m = 2$, isomorphic to $\mathbb{Z}/3\mathbb{Z}$ if $m = 3$ and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $m = 4$. Now that we know that diagonal elements in Δ only appear if generated by hyperplanar elements, Lemma 3.5 tells us that Δ is contained in $E_0^n = B^H$, and more precisely in the "hyperplane"

$$\mathcal{H} := \left\{ (x_1, \dots, x_n) \in E_0^n \mid \sum_{i=1}^{n} x_i = 0 \right\} \subset E_0^n \subset B. \tag{3.3}$$

Indeed, for $m = 3, 4$ the mere presence of a hyperplanar element implies by G-stability that Δ actually contains the whole "hyperplane" and thus the presence of any additional element in Δ would imply the existence of elements of the form $(x, 0, \dots, 0)$, which is forbidden by construction. A similar argument using Proposition 3.6 works for $m = 2$. In this last case, one hyperplanar element does not suffice to generate the whole subgroup $\mathcal{H} \subset E_0^n$ since $E_0 = E[2]$ needs two generators. We prove now that if Δ is not the whole $\mathcal{H}$, things do not work either.

Proposition 3.7. *Assume that $m = 2$, $\Delta \neq \{0\}$ and there exists a hyperplanar 2-torsion element that is not in Δ. Then A/G is not smooth.*

Proof. As before, we can use the Chevalley-Shephard-Todd Theorem on the quotient $B/(\Delta \rtimes G) \cong A/G$.

By the previous proposition, we may assume that Δ has a non-trivial element $\bar{t} = (t, t, 0, \dots, 0) \in \Delta$ with $t \in E[2]$. But since $E[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$, we easily see from the hypothesis that there are no elements of the form $(s, s, 0, \dots, 0)$ for $s \neq 0, t$.

Let $s \in E[2]$ be such an element. Let $t_1 \in E[4]$ be such that $2t_1 = t$ and let $t_2 = t_1 + s \in E[4]$. Let $(x_3, \dots, x_n) \in E^{n-2}$ be a general element and consider the element $\bar{x} = (t_1, t_2, x_3, \dots, x_n) \in B$. Recalling the notations given in page 686, it is easy to see that $(\bar{t}, \rho) \in \Delta \rtimes G$ fixes $\bar{x}$. Since $t_1 \neq \pm t_2$, it is also easy to see that no element in G fixes $\bar{x}$, so that pseudoreflections fixing $\bar{x}$ can only be of type (III), that is either $(\bar{t}, \sigma)$ or $(\bar{t}, \rho\sigma)$. But again, since $t_1 \neq \pm t_2$, we see that neither of these fixes $\bar{x}$. Thus, $\mathrm{Stab}_{\Delta \rtimes G}(\bar{x})$ is not generated by pseudoreflections and hence $B/(\Delta \rtimes G)$ cannot be smooth by the Chevalley-Shephard-Todd Theorem. □

Thus, we are reduced to the "full hyperplanar" case, that is, $\Delta = \mathcal{H}$. We prove then the following:

Proposition 3.8 (Δ is not hyperplanar). *Assume that* $\Delta = \mathcal{H}$ (cf. (3.3)). *Then A/G is not smooth **except** if $G = G(2, 2, 3)$.*

Proof. As always, it will suffice to give an element $\bar{x} \in B = E^n$ such that its stabilizer in $\Delta \rtimes G$ is not generated by pseudoreflections. The idea, as in the last proof, is to exhibit an element whose coordinates x_i are "different enough" so that it is clear that elements in $S_n \subset G$ cannot appear in $\mathrm{Stab}_{\Delta \rtimes G}(\bar{x})$, even after being mixed up with elements of $\Delta \times H \subset \Delta \rtimes G$. This amounts to ensuring that different coordinates do not belong to the same $(E_0 \times \mu_m)$-orbit (this is how $\Delta \times H$ acts on coordinates). Then the stabilizer must be contained in $\Delta \times H$ and hence it is easy to exhibit examples that are not generated by pseudoreflections.

Consider then the following element $\bar{x} \in B$:

- If $G = G(2, p, n)$ and $n \geq 4$, then $\bar{x} = (0, a', b', c', x_5, \dots, x_n)$.
  Here, $(x_5, \dots, x_n) \in E^{n-4}$ is a general element and $2a' = a$, $2b' = b$, $2c' = c$, where $E[2] = \{0, a, b, c\}$;
- If $G = G(2, 1, 3)$, then $\bar{x} = (a', b', c')$, where a', b', c' are as above;
- If $G = G(3, p, n)$, then $\bar{x} = (0, d, 2d, x_4, \dots, x_n)$.
  Here $(x_4, \dots, x_n) \in E^{n-3}$ is a general element and $d \in E[3]$ is not ζ_3-invariant;
- If $G = G(4, p, n)$, then $\bar{x} = (0, d, e', x_4, \dots, x_n)$.
  Here, $(x_4, \dots, x_n) \in E^{n-3}$ is a general element, d and $e = 2e'$ are in $E[2]$, d is not ζ_4-invariant and e is ζ_4-invariant.

The fact that these coordinates are in different $(E_0 \times \mu_m)$-orbits is seen as follows. In the first two cases, multiplication by 2 kills the actions of E_0 and μ_2 on 4-torsion elements and the coordinates are still all different. In the third case, the action of ζ_3 on d is by translation by a ζ_3-invariant element (say, e), so E_0 and μ_3 act in the same way on d. A direct computation tells us then that $0, d$ and $2d$ are in different $(E_0 \times \mu_3)$-orbits. In the fourth case, all coordinates have different torsion.

Thus, $\mathrm{Stab}_{\Delta\rtimes G}(x) \subset \Delta \times H$ as it was explained above. And easy direct computations in $\Delta \times H$ tell us that the stabilizer of $\bar{x}$ is given in each case by:

(m, p, n)	Generators of $\mathrm{Stab}_{\Delta\rtimes G}(\bar{x}) \subset \Delta \times H$	
$(2, p, n), n \geq 4$	$((0, a, b, c, 0, \dots, 0), (-1, -1, -1, -1, 1, \dots, 1))$	
	$(\bar{0}, (-1, 1, \dots, 1))$	(exists only if $p = 1$).
$(2, 1, 3)$	$((a, b, c), (-1, -1, -1))$	
$(3, p, n)$	$((0, 2e, e, 0, \dots, 0), (\zeta_3, \zeta_3, \zeta_3, 1, \dots, 1))$	
	$(\bar{0}, (\zeta_3, 1, \dots, 1))$	(exists only if $p = 1$)
$(4, p, n)$	$((0, e, e, 0, \dots, 0), (\zeta_4, \zeta_4, -1, 1, \dots, 1))$	
	$(\bar{0}, (-1, 1, \dots, 1))$	(exists only if $p \leq 2$)
	$(\bar{0}, (\zeta_4, 1, \dots, 1))$	(exists only if $p = 1$)

In every case, the first element is clearly not a pseudoreflection and it cannot be generated by the others, which proves the proposition. □

The statement of the last proposition hints that the quotient A/G is indeed smooth for $G = G(2, 2, 3)$. This is actually the case, since it is well-known that $G(2, 2, 3)$ is isomorphic, as a complex reflection group, to S_4 and was therefore already considered in the previous section. The proof of (1) $\Rightarrow$ (4) is now complete.

3.3. Sporadic groups

We deal now with complex reflection groups that are not of the type $G(m, p, n)$. As we recalled before, these are 34 sporadic groups with given actions on $\mathbb{C}^n$ where n varies from 2 to 8.

Let G be such a sporadic group. Recall that having an Abelian variety A with an action of G by automorphisms gives us in particular a linear action of G on $T_0(A) \cong \mathbb{C}^n$ that preserves the lattice $\Lambda = \Lambda_A$. We need then a classification of G-invariant lattices up to equivalence. A great part of this work was done by Popov in [10], where he studied infinite complex reflection groups, in particular *crystallographic* complex reflection groups, which turn out to be extensions of a finite complex reflection group G by some lattice Λ in $\mathbb{C}^n$, where the action of G on $\mathbb{C}^n$ is the one given by Shephard-Todd. In order to deal with sporadic groups, we use then some of Popov's results, which we briefly recall here.

First of all, we need the notion of *root lattice*. Given a finite (irreducible) complex reflection group G, we can consider the directions on which the pseudoreflections act. With these one can define an actual (irreducible) root system which in turn is useful for classifying these groups (*cf.* [10, Section 1]). Here, we only care about the lines generated by these roots, that is the eigenspaces of eigenvalue $\neq 1$ for some pseudoreflection $\sigma \in G$, which Popov calls *root lines*. If we consider a G-invariant lattice $\Lambda \subset \mathbb{C}^n$, then the sublattices $\Lambda \cap L$ for L a root line generate a G-invariant sublattice Λ^0 of Λ called the *root lattice* of Λ. Note that this is precisely how we constructed the G-equivariant isogeny $B \to A$ for $G = G(1, 1, n+1) = S_{n+1}$.

We have then the following result, *cf.* [10, Section 2.6]:

Theorem 3.9 (Popov). *The only sporadic groups G in the list of Shephard-Todd that admit a G-invariant lattice are the numbers* 4, 5, 8, 12, 24-26, 28, 29, 31-37. *Their corresponding root lattices are classified up to equivalence by the table in* [10, *Section* 2.6, *pages* 37-44].

Note that Popov's notion of equivalence of G-invariant lattices induces isomorphisms between the corresponding Abelian varieties with G-action, so that we only need to study Abelian varieties $A = \mathbb{C}^n/\Lambda$ for lattices Λ such that its root lattice Λ^0 is in Popov's table. Let us recall then another result that will be useful to classify lattices that are not a root lattice *cf.* [10, Sections 4.2-4.4].

Consider the endomorphism of $\mathbb{C}^n$ defined as $S := n \cdot I_n - \sum_{i=1}^{n} R_i$, where R_i denotes the i-th pseudoreflection of a fixed generating set of pseudoreflections of G.

Theorem 3.10 (Popov). *Let Λ be a G-invariant lattice in $\mathbb{C}^n$ and let Λ^0 be its root lattice. Then $\Lambda^0 \subset \Lambda \subset S^{-1}\Lambda^0$. In particular, if $|\det(S)| = 1$, then every G-invariant lattice is a root lattice.*

All we are left to do then is to explicitly verify, for each lattice Λ^0 in Popov's list and for each G-invariant lattice between Λ^0 and $S^{-1}\Lambda^0$, whether the quotient of the corresponding Abelian variety by G is smooth or not. As it turns out, this is never true, which we summarize in the following proposition:

Proposition 3.11. *Let G be a sporadic group from the Shephard-Todd list. If G acts on an Abelian variety A in such a way that its action on $T_0(A)$ is an irreducible representation, then A/G is not smooth.*

Proof. For every such pair (A, G), we consider the associated pair (Λ, G), where $A = \mathbb{C}^n/\Lambda$. Tables 3.1 and 3.2 give, for every such pair, a point $x_0 \in A$ such that its stabilizer is not generated by pseudoreflections. The result follows then from the Chevalley-Shephard-Todd theorem.

We start with the groups G such that $|\det(S)| = 1$, so that we only need to verify Popov's explicit lattices. For these, Table 3.1 gives:

- The group G (by giving its number in Shephard-Todd's list);
- Popov's name for the group $\Lambda^0 \rtimes G$;
- A rational linear combination v_0 of the $\mathbb{Z}$-basis $\{e_1, \dots, e_{2n}\}$ of $\Lambda = \Lambda^0$;
- The order of the stabilizer $S_0 = \mathrm{Stab}_G(x_0)$ of the image x_0 of v_0 in the Abelian variety $A = \mathbb{C}^n/\Lambda$;
- The order of the subgroup P_0 of S_0 that is generated by pseudoreflections.

We refer to [10, Section 2.6, pages 37-44] for the explicit $\mathbb{Z}$-basis. In each case, the first n elements of the basis are Popov's $e_1, \dots, e_n$ and the $(n+i)$-th element is $\tau_i e_i$ for some explicit $\tau_i \in \mathbb{C}$.

We consider now those groups G in Popov's table for which $|\det(S)| \neq 1$, so that we need to check for new lattices aside from Popov's. These correspond to the numbers 4, 25, 33, 35 and 36 in Shephard-Todd's list. Since we always have

Table 3.1. Examples of non-smooth points in A/G for sporadic groups G such that $|\det(S)| = 1$.

$\#G$	$\Lambda^0 \rtimes G$	$v_0 \in \Lambda^0 \otimes_{\mathbb{Z}} \mathbb{Q}$	$\lvert S_0\rvert$	$\lvert P_0\rvert$
5	$[K_5]$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	3	1
8	$[K_8]$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	3	1
12	$[K_{12}]$	$(0, 0, 0, \frac{1}{2})$	16	8
24	$[K_{24}]$	$(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4})$	4	1
26	$[K_{26}]_1$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$	36	18
26	$[K_{26}]_2$	$(0, 0, -\frac{1}{3}, 0, 0, \frac{1}{3})$	72	24
28	$[F_4]_1^\alpha$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$	12	6
28	$[F_4]_2^\beta$	$(0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0)$	16	8
28	$[F_4]_3^\gamma$	$(0, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, 0)$	16	8
29	$[K_{29}]$	$(\frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0, 0, 0)$	768	384
31	$[K_{31}]$	$(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0)$	384	192
32	$[K_{32}]$	$(\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0)$	1296	648
34	$[K_{34}]$	$(\frac{1}{3}, 0, 0, 0, 0, 0, -\frac{1}{3}, 0, 0, 0, 0, 0)$	155520	51840
37	$[E_8]^\alpha$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, \frac{1}{2})$	103680	51840

$\Lambda^0 \subset \Lambda \subset S^{-1}\Lambda^0$ and $[S^{-1}\Lambda^0 : \Lambda^0] = |\det(S)|^2$, we see that there are finitely many other lattices to look at. Actually, in all five cases we get that the action of G on the quotient $S^{-1}\Lambda^0/\Lambda^0$ is trivial, so that every lattice in between is a G-invariant lattice and needs to be considered. We keep then notations as above (in particular, Popov's $\mathbb{Z}$-basis is given by $\{e_1, \dots, e_{2n}\}$) and we go case by case:

- In case 4, a $\mathbb{Z}$-basis for $S^{-1}\Lambda^0$ is given by $\{d_1, d_2, e_3, e_4\}$, where $d_1 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3$ and $d_2 = \frac{1}{2}e_1 + \frac{1}{2}e_4$. In particular, we see that the quotient $S^{-1}\Lambda^0/\Lambda^0$ is a Klein group and thus, apart from $S^{-1}\Lambda^0$, we have 3 new lattices to consider: $\Lambda_1 := \langle d_1, \Lambda^0\rangle$, $\Lambda_2 := \langle d_2, \Lambda^0\rangle$, $\Lambda_3 := \langle d_1 + d_2, \Lambda^0\rangle$;
- In case 25, a $\mathbb{Z}$-basis for $S^{-1}\Lambda^0$ is given by $\{d_1, e_2, \dots, e_6\}$, where $d_1 = \frac{1}{3}e_1 + \frac{1}{3}e_3 + \frac{2}{3}e_4 + \frac{2}{3}e_6$. Since the index is 3, this is the only new lattice that needs to be checked;
- In case 33, a $\mathbb{Z}$-basis for $S^{-1}\Lambda^0$ is given by $\{d_1, e_2, \dots, e_5, d_6, e_7 \dots, e_{10}\}$, where $d_1 = \frac{1}{2}e_1 + \frac{1}{2}e_3 + \frac{1}{2}e_5$ and $d_6 = \frac{1}{2}e_6 + \frac{1}{2}e_8 + \frac{1}{2}e_{10}$. In particular, we see that the quotient $S^{-1}\Lambda^0/\Lambda^0$ is a Klein group and thus, apart from $S^{-1}\Lambda^0$, we have 3 new lattices to consider: $\Lambda_1 := \langle d_1, \Lambda^0\rangle$, $\Lambda_2 := \langle d_6, \Lambda^0\rangle$, $\Lambda_3 := \langle d_1 + d_6, \Lambda^0\rangle$;
- In case 35, a $\mathbb{Z}$-basis for $S^{-1}\Lambda^0$ is given by $\{d_1, e_2, \dots, e_6, d_7, e_8 \dots, e_{12}\}$, where $d_1 = \frac{1}{3}e_1 - \frac{1}{3}e_3 + \frac{1}{3}e_5 - \frac{1}{3}e_6$ and $d_7 = \frac{1}{3}e_7 - \frac{1}{3}e_9 + \frac{1}{3}e_{11} - \frac{1}{3}e_{12}$. In particular, we see that the quotient $S^{-1}\Lambda^0/\Lambda^0$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and thus, apart from $S^{-1}\Lambda^0$, we have 4 new lattices to consider: $\Lambda_1 := \langle d_1, \Lambda^0\rangle$, $\Lambda_2 := \langle d_7, \Lambda^0\rangle$, $\Lambda_3 := \langle d_1 + d_7, \Lambda^0\rangle$, $\Lambda_4 := \langle d_1 + 2d_7, \Lambda^0\rangle$;

- In case 36, a $\mathbb{Z}$-basis for $S^{-1}\Lambda^0$ is given by $\{e_1, d_2, e_3, \dots, e_8, d_9, e_{10}, \dots, e_{14}\}$, where $d_2 = \frac{1}{2}e_2 + \frac{1}{2}e_5 + \frac{1}{2}e_7$ and $d_9 = \frac{1}{2}e_9 + \frac{1}{2}e_{12} + \frac{1}{2}e_{14}$. In particular, we see that the quotient $S^{-1}\Lambda^0/\Lambda^0$ is a Klein group and thus, apart from $S^{-1}\Lambda^0$, we have 3 new lattices to consider: $\Lambda_1 := \langle d_2, \Lambda^0\rangle$, $\Lambda_2 := \langle d_9, \Lambda^0\rangle$, $\Lambda_3 := \langle d_2 + d_9, \Lambda^0\rangle$.

Table 3.2 gives then, for every pair (A, G) with $A = \mathbb{C}^n/\Lambda$:

- The group G (by giving its number in Shephard-Todd's list);
- The corresponding lattice Λ (as we named them here above);
- A rational linear combination v_0 of the corresponding $\mathbb{Z}$-basis (as given here above);

Table 3.2. Non-smooth points in A/G for sporadic groups G such that $|\det(S)| \neq 1$.

#G	Λ	$v_0 \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$	$\lvert S_0\rvert$	$\lvert P_0\rvert$
4	Λ^0	$(\frac{1}{2}, 0, 0, 0)$	2	1
4	Λ_1	$(0, \frac{1}{2}, 0, 0)$	4	1
4	Λ_2	$(0, 0, \frac{1}{2}, \frac{1}{2})$	4	1
4	Λ_3	$(0, \frac{1}{2}, \frac{1}{2}, 0)$	6	3
4	$S^{-1}\Lambda^0$	$(0, 0, 0, \frac{1}{2})$	8	1
25	Λ^0	$(0, -\frac{1}{3}, 0, 0, \frac{1}{3}, -\frac{1}{3})$	3	1
25	$S^{-1}\Lambda^0$	$(0, 0, 0, \frac{1}{3}, 0, \frac{1}{3})$	72	24
33	Λ^0	$(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2})$	108	54
33	Λ_1	$(\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	1296	648
33	Λ_2	$(0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0)$	1296	648
33	Λ_3	$(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0)$	240	120
33	$S^{-1}\Lambda^0$	$(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0)$	1296	648
35	Λ^0	$(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, 0)$	72	36
35	Λ_1	$(0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0)$	648	216
35	Λ_2	$(0, 0, 0, 0, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0)$	648	216
35	Λ_3	$(0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0)$	648	216
35	Λ_4	$(0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, 0, -\frac{1}{3}, -\frac{1}{3}, 0, -\frac{1}{3}, 0)$	648	216
35	$S^{-1}\Lambda^0$	$(0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0)$	648	216
36	Λ^0	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$	1440	720
36	Λ_1	$(0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	103680	51840
36	Λ_2	$(0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0)$	103680	51840
36	Λ_3	$(0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0)$	3840	1920
36	$S^{-1}\Lambda^0$	$(0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0, 0, 0, 0, 0)$	103680	51840

- The order of the stabilizer $S_0 = \mathrm{Stab}_G(x_0)$ of the image x_0 of v_0 in the Abelian variety A;
- The order of the subgroup P_0 of S_0 that is generated by pseudoreflections.

This concludes the proof of Proposition 3.11. □

Remark 3.12. For each lattice Λ and each "bad" element x_0 analyzed here above, we computed the stabilizer S_0 and its subgroup P_0 by brute force using basic SageMath algorithms (we thank once again Antonio Behn for his enormous help in optimizing our first algorithms). Since these are really basic, readers can certainly write their own (and probably in a more efficient manner than ours!). However, for those who would like to look at our code, it is presented in an appendix to a previous version of this article (*cf.* arxiv.org/abs/1801.00028v2).

The main idea in order to find these elements was to check the stabilizers (and the pseudoreflections therein) of small torsion elements chosen via the following principle: for every element g of the matrix group G, we decomposed $\mathbb{Z}^{2n}$ as $\ker(g - I_{2n}) \oplus \ker(g - I_{2n})^{\perp}$, where the $\perp$ is taken with respect to a G-invariant Hermitian form H on $\mathbb{C}^n$. By restricting g to $\ker(g - I_{2n})^{\perp}$, we obtain an integer-valued matrix $\tilde{g}$ such that $\tilde{g} - I$ is invertible over $\mathbb{Q}$. The columns of $(\tilde{g} - I)^{-1}$ that contain rational, non-integer numbers therefore correspond to fixed points of g in A that do not come from the eigenspace associated to 1 of g. These were the vectors whose stabilizers we calculated and analyzed.

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