Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXI (2020), 569-641

Sign-changing blowing-up solutions for the critical nonlinear heat equation

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Abstract. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and denote the regular part of the Green function on Ω with Dirichlet boundary condition by $H(x, y)$. Assume the integer k_0 is sufficiently large, $q \in \Omega$ and $n \geq 5$. For $k \geq k_0$ we prove that there exist initial data u_0 and smooth parameter functions $\xi(t) \to q$ and $0 < \mu(t) \to 0$ for $t \to +\infty$ such that the solution u_q of the critical nonlinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = u_0 & \text{in } \Omega \end{cases}$$

has the form

$$u_q(x, t) \approx \mu(t)^{-\frac{n-2}{2}} \left(Q_k \left(\frac{x - \xi(t)}{\mu(t)} \right) - H(x, q) \right),$$

where the profile Q_k is the non-radial sign-changing solution of the Yamabe equation

$$\Delta Q + |Q|^{\frac{4}{n-2}} Q = 0 \text{ in } \mathbb{R}^n,$$

constructed in [9]. In dimension 5 and 6 we also investigate the stability of $u_q(x, t)$.

Mathematics Subject Classification (2010): 35K20 (primary); 35J15, 35J61, 35K15 (secondary).

Contents

M. del Pino has been supported by a Royal Society Professorship (UK) and Fondo Basal CMM (Chile). M. Musso has been partly supported by grant Fondecyt 1160135 (Chile). J. Wei has been partially supported by NSERC of Canada. Y. Zheng has been partially supported by NSF of China (11301374) and China Scholarship Council (CSC).

Received November 04, 2018; accepted in revised form March 23, 2019.
Published online December 2020.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$. We consider the following critical nonlinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = u_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

for a function $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ and smooth initial datum u_0 satisfying $u_0|_{\partial\Omega} = 0$.

Problem (1.1) can be viewed as a special case of the well-known Fujita equation

$$u_t = \Delta u + |u|^{p-1} u \tag{1.2}$$

with $p > 1$, which appears in many applied disciplines and becomes a prototype for the analysis of singularity formation in nonlinear parabolic equations. A large amount of literature has been devoted to this problem on the asymptotic behaviour and blowing-up solutions after Fujita's seminal work [18]. See, for example, [1, 2, 11, 12, 19–23, 27–29, 31, 41] and the references therein. We refer the interested readers to [39] for the corresponding background and a comprehensive survey of the results until 2007. Blowing-up phenomena for problem (1.2) are very sensitive to the exponent p, the critical case $p = \frac{n+2}{n-2}$ is special in several ways, positive steady state solutions do not exist if $p < \frac{n+2}{n-2}$. Radial and positive global solutions must go to zero and bounded, see [35, 36, 39], they exist in the case $p > \frac{n+2}{n-2}$ with infinite energy, see [24]. Infinite time blowing-up solutions exist in that case but they exhibit entirely different nature, see [37, 38].

The motivation of this paper is twofolds. In [2], Cortazar, del Pino and Musso proved the following result. Suppose $n > 4$, denote the Green's function of the Laplacian Δ in Ω with Dirichlet boundary value as $G(x, y)$ and $H(x, y)$ is the regular part of $G(x, y)$. Let $q_1, \cdots, q_k$ be k distinct points in Ω such that the matrix

$$\hat{\mathcal{G}}(q) = \begin{bmatrix} H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\ -G(q_2, q_1) & H(q_2, q_2) & \cdots & -G(q_2, q_k) \\ \vdots & \vdots & \ddots & \vdots \\ -G(q_k, q_1) & -G(q_k, q_2) & \cdots & H(q_k, q_k) \end{bmatrix} \tag{1.3}$$

is positive definite. They proved the existence of u_0 and smooth parameter functions $\xi_j(t) \to q_j, 0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \cdots, k$, such that (1.1) has an infinite time blowing-up solution u_q with approximation form

$$u_q \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j^2(t) + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}.$$

Moreover, for some positive constants β_j, $\mu_j(t) = \beta_j t^{-\frac{1}{n-4}}(1 + o(1))$. Note that the profile of u_q is

$$U(x) = \alpha_n \left(\frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}}, \tag{1.4}$$

which is the unique radial symmetrical solution for the Yamabe equation

$$\Delta Q + |Q|^{\frac{4}{n-2}} Q = 0 \text{ in } \mathbb{R}^n. \tag{1.5}$$

On the other hand, much less is known for the sign-changing solutions to (1.5). Pohozaev's identity tells us that any sign-changing solution of (1.5) is non-radial. The existence of non-radial sign-changing and with arbitrary large energy elements of $\Sigma := \left\{ Q \in \mathcal{D}^{1,2}(\mathbb{R}^n) \backslash \{0\} : Q \text{ satisfies (1.5)} \right\}$ was first proved by W. Ding [14] using variational arguments. Indeed, using stereographic projection to S^n, (1.5) transforms into

$$\Delta_{S^n} v + \frac{n(n-2)}{4} \left(|v|^{\frac{4}{n-2}} v - v \right) = 0 \text{ in } S^n,$$

(see, for example, [26, 40]), Ding proved the existence of infinitely many critical points to the corresponding energy functional in the space of functions satisfying

$$v(x) = v(|x_1|, |x_2|), \quad x = (x_1, x_2) \in S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^{n+1-k}, \quad k \geq 2.$$

More explicit constructions of sign-changing solutions to (1.5) were obtained in [9, 10, 30]. Furthermore, [33] proves the rigidity results (non-degeneracy) of the solutions found in [9, 10]. Classification of solutions in Σ plays an important role in the soliton resolution conjecture for energy critical wave equation, for example, [15, 16] and the references therein. Therefore, a natural question is: does the infinite time blowing-up phenomenon for problem (1.1) occurs with sign-changing profiles? The aim of this paper is to show that the sign-changing blowing-up solutions with basic cell constructed in [9] do exist.

Our starting point is the sign-changing solutions Q of (1.5) constructed in [9] and [10]. Let us describe these solutions more precisely. In [9], it was proven that there exists a large positive integer k_0 such that $\forall k \geq k_0$, a solution $Q = Q_k$ of (1.5) exists. Furthermore, if we define the energy functional by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx, \quad p = \frac{n+2}{n-2},$$

then we have

$$E(Q_k) = \begin{cases} (k+1)\, S_n \,\left(1 + O(k^{2-n})\right) & \text{if } n \geq 4 \\ (k+1)\, S_3 \,\left(1 + O(k^{-1}|\log k|^{-1}\right) & \text{if } n = 3 \end{cases}$$

as $k \to \infty$. Here S_n is a positive constant depending on n. $Q = Q_k$ decays like the radial symmetrical solution $U(x)$ defined in (1.4) at infinity, that is to say, we have

$$\lim_{|x|\to\infty} |x|^{n-2}\, Q_k(x) = \left[\frac{4}{n(n-2)}\right]^{\frac{n-2}{4}} 2^{\frac{n-2}{2}}\,(1 + d_k) \tag{1.6}$$

where

$$d_k = \begin{cases} O\big(k^{-1}\big) & \text{if } n \geq 4\,, \\ O\big(k^{-1}|\log k|^2\big) & \text{if } n = 3 \end{cases} \qquad \text{as} \quad k \to \infty.$$

Furthermore, we have

$$Q(x) = [n(n-2)]^{\frac{n-2}{4}} \left(1 - \frac{n-2}{2}|x|^2 + O\big(|x|^3\big)\right) \quad \text{as} \quad |x| \to 0$$

and there exists $\eta > 0$ (depending only on k_0) such that for any k,

$$\eta \leq Q(x) \leq Q(0) \quad \text{for all} \quad |x| \leq \frac{1}{2}.$$

On the other hand, $Q = Q_k$ is invariant under rotation of angle $\frac{2\pi}{k}$ in the x_1, x_2 plane, *i.e.*,

$$Q\Big(e^{\frac{2\pi}{k}}\bar{x}, x'\Big) = Q(\bar{x}, x'), \quad \bar{x} = (x_1, x_3), \quad x' = (x_3, \ldots, x_n). \tag{1.7}$$

It is also even in the x_j-coordinates, for any $j = 2, \cdots, n$ and invariant under the Kelvin's transformation, namely, we have

$$Q(x_1, \ldots, x_j, \ldots, x_n) = Q(x_1, \ldots, -x_j, \ldots, x_n), \quad j = 2, \ldots, n \tag{1.8}$$

and

$$Q(x) \;=\; |x|^{2-n} Q\big(|x|^{-2}x\big). \tag{1.9}$$

It was proved in [33] that these solutions are non-degenerate. More precisely, fix one solution $Q = Q_k$ and define the linearized operator of (1.5) at Q as

$$L(\phi) = \Delta\phi + p|Q|^{p-1}\phi. \tag{1.10}$$

The invariance of any solution of (1.5) under dilation (if u satisfies (1.5), then the function $\mu^{-\frac{n-2}{2}} u(\mu^{-1}x)$ solves (1.5) for all $\mu > 0$), under translation (if u solves

(1.5), then $u(x+\xi)$ also solves (1.5) for $\xi \in \mathbb{R}^n$), together with the invariance (1.7), (1.8), (1.9) produce natural kernel functions φ of L, that is to say, we have

$$L(\varphi) = 0.$$

These are $3n$ linearly independent functions defined as follows:

$$z_0(x) = \frac{n-2}{2} Q(x) + \nabla Q(x) \cdot x, \tag{1.11}$$

$$z_\alpha(x) = \frac{\partial}{\partial x_\alpha} Q(x), \quad \text{for} \quad \alpha = 1, \dots, n, \tag{1.12}$$

$$z_{n+1}(x) = -x_2 \frac{\partial}{\partial x_1} Q(x) + x_1 \frac{\partial}{\partial x_2} Q(x), \tag{1.13}$$

$$z_{n+2}(x) = -2x_1 z_0(x) + |x|^2 z_1(x), \quad z_{n+3}(x) = -2x_2 z_0(x) + |x|^2 z_2(x) \tag{1.14}$$

and, for $l = 3, \dots, n$

$$z_{n+l+1}(x) = -x_l z_1(x) + x_1 z_l(x), \quad z_{2n+l-1}(x) = -x_l z_2(x) + x_2 z_l(x). \tag{1.15}$$

Indeed, direct computations yield that

$$L(z_\alpha) = 0, \quad \text{for all} \quad \alpha = 0, 1, \dots, 3n-1.$$

The function z_0 defined by (1.11) is from the invariance of (1.5) under dilation $\mu^{-\frac{n-2}{2}} Q(\mu^{-1}x)$. z_i, $i = 1, \dots, n$ defined by (1.12) are due to the invariance of (1.5) under translation $Q(x+\xi)$. The function z_{n+1} in (1.13) is generated from the invariance of Q with respect to rotation in the (x_1, x_2)-plane. The functions z_{n+2} and z_{n+3} in (1.14) are generated from the invariance of (1.5) with respect to the Kelvin transformation (1.9). The functions in (1.15) are due to the invariance of (1.5) under rotations in the (x_1, x_l)-plane, (x_2, x_l)-plane respectively.

Let us recall that the Green's function $G(x, y)$ is defined by the following Dirichlet boundary value problem

$$\begin{cases} -\Delta G(x, y) = c(n)\delta(x-y) & \text{in } \Omega \\ G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\delta(x)$ is the Dirac measure at the origin and $c(n)$ is a constant depending on n satisfying

$$-\Delta\Gamma(x) = c(n)\delta(x), \quad \Gamma(x) = \frac{Q(0)}{|x|^{n-2}} = \frac{[n(n-2)]^{\frac{n-2}{4}}}{|x|^{n-2}}.$$

Denote the regular part of $G(x, y)$ as $H(x, y)$, namely, $H(x, y)$ satisfies the following problem

$$\begin{cases} -\Delta H(x, y) = 0 & \text{in } \Omega \\ H(\cdot, y) = \Gamma(\cdot - y) & \text{in } \partial\Omega. \end{cases}$$

Our main result can be stated as follows.

Theorem 1.1. *Assume k_0 is a sufficiently large integer, $n > 4$ and q is a point in Ω such that $H(q, q) > 0$, then for any $k \geq k_0$, there exist an initial datum u_0 and smooth parameter functions $\xi(t) \to q$, $0 < \mu(t) \to 0$ (as $t \to +\infty$) such that the solution u_q to* (1.1) *has form*

$$u_q(x, t) = \mu(t)^{-\frac{n-2}{2}} \left(Q_k \left(\frac{x - \xi(t)}{\mu(t)} \right) - H(x, q) + \varphi(x, t) \right), \tag{1.16}$$

where $\varphi(x, t)$ is a bounded smooth function satisfying $\varphi(x, t) \to 0$ uniformly away from q as $t \to +\infty$.

Theorem 1.1 exhibits new blowing-up phenomena where the profile of bubbling is sign-changing rather than the positive solution for the critical heat equation. In the case of positive bubbling, the linear operator around the basic cell contains exactly $n + 1$ dimensional kernels corresponding to the rigidity motions (translation and dilation). However, in the case of sign-changing (non-radial) blowing-up solution, the kernel of the linearized operators at the basic cell includes not only the functions generated from dilation and translations, but also functions due to rotation around the sub-planes and Kelvin transform. Therefore we have to find enough parameter functions to adjust. Similar to the supercritical Bahri-Coron's problem in [34], our computations indicate that the dominated role played is still scaling and translations. Indeed, (1.16) has a more involved form, see (2.18) below for details. Note that in [43], sign-changing blow-up solutions were also constructed, but their basic cell is the positive radial solution $U(x)$ defined in (1.4).

We believe that this is the *first* example of blowing-up solutions in nonlinear parabolic equations whose core profile is non-radial. In a series of interesting papers, Duyckaerts, Kenig and Merle [15,17] introduced the notion of nondegenracy for nonradial solutions of the equation (1.5) and obtained the profile decomposition for possible blow-up solutions for energy critical wave equation in general setting. Existence of bubbling solutions with the positive radial profile for the energy critical wave equations has been constructed in [13,25]. However as far as we know there are no examples of noradial blow-up for energy critical wave equation.

To prove Theorem 1.1, we will use the *inner-outer gluing scheme* for parabolic problems. Gluing methods have been proven very useful in singular perturbation elliptic problems, for example, [6–8]. Recently, this method has also been developed to various evolution problems, for instance, the construction of infinite time blowing-up solutions for energy critical nonlinear heat equation [2,12], the formation of singularity to harmonic map flow [3], finite time blowing-up solutions for energy critical heat equation [11], vortex dynamics in Euler flows [4] and type II ancient solutions for the Yamabe flow [5].

The proof consists of constructing an approximation to the solution with sufficiently small error, then we solve for a small remainder term using linearization around the bubble and the Schauder fixed-point arguments. In Section 2, we construct the first approximation with form (2.18). To get an approximation with fast decay far away from the point q, we add nonlocal terms to cancel the slow decay parts as in [3]. Then we compute the error, in order to improve the approximation error near the point q, we have to use solvability conditions for the corresponding elliptic linearized operator around the sign-changing bubble. These conditions yield an ODE for the scaling parameter function, from which deduce the blow-up dynamics of our solutions. After the approximate solution has been constructed, the full problem is solved as a small perturbation by the *inner-outer gluing scheme*, see Section 3. This consists of decomposing the perturbation term into form $\eta\tilde{\phi} + \psi$, where η is a smooth cut-off function vanishing away from q. The tuple $(\tilde{\phi}, \psi)$ satisfy a coupled nonlinear parabolic system where the equation for ψ is a small perturbation of the standard heat equation, and $\tilde{\phi}$ satisfies the parabolic linearized equation around the bubble.

When dealing with parabolic problems for $\tilde{\phi}$, a crucial step is to find a solution to the linearized parabolic equation around the bubble with sufficiently fast decay. However, it seems that the argument in [2] for the positive bubbling of the critical heat equation does not work in our sign-changing case since we can not perform Fourier mode expansions. Inspired by the linear theory of [3, 32] and [42], our main contributions in this paper is to use blowing-up arguments based on the nondegeneracy of bubbles proved in [33] and a removable of singularity property for the corresponding limit equation. As pointed out in [15], the term $|Q|^{p-1} = |Q|^{\frac{4}{n-2}}$ in $L(\phi) = \Delta\phi + p|Q|^{p-1}\phi$ is not C^1 when the space dimension $n \geq 7$, as a result of this fact, the solution $\tilde{\phi}$, ψ do not have Lipschitz property with respect to the parameter functions. This is the reason we use Schauder fixe-point theorem rather than Contraction Mapping Theorem to solve the inner-outer gluing parabolic system in Section 4. In dimension 5 and 6, $\tilde{\phi}$ and ψ do have Lipschitz continuity with respect to the parameter functions, Theorem 1.1 as well as a stability result for u_q can be proved using the Contraction Mapping Theorem in the spirit of [2], see Section 8.

2. Construction of the approximation

2.1. The basic cell

Let $O(n)$ be the orthogonal group of $n \times n$ matrices M with real coefficients and $M^T M = I$, $SO(n) \subset O(n)$ be the special orthogonal group of all matrices in $O(n)$ satisfying $\det(M) = 1$. It is well known that $SO(n)$ is a compact group containing all rotations in $\mathbb{R}^n$, and via isometry, it can be identified with a compact subset of

$\mathbb{R}^{\frac{n(n-1)}{2}}$. Let $\hat{S}$ be the subgroup of $SO(n)$ generated by rotations in the (x_1, x_2)-plane and (x_j, x_α)-plane, for any $j = 1, 2, \alpha = 3, \ldots, n$. Then $\hat{S}$ is a compact manifold of dimension $2n-3$ without boundary. That is to say, there exists a smooth injective map $\chi : \hat{S} \to \mathbb{R}^{\frac{n(n-1)}{2}}$ such that $\chi(\hat{S})$ is a compact manifold without boundary of dimension $2n-3$ and $\chi^{-1} : \chi(\hat{S}) \to \hat{S}$ is the smooth parametrization of $\hat{S}$ in a neighborhood of the identity map. Let us write

$$\theta \in K = \chi(\hat{S}), \quad R_\theta = \chi^{-1}(\theta)$$

for a smooth compact manifold K of dimension $2n-3$ and R_θ denotes a rotation map in $\hat{S}$.

Let $A = (\mu, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, define

$$Q_A(x) = \mu^{-\frac{n-2}{2}} \, |\eta_A(x)|^{2-n} \; Q\left(\frac{R_\theta\left(\frac{x-\xi}{\mu} - a|\frac{x-\xi}{\mu}|^2\right)}{|\eta_A(x)|^2}\right), \tag{2.1}$$

where

$$\eta_A(x) = \frac{x-\xi}{|x-\xi|} - a\frac{|x-\xi|}{\mu} \tag{2.2}$$

and Q is the fixed non-degenerate solution to problem (1.5) as described in the introduction. It was proved in [15] that for any choice of A, Q_A still satisfies (1.5), *i.e.*,

$$\Delta Q_A + |Q_A|^{p-1} Q_A = 0, \quad \text{in} \quad \mathbb{R}^n.$$

Direct computations yield the following relations between the differentiation of Q_A with respect to each component of A and z_α defined in (1.11), (1.12), (1.13), (1.14) and (1.15). Precisely, we have

$$z_0(y) = -\frac{\partial}{\partial \mu}\,[Q_A(x)]_{|\mu=1,\xi=0,a=0,\theta=0} \tag{2.3}$$

$$z_\alpha(y) = -\frac{\partial}{\partial \xi_\alpha}\,[Q_A(x)]_{|\mu=1,\xi=0,a=0,\theta=0}\,, \quad \alpha = 1, \ldots, n, \tag{2.4}$$

$$z_{n+2}(y) = \frac{\partial}{\partial a_1}\,[Q_A(x)]_{|\mu=1,\xi=0,a=0,\theta=0}\,, \tag{2.5}$$

$$z_{n+3}(y) = \frac{\partial}{\partial a_2}\,[Q_A(x)]_{|\mu=1,\xi=0,a=0,\theta=0}\,. \tag{2.6}$$

Let $\theta = (\theta_{12}, \theta_{13}, \ldots, \theta_{1n}, \theta_{23}, \ldots, \theta_{2n})$, where θ_{ij} is the rotation in the (i, j)-plane, then we have

$$z_{n+1}(y) = \frac{\partial}{\partial \theta_{12}}\,[Q_A(x)]_{|\mu=1,\xi=0,a=0,\theta=0} \tag{2.7}$$

and, for $l = 3, \ldots, n$,

$$z_{n+l+1}(y) = \frac{\partial}{\partial \theta_{1l}} [Q_A(x)]_{|\mu=1,\xi=0,a=0,\theta=0}, \tag{2.8}$$

$$z_{2n+l-1}(y) = \frac{\partial}{\partial \theta_{2l}} [Q_A(x)]_{|\mu=1,\xi=0,a=0,\theta=0}. \tag{2.9}$$

Following the definition in [15], a solution Q of (1.5) is non-degenerate if

$$\text{Kernel}(L) = \text{Span}\{z_\alpha \ : \ \alpha = 0, 1, 2, \ldots, 3n-1\}, \tag{2.10}$$

or equivalently, any bounded solution of $L(\varphi) = 0$ is a linear combination of z_α, $\alpha = 0, \ldots, 3n-1$. It was proved in [33] that, the solution Q is non-degenerate when the dimension satisfies some extra conditions. Indeed, the authors showed that for all dimensions $n \le 48$, any solution $Q = Q_k$ is non-degenerate, for dimension $n \ge 49$, there exists a subsequence of solutions Q_{k_j} which is non-degenerate in the sense (2.10).

2.2. Setting up the problem

Let $t_0 > 0$ be a sufficiently large constant, let us consider the heat equation

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in } \Omega \times (t_0, \infty) \\ u = 0 & \text{in } \partial\Omega \times (t_0, \infty). \end{cases} \tag{2.11}$$

Observe that the solution of (2.11) provides a solution $u(x,t) = u(x, t - t_0)$ to (1.1). Given a fixed point $q \in \Omega$, we will find a solution $u(x,t)$ of equation (2.11) with approximate form

$$u(x,t) \approx \mu(t)^{-\frac{n-2}{2}} Q\left(\frac{x - \xi(t)}{\mu(t)}\right).$$

More precisely, let $A = A(t) = (\mu(t), \xi(t), a(t), \theta(t)) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$ be the parameter functions and define the function

$$Q_{A(t)}(x) = \mu(t)^{-\frac{n-2}{2}} \left|\eta_{A(t)}(x)\right|^{2-n} Q\left(\frac{R_{\theta(t)}\left(\frac{x-\xi(t)}{\mu(t)} - a(t)\left|\frac{x-\xi(t)}{\mu(t)}\right|^2\right)}{|\eta_{A(t)}(x)|^2}\right), \tag{2.12}$$

where

$$\eta_{A(t)}(x) = \frac{x - \xi(t)}{|x - \xi(t)|} - a(t)\frac{|x - \xi(t)|}{\mu(t)} \tag{2.13}$$

and Q is the non-degenerate solution for (1.5) described in Section 2.1. With abuse of notation when there is no ambiguity, here and in what follows, $A(t) =$

$(\mu(t), \xi(t), a(t), \theta(t))$ will be abbreviated as $A = (\mu, \xi, a, \theta)$, a is a vector in $\mathbb{R}^2$, $a = \binom{a_1}{a_2} \in \mathbb{R}^2$, it is also a vector in $\mathbb{R}^n$, namely,

$$a = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

To begin with, we assume that for a fixed positive function $\mu_0(t) \to 0$ $(t \to +\infty)$ and a constant $\sigma > 0$, there hold

$$\begin{aligned} \mu(t) &= \mu_0(t) + O(\mu_0^{1+\sigma}(t)) && \text{as} \quad t \to +\infty, \\ \xi(t) &= q + O(\mu_0^{1+\sigma}(t)) && \text{as} \quad t \to +\infty, \\ a(t) &= O(\mu_0^{\sigma}(t)) && \text{as} \quad t \to +\infty, \\ \theta(t) &= O(\mu_0^{\sigma}(t)) && \text{as} \quad t \to +\infty. \end{aligned}$$

In [15], it was proven that for any choice of A, the function Q_A still satisfies (1.5), namely

$$\Delta Q_A + |Q_A|^{p-1} Q_A = 0 \text{ in } \mathbb{R}^n.$$

Let $\tilde{y} = \frac{R_{\theta(t)}\left(\frac{x-\xi(t)}{\mu(t)} - a(t)\left|\frac{x-\xi(t)}{\mu(t)}\right|^2\right)}{|\eta|^2}$ and $\eta = \frac{x-\xi(t)}{|x-\xi(t)|} - a(t)\frac{|x-\xi(t)|}{\mu(t)}$, then we have the following expansions

$$\begin{aligned} |\eta|^2 &= \left| \frac{x-\xi(t)}{|x-\xi(t)|} - a(t)\frac{|x-\xi(t)|}{\mu(t)} \right|^2 \\ &= 1 - 2a(t) \cdot \left(\frac{x-\xi(t)}{\mu(t)} \right) + |a(t)|^2 \frac{|x-\xi(t)|^2}{\mu^2(t)}, \\ \frac{1}{|\eta|^2} &= \frac{1}{1 - 2a(t) \cdot \left(\frac{x-\xi(t)}{\mu(t)} \right) + |a(t)|^2 \frac{|x-\xi(t)|^2}{\mu^2(t)}} \\ &= 1 + 2a(t) \cdot \left(\frac{x-\xi(t)}{\mu(t)} \right) + O\left(|a(t)|^2 \frac{|x-\xi(t)|^2}{\mu^2(t)} \right), \\ \tilde{y} &= \frac{R_{\theta(t)}\left(\frac{x-\xi(t)}{\mu(t)} - a(t)\left| \frac{x-\xi(t)}{\mu(t)} \right|^2 \right)}{|\eta|^2} \\ &= R_{\theta(t)}\left(\frac{x-\xi(t)}{\mu(t)} \right) + R_{\theta(t)} a(t) \left| \frac{x-\xi(t)}{\mu(t)} \right|^2 + O\left(|a|^2 \frac{|x-\xi(t)|^3}{\mu^3(t)} \right). \end{aligned}$$

Denote the error operator as

$$S(u) := -u_t + \Delta u + |u|^{p-1} u,$$

with $p = \frac{n+2}{n-2}$. Then the error of the first approximation $Q_A(x,t)$ can be computed as

$$S(Q_A) = -\frac{\partial}{\partial t}\left(Q_A(x,t)\right) = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3.$$

For $y = \frac{x-\xi(t)}{\mu(t)}$, using Taylor expansion, the expressions of $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are given below explicitly.

$$\begin{aligned}
\mathcal{E}_0 &= \frac{\dot{\mu}(t)}{\mu(t)}\mu^{-\frac{n-2}{2}}(t)\,|\eta|^{2-n}\,z_0(\tilde{y}) + \frac{\dot{\mu}(t)}{\mu(t)}\mu^{-\frac{n-2}{2}}(t)\,|\eta|^{2-n}\,z_0(\tilde{y})\,(2\tilde{y}\cdot R_\theta a)\\
&\quad - \frac{\dot{\mu}(t)}{\mu(t)}\mu^{-\frac{n-2}{2}}(t)\,|\eta|^{2-n}\left(\nabla Q(\tilde{y})\cdot\frac{R_\theta a}{|\eta|^2}\right)\left(\frac{|x-\xi(t)|^2}{\mu^2(t)}\right)\\
&= \frac{\dot{\mu}(t)}{\mu(t)}\mu^{-\frac{n-2}{2}}(t) z_0\,(y)\,(1+(y\cdot a)\,F_0(\mu,\xi,a,\theta,y)),
\end{aligned}$$

where f are generic smooth bounded functions of the tuple (μ,ξ,a,θ,y) which may different from one place to another, $F_0(\mu,\xi,a,\theta,y)$ is a smooth bounded function depending on (μ,ξ,a,θ,y). Similarly, we have

$$\begin{aligned}
\mathcal{E}_1 &= \mu(t)^{-\frac{n-2}{2}}(n-2)|\eta|^{-n}\,(\eta\cdot a)\left(\frac{x-\xi}{|x-\xi|}\cdot\frac{\dot{\xi}}{\mu}\right)Q(\tilde{y})\\
&\quad + \mu^{-\frac{n-2}{2}}|\eta|^{2-n}\nabla Q(\tilde{y})\cdot\left[\frac{1}{|\eta|^2}R_\theta\left(\frac{\dot{\xi}}{\mu} - \frac{2a(x-\xi)\cdot\dot{\xi}}{\mu^2}\right)\right]\\
&\quad + \mu^{-\frac{n-2}{2}}|\eta|^{2-n}\nabla Q(\tilde{y})\cdot\left(\tilde{y}\frac{2\eta}{|\eta|^2}\left(a\left(\frac{x-\xi}{|x-\xi|}\frac{\dot{\xi}}{\mu}\right)\right)\right)\\
&= \mu^{-\frac{n-2}{2}}\nabla Q\,(y)\cdot\frac{\dot{\xi}}{\mu(t)}\,(1+(y\cdot a)\,F_1(\mu,\xi,a,\theta,y)),
\end{aligned}$$

where f are generic smooth bounded functions of the tuple (μ,ξ,a,θ,y) which may different from one place to another, $F_1(\mu,\xi,a,\theta,y)$ is a smooth bounded function depending on (μ,ξ,a,θ,y). Furthermore, $\mathcal{E}_2 = \mathcal{E}_{21} + \mathcal{E}_{22}$, where

$$\begin{aligned}
\mathcal{E}_{21} &= -\mu^{-\frac{n-2}{2}}|\eta|^{-n}2\,(\dot{a}_1\cdot y)\left[\frac{n-2}{2}Q\,(\tilde{y}) + \nabla Q(\tilde{y})\cdot\tilde{y}\right]\\
&\quad + \mu^{-\frac{n-2}{2}}|\eta|^{-n}R_\theta\dot{a}_1\cdot\nabla Q(\tilde{y})\left|\frac{x-\xi}{\mu}\right|^2\\
&\quad + \mu^{-\frac{n-2}{2}}|\eta|^{-n}2\left[\frac{n-2}{2}Q(\tilde{y}) + \nabla Q(\tilde{y})\cdot\tilde{y}\right]\left|\frac{x-\xi}{\mu}\right|^2 a_1\dot{a}_1\\
&= \mu^{-\frac{n-2}{2}}\left\{-2\,(\dot{a}_1\cdot y)\left[\frac{n-2}{2}Q\,(y) + \nabla Q\,(y)\cdot y\right] + \dot{a}_1\cdot\nabla Q\,(y)\,|y|^2\right\}\\
&\quad \times\big(1+(y\cdot a)\,F_{21}(\mu,\xi,a,\theta,y)\big)\\
&= \mu^{-\frac{n-2}{2}}z_{n+2}(y)\dot{a}_1\big(1+(y\cdot a)\,F_{21}(\mu,\xi,a,\theta,y)\big)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{22} &= -\mu^{-\frac{n-2}{2}}|\eta|^{-n}2\left(\dot{a}_2\cdot y\right)\left[\frac{n-2}{2}Q\left(\tilde{y}\right)+\nabla Q(\tilde{y})\cdot\tilde{y}\right] \\
&\quad+\mu^{-\frac{n-2}{2}}|\eta|^{-n}R_\theta\dot{a}_2\cdot\nabla Q(\tilde{y})\left|\frac{x-\xi}{\mu}\right|^2 \\
&\quad+\mu^{-\frac{n-2}{2}}|\eta|^{-n}2\left[\frac{n-2}{2}Q(\tilde{y})+\nabla Q(\tilde{y})\cdot\tilde{y}\right]\left|\frac{x-\xi}{\mu}\right|^2 a_2\dot{a}_2 \\
&= \mu^{-\frac{n-2}{2}}\left\{-2\left(\dot{a}_2\cdot y\right)\left[\frac{n-2}{2}Q\left(y\right)+\nabla Q\left(y\right)\cdot y\right]+\dot{a}_2\cdot\nabla Q\left(y\right)|y|^2\right\} \\
&\quad\times\big(1+(y\cdot a)\,F_{22}(\mu,\xi,a,\theta,y)\big) \\
&= \mu^{-\frac{n-2}{2}}z_{n+3}(y)\dot{a}_2\big(1+(y\cdot a)\,F_{22}(\mu,\xi,a,\theta,y)\big).
\end{aligned}$$

Here we identify the component a_1 of a with the vector

$$\begin{pmatrix} a_1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \in \mathbb{R}^n,$$

the component a_2 with the vector

$$\begin{pmatrix} 0 \\ a_2 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \in \mathbb{R}^n,$$

f are generic smooth bounded functions of the tuple (μ,ξ,a,θ,y) which may different from one place to another, $F_{21}(\mu,\xi,a,\theta,y)$ and $F_{22}(\mu,\xi,a,\theta,y)$ are smooth bounded functions depending on (μ,ξ,a,θ,y). Finally, $\mathcal{E}_3 = \mathcal{E}_{3,12} + \sum_{j=3}^n \mathcal{E}_{3,1j} + \sum_{j=3}^n \mathcal{E}_{3,2j}$, where

$$\begin{aligned}
\mathcal{E}_{3,12} &= \mu^{-\frac{n-2}{2}}|\eta|^{2-n}\nabla Q(\tilde{y})\cdot(i\tilde{y})\dot{\theta}_{12} \\
&= \mu^{-\frac{n-2}{2}}z_{n+1}(y)\dot{\theta}_{12}\left(1+(y\cdot R_\theta a)\,F_{3,21}(\mu,\xi,a,\theta,y)\right)
\end{aligned}$$

and similarly, for $j=3,\cdots,n$,

$$\begin{aligned}
\mathcal{E}_{3,1j} &= \mu^{-\frac{n-2}{2}}z_{n+j+1}(y)\dot{\theta}_{1j}\left(1+(y\cdot R_\theta a)\,F_{3,1j}(\mu,\xi,a,\theta,y)\right), \\
\mathcal{E}_{3,2j} &= \mu^{-\frac{n-2}{2}}z_{2n+l-1}(y)\dot{\theta}_{2j}\left(1+(y\cdot R_\theta a)\,F_{3,2j}(\mu,\xi,a,\theta,y)\right),
\end{aligned}$$

where i is the rotation matrix with angle $\frac{\pi}{2}$ around the axes x_1, x_2 in $\mathcal{E}_{3,12}$, around the axes x_1, x_j in $\mathcal{E}_{3,1j}$ and around the axes x_2, x_j in $\mathcal{E}_{3,2j}$ respectively, $F_{3,12}(\mu, \xi, a, \theta, y)$, $F_{3,1j}(\mu, \xi, a, \theta, y)$ and $F_{3,2j}(\mu, \xi, a, \theta, y)$, $j = 3, \cdots, n$, are smooth bounded functions depending on (μ, ξ, a, θ, y).

To perform the gluing method, the terms $\mu^{-\frac{n-2}{2}-1}\dot{\mu}z_0(y)$, $\mu^{-\frac{n-2}{2}-1}\dot{\xi}\cdot\nabla Q(y)$ and $\mu^{-\frac{n-2}{2}-1}\nabla Q(y)\cdot(iR_\theta\xi)\,\dot{\theta}$ do not have enough decay, inspired by [3], we should add nonlocal terms to cancel them out at main order. By the detailed construction of Q (see [9]) and (1.6) we know that the main order of $z_0(y)$ is

$$\frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}$$

with $D_{n,k} = -\frac{n-2}{2}\left[\frac{4}{n(n-2)}\right]^{\frac{n-2}{4}} 2^{\frac{n-2}{2}}(1+d_k)$. Therefore, we consider the following heat equation

$$-\varphi_t + \Delta\varphi + \frac{\dot{\mu}}{\mu}\mu^{-(n-2)}\frac{D_{n,k}\left(2-\left|\frac{x-\xi}{\mu}\right|^2\right)}{\left(1+\left|\frac{x-\xi}{\mu}\right|^2\right)^{\frac{n}{2}}} = 0 \text{ in } \mathbb{R}^n\times(t_0,+\infty). \qquad (2.14)$$

By the Duhamel principle, we known

$$\Phi^0(x,t) = -\int_{t_0}^{t}\int_{\mathbb{R}^n} p(t-\tilde{s}, x-y)\frac{\dot{\mu}(\tilde{s})}{\mu(\tilde{s})}\mu^{-(n-2)}(\tilde{s})\frac{D_{n,k}\left(2-\left|\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2\right)}{\left(1+\left|\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2\right)^{\frac{n}{2}}}dy\,d\tilde{s} \qquad (2.15)$$

provides a bounded solution for (2.14). Here $p(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{-|x|^2}{4t}}$ is the standard heat kernel for the heat operator $-\frac{\partial}{\partial t}+\Delta$ on $\mathbb{R}^n\times(t_0,+\infty)$. By the super-sub solution argument, $\Phi^0(x,t)$ satisfies the estimate $\Phi^0(x,t)\sim\frac{\dot{\mu}}{\mu}\frac{\mu^{-n+4}}{1+|y|^{n-4}}$ (see Lemma 4.3).

To cancel the main order $\mu^{-\frac{n-2}{2}-1}\dot{\xi}\cdot\frac{E_{n,k}y}{(1+|y|^2)^{\frac{n}{2}}}$ of $\mu^{-\frac{n-2}{2}-1}\dot{\xi}\cdot\nabla Q(y)$ where $E_{n,k}$ is a constant depending on n and k, for $y=\frac{x-\xi}{\mu}$, we consider the following heat equation

$$-\varphi_t + \Delta\varphi + E_{n,k}\mu^{-(n-2)}\frac{1}{\left(1+|y|^2\right)^{\frac{n}{2}}}\frac{\dot{\xi}}{\mu}\cdot y = 0 \text{ in } \mathbb{R}^n\times(t_0,+\infty). \qquad (2.16)$$

The solution defined from the Duhamel principle

$$\Phi^1(x,t) = -E_{n,k}\int_{t_0}^{t}\int_{\mathbb{R}^n} p(t-\tilde{s}, x-y)\mu^{-(n-2)}(\tilde{s})\frac{\dot{\xi}(\tilde{s})\cdot\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}}{\mu(\tilde{s})} \times \frac{1}{\left(1+\left|\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2\right)^{\frac{n}{2}}} dy d\tilde{s}$$

satisfies the estimate $\Phi^1(x,t) \sim \frac{|\dot{\xi}|}{\mu}\frac{\mu^{-n+4}}{1+|y|^{n-3}}$.

Similarly, for $i = 1, 2$, we consider the heat equation

$$-\varphi_t+\Delta\varphi+\mu^{-(n-2)}\frac{E_{n,k}|y|^2-2D_{n,k}\left(2-|y|^2\right)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\dot{a}_i y_i = 0 \text{ in } \mathbb{R}^n\times(t_0,+\infty), \quad (2.17)$$

which has a bounded solution given by

$$\Phi^{2,i}(x,t) = -\int_{t_0}^{t}\int_{\mathbb{R}^n} p(t-\tilde{s}, x-y)\mu^{-(n-2)}(\tilde{s})\dot{a}_i(\tilde{s})\left(\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right)_i \times \frac{E_{n,k}\left|\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2 - 2D_{n,k}\left(2-\left|\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2\right)}{\left(1+\left|\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2\right)^{\frac{n}{2}}} dy d\tilde{s}$$

satisfies the estimate $\Phi^{2,i}(x,t) \sim |\dot{a}_i|\frac{\mu^{-n+4}}{1+|y|^{n-5}}$.

Now we define $\Phi^*(x,t) = \Phi^0(x,t) + \Phi^1(x,t) + \sum_{i=1}^{2}\Phi^{2,i}(x,t)$. Since the final solution must sasify $u = 0$ in $\partial\Omega$, a better approximation than $Q_A(x,t)$ should be

$$u_A(x,t) = Q_A(x,t) + \mu^{\frac{n-2}{2}}\Phi^*(x,t) - \mu^{\frac{n-2}{2}}H(x,q). \quad (2.18)$$

The error of u_A can be computed as follows,

$$S(u_A) = -\partial_t u_A + |u_A|^{p-1}u_A - |Q_A|^{p-1}Q_A + \mu^{\frac{n-2}{2}}\Delta\Phi^*(x,t). \quad (2.19)$$

2.3. The error $S(u_A)$

Near the given point q, the following expansion holds.

Lemma 2.1. *Consider the region $|x-q| \le \varepsilon$ for ε small enough, we have*

$$S(u_A) = \mu^{-\frac{n+2}{2}}(\mu E_0 + \mu E_1 + \mu E_2 + \mu E_3 + \mathcal{R})$$

with

$$E_0 = p|Q|^{p-1}\Big[-\mu^{n-3}H(q,q)+\mu^{n-3}\Phi^0(q,t)\Big]+\dot\mu(t)\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\big(1+|y|^2\big)^{\frac{n}{2}}}\right),$$

$$E_1 = p|Q|^{p-1}\Big[-\mu^{n-2}\nabla H(q,q)\Big]\cdot y + p|Q|^{p-1}\Big[\mu^{n-3}\Phi^1(q,t)\Big] + \left(\nabla Q(y)-\frac{E_{n,k}y}{\big(1+|y|^2\big)^{\frac{n}{2}}}\right)\cdot\dot\xi,$$

$$E_2 = p|Q|^{p-1}\Big[\mu^{n-3}\Phi^{2,1}(q,t)+\mu^{n-3}\Phi^{2,2}(q,t)\Big] + \mu(t)\dot a_1\left(-2y_1\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\big(1+|y|^2\big)^{\frac{n}{2}}}\right)+|y|^2\left(\frac{\partial}{\partial y_1}Q(y)-\frac{E_{n,k}y_1}{\big(1+|y|^2\big)^{\frac{n}{2}}}\right)\right) + \mu(t)\dot a_2\left(-2y_2\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\big(1+|y|^2\big)^{\frac{n}{2}}}\right)+|y|^2\left(\frac{\partial}{\partial y_2}Q(y)-\frac{E_{n,k}y_2}{\big(1+|y|^2\big)^{\frac{n}{2}}}\right)\right),$$

$$E_3 = z_{n+1}(y)\mu\dot\theta_{12} + \sum_{j=3}^{n}\big(z_{n+j+1}(y)\mu\dot\theta_{1j} + z_{2n+j-1}(y)\mu\dot\theta_{2j}\big),$$

$$\mathcal{R} = \Big(\mu_0^{n+2}+\mu_0^{n-1}\dot\mu\Big)f + \frac{\mu_0^{n-1}\vec f}{1+|y|^2}\cdot a + \frac{\mu_0^{n-2}\vec g}{1+|y|^4}\cdot(\xi-q) + \mu_0^n\dot\xi\cdot\vec h,$$

where f, $\vec f$, $\vec g$ *and* $\vec h$ *are smooth and bounded functions depending on the tuple of variables* $(\mu_0^{-1}\mu, \xi, a, \theta, x-\xi)$.

Proof. Set

$$\tilde y = \frac{R_\theta\left(\frac{x-\xi(t)}{\mu(t)} - a\left|\frac{x-\xi(t)}{\mu(t)}\right|^2\right)}{|\eta|^2},$$

we have

$$u_A(x,t) = \mu(t)^{-\frac{n-2}{2}}\,|\eta|^{2-n}\,Q(\tilde y) + \mu^{\frac{n-2}{2}}\Phi^*(x,t) - \mu^{\frac{n-2}{2}}H(x,q)$$

and

$$S(u_A) = S_1 + S_2,$$

where

$$\begin{aligned} S_1 :=& \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \frac{n-2}{2}\mu^{\frac{n-4}{2}}\dot{\mu}H(x,q) \\ &- \frac{n-2}{2}\mu^{\frac{n-4}{2}}\dot{\mu}\Phi^*(x,t) - \mu^{\frac{n-2}{2}}\partial_t\Phi^*(x,t), \\ S_2 :=& \left|\mu(t)^{-\frac{n-2}{2}}|\eta|^{2-n}Q(\tilde{y}) + \mu^{\frac{n-2}{2}}\Phi^*(x,t) - \mu^{\frac{n-2}{2}}H(x,q)\right|^{p-1} \\ &\times\left(\mu(t)^{-\frac{n-2}{2}}|\eta|^{2-n}Q(\tilde{y}) + \mu^{\frac{n-2}{2}}\Phi^*(x,t) - \mu^{\frac{n-2}{2}}H(x,q)\right) \\ &- \mu(t)^{-\frac{n+2}{2}}|\eta|^{-2-n}|Q(\tilde{y})|^{p-1}Q(\tilde{y}) + \mu^{\frac{n-2}{2}}\Delta\Phi^*(x,t). \end{aligned}$$

Let

$$S_2 = \mu^{-\frac{n+2}{2}}|\eta|^{-2-n}\left[|Q(\tilde{y})+\Theta|^{p-1}(Q(\tilde{y})+\Theta) - |Q(\tilde{y})|^{p-1}Q(\tilde{y})\right],$$

and

$$\Theta = \mu^{n-2}|\eta|^{n-2}\Phi^*(x,t) - \mu^{n-2}|\eta|^{n-2}H(x,q). \tag{2.20}$$

Observe that $|\Theta| \lesssim \mu_0^{n-2}$ when ε is small enough, we may assume $Q(y)^{-1}|\Theta| < \frac{1}{2}$ in the considered region $|x-q| < \varepsilon$. Using Taylor's expansion, we obtain the following

$$S_2 = \mu^{-\frac{n+2}{2}}|\eta|^{-2-n}\left[p\,|Q(\tilde{y})|^{p-1}\Theta + p(p-1)\int_0^1 (1-s)\,|Q(\tilde{y})+s\Theta|^{p-2}\,ds\,\Theta^2\right].$$

Hence we have

$$\begin{aligned} \Theta =& \mu^{n-2}|\eta|^{n-2}\Phi^*\big((|\eta|^2 R_{-\theta}\tilde{y} + a|y|^2)\mu + \xi, t\big) \\ &- \mu^{n-2}|\eta|^{n-2}H\big((|\eta|^2 R_{-\theta}\tilde{y} + a|y|^2)\mu + \xi, q\big). \end{aligned}$$

We further expand as

$$\begin{aligned} \Theta =& -\mu^{n-2}|\eta|^{n-2}\left(H(q,q) - \Phi^*(q,t)\right) \\ &+ \big((|\eta|^2 R_{-\theta}\tilde{y} + a|y|^2)\mu + \xi - q\big)\cdot\left[-\mu^{n-2}|\eta|^{n-2}\nabla\left(H(q,q) - \Phi^*(q,t)\right)\right] \\ &+ \int_0^1 \left\{-\mu^{n-2}|\eta|^{n-2}D_x^2 H\big(q + s\big((|\eta|^2 R_{-\theta}\tilde{y} + a|y|^2)\mu + \xi - q\big), q\big)\right\} \\ &\times\left[(|\eta|^2 R_{-\theta}\tilde{y} + a|y|^2)\mu + \xi - q\right]^2(1-s)ds \\ &+ \int_0^1 \left\{\mu^{n-2}|\eta|^{n-2}D_x^2\Phi^*\big(q + s\big((|\eta|^2 R_{-\theta}\tilde{y} + a|y|^2)\mu + \xi - q\big), t\big)\right\} \\ &\times\left[(|\eta|^2 R_{-\theta}\tilde{y} + a|y|^2)\mu + \xi - q\right]^2(1-s)ds. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\Theta &= -\mu^{n-2}|\eta|^{n-2}H(q,q) - \mu^{n-1}|\eta|^{n}\nabla H(q,q)\cdot R_{-\theta}\tilde{y} \\
&\quad - \mu^{n-2}|\eta|^{n-2}\nabla H(q,q)\cdot(\xi-q) - \mu^{n-1}|\eta|^{n-2}\nabla H(q,q)\cdot a|y|^2 \\
&\quad + \mu^{n-2}|\eta|^{n-2}\Phi^*(q,t) + \mu^{n-1}|\eta|^{n}\nabla\Phi^*(q,t)\cdot R_{-\theta}\tilde{y} \\
&\quad + \mu^{n-2}|\eta|^{n-2}\nabla\Phi^*(q,t)\cdot(\xi-q) - \mu^{n-1}|\eta|^{n-2}\nabla\Phi^*(q,t)\cdot a|y|^2 \\
&\quad + \mu_0^n F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi) \\
&= -\mu^{n-2}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n-2}{2}} H(q,q) \\
&\quad - \mu^{n-1}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n}{2}} \nabla H(q,q)\cdot R_{-\theta}\tilde{y} \\
&\quad - \mu^{n-2}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n-2}{2}} \nabla H(q,q)\cdot(\xi-q) \\
&\quad - \mu^{n-1}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n-2}{2}} \nabla H(q,q)\cdot a|y|^2 \\
&\quad + \mu^{n-2}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n-2}{2}} \Phi^*(q,t) \\
&\quad + \mu^{n-1}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n}{2}} \nabla\Phi^*(q,t)\cdot R_{-\theta}\tilde{y} \\
&\quad + \mu^{n-2}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n-2}{2}} \nabla\Phi^*(q,t)\cdot(\xi-q) \\
&\quad + \mu^{n-1}\left(1-2a\cdot\frac{x-\xi}{\mu(t)}+|a|^2\frac{|x-\xi|^2}{\mu^2(t)}\right)^{\frac{n-2}{2}} \nabla\Phi^*(q,t)\cdot a|y|^2 \\
&\quad + \mu_0^n F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi) \\
&= -\mu^{n-2}\left(1+O(|a||y|)\right)H(q,q) \\
&\quad - \mu^{n-1}\left(1+O(|a||y|)\right)\nabla H(q,q)\cdot y\left(1+O(|a||y|)\right) \\
&\quad - \mu^{n-2}\left(1+O(|a||y|)\right)\nabla H(q,q)\cdot(\xi-q) - \mu^{n-1}\left(1+O(|a||y|)\right)\nabla H(q,q)\cdot a|y|^2 \\
&\quad + \mu^{n-2}\left(1+O(|a||y|)\right)\Phi^*(q,t) + \mu^{n-1}\left(1+O(|a||y|)\right)\nabla\Phi^*(q,t)\cdot y\left(1+O(|a||y|)\right) \\
&\quad + \mu^{n-2}\left(1+O(|a||y|)\right)\nabla\Phi^*(q,t)\cdot(\xi-q) \\
&\quad + \mu^{n-1}\left(1+O(|a||y|)\right)\nabla\Phi^*(q,t)\cdot a|y|^2 + \mu_0^n F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi) \\
&= -\mu^{n-2}H(q,q) - \mu^{n-1}\nabla H(q,q)\cdot y - \mu^{n-2}\nabla H(q,q)\cdot(\xi-q) \\
&\quad - \mu^{n-1}\nabla H(q,q)\cdot a|y|^2 + \mu^{n-2}\Phi^*(q,t) \\
&\quad + \mu^{n-1}\nabla\Phi^*(q,t)\cdot y + \mu^{n-2}\nabla\Phi^*(q,t)\cdot(\xi-q) + \mu^{n-1}\nabla\Phi^*(q,t)\cdot a|y|^2 \\
&\quad + \mu_0^n F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi) + \mu_0^{n-2}|a||y|F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
& p\,|Q(\tilde{y})|^{p-1}\,\Theta \\
&= p\left|Q\left(R_\theta y + a\,|y|^2 + O\left(|a|^2|y|^3\right)\right)\right|^{p-1}\Theta \\
&= p\left|Q(y) + \nabla Q(y)\cdot\left(a\,|y|^2 + (R_\theta y - y)\right) + O(|a|^2|y|^2)\right|^{p-1}\Theta \\
&= p\left(|Q|^{p-1}(y) + O(|a||y|)\right)\Theta \\
&= p\left(|Q|^{p-1}(y) + O(|a||y|)\right)\Big(-\mu^{n-2}H(q,q) - \mu^{n-1}\nabla H(q,q)\cdot y \\
&\quad - \mu^{n-2}\nabla H(q,q)\cdot(\xi - q) - \mu^{n-1}\nabla H(q,q)\cdot a|y|^2 \\
&\quad + \mu^{n-2}\Phi^*(q,t) + \mu^{n-1}\nabla\Phi^*(q,t)\cdot y + \mu^{n-2}\nabla\Phi^*(q,t)\cdot(\xi - q) \\
&\quad + \mu^{n-1}\nabla\Phi^*(q,t)\cdot a|y|^2 + \mu_0^n F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi) \\
&\quad + \mu_0^{n-2}|a||y|F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi)\Big) \\
&= p\,|Q|^{p-1}(y)\Big(-\mu^{n-2}H(q,q) - \mu^{n-1}\nabla H(q,q)\cdot y \\
&\quad - \mu^{n-2}\nabla H(q,q)\cdot(\xi - q) - \mu^{n-1}\nabla H(q,q)\cdot a|y|^2 \\
&\quad + \mu^{n-2}\Phi^*(q,t) + \mu^{n-1}\nabla\Phi^*(q,t)\cdot y + \mu^{n-2}\nabla\Phi^*(q,t)\cdot(\xi - q) \\
&\quad + \mu^{n-1}\nabla\Phi^*(q,t)\cdot a|y|^2\Big) + \frac{\mu_0^{n-2}|a||y|}{1+|y|^4}F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi),
\end{aligned}
$$

where the smooth functions F are bounded in its arguments which may different from line to line.

Decompose S_1 as $S_1 = S_{11} + S_{12}$, where

$$
\begin{aligned}
S_{11} &:= \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 - \mu^{\frac{n-2}{2}}\partial_t\Phi^*(x,t), \\
S_{12} &:= \frac{n-2}{2}\mu^{\frac{n-4}{2}}\dot{\mu}H(x,q) - \frac{n-2}{2}\mu^{\frac{n-4}{2}}\dot{\mu}\Phi^*(x,t).
\end{aligned}
$$

Observe that

$$
S_{12} = \mu_0^{\frac{n-2}{2}-1}\dot{\mu}F(\mu_0^{-1}\mu,\xi,a,\theta,x-\xi)
$$

holds for a function F smooth and bounded in their arguments. This proves the lemma. □

Recall that we are trying to find a solution with form

$$
u(x,t) = u_A(x,t) + \tilde{\phi}(x,t),
$$

where $\tilde{\phi}$ is a small term compared with $u_A(x,t)$. By the relation $S(u_A+\tilde{\phi})=0$, the main equation can be written as

$$-\partial_t\tilde{\phi}+\Delta\tilde{\phi}+p\,|u_A|^{p-1}\,\tilde{\phi}+S(u_A)+\tilde{N}_A(\tilde{\phi}), \tag{2.21}$$

where

$$\tilde{N}_A(\tilde{\phi})=\left|u_A+\tilde{\phi}\right|^{p-1}(u_A+\tilde{\phi})-|u_A|^{p-1}(u_A+\tilde{\phi})-p\,|u_A|^{p-1}\,\tilde{\phi}. \tag{2.22}$$

Note that around q it is more convenient to use the self-similar form, so we write $\tilde{\phi}(x,t)$ as

$$\tilde{\phi}(x,t)=\mu(t)^{-\frac{n-2}{2}}\phi\left(\frac{x-\xi(t)}{\mu(t)}\right). \tag{2.23}$$

2.4. Improvement of the approximation

The largest term in the expansion for $\mu^{\frac{n+2}{2}}S(u_A)$ is μE_0. To improve the approximation error near the point q, $\phi(y,t)$ should be the solution of the elliptic equation (at main order)

$$\Delta_y\phi_0+p|Q|^{p-1}(y)\phi_0=-\mu_0E_0\quad\text{in }\mathbb{R}^n,\quad \phi_0(y,t)\to 0\quad\text{as }|y|\to\infty. \tag{2.24}$$

Equation (2.24) is an elliptic equation of the form

$$L[\psi]:=\Delta_y\psi+p|Q|^{p-1}(y)\psi=h(y)\quad\text{in }\mathbb{R}^n,\quad \psi(y)\to 0\quad\text{as }|y|\to\infty. \tag{2.25}$$

By the nondegeneracy of the basic cell Q (see [33]), we know that each bounded solution of $L[\psi]=0$ in $\mathbb{R}^n$ is contained in the space

$$\operatorname{span}\{z_0,\cdots,z_{3n-1}\}.$$

Standard elliptic theory tells us that problem (2.25) is solvable for $h(y)=O(|y|^{-m})$, $m>2$, if and only if the L^2 orthogonal identities

$$\int_{\mathbb{R}^n}h(y)z_i(y)dy=0\quad\text{for all } i=0,\cdots,3n-1$$

hold.

For (2.24), we first consider the following condition,

$$\int_{\mathbb{R}^n}\mu^{\frac{n+2}{2}}S(u_A)(y,t)z_0(y)dy=0. \tag{2.26}$$

We claim that, for suitable positive constant b and a positive constant c_n depending only on n, choosing $\mu=b\mu_0(t)$, $\mu_0(t)=c_nt^{-\frac{1}{n-4}}$, (2.26) can be achieved at main order. Observe that $\dot{\mu}_0(t)=-\frac{1}{(n-4)c_n^{n-4}}\mu_0^{n-3}(t)$ and the main contribution to the left of (2.26) comes from the following term

$$E_{0j}=p|Q|^{p-1}\left[\mu^{n-3}\left(\Phi^0(q,t)-H(q,q)\right)\right]+\dot{\mu}(t)\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right).$$

Now let us compute the term $\Phi^0(q,t)$ which is given by (2.15). Note that the heat kernel function $p(t,x)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{-|x|^2}{4t}}$ satisfies the following transformation law

$$p(t-\tilde{s},q-y)=(t-\tilde{s})^{-\frac{n}{2}}p\left(1,\frac{|q-y|}{(t-\tilde{s})^{\frac{1}{2}}}\right),$$

therefore we have

$$\begin{aligned}
&\Phi^0(q,t)\\
&=-\int_{t_0}^{t}\int_{\mathbb{R}^n}p(t-\tilde{s},q-y)\frac{\dot{\mu}(\tilde{s})}{\mu(\tilde{s})}\frac{\mu^{-(n-2)}(\tilde{s})D_{n,k}\left(2-\left|\frac{y-\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2\right)}{\left(1+\left|\frac{y-R_{\theta(\tilde{s})}\xi(\tilde{s})}{\mu(\tilde{s})}\right|^2\right)^{\frac{n}{2}}}dyd\tilde{s}\\
&=-(1+o(1))\int_{t_0}^{t}\int_{\mathbb{R}^n}p(t-\tilde{s},q-y)\frac{\dot{\mu}(\tilde{s})}{\mu(\tilde{s})}\frac{\mu^{-(n-2)}(\tilde{s})D_{n,k}\left(2-\left|\frac{y-q}{\mu(\tilde{s})}\right|^2\right)}{\left(1+\left|\frac{y-q}{\mu(\tilde{s})}\right|^2\right)^{\frac{n}{2}}}dyd\tilde{s}\\
&=-(1+o(1))\int_{t_0}^{t}\frac{1}{(t-\tilde{s})^{\frac{n}{2}}}d\tilde{s}\int_{\mathbb{R}^n}p\left(1,\frac{q-y}{(t-\tilde{s})^{\frac{1}{2}}}\right)\frac{\dot{\mu}(\tilde{s})}{\mu(\tilde{s})}\\
&\qquad\times\frac{\mu^{-(n-2)}(\tilde{s})\,(t-\tilde{s})^{\frac{n}{2}}\,D_{n,k}\left(2-\left|\frac{(t-\tilde{s})^{\frac{1}{2}}}{\mu(\tilde{s})}\frac{q-y}{(t-\tilde{s})^{\frac{1}{2}}}\right|^2\right)}{\left(1+\left|\frac{(t-\tilde{s})^{\frac{1}{2}}}{\mu(\tilde{s})}\frac{q-y}{(t-\tilde{s})^{\frac{1}{2}}}\right|^2\right)^{\frac{n}{2}}}d\frac{y-q_j}{(t-\tilde{s})^{\frac{1}{2}}}\\
&=-(1+o(1))\int_{t_0}^{t}\frac{\dot{\mu}(\tilde{s})}{\mu(\tilde{s})}\mu^{-(n-2)}(\tilde{s})d\tilde{s}\int_{\mathbb{R}^n}p\left(1,\frac{q-y}{(t-\tilde{s})^{\frac{1}{2}}}\right)\\
&\qquad\times\frac{D_{n,k}\left(2-\left|\frac{(t-\tilde{s})^{\frac{1}{2}}}{\mu(\tilde{s})}\frac{q-y}{(t-\tilde{s})^{\frac{1}{2}}}\right|^2\right)}{\left(1+\left|\frac{(t-\tilde{s})^{\frac{1}{2}}}{\mu(\tilde{s})}\frac{q-y}{(t-\tilde{s})^{\frac{1}{2}}}\right|^2\right)^{\frac{n}{2}}}d\frac{y-q}{(t-\tilde{s})^{\frac{1}{2}}}\\
&=-(1+o(1))\int_{t_0}^{t}\frac{\dot{\mu}(\tilde{s})}{\mu(\tilde{s})}\mu^{-(n-2)}(\tilde{s})F\left(\frac{(t-\tilde{s})^{\frac{1}{2}}}{\mu(\tilde{s})}\right)d\tilde{s},
\end{aligned}$$

with

$$F(a)=\int_{\mathbb{R}^n}p\,(1,x)\,\frac{D_{n,k}\left(2-a^2|x|^2\right)}{\left(1+a^2|x|^2\right)^{\frac{n}{2}}}dx.$$

We claim that, for a suitable positive constant c depending on n and b, it holds that

$$\Phi^0(q,t)=-(1+o(1))\int_{t_0}^{t}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s=c(1+o(1)). \quad (2.27)$$

Indeed, for a small positive constant δ, decompose the integral

$$\int_{t_0}^{t}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s$$

as

$$\begin{aligned}\int_{t_0}^{t}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s &= \int_{t_0}^{t-\delta}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s\\ &\quad+\int_{t-\delta}^{t}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s\\ &=: I_1+I_2.\end{aligned}$$

For I_1, we have $t-\tilde s>\delta$, therefore

$$\begin{aligned}0\le -I_1 &\le \frac{b^{4-n}}{(n-4)c_n^{n-4}}\int_{t_0}^{t-\delta}\mu^{-2}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s\\ &\le C\frac{b^{4-n}}{(n-4)c_n^{n-4}}\int_{t_0}^{t-\delta}\mu^{-2}(\tilde s)\left|\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right|^{-(n-2)}d\tilde s\\ &=\frac{C}{n-4}\int_{t_0}^{t-\delta}\frac{1}{\tilde s}\frac{1}{(t-\tilde s)^{\frac{n-2}{2}}}d\tilde s\le\frac{C}{(n-4)t_0}\frac{2}{n-4}\frac{1}{\delta^{\frac{n-4}{2}}}.\end{aligned}$$

Note that we have used the definition $\mu_0=bc_nt^{-\frac{1}{n-4}}$ and the fact $|a|^{n-2}F(a)\le C$. For

$$I_2=\int_{t-\delta}^{t}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s,$$

after the change of variables $\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}=\hat s$, we have

$$\begin{aligned}d\tilde s&=-\frac{\mu(\tilde s)}{\frac12(t-\tilde s)^{-\frac12}+\dot\mu(\tilde s)\hat s}d\hat s,\\ I_2&=\int_{t-\delta}^{t}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\frac{(t-\tilde s)^{\frac12}}{\mu(\tilde s)}\right)d\tilde s\\ &=\int_0^{\frac{\delta^{\frac12}}{\mu(t-\delta)}}\frac{\dot\mu(\tilde s)}{\mu(\tilde s)}\mu^{-(n-2)}(\tilde s)F\left(\hat s\right)\frac{\mu(\tilde s)}{\frac12(t-\tilde s)^{-\frac12}+\dot\mu(\tilde s)\hat s}d\hat s.\end{aligned}$$

Observe that for small δ, $\frac{1}{2}(t-\tilde{s})^{-\frac{1}{2}}+\dot{\mu}(\tilde{s})\hat{s}=\frac{1}{2}(t-\tilde{s})^{-\frac{1}{2}}(1-\frac{2}{(n-4)\tilde{s}}(t-\tilde{s}))>\frac{1}{2}(t-\tilde{s})^{-\frac{1}{2}}(1-\frac{2}{(n-4)\tilde{s}}\delta)$, $d\tilde{s}=\frac{\mu(\tilde{s})}{\frac{1}{2}(t-\tilde{s})^{-\frac{1}{2}}}(1+O(\delta))d\hat{s}$, hence

$$I_2=-\frac{2b^{4-n}}{(n-4)c_n^{n-4}}\left(\int_0^{\frac{\delta^{\frac{1}{2}}}{\mu(t-\delta)}}\hat{s}F\left(\hat{s}\right)d\hat{s}+o(1)\right)=-\frac{2b^{4-n}}{(n-4)c_n^{n-4}}A+o(1)$$

when $\frac{\delta^{\frac{1}{2}}}{\mu(t-\delta)}$ is large enough. Here the constant $A=\int_0^\infty \tilde{s}F(\tilde{s})d\tilde{s}<+\infty$ since the dimension of the space satisfies $n>4$. Hence we have

$$\begin{aligned}\Phi^0(q,t)&=-(1+o(1))\int_{t_0}^t\frac{\dot{\mu}(\tilde{s})}{\mu(\tilde{s})}\mu^{-(n-2)}(\tilde{s})F\left(\frac{(t-\tilde{s})^{\frac{1}{2}}}{\mu(\tilde{s})}\right)d\tilde{s}\\&=\frac{2b^{4-n}}{(n-4)c_n^{n-4}}A+o(1):=Bb^{4-n}+o(1)\end{aligned}\tag{2.28}$$

when t_0 is sufficiently large. Here the constant B is $B=B_n:=\frac{2}{(n-4)c_n^{n-4}}A$. This is (2.27).

Direct computations yield that

$$\mu_0^{-(n-3)}(t)\int_{\mathbb{R}^n}E_0(y,t)z_0(y)dy\approx c_1b^{n-3}H(q,q)-\frac{2c_1A+c_2}{(n-4)c_n^{n-4}}b\tag{2.29}$$

with

$$\begin{aligned}c_1&=-p\int_{\mathbb{R}^n}|Q|^{p-1}(y)z_0(y)dy\in(0,+\infty),\\c_2&=\int_{\mathbb{R}^n}\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)z_0(y)dy\in(0,+\infty).\end{aligned}$$

Note that $c_1<+\infty$ and $c_2<+\infty$ are due to the assumption $n>4$. We will prove this

$$c_1>0,\quad c_2>0\tag{2.30}$$

in the Appendix. Write

$$\mu(t)=b\mu_0(t)=bc_nt^{-\frac{1}{n-4}}.$$

Then (2.26) can be satisfied at main order if the following holds

$$b^{n-2}H(q,q)-\frac{2c_1A+c_2}{(n-4)c_n^{n-4}c_1}b^2=0.\tag{2.31}$$

Imposing $\frac{2c_1A+c_2}{(n-4)c_n^{n-4}c_1} = \frac{2}{n-2}$, *i.e.*,

$$c_n = \left[\frac{(2c_1A+c_2)(n-2)}{2(n-4)c_1}\right]^{\frac{1}{n-4}},$$

we get

$$\dot{\mu}_0(t) = -\frac{2c_1}{(2c_1A+c_2)(n-2)}\mu_0^{n-3}(t). \tag{2.32}$$

By (2.31) and (2.32), the constants b should satisfy the relation

$$H(q,q)b^{n-3} = \frac{2b}{n-2}. \tag{2.33}$$

It is clear that (2.33) can be uniquely solved if and only if

$$H(q,q) > 0, \tag{2.34}$$

which holds from the maximum principle. Under the assumption (2.34),

$$b = \left(\frac{2}{(n-2)H(q,q)}\right)^{\frac{1}{n-4}}. \tag{2.35}$$

Similarly, the relations

$$\int_{\mathbb{R}^n} \mu^{\frac{n+2}{2}} S(u_A)(y,t)z_i(y)dy = 0, \quad i = 1, \cdots, 3n-1 \tag{2.36}$$

can be achieved at main order by choosing $\xi_0 = q, a_0 = (0,0)$ and $\theta_0 = (0, \cdots, 0)$.

Now fix $\mu_0(t)$ and the constant b satisfying (2.35), denote

$$\bar{\mu}_0 = b\mu_0(t).$$

Let Φ be the solution for (2.24) for $\mu = \bar{\mu}_0$ which is unique, then we have the following

$$\Delta_y\Phi + p|Q|^{p-1}(y)\Phi = -\mu_0E_0[\mu_0, \dot{\mu}_0] \text{ in } \mathbb{R}^n, \ \ \Phi(y,t) \to 0 \text{ as } |y| \to \infty.$$

From the definitions for μ_0 and b, we obtain

$$\mu_0E_0 = -\gamma\mu_0^{n-2}q_0(y),$$

where γ is positive,

$$q_0(y) := \frac{p|Q|^{p-1}(y)c_2b^2}{(n-4)c_n^{n-4}c_1} + \frac{b^2}{(n-4)c_n^{n-4}}\left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right) \tag{2.37}$$

and $\int_{\mathbb{R}^n} q_0(y)z_0dy = 0$.

Let $p_0 = p_0(|y|)$ be the solution for $L(p_0) = q_0$. Then $p_0(y) = O(|y|^{-2})$ as $|y| \to \infty$ since (2.37) holds. Therefore,

$$\Phi(y,t) = \gamma \mu_0^{n-2} p_0(y). \tag{2.38}$$

Thus the corrected approximation should be

$$u_A^*(x,t) = u_A(x,t) + \tilde{\Phi}(x,t) \tag{2.39}$$

with

$$\tilde{\Phi}(x,t) = \mu(t)^{-\frac{n-2}{2}} \Phi\left(\frac{x-\xi(t)}{\mu(t)}\right).$$

2.5. Estimating the error $S(u_A^*)$

In the region $|x-q| > \delta$, $S(u_A^*)$ can be described as

$$S(u_A^*)(x,t) = \mu_0^{\frac{n-2}{2}-1} \dot{\mu} f_1 + \mu_0^{\frac{n+2}{2}} f_2 + \mu_0^{\frac{n-2}{2}} \dot{\xi} \cdot \vec{f}_1 + \mu_0^{\frac{n}{2}} \dot{a} \cdot \vec{f}_2 + \mu_0^{\frac{n}{2}} \dot{\theta} \cdot \vec{f}_3, \tag{2.40}$$

where f_1, f_2, $\vec{f}_1$, $\vec{f}_2$ and $\vec{f}_3$ are smooth bounded functions depending on the tuple $(x, \mu_0^{-1}\mu, \xi, a, \theta)$.

In the region near the point q, direct computations yields that

$$\begin{aligned} S(u_A^*) &= S(u_A) - \mu^{-\frac{n+2}{2}} \mu_0 E_0[\bar{\mu}_0, \dot{\mu}_0] \\ &\quad + \mu^{-\frac{n+2}{2}} \left\{ -\mu^2 \partial_t \Phi(y,t) + \mu\dot{\mu} \left[\frac{n-2}{2} \Phi(y,t) + y \cdot \nabla_y \Phi \right] + \nabla_y \Phi(y,t) \cdot \mu \dot{\xi} \right\} \\ &\quad + \left| u_A + \tilde{\Phi} \right|^{p-1} (u_A + \tilde{\Phi}) - |u_A|^{p-1} u_A - p\mu^{-\frac{n+2}{2}} |Q(y)|^{p-1} \Phi(y,t), \end{aligned} \tag{2.41}$$

where $y = \frac{x-\xi}{\mu}$. If $|x-q| \le \delta$,

$$\mu^{\frac{n+2}{2}} S(u_A^*) = \mu^{\frac{n+2}{2}} S(u_A) - \mu_0 E_0[\bar{\mu}_0, \dot{\mu}_0] + A(y), \tag{2.42}$$

where

$$A = \mu_0^{n+4} f(\mu_0^{-1}\mu, \xi, a, \theta, \mu y) + \frac{\mu_0^{2n-4}}{1+|y|^2} g(\mu_0^{-1}\mu, \xi, a, \theta, \mu y), \quad y = \frac{x-\xi}{\mu} \tag{2.43}$$

for smooth and bounded functions f and g.

Now we write $\mu(t)$ as

$$\mu(t) = \bar{\mu}_0 + \lambda(t).$$

From (2.42),

$$S(u_A^*)=\mu^{-\frac{n+2}{2}}\left\{\mu_0\big(E_0[\mu,\dot\mu]-E_0[\bar\mu_0,\dot\mu_0]\big)+\lambda E_0[\mu,\dot\mu]+\mu E_1[\mu,\dot\xi]+R+A\right\}.$$

Observe that Φ^0 is a nonlocal term depending on $\mu,\ \dot\mu$ and we have

$$\begin{aligned}&\mu^{n-3}\Phi^0[\bar\mu_0+\lambda, b\dot\mu_0+\dot\lambda](q,t)-\mu^{n-3}\Phi^0[\bar\mu_0, b\dot\mu_0](q,t)\\&=-2A\dot\lambda-\mu_0^{n-4}(n-3)B\lambda\end{aligned}$$

which can be deduced by similar arguments as (2.28), one gets

$$\begin{aligned}&E_0[\bar\mu_0+\lambda, b\dot\mu_0+\dot\lambda]-E_0[\bar\mu_0, b\dot\mu_0]\\&=\dot\lambda\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\big(1+|y|^2\big)^{\frac n2}}\right)-\mu_0^{n-4}p|Q|^{p-1}(y)\Big[(n-3)b^{n-4}H(q,q)\lambda\Big]\\&+\mu_0^{n-4}p|Q|^{p-1}(y)(n-3)B\lambda-p|Q|^{p-1}(y)2A\dot\lambda\\&-\mu_0^{n-4}p|Q|^{p-1}(y)(n-3)B\lambda,\end{aligned}$$

As for $\lambda E_0[\mu,\dot\mu]$, we have

$$\begin{aligned}\lambda E_0[\mu,\dot\mu]=\lambda\dot\lambda&\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\big(1+|y|^2\big)^{\frac n2}}\right)\\&+\lambda b\Bigg[\dot\mu_0\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\big(1+|y|^2\big)^{\frac n2}}\right)\\&+p|Q|^{p-1}(y)\mu_0^{n-3}\big(-b^{n-4}H(q,q)\big)\Bigg]\\&+p|Q|^{p-1}(y)b\mu_0^{n-3}B\lambda\\&-\mu_0^{n-4}p|Q|^{p-1}(y)f(\mu_0^{-1}\lambda)\lambda^2,\end{aligned}$$

where f is smooth and bounded in its arguments.

Combining all the estimates above, we get the expansion for $S(u_A^*)$.

Lemma 2.2. *In the region $|x-q|\le\delta$ for a fixed small $\delta>0$, set $\mu=\bar\mu_0+\lambda$ with $|\lambda(t)|\le\mu_0(t)^{1+\sigma}$ for some positive number $\sigma\in(0,n-4)$. When t is large* *enough, we have the expansion of $S(u_A^*)$ as*

$$
\begin{aligned}
&S(u_A^*)\\
&= \mu^{-\frac{n+2}{2}} \Big\{ \mu_0 \dot\lambda \Big(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}} - 2Ap|Q|^{p-1}(y) \Big) \\
&\quad - \mu_0 \mu_0^{n-4} p|Q|^{p-1}(y) \Big[(n-3) b^{n-4} H(q,q) \lambda \Big] \\
&\quad + \Big(\nabla Q(y) - \frac{E_{n,k} y}{\left(1+|y|^2\right)^{\frac{n}{2}}} \Big) \cdot \dot\xi + p|Q|^{p-1} \Big[-\mu^{n-2} \nabla H(q,q) \Big] \cdot y \\
&\quad + \mu^2(t) \dot a_1 \Big(-2y_1 \Big(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}} \Big) + |y|^2 \Big(\frac{\partial}{\partial y_1} Q(y) - \frac{E_{n,k} y_1}{\left(1+|y|^2\right)^{\frac{n}{2}}} \Big) \Big) \\
&\quad + \mu^2(t) \dot a_2 \Big(-2y_2 \Big(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}} \Big) + |y|^2 \Big(\frac{\partial}{\partial y_2} Q(y) - \frac{E_{n,k} y_2}{\left(1+|y|^2\right)^{\frac{n}{2}}} \Big) \Big) \\
&\quad + \mu^2(t) \dot\theta_{12} z_{n+1}(y) + \sum_{j=3}^{n} \Big(\mu^2(t) \dot\theta_{1j} z_{n+j+1}(y) + \mu^2(t) \dot\theta_{2j} z_{2n+j-1}(y) \Big) \Big\} \\
&\quad + \mu^{-\frac{n+2}{2}} \lambda b \Big[\dot\mu_0 \Big(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}} \Big) \\
&\qquad\qquad + p|Q|^{p-1}(y) \mu_0^{n-3} \Big(-b^{n-4} H(q,q) + B \Big) \Big] \\
&\quad + \mu_0^{-\frac{n+2}{2}} \Big[\mu_0^{n-4} p|Q|^{p-1}(y) f_1 \lambda^2 + \frac{f_2}{1+|y|^{n-2}} \lambda \dot\lambda + \mu_0^{n+2} f_3 + \mu_0^{n-1} \dot\mu f_4 \Big] \\
&\quad + \mu_0^{-\frac{n+2}{2}} \Big[\dot\xi \vec f_1 + \dot\xi \vec f_2 + \dot\xi \vec f_3 \Big] \\
&\quad + \mu_0^{-\frac{n+2}{2}} \Big[\frac{\mu_0^n g_1}{1+|y|^2} + \frac{\mu_0^{2n-4} g_2}{1+|y|^2} + \frac{\mu_0^{n-2} g_3}{1+|y|^4} \lambda + \frac{\mu_0^{n-1} \vec g_1}{1+|y|^2} \cdot a + \frac{\mu_0^{n-2} \vec g_2}{1+|y|^4} \cdot (\xi - q) \Big],
\end{aligned}
$$

where $x = \xi + \mu y$, f_1, f_2, f_3, f_4, $\vec f_1$, $\vec f_2$, $\vec f_3$, g_1, g_2, g_3 and $\vec g_1$, $\vec g_2$ are smooth bounded (vector) functions depending on the tuple of variables $(\mu_0^{-1}\mu, \xi, a, \theta, x)$.

3. The inner-outer gluing procedure

We will find a solution for (2.11) with form

$$u = u_A^* + \tilde\phi$$

when t_0 is large enough, the function $\tilde\phi(x,t)$ is small compared to u_A^*. To this aim, we use the **inner-outer gluing procedure**.

Write

$$\tilde{\phi}(x,t) = \psi(x,t) + \phi^{in}(x,t) \quad \text{where} \quad \phi^{in}(x,t) := \eta_R(x,t)\tilde{\phi}(x,t)$$

with

$$\tilde{\phi}(x,t) := \mu_0^{-\frac{n-2}{2}} \phi\left(\frac{x-\xi}{\mu_0}, t\right), \quad \mu_0(t) = b\mu_0(t)$$

and

$$\eta_R(x,t) = \eta\left(\frac{|x-\xi|}{R\mu_0}\right).$$

In above, $\eta(\tau)$ is a (smooth) cut-off function defined on the interval $[0,+\infty)$, $\eta(\tau) = 1$ for $0 \le \tau < 1$ and $\eta(\tau) = 0$ for $\tau > 2$. R is a fixed number defined as

$$R = t_0^{\rho} \quad \text{with} \quad 0 < \rho \ll 1. \tag{3.1}$$

Under this ansatz, problem (2.11) can be written as

$$\begin{cases} \partial_t \tilde{\phi} = \Delta\tilde{\phi} + p(u_A^*)^{p-1}\tilde{\phi} + \tilde{N}(\tilde{\phi}) + S(u_A^*) & \text{in} \quad \Omega \times (t_0, \infty) \\ \tilde{\phi} = -u_A^* & \text{in} \quad \partial\Omega \times (t_0, \infty) \end{cases} \tag{3.2}$$

where $\tilde{N}_A(\tilde{\phi}) = |u_A^* + \tilde{\phi}|^{p-1}(u_A^* + \tilde{\phi}) - p|u_A^*|^{p-1}\tilde{\phi} - |u_A^*|^{p-1}u_A^*$, $S(u_A^*) = -\partial_t \mu_A^* + \Delta u_A^* + |u_A^*|^{p-1}u_A^*$. Let us write $S(u_A^*)$ as

$$S(u_A^*) = S_A + S_A^{(2)},$$

where

$$\begin{aligned}
& S_A \\
&= \mu^{-\frac{n+2}{2}} \Bigg\{ \mu_0\dot{\lambda}\left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{(1+|y|^2)^{\frac{n}{2}}} - 2Ap|Q|^{p-1}(y) \right) \\
&\quad - \mu_0 \mu_0^{n-4} p|Q|^{p-1}(y) \left[(n-3)b^{n-4}H(q,q)\lambda \right] \\
&\quad + \lambda b \left[\dot{\mu}_0 \left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{(1+|y|^2)^{\frac{n}{2}}} \right) + p|Q|^{p-1}(y)\mu_0^{n-3}(-b^{n-4}H(q,q) + B) \right] \\
&\quad + \left(\nabla Q(y) - \frac{E_{n,k}y}{(1+|y|^2)^{\frac{n}{2}}} \right) \cdot \dot{\xi} + p|Q|^{p-1}\Big[-\mu^{n-2}\nabla H(q,q) \Big] \cdot y \\
&\quad + \mu^2(t)\dot{a}_1 \left(-2y_1 \left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{(1+|y|^2)^{\frac{n}{2}}} \right) + |y|^2 \left(\frac{\partial}{\partial y_1} Q(y) - \frac{E_{n,k}y_1}{(1+|y|^2)^{\frac{n}{2}}} \right) \right) \\
&\quad + \mu^2(t)\dot{a}_2 \left(-2y_2 \left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{(1+|y|^2)^{\frac{n}{2}}} \right) + |y|^2 \left(\frac{\partial}{\partial y_2} Q(y) - \frac{E_{n,k}y_2}{(1+|y|^2)^{\frac{n}{2}}} \right) \right) \\
&\quad + \mu^2(t)\dot{\theta}_{12} z_{n+1}(y) + \sum_{j=3}^{n} \left(\mu^2(t)\dot{\theta}_{1j} z_{n+j+1}(y) + \mu^2(t)\dot{\theta}_{2j} z_{2n+j-1}(y) \right) \Bigg\}.
\end{aligned}$$

Define

$$V_A = p\left(|u_A^*|^{p-1} - \left|\mu^{-\frac{n-2}{2}} Q\left(\frac{x-\xi}{\mu}\right)\right|^{p-1}\right)\eta_R + p(1-\eta_R)|u_A^*|^{p-1}, \quad (3.3)$$

then $\tilde{\phi}$ satisfies problem (3.2) if

(1) ψ solves the **outer problem**

$$\begin{cases} \partial_t \psi = \Delta\psi + V_A\psi + 2\nabla\eta_R\nabla\tilde{\phi} \\ \qquad\qquad +\tilde{\phi}\big(\Delta - \partial_t\big)\eta_R + \tilde{N}_A(\tilde{\phi}) + S_{\text{out}} & \text{in } \Omega\times(t_0,\infty) \\ \psi = -u_A^* & \text{on } \quad \partial\Omega\times(t_0,\infty), \end{cases} \quad (3.4)$$

with

$$S_{\text{out}} = S_A^{(2)} + (1-\eta_R)S_A. \quad (3.5)$$

(2) $\tilde{\phi}$ solves

$$\begin{aligned} \eta_R\partial_t\tilde{\phi} = \eta_R\Big[\Delta\tilde{\phi} + p|Q_{\mu,\xi,\theta}|^{p-1}\tilde{\phi} + p|Q_{\mu,\xi}|^{p-1}\psi + S_A\Big] \\ \text{in } B_{2R\mu}(\xi)\times(t_0,\infty), \end{aligned} \quad (3.6)$$

for $Q_{\mu,\xi} := \mu^{-\frac{n-2}{2}} Q\left(\frac{x-\xi}{\mu}\right)$. In the self-similar form, (3.6) becomes the so-called **inner problem**

$$\begin{aligned} \mu_0^2\partial_t\phi = {} & \Delta_y\phi + p|Q|^{p-1}(y)\phi + \mu_0^{\frac{n+2}{2}} S_A(\xi+\mu_0 y, t) \\ & + p\mu_0^{\frac{n-2}{2}}\frac{\mu_0^2}{\mu^2}|Q|^{p-1}(\frac{\mu_0}{\mu}y)\psi(\xi+\mu_0 y, t) + B[\phi] + B^0[\phi] \\ & \text{in} \quad B_{2R}(0)\times(t_0,\infty), \end{aligned} \quad (3.7)$$

where

$$B[\phi] := \mu_0\dot{\mu}_0\left(\frac{n-2}{2}\phi + y\cdot\nabla_y\phi\right) + \mu_0\nabla\phi\cdot\dot{\xi} \quad (3.8)$$

and

$$\begin{aligned} B^0[\phi] := {} & p\left[|Q|^{p-1}\left(\frac{\mu_0}{\mu}y\right) - |Q|^{p-1}(y)\right]\phi \\ & + p\left[\mu_0^2|u_A^*|^{p-1} - |Q|^{p-1}\left(\frac{\mu_0}{\mu}y\right)\right]\phi. \end{aligned} \quad (3.9)$$

4. Scheme of the proof

To find a solution (ϕ, ψ) satisfying (3.4) and (3.7), we proceed with the following steps.

4.1. Linear theory for (3.7)

Let us rewrite problem (3.7) as

$$\mu_0^2 \partial_t \phi = \Delta_y \phi + p|Q|^{p-1}(y)\phi \\ + H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t),\ y \in B_{2R}(0), \tag{4.1}$$

for $t \geq t_0$, where

$$H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi] := \mu_0^{\frac{n+2}{2}} S_A(\xi + \mu_0 y, t) + B[\phi] + B^0[\phi] \\ + p\mu_0^{\frac{n-2}{2}} \frac{\mu_0^2}{\mu^2} |Q|^{p-1}\left(\frac{\mu_0}{\mu} y\right) \psi(\xi + \mu_0 y, t), \tag{4.2}$$

the terms $B[\phi]$, $B^0[\phi]$ are defined in (3.8), (3.9) respectively. Using change of variables

$$t = t(\tau), \quad \frac{dt}{d\tau} = \mu_0^2(t),$$

(4.1) becomes

$$\partial_\tau \phi = \Delta_y \phi + p|Q|^{p-1}(y)\phi + H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau)) \tag{4.3}$$

for $y \in B_{2R}(0)$, $\tau \geq \tau_0$. Here τ_0 the (unique) positive number such that $t(\tau_0) = t_0$. We try to find a solution ϕ to the following equation:

$$\begin{cases} \partial_\tau \phi = \Delta_y \phi + p|Q|^{p-1}(y)\phi \\ \qquad +H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau)) & y \in B_{2R}(0),\ \tau \geq \tau_0 \\ \phi(y, \tau_0) = \sum_{l=1}^{K} e_l Z_l(y) & y \in B_{2R}(0), \end{cases} \tag{4.4}$$

for suitable constants e_l, $l = 1, \cdots, K$. Here Z_l are eigenfunctions associated to negative eigenvalues of the problem

$$L(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n).$$

It was proved in [15] that K is finite and Z_l satisfies

$$Z_l(x) \sim \frac{e^{-\sqrt{-\lambda}|x|}}{|x|^{\frac{N-1}{2}}} \text{ as } |x| \to \infty.$$

Next, we prove that (4.4) is solvable for ϕ, provided ψ is in suitable weighted spaces and the parameter functions λ, ξ, a, θ are chosen so that the term $H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau))$ in the right-hand side of (4.4) satisfies the following L^2 orthogonality conditions

$$\int_{B_{2R}} H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau)) z_l(y) dy = 0, \tag{4.5}$$

for all $\tau \geq \tau_0$, $l = 0, 1, 2, \cdots, 3n-1$. These conditions will impose highly nonlinearity to (4.4), to get a solution ϕ, we apply the Schauder fixed-point theorem. We first need a linear theory for (4.4).

For $R > 0$ large but fixed, consider the following initial value problem

$$\begin{cases} \partial_\tau \phi = \Delta \phi + p|Q|^{p-1}(y)\phi + h(y,\tau) & y \in B_{2R}(0)\ \tau \geq \tau_0 \\ \phi(y,\tau_0) = \sum_{l=1}^{K} e_l Z_l(y), & y \in B_{2R}(0). \end{cases} \tag{4.6}$$

Set

$$\nu = 1 + \frac{\sigma}{n-2},$$

then we have $\mu_0^{n-2+\sigma} \sim \tau^{-\nu}$. Define the weighted norm for h as

$$\|h\|_{\alpha,\nu} := \sup_{\tau > \tau_0} \sup_{y \in B_{2R}} \tau^{\nu}(1+|y|^{\alpha})|h(y,\tau)|.$$

Then the following estimates for (4.6) hold.

Proposition 4.1. *Suppose* $\alpha \in (2, n-2)$, $\nu > 0$, $\|h\|_{2+\alpha,\nu} < +\infty$ *and*

$$\int_{B_{2R}} h(y,\tau) z_j(y) dy = 0 \text{ for all } \tau \in (\tau_0, \infty),\ j = 0, 1, \cdots, 3n-1.$$

Then there exist functions $\phi = \phi[h](y,\tau)$ *and* $(e_1, \cdots, e_K) = (e_1[h](\tau), \cdots \cdots, e_K[h](\tau))$ *satisfying* (4.6). *Furthermore, for* $\tau \in (\tau_0, +\infty)$, $y \in B_{2R}(0)$, *there hold*

$$(1+|y|)|\nabla_y \phi(y,\tau)| + |\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-\alpha}\|h\|_{2+\alpha,\nu} \tag{4.7}$$

and

$$|e_l[h]| \lesssim \|h\|_{2+\alpha,\nu} \text{ for } l = 1, \cdots, K. \tag{4.8}$$

Here and in the following of this paper, the symbol $a \lesssim b$ means $a \leq Cb$ for some positive constant C which is independent of t and t_0. The proof of Proposition 4.1 is given in Section 5.

4.2. The orthogonality conditions (4.5)

To apply Proposition 4.1, we should choose the parameter functions λ, ξ, a and θ such that (4.5) hold.

Let us fix a $\sigma \in (0, n-4)$. Given $h(t) : (t_0, \infty) \to \mathbb{R}^k$ and $\delta > 0$, the weighted L^∞ norm is defined as

$$\|h\|_\delta := \|\mu_0(t)^{-\delta} h(t)\|_{L^\infty(t_0,\infty)}.$$

In what follows, α is always a positive constant such that $\alpha > 2$ and $\alpha - 2$ is small enough. Also assume the parameter functions $\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}$ and $\dot{\theta}$ satisfy the following constraints,

$$\|\dot{\lambda}(t)\|_{n-3+\sigma} + \|\dot{\xi}(t)\|_{n-3+\sigma} + \|\dot{a}(t)\|_{n-4+\sigma} + \|\dot{\theta}(t)\|_{n-4+\sigma} \le \frac{c}{R^{\alpha-2}}, \tag{4.9}$$

$$\|\lambda(t)\|_{1+\sigma} + \|\xi(t) - q\|_{1+\sigma} + \|a(t)\|_{\sigma} + \|\theta(t)\|_{\sigma} \le \frac{c}{R^{\alpha-2}}, \tag{4.10}$$

here c is a positive constant which is independent of R, t and t_0. Let us define the norm $\|\phi\|_{n-2+\sigma,\alpha}$ of ϕ as the least number $M > 0$ such that

$$(1+|y|)|\nabla_y \phi(y,t)| + |\phi(y,t)| \le M \frac{\mu_0^{n-2+\sigma}}{1+|y|^{\alpha}} \tag{4.11}$$

and $\|\psi\|_{**,\beta,\alpha}$ is the least $M > 0$ such that

$$|\psi(x,t)| \le M \frac{t^{-\beta}}{1+|y|^{\alpha-2}}, \quad y = \frac{|x-\xi|}{\mu} \tag{4.12}$$

holds. Here $\beta = \frac{n-2}{2(n-4)} + \frac{\sigma}{n-4}$. We suppose ϕ and ψ satisfy

$$\|\phi\|_{n-2+\sigma,\alpha} \le c t_0^{-\varepsilon} \tag{4.13}$$

and

$$\|\psi\|_{**,\beta,\alpha} \le \frac{c t_0^{-\varepsilon}}{R^{\alpha-2}}$$

for some small $\varepsilon > 0$, respectively.

Then we have the following result.

Proposition 4.2. (4.5) *is equivalent to*

$$\begin{cases} \dot{\lambda} + \frac{1+(n-4)}{(n-4)t}\lambda = \Pi_0[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{\xi}_l = \Pi_l[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), & l = 1, \cdots, n, \\ \dot{\theta}_{12} = \mu_0^{-1}\Pi_{n+1}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{a}_1 = \mu_0^{-1}\Pi_{n+2}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{a}_2 = \mu_0^{-1}\Pi_{n+3}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{\theta}_{1l} = \mu_0^{-1}\Pi_{n+l+1}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), & l = 3, \cdots, n \\ \dot{\theta}_{2l} = \mu_0^{-1}\Pi_{2n+l-1}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), & l = 3, \cdots, n. \end{cases} \tag{4.14}$$

The terms in the right-hand side of (4.14) *can be written as*

$$\begin{aligned} &\Pi_0[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t) \\ &= \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0^{n-3+\sigma}(t) f_0(t) + \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \\ &\times \Theta_0\left[\dot{\lambda}, \dot{\xi}, \mu_0\dot{a}, \mu_0\dot{\theta}, \mu_0^{n-4}(t)\lambda, \mu_0^{n-4}(\xi - q), \mu_0^{n-3}a, \mu_0^{n-3}\theta, \mu_0^{n-3+\sigma}\phi, \mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t) \end{aligned}$$

and for $l = 1, \cdots, 3n-1$,

$$\begin{aligned}
&\Pi_l[\lambda, \xi, a, \theta, \dot\lambda, \dot\xi, \dot a, \dot\theta, \phi, \psi](t)\\
&= \mu_0^{n-2} c_l \left[b^{n-2} \nabla H(q,q) \right] + \mu_0^{n-2+\sigma}(t) f_l(t) + \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\\
&\times \Theta_l \left[\dot\lambda, \dot\xi, \mu_0 \dot a, \mu_0 \dot\theta, \mu_0^{n-4}(t)\lambda, \mu_0^{n-4}(\xi - q), \mu_0^{n-3} a, \mu_0^{n-3}\theta, \mu_0^{n-3+\sigma}\phi, \mu_0^{\frac{n-2}{2}+\sigma}\psi \right](t),
\end{aligned}$$

where c_l *are suitable constants,* $f_l(t)$ *and* $\Theta_l[\cdots](t)$ *(*$l = 0, \cdot, 3n-1$*) are bounded smooth functions for* $t \in [t_0, \infty)$.

The proof of Proposition 4.2 is given in Section 6.

4.3. The outer problem

Let us consider the out problem (3.4),

$$\begin{cases} \partial_t \psi = \Delta\psi + V_A \psi + 2\nabla\eta_R \nabla\tilde\phi + \tilde\phi(\Delta - \partial_t)\eta_R \\ \qquad\qquad + \tilde N_A(\tilde\phi) + S_{\text{out}}, & \text{in } \Omega \times (t_0, \infty), \\ \psi = -u_A^* \quad \text{on} \quad \partial\Omega \times (t_0,\infty), \quad \psi(t_0, \cdot) = \psi_0 & \text{in } \Omega, \end{cases} \tag{4.15}$$

with a smooth and small initial datum ψ_0.

To apply the Schauder fixed-point theorem to (4.15) and get a solution ψ, we first consider the corresponding linear problem

$$\begin{cases} \partial_t \psi = \Delta\psi + V_A \psi + f(x,t) & \text{in} \quad \Omega \times (t_0,\infty), \\ \psi = g & \text{on} \quad \partial\Omega \times (t_0, \infty), \\ \psi(t_0, \cdot) = h & \text{in} \quad \Omega, \end{cases} \tag{4.16}$$

where $f(x,t)$, $g(x,t)$ and $h(x)$ are smooth functions, $V_{\mu,\xi}$ is defined in (3.3). We denote $\|f\|_{*,\gamma,2+\alpha}$ as the least $M > 0$ such that

$$|f(x,t)| \le M \frac{\mu^{-2} t^{-\gamma}}{1 + |y|^{2+\varsigma}}, \quad y = \frac{x - \xi}{\mu} \tag{4.17}$$

for given $\varsigma, \gamma > 0$. Then the following *a priori* estimate holds for problem (4.16).

Proposition 4.3. *Suppose* $\|f\|_{*,\gamma,2+\varsigma} < +\infty$ *for some constants* $\varsigma, \gamma > 0$, $0 < \varsigma \ll 1$, $\|h\|_{L^\infty(\Omega)} < +\infty$ *and* $\|\tau^\gamma g(x,\tau)\|_{L^\infty(\partial\Omega \times (t_0,\infty))} < +\infty$. *Let* $\phi = \psi[f, g, h]$ *be the unique solution of* (4.16)*, then there exists* $\delta = \delta(\Omega) > 0$ *small such that, for all* (x,t)*, one has*

$$\begin{aligned} |\psi(x,t)| \lesssim \|f\|_{*,\gamma,2+\varsigma} \frac{t^{-\gamma}}{1 + |y|^\varsigma} + e^{-\delta(t - t_0)} \|h\|_{L^\infty(\Omega)} \\ + t^{-\gamma} \|\tau^\gamma g(x,\tau)\|_{L^\infty(\partial\Omega \times (t_0,\infty))}, \quad y = \frac{x-\xi}{\mu} \end{aligned} \tag{4.18}$$

and

$$|\nabla\psi(x,t)| \lesssim \|f\|_{*,\gamma,2+\varsigma}\frac{\mu^{-1}t^{-\gamma}}{1+|y|^{\varsigma+1}} \text{ for } |y|\le R. \tag{4.19}$$

The proof is the same as Lemma 4.1 in [2], so we omit it. This result will be applied to problem (4.15), as a first step, we establish the following estimates for

$$f[\psi](x,t) = 2\nabla\eta_R\nabla\tilde{\phi} + \tilde{\phi}\big(\Delta-\partial_t\big)\eta_R + \tilde{N}_A(\tilde{\phi}) + S_{\text{out}}.$$

Proposition 4.4. *We have*

(1)

$$|S_{\text{out}}(x,t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha}}, \tag{4.20}$$

(2)

$$\Big|2\nabla\eta_R\nabla\tilde{\phi} + \tilde{\phi}\big(\Delta-\partial_t\big)\eta_R\Big| \lesssim \frac{1}{R^{\alpha-2}}\|\phi\|_{n-2+\sigma,\alpha}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha}}, \tag{4.21}$$

(3)

$$\tilde{N}_A(\tilde{\phi})$$

$$\lesssim \begin{cases} t_0^{-\varepsilon}\big(\|\phi\|^2_{n-2+\sigma,\alpha} + \|\psi\|^2_{**,\beta,\alpha}\big)\frac{1}{R^{\alpha-2}}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha}}, & \text{when } 6\ge n,\\ t_0^{-\varepsilon}\big(\|\phi\|^p_{n-2+\sigma,\alpha} + \|\psi\|^p_{**,\beta,\alpha}\big)\frac{1}{R^{\alpha-2}}\frac{\mu_j^{-2}\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha}}, & \text{when } 6< n. \end{cases} \tag{4.22}$$

The proof of Proposition 4.4 is given in Section 7.

4.4. Proof of Theorem 1.1: solving the inner-outer gluing system

Let us formulate the whole problem into a fixed point problem.

Fact 1. Let h be a function satisfying $\|h\|_{n-3+\sigma} \lesssim \frac{1}{R^{\alpha-2}}$. The solution for

$$\dot{\lambda} + \frac{1+(n-4)}{(n-4)t}\lambda = h(t) \tag{4.23}$$

can be expressed as follows

$$\lambda(t) = t^{-\frac{1+(n-4)}{(n-4)}}\left[d + \int_{t_0}^{t}\tau^{\frac{1+(n-4)}{(n-4)}}h(\tau)d\tau\right], \tag{4.24}$$

with d be an arbitrary constant. Therefore, it holds that

$$\|t^{\frac{1+\sigma}{n-4}}\lambda(t)\|_{L^\infty(t_0,\infty)} \lesssim t_0^{-\frac{(n-4)-\sigma}{n-4}} d + \|h\|_{n-3+\sigma}$$

and

$$\|\dot{\lambda}(t)\|_{n-3+\sigma} \lesssim t_0^{-\frac{(n-4)-\sigma}{n-4}} d + \|h\|_{n-3+\sigma}.$$

Set $\Lambda(t) = \dot{\lambda}(t)$, then we have

$$\Lambda + \frac{1}{t}\frac{1+(n-4)}{(n-4)}\int_t^\infty \Lambda(s)ds = h(t), \tag{4.25}$$

which defines a bounded linear operator $\mathcal{L}_1 : h \to \Lambda$ associating the solution Λ of (4.25) to any h satisfying $\|h\|_{n-3+\sigma} < +\infty$. Moreover, the operator $\mathcal{L}_1$ is continuous between the space $L^\infty(t_0, \infty)$ endowed with the $\|\cdot\|_{n-3+\sigma}$-topology.

For any $h : [t_0, \infty) \to \mathbb{R}^n$ with $\|h\|_{n-3+\sigma} < +\infty$, the solution of

$$\dot{\xi} = \mu_0^{n-2} c\left[b^{n-2}\nabla H(q,q)\right] + h(t) \tag{4.26}$$

can be written as

$$\xi(t) = \xi^0(t) + \int_t^\infty h(s)ds, \tag{4.27}$$

where

$$\xi^0(t) = q + c\left[-b^{n-2}\nabla H(q,q)\right]\int_t^\infty \mu_0^{n-2}(s)ds.$$

Thus

$$|\xi(t) - q| \lesssim t^{-\frac{2}{n-4}} + t^{-\frac{1+\sigma}{n-4}}\|h\|_{n-3+\sigma}$$

and

$$\|\dot{\xi} - \dot{\xi}^0\|_{n-3+\sigma} \lesssim \|h\|_{n-3+\sigma}.$$

Define $\Xi(t) = \dot{\xi}(t) - \dot{\xi}^0$, then (4.27) defines a continuous linear operator $\mathcal{L}_2 : h \to \Xi$ in the $\|\cdot\|_{n-3+\sigma}$-topology.

Similarly, from Proposition 4.2, we can define $\mathcal{L}_3 : h \to \Gamma := \dot{a}(t)$ and $\mathcal{L}_4 : h \to \Upsilon := \dot{\theta}(t)$ which are continuous linear operators in the $\|\cdot\|_{n-4+\sigma}$-topology.

Note that $(\lambda, \xi, a, \theta)$ is a solution of (4.14) if $(\Lambda = \dot{\lambda}(t),\ \Xi = \dot{\xi}(t) - \dot{\xi}^0(t),\ \Gamma := \dot{a}(t),\ \Upsilon := \dot{\theta}(t))$ is a fixed point of the following problem

$$(\Lambda, \Xi, \Gamma, \Upsilon) = \mathcal{T}_0(\Lambda, \Xi, \Gamma, \Upsilon), \tag{4.28}$$

where

$$\begin{aligned}\mathcal{T}_0 :=& \Big(\mathcal{L}_1(\hat{\Pi}_1[\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi], \mathcal{L}_2(\hat{\Pi}_2[\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi]),\\ &\qquad \mathcal{L}_3(\hat{\Pi}_3[\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi], \mathcal{L}_4(\hat{\Pi}_4[\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi]\Big)\\ :=& \big(\bar{A}_1(\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi), \bar{A}_2(\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi), \bar{A}_3(\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi),\\ &\qquad \bar{A}_4(\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi)\big)\end{aligned}$$

with

$$\hat{\Pi}_l[\Lambda, \Xi, \Gamma, \Upsilon, \phi, \psi]$$
$$:= \Pi_l\left[\int_t^\infty \Lambda, q + \int_t^\infty \Xi, \mu_0\int_t^\infty \Gamma, \int_t^\infty \Upsilon, \Lambda, \Xi, \mu_0\Gamma, \Upsilon, \phi, \psi\right]$$

for $l = 0, 1, \cdots, 3n-1$.

Fact 2. Proposition 4.1 tells us that there exists a linear operator $\mathcal{T}_1$ associating to the solution of (4.6) for any function $h(y,\tau)$ with $\|h\|_{2+\alpha,\nu}$-bounded. Thus the solution of problem (4.3) is a fixed point of the problem

$$\phi = \mathcal{T}_1(H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau))). \tag{4.29}$$

Fact 3. Proposition 4.3 defines a linear operator $\mathcal{T}_2$ which associates any given functions $f(x,t)$, $g(x,t)$ and $h(x)$ to the corresponding solution $\psi = \mathcal{T}_2(f, g, h)$ for problem (4.16). Denote $\psi_1(x,t) := \mathcal{T}_2(0, -u_A^*, \psi_0)$. From (2.39), (2.18) and (2.38), $\forall x \in \partial\Omega$, one has

$$|u_A^*(x,t)| \lesssim \mu_0^{\frac{n+2}{2}}(t).$$

From Lemma 4.3,

$$|\psi_1| \lesssim e^{-\delta(t-t_0)}\|\psi_0\|_{L^\infty(\mathbb{R}^n)} + t^{-\beta}\mu_0(t_0)^{2-\sigma} \text{ where } \beta = \frac{n-2}{2(n-4)} + \frac{\sigma}{n-4}.$$

Therefore, $\psi + \psi_1$ is a solution to (4.15) if ψ is a fixed point of the following operator

$$\mathcal{A}(\psi) := \mathcal{T}_2(f[\psi], 0, 0),$$

with

$$f[\psi] = 2\nabla\eta_R\nabla\tilde{\phi} + \tilde{\phi}(\Delta - \partial_t)\eta_R + \tilde{N}_A(\tilde{\phi}) + S_{\text{out}}. \tag{4.30}$$

That is to say, we have to solve the fixed point problem

$$\psi = \mathcal{T}_2(f[\psi], 0, 0). \tag{4.31}$$

From **Fact 1-3**, to prove Theorem 1.1, we should solve the following fixed point problem with unknowns $(\phi, \psi, \lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta})$,

$$\begin{cases} (\Lambda, \Xi, \Gamma, \Upsilon) = \mathcal{T}_0(\Lambda, \Xi, \Gamma, \Upsilon), \\ \phi = \mathcal{T}_1(H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau))), \\ \psi = \mathcal{T}_2(f(\psi), 0, 0), \end{cases} \tag{4.32}$$

where

$$f(\psi) = 2\nabla\eta_R\nabla\tilde{\phi} + \tilde{\phi}(\Delta - \partial_t)\eta_R + \tilde{N}_A(\tilde{\phi}) + S_{\text{out}}.$$

To find a fixed point, we will use the Schauder fixed-point theorem in the set

$$\mathcal{B}=\Big\{(\phi,\psi,\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta): R^{\alpha-2}\|\dot\lambda(t)\|_{n-3+\sigma}+R^{\alpha-2}\|\dot\xi(t)\|_{n-3+\sigma} + R^{\alpha-2}\|\dot a(t)\|_{n-4+\sigma}+R^{\alpha-2}\|\dot\theta(t)\|_{n-4+\sigma}+R^{\alpha-2}\|\lambda(t)\|_{1+\sigma} + R^{\alpha-2}\|\xi(t)-q\|_{1+\sigma}+R^{\alpha-2}\|a\|_{\sigma}+R^{\alpha-2}\|\theta\|_{\sigma}+t_0^{\varepsilon}R^{\alpha-2}\|\psi\|_{**,\beta,\alpha} + t_0^{\varepsilon}\|\phi\|_{n-2+\sigma,\alpha}\le c\Big\}$$

for some large but fixed positive constant c.

Let

$$K:=\max\{\|f_0\|_{n-3+\sigma},\|f_1\|_{n-3+\sigma},\cdots,\|f_{3n-1}\|_{n-3+\sigma}\}$$

where $f_0, f_1,\cdots,f_{3n-1}$ are the functions defined in Lemma 4.2. Then we have

$$\Big|t^{\frac{n-3+\sigma}{n-4}}\bar A_i(\Lambda,\Xi,\Gamma,\Upsilon,\phi,\psi)\Big| \lesssim t_0^{-\frac{(n-4)-\sigma}{n-4}}d+\frac{1}{R^{\alpha-2}}\|\phi\|_{n-2+\sigma,a}+\frac{1}{R^{\alpha-2}}\|\psi\|_{**,\beta,\alpha}+\frac{K}{R^{\alpha-2}} +\frac{1}{R^{\alpha-2}}\|\Lambda\|_{n-3+\sigma}+\frac{1}{R^{\alpha-2}}\|\Xi\|_{n-3+\sigma}.$$

Thus, for d satisfying $t_0^{-\frac{(n-4)-\sigma}{n-4}}d<\frac{K}{R^{\alpha-2}}$, $\mathcal{T}_0(\mathcal{B})\subset\mathcal{B}$ (choose the constant ρ in (3.1) sufficiently small).

On the set $\mathcal{B}$, it is clear that

$$\big|H[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](y,t(\tau))\big|\lesssim t_0^{-\varepsilon}\frac{\mu_0^{n-2+\sigma}}{1+|y|^{2+\alpha}}.$$

From Proposition 4.1, $\mathcal{T}_1(\mathcal{B})\subset\mathcal{B}$ holds.

Similarly, Proposition 4.4 ensures that $\mathcal{T}_2(\mathcal{B})\subset\mathcal{B}$. Therefore the operator $\mathcal{T}$ defined in (4.32) maps the set $\mathcal{B}$ into itself. Since $\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi$ and ψ decay uniformly when $t\to+\infty$, this fact combines with the standard parabolic estimate ensures that $\mathcal{T}$ is compact. By the Schauder fixed-point theorem, we conclude that (4.32) has a fixed point in $\mathcal{B}$. That is to say, we find a solution to the system of outer problem (3.4) and inner problem (3.7), which provides a solution to (1.1). This completes the proof of Theorem 1.1.

5. Proof of Proposition 4.1

In the following, we assume that $h=h(y,\tau)$ is a function defined on $\mathbb{R}^n$ which is zero outside the ball $B_{2R}(0)$ for all $\tau>\tau_0$. As a first step to the proof of proposition 4.1, we have the following:

Lemma 5.1. *Suppose* $\alpha \in (2, n-2)$, $\nu > 0$, $\|h\|_{2+\alpha,\nu} < +\infty$ *and*

$$\int_{\mathbb{R}^n} h(y,\tau) z_j(y) dy = 0 \text{ for all } \tau \in (\tau_0, \infty),\ j = 0, 1, \cdots, 3n-1.$$

Then for any $\tau_1 > \tau_0$ *large enough, the solution* $(\phi(y,\tau), c_1(\tau), \cdots, c_K(\tau))$ *to the following problem*

$$\begin{cases} \partial_\tau \phi = \Delta\phi + p|Q|^{p-1}(y)\phi + h(y,\tau) - \displaystyle\sum_{l=1}^{K} c_l(\tau) Z_l(y),\ y \in \mathbb{R}^n,\ \tau \geq \tau_0, \\ \displaystyle\int_{\mathbb{R}^n} \phi(y,\tau) Z_l(y) dy = 0 \text{ for all } \tau \in (\tau_0, +\infty),\ l = 1, \cdots, K, \\ \phi(y, \tau_0) = 0,\ y \in \mathbb{R}^n, \end{cases} \tag{5.1}$$

satisfies

$$\|\phi(y,\tau)\|_{\alpha,\tau_1} \lesssim \|h\|_{2+\alpha,\tau_1} \tag{5.2}$$

and $\forall l = 1, \cdots, K$,

$$|c_l(\tau)| \lesssim \tau^{-\nu} R^\alpha \|h\|_{2+\alpha,\tau_1} \text{ for } \tau \in (\tau_0, \tau_1).$$

Here $\|h\|_{b,\tau_1} := \sup_{\tau\in(\tau_0,\tau_1)} \tau^\nu \|(1+|y|^b)h\|_{L^\infty(\mathbb{R}^n)}$.

Proof. (5.1) is equivalent to

$$\begin{cases} \partial_\tau \phi = \Delta\phi + p|Q|^{p-1}(y)\phi + h(y,\tau) - \displaystyle\sum_{l=1}^{K} c_l(\tau) Z_l(y),\ y \in \mathbb{R}^n,\ \tau \geq \tau_0, \\ \phi(y, \tau_0) = 0,\ y \in \mathbb{R}^n \end{cases} \tag{5.3}$$

with $c_l(\tau)$ given by the following relation

$$c_l(\tau) \int_{\mathbb{R}^n} |Z_l(y)|^2 dy = \int_{\mathbb{R}^n} h(y,\tau) Z_l(y) dy, \quad l = 1, \cdots, K.$$

Then

$$|c_l(\tau)| \lesssim \tau^{-\nu} R^\alpha \|h\|_{2+\alpha,\tau_1} \tag{5.4}$$

holds for $\tau \in (\tau_0, \tau_1)$. Therefore we are left with the proof of (5.2) for the solution ϕ of equation (5.3). Inspired by Lemma 4.5 of [3], the linear theory of [32] and [42], we use the blowing-up argument.

First, we have **Claim**: given $\tau_1 > \tau_0$, $\|\phi\|_{\alpha,\tau_1} < +\infty$ holds. Indeed, given $R_0 > 0$, the standard parabolic theory ensures that there is a constant $K_1 = K_1(R_0, \tau_1)$ such that

$$|\phi(y,\tau)| \leq K_1 \quad \text{in } B_{R_0}(0) \times (\tau_0, \tau_1].$$

Let us fix $R_0 > 0$ large enough and take $K_2 > 0$ large enough, then $K_2\rho^{-\alpha}$ is a super-solution of (5.3) when $\rho > R_0$. Therefore, for any $\tau_1 > 0$, $|\phi| \le 2K_2\rho^{-\alpha}$ and $\|\phi\|_{\alpha,\tau_1} < +\infty$. Next, we prove the following identities,

$$\int_{\mathbb{R}^n} \phi(y,\tau)z_j(y)dy = 0 \text{ for all } \tau \in (\tau_0,\tau_1),\ j = 0,1,\cdots,3n-1 \tag{5.5}$$

and

$$\int_{\mathbb{R}^n} \phi(y,\tau)Z_l(y)dy = 0 \text{ for all } \tau \in (\tau_0,\tau_1),\ l = 1,\cdots,K. \tag{5.6}$$

Indeed, (5.6) follows from the definition of $c_l(\tau)$. Let us test (5.3) with $z_j\eta$, where $\eta(y) = \eta_0(|y|/\tilde{R})$, $j = 0,1,\cdots,3n-1$, $\tilde{R}$ is a positive constant and η_0 is a smooth cut-off function defined by

$$\eta_0(r) = \begin{cases} 1, & \text{for } r < 1 \\ 0, & \text{for } r > 2. \end{cases}$$

Then we have

$$\int_{\mathbb{R}^n} \phi(\cdot,\tau)z_j\eta = \int_0^\tau ds \int_{\mathbb{R}^n} (\phi(\cdot,s)L_0[\eta z_j] + hz_j\eta - \sum_{l=1}^K c_l(s)Z_l z_j\eta).$$

Furthermore,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\phi L_0[\eta z_j] + hz_j\eta - \sum_{l=1}^K c_l(s)Z_l z_j\eta\right) \\ =& \int_{\mathbb{R}^n} \phi\left(z_j\Delta\eta + 2\nabla\eta\nabla z_j\right) \\ &- hz_j(1-\eta) + \sum_{l=1}^K c_l(s)Z_l z_j(1-\eta) \\ =& O(\tilde{R}^{-\varepsilon}) \end{aligned}$$

holds uniformly on $\tau \in (\tau_0,\tau_1)$ for a small positive number ε. Letting $\tilde{R} \to +\infty$, we get (5.5). Finally, we claim that when $\tau_1 > \tau_0$ is large enough, for any solution ϕ of (5.3) satisfying $\|\phi\|_{\alpha,\tau_1} < +\infty$, (5.5) and (5.6), there holds

$$\|\phi\|_{\alpha,\tau_1} \lesssim \|h\|_{2+\alpha,\tau_1}. \tag{5.7}$$

This proves (5.2).

To prove estimate (5.7), we use the contradiction arguments. Suppose there are sequences $\tau_1^k \to +\infty$ and ϕ_k, h_k, c_l^k $(l = 1,\cdots,K)$ satisfying the following parabolic problem

$$\begin{cases} \partial_\tau \phi_k = \Delta \phi_k + p|Q|^{p-1}(y)\phi_k + h_k - \sum_{l=1}^{K} c_l^k(s) Z_l(y), \ y \in \mathbb{R}^n, \ \tau \geq \tau_0, \\ \int_{\mathbb{R}^n} \phi_k(y,\tau) z_j(y) dy = 0 \text{ for all } \tau \in (\tau_0, \tau_1^k), \ j = 0, 1, \cdots, 3n-1, \\ \int_{\mathbb{R}^n} \phi_k(y,\tau) Z_l(y) dy = 0 \text{ for all } \tau \in (\tau_0, \tau_1), \ l = 1, \cdots, K, \\ \phi_k(y, \tau_0) = 0, \ y \in \mathbb{R}^n \end{cases}$$

and

$$\|\phi_k\|_{\alpha,\tau_1^k} = 1, \quad \|h_k\|_{2+\alpha,\tau_1^k} \to 0. \tag{5.8}$$

By (5.4), we obtain $\sup_{\tau \in (\tau_0, \tau_1^k)} \tau^\nu c_l^k(\tau) \to 0, l = 1, \ldots, K$. First, we claim that the following holds

$$\sup_{\tau_0 < \tau < \tau_1^k} \tau^\nu |\phi_k(y,\tau)| \to 0 \tag{5.9}$$

uniformly on compact subsets of $\mathbb{R}^n$. Indeed, if for some $|y_k| \leq M$, $\tau_0 < \tau_2^k < \tau_1^k$,

$$(\tau_2^k)^\nu |\phi_k(y_k, \tau_2^k)| \geq \frac{1}{2},$$

then we have $\tau_2^k \to +\infty$. Now, define

$$\tilde{\phi}_n(y,\tau) = (\tau_2^k)^\nu \phi_n(y, \tau_2^k + \tau).$$

Then

$$\partial_\tau \tilde{\phi}_k = L[\tilde{\phi}_k] + \tilde{h}_k - \sum_{l=1}^{K} \tilde{c}_l^k(\tau) Z_l(y) \text{ in } \mathbb{R}^n \times (\tau_0 - \tau_2^k, 0],$$

with $\tilde{h}_k \to 0$, $\tilde{c}_l^k \to 0$ $(l = 1, \cdots, K)$ uniformly on compact subsets in $\mathbb{R}^n \times (-\infty, 0]$, moreover, we have

$$|\tilde{\phi}_k(y,\tau)| \leq \frac{1}{1+|y|^\alpha} \text{ in } \mathbb{R}^n \times (\tau_0 - \tau_2^k, 0].$$

Using the dominant convergence theorem and the fact that $\alpha \in (2, n-2)$, $\tilde{\phi}_k \to \tilde{\phi}$ uniformly on compact subsets in $\mathbb{R}^n \times (-\infty, 0]$ for a function $\tilde{\phi} \neq 0$ satisfying

$$\begin{cases} \partial_\tau \tilde{\phi} = \Delta \tilde{\phi} + p|Q|^{p-1}(y)\tilde{\phi} & \text{in } \mathbb{R}^n \times (-\infty, 0] \\ \int_{\mathbb{R}^n} \tilde{\phi}(y,\tau) z_j(y) dy = 0 & \text{for all } \tau \in (-\infty, 0], \ j = 0, 1, \cdots, 3n-1 \\ \int_{\mathbb{R}^n} \tilde{\phi}(y,\tau) Z_l(y) dy = 0 & \text{for all } \tau \in (-\infty, 0], \ l = 1, \cdots, K \\ |\tilde{\phi}(y,\tau)| \leq \frac{1}{1+|y|^\alpha} & \text{in } \mathbb{R}^n \times (-\infty, 0] \\ \tilde{\phi}(y, \tau_0) = 0 & y \in \mathbb{R}^n. \end{cases} \tag{5.10}$$

Now we claim that $\tilde{\phi} = 0$, which contradicts the fact that $\tilde{\phi} \neq 0$. Standard parabolic regularity tells us that $\tilde{\phi}(y, \tau)$ is $C^{2,\varrho}$ for some $\varrho \in (0, 1)$. Then a scaling argument shows that

$$(1 + |y|)|\nabla\tilde{\phi}| + |\tilde{\phi}_\tau| + |\Delta\tilde{\phi}| \lesssim (1 + |y|)^{-2-\alpha}.$$

Differentiating (5.10) with respect to τ, we have $\partial_\tau\tilde{\phi}_\tau = \Delta\tilde{\phi}_\tau + p|Q|^{p-1}(y)\tilde{\phi}_\tau$ and

$$(1 + |y|)|\nabla\tilde{\phi}_\tau| + |\tilde{\phi}_{\tau\tau}| + |\Delta\tilde{\phi}_\tau| \lesssim (1 + |y|)^{-4-\alpha}.$$

Furthermore, it holds that

$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 + B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) = 0,$$

where

$$B(\tilde{\phi}, \tilde{\phi}) = \int_{\mathbb{R}^n} \Big[|\nabla\tilde{\phi}|^2 - p|Q|^{p-1}(y)|\tilde{\phi}|^2\Big] dy.$$

Since $\int_{\mathbb{R}^n} \tilde{\phi}(y, \tau)z_j(y)dy = 0$ and $\int_{\mathbb{R}^n} \tilde{\phi}(y, \tau)Z_l(y)dy = 0$ hold $\forall\tau \in (-\infty, 0]$, $j = 0, 1, \cdots, 3n - 1, l = 1, \cdots, K$, we have $B(\tilde{\phi}, \tilde{\phi}) \geq 0$. Note that

$$\int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 = -\frac{1}{2}\partial_\tau B(\tilde{\phi}, \tilde{\phi}).$$

Combine the above facts, we get

$$\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 \leq 0, \quad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 < +\infty.$$

Hence $\tilde{\phi}_\tau = 0$. Thus $\tilde{\phi}$ is independent of τ, $L[\tilde{\phi}] = 0$. Since $\tilde{\phi}$ is bounded, from the nondegeneracy of L, $\tilde{\phi}$ is a linear combination of the kernel functions z_j, $j = 0, 1, \cdots, 3n - 1$. But $\int_{\mathbb{R}^n} \tilde{\phi}z_j = 0$, $j = 0, 1, \cdots, 3n - 1$, we get $\tilde{\phi} = 0$, a contradiction. Therefore (5.9) holds.

From (5.8), there exists a sequence y_k with $|y_k| \to +\infty$ such that

$$(\tau_2^k)^\nu(1 + |y_k|^\alpha)|\phi_k(y_k, \tau_2^k)| \geq \frac{1}{2}.$$

Let

$$\tilde{\phi}_k(z, \tau) := (\tau_2^k)^\nu|y_k|^\alpha\phi_k(y_k + |y_k|z, |y_k|^2\tau + \tau_2^k).$$

Then

$$\partial_\tau\tilde{\phi}_k = \Delta\tilde{\phi}_k + a_k\tilde{\phi}_k + \tilde{h}_k(z, \tau),$$

with

$$\tilde{h}_k(z, \tau) = (\tau_2^k)^\nu|y_k|^{2+\alpha}h_k(y_k + |y_k|z, |y_k|^2\tau + \tau_2^k).$$

From the assumptions on h_k, one gets

$$|\tilde{h}_k(z,\tau)| \lesssim o(1)|\hat{y}_k + z|^{-2-\alpha}((\tau_2^k)^{-1}|y_k|^2\tau + 1)^{-\nu}$$

with

$$\hat{y}_k = \frac{y_k}{|y_k|} \to -\hat{e}$$

and $|\hat{e}| = 1$. Hence $\tilde{h}_k(z,\tau) \to 0$ uniformly on compact subsets in $\mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]$. a_k has the same property as $\tilde{h}_k(z,\tau)$. Furthermore, $|\tilde{\phi}_k(0,\tau_0)| \geq \frac{1}{2}$ and

$$|\tilde{\phi}_k(z,\tau)| \lesssim |\hat{y}_k + z|^{-\alpha}\left((\tau_2^k)^{-1}|y_k|^2\tau + 1\right)^{-\nu}.$$

Hence one may assume $\tilde{\phi}_k \to \tilde{\phi} \neq 0$ uniformly on compact subsets in $\mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]$ for $\tilde{\phi}$ satisfying

$$\tilde{\phi}_\tau = \Delta\tilde{\phi} \quad \text{in } \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0] \tag{5.11}$$

and

$$|\tilde{\phi}(z,\tau)| \leq |z - \hat{e}|^{-\alpha} \quad \text{in } \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]. \tag{5.12}$$

Similar to Lemma 5.2 of [42], functions $\tilde{\phi}$ satisfying (5.11) and (5.12) is zero, which is a contradiction to the fact that $\tilde{\phi} \neq 0$. This concludes the validity of (5.7). Indeed, set

$$u(\rho,t) = (\rho^2 + Ct)^{-\alpha/2} + \frac{\varepsilon}{\rho^{n-2}}.$$

Then

$$-u_t + \Delta u < (\rho^2 + Ct)^{-\alpha/2-1}[\alpha(\alpha + 2 - n) + \frac{C}{2}\alpha] < 0, \text{ if } \alpha < n - 2 - \frac{C}{2}.$$

For any $\alpha < n-2$, we can always find a fixed $C > 0$ such that $\alpha < n - 2 - \frac{C}{2}$. Hence $u(|z - \hat{e}|, \tau + M)$ is a positive super-solution of (5.12) in $(0,\infty) \times [-M, 0]$. Via the comparison principle, $|\tilde{\phi}(z,\tau)| \leq 2u(|z - \hat{e}|, \tau + M)$. Letting $M \to +\infty$ we get

$$|\tilde{\phi}(z,\tau)| \leq \frac{2\varepsilon}{|z - \hat{e}|^{n-2}}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\tilde{\phi} = 0$. The proof is complete. □

Proof of Proposition 4.1. First let us consider the following problem

$$\begin{cases} \partial_\tau \phi = \Delta\phi + p|Q|^{p-1}(y)\phi + h(y,\tau) - \displaystyle\sum_{l=1}^{K} c_l(\tau) Z_l, \ y \in \mathbb{R}^n, \ \tau \geq \tau_0, \\ \phi(y,\tau_0) = 0, \ y \in \mathbb{R}^n. \end{cases}$$

Let $(\phi(y,\tau), c_1(\tau), \cdots, c_K(\tau))$ be the unique solution to problem (5.1). By Lemma 5.1, for $\tau_1 > \tau_0$ large enough, there hold

$$|\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-\alpha}\|h\|_{2+\alpha,\tau_1} \text{ for all } \tau \in (\tau_0,\tau_1),\ \ y \in \mathbb{R}^n$$

and

$$|c_l(\tau)| \le \tau^{-\nu}R^{\alpha}\|h\|_{2+\alpha,\tau_1} \text{ for all } \tau \in (\tau_0,\tau_1), \quad l = 1,\cdots,K.$$

From the assumptions of the proposition, for an arbitrary τ_1, $\|h\|_{2+\alpha,\nu} < +\infty$ and $\|h\|_{2+\alpha,\tau_1} \le \|h\|_{2+\alpha,\nu}$ hold. Therefore, one has

$$|\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-\alpha}\|h\|_{2+\alpha,\nu} \text{ for all } \tau \in (\tau_0,\tau_1),\ \ y \in \mathbb{R}^n$$

and

$$|c_l(\tau)| \le \tau^{-\nu}R^{\alpha}\|h\|_{2+\alpha,\nu} \text{ for all } \tau \in (\tau_0,\tau_1), \quad l = 1\cdots,K.$$

From the arbitrariness of τ_1, we have

$$|\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-\alpha}\|h\|_{2+\alpha,\nu} \text{ for all } \tau \in (\tau_0,+\infty),\ \ y \in \mathbb{R}^n$$

and

$$|c_l(\tau)| \le \tau^{-\nu}R^{\alpha}\|h\|_{2+\alpha,\nu} \text{ for all } \tau \in (\tau_0,+\infty), \quad l = 1\cdots,K.$$

Using the parabolic regularity results and a scaling argument, we get (4.7) and (4.8). □

6. Proof of Proposition 4.2

The following integral identities will be useful in the computation of this section.

Lemma 6.1. *As $k \to +\infty$, for $j = 0,\cdots,3n-1$, we have*

$$\int_{\mathbb{R}^n}\left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{(1+|y|^2)^{\frac{n}{2}}}\right)z_j(y)dy = \begin{cases} a_{0,0} + O(k^{-1}) & \text{if } j = 0\\ O(k^{-1}) & \text{if } j \ne 0\end{cases}$$

$$\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial y_1}Q(y) - \frac{E_{n,k}y_1}{(1+|y|^2)^{\frac{n}{2}}}\right)z_j(y)dy = \begin{cases} a_{1,1} + O(k^{-1}) & \text{if } j = 1\\ a_{1,n+2} + O(k^{-1}) & \text{if } j = n+2\\ O(k^{-1}) & \text{if } j \ne 1, n+2\end{cases}$$

$$\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial y_2}Q(y) - \frac{E_{n,k}y_2}{(1+|y|^2)^{\frac{n}{2}}}\right)z_j(y)dy = \begin{cases} a_{2,2} + O(k^{-1}) & \text{if } j = 2\\ a_{2,n+3} + O(k^{-1}) & \text{if } j = n+3\\ O(k^{-1}) & \text{if } j \ne 2, n+3.\end{cases}$$

For $i = 3,\cdots,n$, $j = 0,\cdots,3n-1$, we have

$$\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial y_i}Q(y) - \frac{E_{n,k}y_i}{(1+|y|^2)^{\frac{n}{2}}}\right)z_j(y)dy = \begin{cases} a_{i,i} + O(k^{-1}) & \text{if } j = i\\ O(k^{-1}) & \text{if } j \ne i.\end{cases}$$

Furthermore,

$$\int_{\mathbb{R}^n} z_{n+1}(y) z_j(y) dy = \begin{cases} a_{n+1,n+1} + O(k^{-1}) & \text{if } j = n+1 \\ O(k^{-1}) & \text{if } j \neq n+1, \end{cases}$$

$$\int_{\mathbb{R}^n} \left(-2y_1 \left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}} \right) + |y|^2 \left(\frac{\partial}{\partial y_1} Q(y) - \frac{E_{n,k} y_1}{\left(1+|y|^2\right)^{\frac{n}{2}}} \right) \right) z_j(y) dy$$
$$= \begin{cases} a_{n+2,1} + O(k^{-1}) & \text{if } j = 1 \\ a_{n+2,n+2} + O(k^{-1}) & \text{if } j = n+2 \\ O(k^{-1}) & \text{if } j \neq 1, n+2, \end{cases}$$

$$\int_{\mathbb{R}^n} \left(-2y_2 \left(z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}} \right) + |y|^2 \left(\frac{\partial}{\partial y_2} Q(y) - \frac{E_{n,k} y_2}{\left(1+|y|^2\right)^{\frac{n}{2}}} \right) \right) z_j(y) dy$$
$$= \begin{cases} a_{n+3,2} + O(k^{-1}) & \text{if } j = 2 \\ a_{n+3,n+3} + O(k^{-1}) & \text{if } j = n+3 \\ O(k^{-1}) & \text{if } j \neq 2, n+3. \end{cases}$$

For $i = 3, \cdots, n$,

$$\int_{\mathbb{R}^n} z_{n+i+1}(y) z_j(y) dy = \begin{cases} a_{n+i+1,n+i+1} + O(k^{-1}) & \text{if } j = n+i+1 \\ O(k^{-1}) & \text{if } j \neq n+i+1 \end{cases}$$

$$\int_{\mathbb{R}^n} z_{2n+i-1}(y) z_j(y) dy = \begin{cases} a_{2n+i-1,2n+i-1} + O(k^{-1}) & \text{if } j = 2n+i-1 \\ O(k^{-1}) & \text{if } j \neq 2n+i-1. \end{cases}$$

In the above, $a_{i,j}$ *are positive constants depending on* n *and* k, *the matrices*

$$\begin{pmatrix} a_{1,1} & a_{1,n+2} \\ a_{n+2,1} & a_{n+2,n+2} \end{pmatrix}, \quad \begin{pmatrix} a_{2,2} & a_{2,n+3} \\ a_{n+3,2} & a_{n+3,n+3} \end{pmatrix}$$

are invertible.

The proof of this lemma is given in the Appendix.

6.1. The equation for λ

We consider (4.5) for $l = 0$.

Lemma 6.2. *When* $l = 0$, (4.5) *is equivalent to*

$$\begin{aligned} &\dot{\lambda} + \frac{1+(n-4)}{(n-4)t}\lambda + O\left(\frac{1}{k}\right)\dot{\xi} + O\left(\frac{1}{k}\right)\mu_0\left(\dot{a}_1 + \dot{a}_2\right) \\ &+ O\left(\frac{1}{k}\right)\mu_0\left(\dot{\theta}_{12} + \sum_{j=3}^{n}(\dot{\theta}_{1j} + \dot{\theta}_{2j})\right) = \Pi_0[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t). \end{aligned} \tag{6.1}$$

The right-hand side term of (6.1) *can be expressed as*

$$\Pi_0[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](t)=\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0^{n-3+\sigma}(t)f_0(t)+\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}$$
$$\times\Theta_0\left[\dot\lambda,\dot\xi,\mu_0\dot a,\mu_0\dot\theta,\mu_0^{n-4}(t)\lambda,\mu_0^{n-4}(\xi-q),\mu_0^{n-3}a,\mu_0^{n-3}\theta,\mu_0^{n-3+\sigma}\phi,\mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t)$$

where $f_0(t)$ *and* $\Theta_0[\cdots](t)$ *are bounded smooth functions for* $t\in[t_0,\infty)$.

Proof. We compute

$$\int_{B_{2R}}H[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](y,t(\tau))z_0(y)dy,$$

where $H[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](y,t(\tau))$ is defined in (4.2). Write

$$\begin{aligned}
&\mu_0^{\frac{n+2}{2}}S_A(\xi+\mu_0 y,t)\\
&=\left(\frac{\mu_0}{\mu}\right)^{\frac{n+2}{2}}\Big[\mu_0S_1(z,t)+\lambda bS_2(z,t)+\mu S_3(z,t)+\mu^2S_4(z,t)\\
&\qquad+\mu^2S_5(z,t)\Big]_{z=\xi+\mu y}\\
&\quad+\left(\frac{\mu_0}{\mu}\right)^{\frac{n+2}{2}}\mu_0[S_1(\xi+\mu_0y,t)-S_1(\xi+\mu y,t)]\\
&\quad+\left(\frac{\mu_0}{\mu}\right)^{\frac{n+2}{2}}\lambda b[S_2(\xi+\mu_0y,t)-S_2(\xi+\mu y,t)]\\
&\quad+\left(\frac{\mu_0}{\mu}\right)^{\frac{n+2}{2}}\mu[S_3(\xi+\mu_0y,t)-S_3(\xi+\mu y,t)]\\
&\quad+\left(\frac{\mu_0}{\mu}\right)^{\frac{n+2}{2}}\mu^2[S_4(\xi+\mu_0y,t)-S_4(\xi+\mu y,t)]\\
&\quad+\left(\frac{\mu_0}{\mu}\right)^{\frac{n+2}{2}}\mu^2[S_5(\xi+\mu_0y,t)-S_5(\xi+\mu y,t)],
\end{aligned}\tag{6.2}$$

where

$$S_1(z) = \dot{\lambda}\left(z_0\Big(\frac{z-\xi}{\mu}\Big) - \frac{D_{n,k}\Big(2-\Big|\frac{z-\xi}{\mu}\Big|^2\Big)}{\Big(1+\Big|\frac{z-\xi}{\mu}\Big|^2\Big)^{\frac{n}{2}}} - 2Ap|Q|^{p-1}\Big(\frac{z-\xi}{\mu}\Big)\right)$$

$$- \mu_0^{n-4} p|Q|^{p-1}\Big(\frac{z-\xi}{\mu}\Big)\Big[(n-3)b^{n-4}H(q,q)\lambda\Big],$$

$$S_2(z) = \dot{\mu}_0\left(z_0\Big(\frac{z-\xi}{\mu}\Big) - \frac{D_{n,k}\Big(2-\Big|\frac{z-\xi}{\mu}\Big|^2\Big)}{\Big(1+\Big|\frac{z-\xi}{\mu}\Big|^2\Big)^{\frac{n}{2}}}\right)$$

$$+ p|Q|^{p-1}\Big(\frac{z-\xi}{\mu}\Big)\mu_0^{n-3}\Big(-b^{n-4}H(q,q)+B\Big),$$

$$S_3(z) = \left(\nabla Q\Big(\frac{z-\xi}{\mu}\Big) - \frac{E_{n,k}\frac{z-\xi}{\mu}}{\Big(1+\Big|\frac{z-\xi}{\mu}\Big|^2\Big)^{\frac{n}{2}}}\right)\cdot\dot{\xi}$$

$$+ p|Q|^{p-1}\Big(\frac{z-\xi}{\mu}\Big)\Big[-\mu^{n-2}\nabla H(q,q)\Big]\cdot\Big(\frac{z-\xi}{\mu}\Big),$$

$$S_4(z) = \dot{a}_1\left(-2\Big(\frac{z-\xi}{\mu}\Big)_1\left(z_0\Big(\frac{z-\xi}{\mu}\Big) - \frac{D_{n,k}\Big(2-\Big|\frac{z-\xi}{\mu}\Big|^2\Big)}{\Big(1+\Big|\frac{z-\xi}{\mu}\Big|^2\Big)^{\frac{n}{2}}}\right)\right.$$

$$\left.+\Big|\frac{z-\xi}{\mu}\Big|^2\left(\frac{\partial}{\partial y_1}Q\Big(\frac{z-\xi}{\mu}\Big) - \frac{E_{n,k}\Big(\frac{z-\xi}{\mu}\Big)_1}{\Big(1+\Big|\frac{z-\xi}{\mu}\Big|^2\Big)^{\frac{n}{2}}}\right)\right)$$

$$+\dot{a}_2\left(-2\Big(\frac{z-\xi}{\mu}\Big)_2\left(z_0\Big(\frac{z-\xi}{\mu}\Big) - \frac{D_{n,k}\Big(2-\Big|\frac{z-\xi}{\mu}\Big|^2\Big)}{\Big(1+\Big|\frac{z-\xi}{\mu}\Big|^2\Big)^{\frac{n}{2}}}\right)\right.$$

$$\left.+\Big|\frac{z-\xi}{\mu}\Big|^2\left(\frac{\partial}{\partial y_2}Q\Big(\frac{z-\xi}{\mu}\Big) - \frac{E_{n,k}\Big(\frac{z-\xi}{\mu}\Big)_2}{\Big(1+\Big|\frac{z-\xi}{\mu}\Big|^2\Big)^{\frac{n}{2}}}\right)\right),$$

$$S_5(z) = z_{n+1}\Big(\frac{z-\xi}{\mu}\Big)\dot{\theta}_{12} + \sum_{j=3}^{n}\Big(z_{n+j+1}\Big(\frac{z-\xi}{\mu}\Big)\dot{\theta}_{1j} + z_{2n+j-1}\Big(\frac{z-\xi}{\mu}\Big)\dot{\theta}_{2j}\Big).$$

Direct computations yield that

$$\begin{aligned}\int_{B_{2R}} S_1(\xi+\mu y)z_0(y)dy &= (2Ac_1+c_2)(1+O(R^{2-n})+O(R^{-2}))\dot{\lambda}\\ &\quad + c_1(1+O(R^{-2}))\mu_0^{n-4}\Big[(n-3)b^{n-4}H(q,q)\lambda\Big],\\ \int_{B_{2R}} S_2(\xi+\mu y)z_0(y)dy &= O(R^{2-n}+R^{-2})\mu_0^{n-3},\\ \int_{B_{2R}} S_3(\xi+\mu y)z_0(y)dy &= O\left(\frac{1}{k}\right)\dot{\xi}+O(1+R^{-2})\mu_0^{n-2},\\ \int_{B_{2R}} S_4(\xi+\mu y)z_0(y)dy &= O\left(\frac{1}{k}\right)(\dot{a}_1+\dot{a}_2),\\ \int_{B_{2R}} S_5(\xi+\mu y)z_0(y)dy &= O\left(\frac{1}{k}\right)\left(\dot{\theta}_{12}+\sum_{j=3}^{n}\left(\dot{\theta}_{1j}+\dot{\theta}_{2j}\right)\right).\end{aligned}$$

Since $\frac{\mu_0}{\mu}=(1+\frac{\lambda}{\mu_0})^{-1}$, for $l=1,2,3,4,5$, we have the following estimates

$$\begin{aligned}&\int_{B_{2R}}[S_l(\xi+\mu_0 y,t)-S_l(\xi+\mu y,t)]z_0(y)dy\\ &= g\left(t,\frac{\lambda}{\mu_0}\right)\dot{\lambda}+g\left(t,\frac{\lambda}{\mu_0}\right)\dot{\xi}+g\left(t,\frac{\lambda}{\mu_0}\right)(\dot{a}_1+\dot{a}_2)\\ &\quad+g\left(t,\frac{\lambda}{\mu_0}\right)\left(\dot{\theta}_{12}+\sum_{j=3}^{n}\left(\dot{\theta}_{1j}+\dot{\theta}_{2j}\right)\right)\\ &\quad+g\left(t,\frac{\lambda}{\mu_0}\right)\mu_0^{n-4}\left(\lambda+(\xi-q)+a_1+a_2+\theta_{12}+\sum_{j=3}^{n}\left(\theta_{1j}+\theta_{2j}\right)\right)+\mu_0^{n-3+\sigma}f(t),\end{aligned}$$

where f and g are smooth, bounded functions satisfying $g(\cdot,s)\sim s$ as $s\to 0$. Thus

$$\begin{aligned}&c\Big(\frac{\mu}{\mu_0}\Big)^{\frac{n+2}{2}}\mu_0^{-1}\int_{B_{2R}}\mu_0^{\frac{n+2}{2}}S_A(\xi+\mu_0 y,t)z_0(y)dy\\ &=\left[\dot{\lambda}+\frac{1+(n-4)}{(n-4)t}\lambda\right]+\left(O\Big(\frac{1}{k}\Big)+t_0^{-\varepsilon}g\Big(t,\frac{\lambda}{\mu_0}\Big)\right)\dot{\xi}\\ &\quad+\left(O\Big(\frac{1}{k}\Big)+t_0^{-\varepsilon}g\Big(t,\frac{\lambda}{\mu_0}\Big)\right)\mu\Big(\dot{a}_1+\dot{a}_2\Big)\\ &\quad+\left(O\Big(\frac{1}{k}\Big)+t_0^{-\varepsilon}g\Big(t,\frac{\lambda}{\mu_0}\Big)\right)\mu\left(\dot{\theta}_{12}+\sum_{j=3}^{n}\left(\dot{\theta}_{1j}+\dot{\theta}_{2j}\right)\right)\\ &\quad+g\Big(t,\frac{\lambda}{\mu_0}\Big)\mu_0^{n-4}\left(\lambda+(\xi-q)+\mu a_1+\mu a_2+\mu\theta_{12}+\mu\sum_{j=3}^{n}\left(\theta_{1j}+\theta_{2j}\right)\right)\end{aligned}$$

for smooth bounded functions g satisfying $g(\cdot,s)\sim s$ as $s\to 0$.

Let us compute the term

$$p\mu_0^{\frac{n-2}{2}}\left(1+\frac{\lambda}{\mu_0}\right)^{-2}\int_{B_{2R}}|Q|^{p-1}\left(\frac{\mu_0}{\mu}y\right)\psi(\xi+\mu_0 y,t)z_0(y)dy.$$

Its principal part is $I:=\int_{B_{2R}}|Q|^{p-1}(y)\psi(\xi+\mu_0 y,t)z_0(y)dy$. From (4.12), we have $I=\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0^{\frac{n-2}{2}+\sigma}f(t)$ for a smooth bounded function f.

Furthermore, we have

$$\int_{B_{2R}}B[\phi](y,t)z_0(y)dy=\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\Big[\mu_0^{n-3+\sigma}(t)\ell[\phi](t)+\dot{\xi}\ell[\phi](t)\Big]$$

and

$$\int_{B_{2R}}B^0[\phi](y,t)z_0(y)dy=\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0^{n-2+\sigma}g\left(\frac{\lambda}{\mu_0}\right)[\phi](t)$$

for smooth bouned function $g(s)$ with $g(s)\sim s$ $(s\to 0)$ and $\ell[\phi](t)$ is bounded smooth in t.

Combine the above estimations, we have the validity of the lemma. ☐

6.2. The equation for ξ

Now we compute (4.5) for $l=1,\cdots,n$.

Lemma 6.3. *For* $l=1$, (4.5) *is equivalent to*

$$\begin{aligned}&a_{1,1}\dot{\xi}_1+a_{n+2,1}\mu_0\dot{a}_1+O\left(\frac{1}{k}\right)\dot{\lambda}+O\left(\frac{1}{k}\right)\mu_0\dot{a}_2\\&+O\left(\frac{1}{k}\right)\mu_0\left(\dot{\theta}_{12}+\sum_{j=3}^{n}(\dot{\theta}_{1j}+\dot{\theta}_{2j})\right)\\&=\Pi_1[\lambda,\xi,a,\theta,\dot{\lambda},\dot{\xi},\dot{a},\dot{\theta},\phi,\psi](t).\end{aligned}\tag{6.3}$$

For $l=2$, (4.5) *is equivalent to*

$$\begin{aligned}&a_{2,2}\dot{\xi}_2+a_{n+3,2}\mu_0\dot{a}_2+O\left(\frac{1}{k}\right)\dot{\lambda}+O\left(\frac{1}{k}\right)\mu_0\dot{a}_1\\&+O\left(\frac{1}{k}\right)\mu_0\left(\dot{\theta}_{12}+\sum_{j=3}^{n}(\dot{\theta}_{1j}+\dot{\theta}_{2j})\right)\\&=\Pi_2[\lambda,\xi,a,\theta,\dot{\lambda},\dot{\xi},\dot{a},\dot{\theta},\phi,\psi](t).\end{aligned}\tag{6.4}$$

For $l = 3, \cdots, n$, (4.5) *is equivalent to*

$$\begin{aligned}&\dot{\xi}_l + O\left(\frac{1}{k}\right)\dot{\lambda} + O\left(\frac{1}{k}\right)\mu_0\left(\dot{a}_1 + \dot{a}_2\right)\\&+ O\left(\frac{1}{k}\right)\mu_0\left(\dot{\theta}_{12} + \sum_{j=3}^{n}(\dot{\theta}_{1j} + \dot{\theta}_{2j})\right)\\&= \Pi_l[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t).\end{aligned} \tag{6.5}$$

For $l = 1, \cdots, n$,

$$\begin{aligned}&\Pi_l[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t)\\&= \mu_0^{n-2}c_l\left[b^{n-2}\nabla H(q,q)\right] + \mu_0^{n-2+\sigma}(t)f(t) + \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\Theta_l\\&\times\left[\dot{\lambda}, \dot{\xi}, \mu_0\dot{a}, \mu_0\dot{\theta}, \mu_0^{n-4}(t)\lambda, \mu_0^{n-4}(\xi - q), \mu_0^{n-3}a, \mu_0^{n-3}\theta, \mu_0^{n-3+\sigma}\phi, \mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t),\end{aligned}$$

where c_l is a positive constant, $f(t)$ and Θ_l are smooth bounded for $t \in [t_0, \infty)$.

Proof. We compute

$$\int_{B_{2R}} H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau))z_l(y)dy,$$

where $H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau))$ is defined in (4.2). Expanding $\mu_0^{\frac{n+2}{2}} S_A(\xi + \mu_0 y, t)$ as (6.2), by direct computations, we have

$$\int_{B_{2R}} S_1(\xi + \mu y)z_l(y)dy = O\left(\frac{1}{k}\right)\left(\dot{\lambda} + \mu_0^{n-4}\lambda\right),$$

$$\int_{B_{2R}} S_2(\xi + \mu y)z_l(y)dy = O\left(\frac{1}{k}\right)\left(\dot{\mu}_0 + \mu_0^{n-3}\right),$$

$$\begin{aligned}\int_{B_{2R}} S_3(\xi + \mu y)z_l(y)dy &= (1 + O(R^{-n}))a_{l,l}\dot{\xi}_l\\&\quad - (1+O(R^{-2}))p\int_{\mathbb{R}^n}|Q|^{p-1}y_l z_l(y)dy\mu^{n-2}\nabla H(q,q),\end{aligned}$$

$$\int_{B_{2R}} S_4(\xi + \mu y)z_l(y)dy = \begin{cases} a_{n+2,1}(1 + O(R^{4-n}))\dot{a}_1 \\ \qquad +O\left(\frac{1}{k}\right)(1 + O(R^{4-n}))\dot{a}_2 & \text{if } l = 1, \\ a_{n+2,2}(1 + O(R^{4-n}))\dot{a}_2 \\ \qquad +O\left(\frac{1}{k}\right)(1 + O(R^{4-n}))\dot{a}_1 & \text{if } l = 2, \\ O\left(\frac{1}{k}\right)(1 + O(R^{4-n}))\left(\dot{a}_1 + \dot{a}_2\right) & \text{if } l = 3, \cdots, n \end{cases}$$

and

$$\int_{B_{2R}} S_5(\xi + \mu y) z_l(y) dy = O\left(\frac{1}{k}\right)\left(\dot\theta_{12} + \sum_{j=3}^{n}\left(\dot\theta_{1j} + \dot\theta_{2j}\right)\right).$$

Since $\frac{\mu_0}{\mu} = (1 + \frac{\lambda}{\mu_0})^{-1}$, for $j = 1, 2, 3, 4, 5$, we have

$$\begin{aligned}
&\int_{B_{2R}} [S_j(\xi + \mu_0 y, t) - S_j(\xi + \mu y, t)] z_l(y) dy \\
&= g(t, \frac{\lambda}{\mu_0})\dot\lambda + g(t, \frac{\lambda}{\mu_0})\dot\xi + g(t, \frac{\lambda}{\mu_0})\,(\dot a_1 + \dot a_2) \\
&\quad + g(t, \frac{\lambda}{\mu_0})\left(\dot\theta_{12} + \sum_{j=3}^{n}\left(\dot\theta_{1j} + \dot\theta_{2j}\right)\right) \\
&\quad + g(t, \frac{\lambda}{\mu_0})\mu_0^{n-4}\left(\lambda + (\xi - q) + a_1 + a_2 + \theta_{12} + \sum_{j=3}^{n}\left(\theta_{1j} + \theta_{2j}\right)\right) \\
&\quad + \mu_0^{n-3+\sigma} f(t),
\end{aligned}$$

where f and g are smooth, bounded functions satisfying $g(\cdot, s) \sim s$ as $s \to 0$. Thus

$$\begin{aligned}
&c\left(\frac{\mu}{\mu_0}\right)^{\frac{n+2}{2}} \mu_0^{-1} \int_{B_{2R}} \mu_0^{\frac{n+2}{2}} S_A(\xi + \mu_0 y, t) z_l(y) dy \\
&= \left[\dot\xi + \frac{-p\int_{\mathbb{R}^n} |Q|^{p-1} y_l z_l(y) dy}{\int_{\mathbb{R}^n} |z_l|^2 dy} b^{n-2}\mu_0^{n-2}\right] + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g\left(t, \frac{\lambda}{\mu_0}\right)\right)\dot\xi \\
&\quad + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g\left(t, \frac{\lambda}{\mu_0}\right)\right)\mu_0\,(\dot a_1 + \dot a_2) \\
&\quad + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g\left(t, \frac{\lambda}{\mu_0}\right)\right)\mu_0\left(\dot\theta_{12} + \sum_{j=3}^{n}\left(\dot\theta_{1j} + \dot\theta_{2j}\right)\right) \\
&\quad + g\left(t, \frac{\lambda}{\mu_0}\right)\mu_0^{n-4}\left(\lambda + (\xi - q) + \mu_0 a_1 + \mu_0 a_2 + \mu_0\theta_{12} + \mu_0\sum_{j=3}^{n}\left(\theta_{1j} + \theta_{2j}\right)\right),
\end{aligned}$$

for smooth bounded functions g satisfying $g(\cdot, s) \sim s$ as $s \to 0$.

The computations for the term

$$p\mu_0^{\frac{n-2}{2}}\left(1 + \frac{\lambda}{\mu_0}\right)^{-2}\int_{B_{2R}} |Q|^{p-1}(\frac{\mu_0}{\mu} y)\psi(\xi + \mu_0 y, t) z_l(y) dy,$$

$B[\phi]$ and $B^0[\phi]$ are similar to that of Lemma 6.2. □

6.3. The equation for θ_{12}

Now we compute (4.5) for $l = n+1$.

Lemma 6.4. *For $l = n+1$, (4.5) is equivalent to*

$$\begin{aligned}&\mu_0\dot\theta_{12} + O\left(\frac{1}{k}\right)\dot\lambda + O\left(\frac{1}{k}\right)\dot\xi + O\left(\frac{1}{k}\right)\mu_0\left(\dot a_1+\dot a_2\right)\\ &+O\left(\frac{1}{k}\right)\mu_0\left(\sum_{j=3}^{n}\dot\theta_{1j}+\dot\theta_{2j}\right)\\ =\;&\Pi_{n+1}[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](t),\end{aligned} \tag{6.6}$$

$$\begin{aligned}&\Pi_{n+1}[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](t)\\ =\;&\mu_0^{n-2+\sigma}(t)f(t)+\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\Theta_{n+1}\\ \times&\left[\dot\lambda,\dot\xi,\mu_0\dot a,\mu_0\dot\theta,\mu_0^{n-4}(t)\lambda,\mu_0^{n-4}(\xi-q),\mu_0^{n-3}a,\mu_0^{n-3}\theta,\mu_0^{n-3+\sigma}\phi,\mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t),\end{aligned}$$

where $f(t)$ and Θ_{n+1} are smooth bounded for $t\in[t_0,\infty)$.

Proof. We compute

$$\int_{B_{2R}} H[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](y,t(\tau))z_{n+1}(y)dy,$$

where $H[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](y,t(\tau))$ is defined in (4.2). Expand $\mu_0^{\frac{n+2}{2}}S_A(\xi+\mu_0 y,t)$ as (6.2), by direct computations, we have

$$\begin{aligned}\int_{B_{2R}} S_1(\xi+\mu y)z_{n+1}(y)dy &= O\left(\frac{1}{k}\right)\left(\dot\lambda+\mu_0^{n-4}\lambda\right),\\ \int_{B_{2R}} S_2(\xi+\mu y)z_{n+1}(y)dy &= O\left(\frac{1}{k}\right)\left(\dot\mu_0+\mu_0^{n-3}\right),\\ \int_{B_{2R}} S_3(\xi+\mu y)z_{n+1}(y)dy &= O\left(\frac{1}{k}\right)\dot\xi+O(1+R^{-1})\mu_0^{n-2},\\ \int_{B_{2R}} S_4(\xi+\mu y)z_{n+1}(y)dy &= O\left(\frac{1}{k}\right)(1+O(R^{-2}))\left(\dot a_1+\dot a_2\right)\end{aligned}$$

and

$$\int_{B_{2R}} S_5(\xi+\mu y)z_{n+1}(y)dy = a_{n+1,n+1}(1+O(R^{2-n}))\dot\theta_{12}+O\left(\frac{1}{k}\right)\sum_{j=3}^{n}\left(\dot\theta_{1j}+\dot\theta_{2j}\right).$$

Since $\frac{\mu_0}{\mu} = (1 + \frac{\lambda}{\mu_0})^{-1}$, for $j = 1, 2, 3, 4, 5$, we have

$$\begin{aligned}
&\int_{B_{2R}} [S_l(\xi + \mu_0 y, t) - S_l(\xi + \mu y, t)] z_{n+1}(y) dy \\
&= g(t, \frac{\lambda}{\mu_0})\dot{\lambda} + g(t, \frac{\lambda}{\mu_0})\dot{\xi} + g(t, \frac{\lambda}{\mu_0})\, (\dot{a}_1 + \dot{a}_2) \\
&\quad + g\left(t, \frac{\lambda}{\mu_0}\right)\left(\dot{\theta}_{12} + \sum_{j=3}^{n} \left(\dot{\theta}_{1j} + \dot{\theta}_{2j}\right)\right) \\
&\quad + g\left(t, \frac{\lambda}{\mu_0}\right)\mu_0^{n-4}\left(\lambda + (\xi - q) + a_1 + a_2 + \theta_{12} + \sum_{j=3}^{n}\left(\theta_{1j} + \theta_{2j}\right)\right) \\
&\quad + \mu_0^{n-3+\sigma} f(t),
\end{aligned}$$

where f and g are smooth, bounded functions satisfying $g(\cdot, s) \sim s$ as $s \to 0$. Thus

$$\begin{aligned}
&c\left(\frac{\mu}{\mu_0}\right)^{\frac{n+2}{2}} \mu_0^{-1} \int_{B_{2R}} \mu_0^{\frac{n+2}{2}} S_A(\xi + \mu_0 y, t) z_{n+1}(y) dy \\
&= \mu_0 \dot{\theta}_{12} + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g(t, \frac{\lambda}{\mu_0})\right)\dot{\xi} \\
&\quad + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g(t, \frac{\lambda}{\mu_0})\right)\mu_0\, (\dot{a}_1 + \dot{a}_2) \\
&\quad + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g(t, \frac{\lambda}{\mu_0})\right)\mu_0 \sum_{j=3}^{n} \left(\dot{\theta}_{1j} + \dot{\theta}_{2j}\right) \\
&\quad + g\left(t, \frac{\lambda}{\mu_0}\right)\mu_0^{n-4}\left(\lambda + (\xi - q) + \mu_0 a_1 + \mu_0 a_2 + \mu_0 \theta_{12} + \mu_0 \sum_{j=3}^{n}\left(\theta_{1j} + \theta_{2j}\right)\right),
\end{aligned}$$

for smooth bounded functions g satisfying $g(\cdot, s) \sim s$ as $s \to 0$.

The computations for the term

$$p\mu_0^{\frac{n-2}{2}} (1 + \frac{\lambda}{\mu_0})^{-2} \int_{B_{2R}} |Q|^{p-1}(\frac{\mu_0}{\mu} y)\psi(\xi + \mu_0 y, t) z_{n+1}(y) dy,$$

$B[\phi]$ and $B^0[\phi]$ are similar to that of Lemma 6.2. □

6.4. The equation for a_1 and a_2

Now we compute (4.5) for $l = n + 2, n + 3$.

Lemma 6.5. *For $l = n+2, n+3$, (4.5) is equivalent to*

$$\begin{aligned}
&a_{1,n+2}\dot{\xi}_1 + a_{n+2,n+2}\mu_0\dot{a}_1 + O\left(\frac{1}{k}\right)\dot{\lambda} + O\left(\frac{1}{k}\right)\dot{\xi} + O\left(\frac{1}{k}\right)\mu_0\dot{a}_2 \\
&+ O\left(\frac{1}{k}\right)\mu_0\left(\dot{\theta}_{12} + \sum_{j=3}^{n}(\dot{\theta}_{1j} + \dot{\theta}_{2j})\right) \qquad (6.7)\\
&= \Pi_{n+2}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t),
\end{aligned}$$

$$\begin{aligned}
&a_{2,n+3}\dot{\xi}_2 + a_{n+3,n+3}\mu_0\dot{a}_2 + O\left(\frac{1}{k}\right)\dot{\lambda} + O\left(\frac{1}{k}\right)\dot{\xi} + O\left(\frac{1}{k}\right)\mu_0\dot{a}_1 \\
&+ O\left(\frac{1}{k}\right)\mu_0\left(\dot{\theta}_{12} + \sum_{j=3}^{n}(\dot{\theta}_{1j} + \dot{\theta}_{2j})\right) \qquad (6.8)\\
&= \Pi_{n+3}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t),
\end{aligned}$$

$$\begin{aligned}
&\Pi_{n+2}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t) \\
&= \mu_0^{n-2+\sigma}(t)f(t) + \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\Theta_{n+2} \\
&\times\left[\dot{\lambda},\dot{\xi},\mu_0\dot{a},\mu_0\dot{\theta},\mu_0^{n-4}(t)\lambda,\mu_0^{n-4}(\xi-q),\mu_0^{n-3}a,\mu_0^{n-3}\theta,\mu_0^{n-3+\sigma}\phi,\mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t),
\end{aligned}$$

$$\begin{aligned}
&\Pi_{n+3}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t) \\
&= \mu_0^{n-2+\sigma}(t)f(t) + \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\Theta_{n+3} \\
&\times\left[\dot{\lambda},\dot{\xi}, \mu_0\dot{a},\mu_0\dot{\theta},\mu_0^{n-4}(t)\lambda,\mu_0^{n-4}(\xi-q),\mu_0^{n-3}a,\mu_0^{n-3}\theta, \mu_0^{n-3+\sigma}\phi,\mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t),
\end{aligned}$$

where $f(t)$ and Θ_{n+2}, Θ_{n+3} are smooth bounded functions for $t \in [t_0, \infty)$.

Proof. We compute

$$\int_{B_{2R}} H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](y, t(\tau))z_{n+2}(y)dy,$$

where $H[\lambda,\xi,a,\theta,\dot{\lambda},\dot{\xi},\dot{a},\dot{\theta},\phi,\psi](y, t(\tau))$ is defined in (4.2). Expand $\mu_0^{\frac{n+2}{2}} S_A(\xi +$

$\mu_0 y, t)$ as (6.2), by direct computations, we have

$$\int_{B_{2R}} S_1(\xi+\mu y) z_{n+2}(y)dy = O\left(\frac{1}{k}\right)\left(\dot\lambda + \mu_0^{n-4}\lambda\right),$$
$$\int_{B_{2R}} S_2(\xi+\mu y) z_{n+2}(y)dy = O\left(\frac{1}{k}\right)\left(\dot\mu_0 + \mu_0^{n-3}\right),$$
$$\int_{B_{2R}} S_3(\xi+\mu y) z_{n+2}(y)dy = a_{2,n+2}\dot\xi + O(1+\log R)\mu_0^{n-2},$$
$$\int_{B_{2R}} S_4(\xi+\mu y) z_{n+2}(y)dy = a_{n+2,n+2}\dot a_1 + O\left(\frac{1}{k}\right)(1+O(R^{-2}))\dot a_2,$$
$$\int_{B_{2R}} S_5(\xi+\mu y) z_{n+2}(y)dy = O\left(\frac{1}{k}\right)\left(\dot\theta_{12} + \sum_{j=3}^{n}\left(\dot\theta_{1j}+\dot\theta_{2j}\right)\right).$$

Since $\frac{\mu_0}{\mu} = (1+\frac{\lambda}{\mu_0})^{-1}$, for $l = 1, 2, 3, 4, 5$, we have

$$\begin{aligned}
&\int_{B_{2R}} [S_l(\xi+\mu_0 y, t) - S_l(\xi+\mu y, t)] z_{n+2}(y)dy \\
&= g(t, \frac{\lambda}{\mu_0})\dot\lambda + g(t, \frac{\lambda}{\mu_0})\dot\xi + g(t, \frac{\lambda}{\mu_0})\,(\dot a_1 + \dot a_2) \\
&\quad + g(t, \frac{\lambda}{\mu_0})\left(\dot\theta_{12} + \sum_{j=3}^{n}\left(\dot\theta_{1j}+\dot\theta_{2j}\right)\right) \\
&\quad + g(t, \frac{\lambda}{\mu_0})\mu_0^{n-4}\left(\lambda + (R_\theta\xi - q) + a_1 + a_2 + \theta_{12} + \sum_{j=3}^{n}\left(\theta_{1j}+\theta_{2j}\right)\right) \\
&\quad + \mu_0^{n-3+\sigma} f(t),
\end{aligned}$$

where f and g are smooth, bounded functions satisfying $g(\cdot, s) \sim s$ as $s \to 0$. Thus

$$\begin{aligned}
&c\left(\frac{\mu}{\mu_0}\right)^{\frac{n+2}{2}} \mu_0^{-2}\int_{B_{2R}} \mu_0^{\frac{n+2}{2}} S_A(\xi+\mu_0 y, t) z_{n+2}(y)dy \\
&= a_{2,n+3}\dot\xi + a_{n+2,n+2}\mu_0\dot a_1 \\
&\quad + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g(t, \frac{\lambda}{\mu_0})\right)\dot\xi + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g(t, \frac{\lambda}{\mu_0})\right)\mu_0\dot a_2 \\
&\quad + \left(O\left(\frac{1}{k}\right) + t_0^{-\varepsilon} g(t, \frac{\lambda}{\mu_0})\right)\mu_0\left(\dot\theta_{12} + \sum_{j=3}^{n}\left(\dot\theta_{1j}+\dot\theta_{2j}\right)\right) \\
&\quad + g\left(t, \frac{\lambda}{\mu_0}\right)\mu_0^{n-4}\left(\lambda + (\xi - q) + \mu_0 a_1 + \mu_0 a_2 + \mu_0\theta_{12} + \mu_0\sum_{j=3}^{n}\left(\theta_{1j}+\theta_{2j}\right)\right),
\end{aligned}$$

for smooth bounded functions g satisfying $g(\cdot, s) \sim s$ as $s \to 0$.

The computations for the term

$$p\mu_0^{\frac{n-2}{2}}\left(1+\frac{\lambda}{\mu_0}\right)^{-2}\int_{B_{2R}}|Q|^{p-1}\left(\frac{\mu_0}{\mu}y\right)\psi(\xi+\mu_0 y,t)z_{n+2}(y)dy,$$

$B[\phi]$ and $B^0[\phi]$ are similar to that of Lemma 6.2. This proves (6.7). The proof of (6.8) is similar. □

6.5. The equation for θ_{1l} and $\theta_{2l}, l = 3, \cdots, n$

Now we compute (4.5) for $l = n+4, \cdots, 3n-1$.

Lemma 6.6. *For $l = 3, \cdots, n$, (4.5) is equivalent to*

$$\begin{aligned}&\mu_0\dot\theta_{1l}+O\left(\frac{1}{k}\right)\dot\lambda+O\left(\frac{1}{k}\right)\dot\xi+O\left(\frac{1}{k}\right)\mu_0\left(\dot a_1+\dot a_2\right)\\&+O\left(\frac{1}{k}\right)\mu_0\left(\dot\theta_{12}+\sum_{j\neq l}\dot\theta_{1j}+\sum_{j=3}^{n}\dot\theta_{2j}\right)\\&=\Pi_{n+l+1}[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](t),\end{aligned}\tag{6.9}$$

$$\begin{aligned}&\mu_0\dot\theta_{2l}+O\left(\frac{1}{k}\right)\dot\lambda+O\left(\frac{1}{k}\right)\dot\xi+O\left(\frac{1}{k}\right)\mu_0\left(\dot a_1+\dot a_2\right)\\&+O\left(\frac{1}{k}\right)\mu_0\left(\dot\theta_{12}+\sum_{j\neq l}\dot\theta_{2j}+\sum_{j=3}^{n}\dot\theta_{1j}\right)\\&=\Pi_{2n+l-1}[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](t),\end{aligned}\tag{6.10}$$

$$\begin{aligned}&\Pi_{n+l+1}[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](t)\\&=\mu_0^{n-2+\sigma}(t)f(t)+\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\Theta_{n+l+1}\\&\times\left[\dot\lambda,\dot\xi,\mu_0\dot a,\mu_0\dot\theta,\mu_0^{n-4}(t)\lambda,\mu_0^{n-4}(\xi-q),\mu_0^{n-3}a,\mu_0^{n-3}\theta,\mu_0^{n-3+\sigma}\phi,\mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t),\end{aligned}$$

$$\begin{aligned}&\Pi_{2n+l-1}[\lambda,\xi,a,\theta,\dot\lambda,\dot\xi,\dot a,\dot\theta,\phi,\psi](t)\\&=\mu_0^{n-2+\sigma}(t)f(t)+\frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\Theta_{2n+l-1}\\&\times\left[\dot\lambda,\dot\xi,\mu_0\dot a,\mu_0\dot\theta,\mu_0^{n-4}(t)\lambda,\mu_0^{n-4}(\xi-q),\mu_0^{n-3}a,\mu_0^{n-3}\theta,\mu_0^{n-3+\sigma}\phi,\mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t),\end{aligned}$$

where $f(t)$ and Θ_{n+l+1}, Θ_{2n+l-1} are smooth bounded for $t \in [t_0, \infty)$.

The proof is similar to Lemma 6.4. Since the matrices

$$\begin{pmatrix} a_{1,1} & a_{1,n+2} \\ a_{n+2,1} & a_{n+2,n+2} \end{pmatrix}, \quad \begin{pmatrix} a_{2,2} & a_{2,n+3} \\ a_{n+3,2} & a_{n+3,n+3} \end{pmatrix}$$

are invertible, equations (6.3), (6.4), (6.7) and (6.8) can be decoupled by inverting the coefficient matrices. Combine Lemmas 6.3, 6.4, 6.5, 6.6 and 6.1, we get the result of Proposition 4.2.

7. Proof of Proposition 4.4

Proof of (4.20). Let us recall from (3.5) that

$$S_{\text{out}} = S_A^{(2)} + (1 - \eta_R) S_A.$$

From (2.40) and Lemma 2.2, in the region $|x - q| > \delta$ with $\delta > 0$, we have the following estimate for S_{out},

$$|S_{\text{out}}(x,t)| \lesssim \mu_0^{\frac{n-2}{2}} (\mu_0^2 + \mu_0^{n-4}) \lesssim \mu_0^{\min(n-4,2)-(\alpha-2)-\sigma}(t_0) \frac{\mu^{-2} \mu_0^{\frac{n-2}{2}+\sigma}}{1+|y|^\alpha}. \quad (7.1)$$

In the region $|x - q| \le \delta$ with $\delta > 0$ sufficiently small, Lemma 2.2 tells us that

$$\left| S_A^{(2)}(x,t) \right| \lesssim \mu_0^{-\frac{n+2}{2}} \frac{\mu_0^n}{1+|y|^2} \lesssim \mu_0^{2-(\alpha-2)-\sigma}(t_0) \frac{\mu^{-2} \mu_0^{\frac{n-2}{2}+\sigma}}{1+|y|^\alpha}. \quad (7.2)$$

By the definition of η_R, if $|x - \xi| > \mu_0 R$, $(1 - \eta_R) \neq 0$. Therefore we have

$$|(1-\eta_R) S_A| \lesssim \left(\frac{1}{R^{n-2-\alpha}} + \frac{1}{R^{4-\alpha}} \right) \frac{1}{R^{a-2}} \frac{\mu^{-2} \mu_0^{\frac{n-2}{2}+\sigma}}{1+|y|^\alpha}. \quad (7.3)$$

Here the decaying assumptions (4.9) and (4.10) are used, respectively. This proves the validity of (4.20). □

Proof of (4.21). For the term $2\nabla \eta_R \nabla \tilde{\phi}$, recalling that

$$\tilde{\phi}(x,t) := \mu_0^{-\frac{n-2}{2}} \phi \left(\frac{x - \xi}{\mu_0}, t \right)$$

and the assumptions (4.11) and (4.13), we have

$$\begin{aligned}
&\left|\left(\nabla\eta_R\cdot\nabla\tilde{\phi}\right)(x,t)\right| \\
&\lesssim \frac{\eta'\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)}{R\mu_0}\mu_0^{-\frac{n-2}{2}}\frac{|\nabla_y\phi|}{\mu_0} \\
&\lesssim \frac{\eta'\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)}{R\mu_0^2}\frac{\mu_0^{\frac{n-2}{2}+\sigma}}{(1+|y|^{1+\alpha})}\|\phi\|_{n-2+\sigma,\alpha} \\
&\lesssim \frac{1}{R^{a-2}}\|\phi\|_{n-2+\sigma,\alpha}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}}{(1+|y|^{\alpha})},
\end{aligned} \tag{7.4}$$

where, in the region $\eta'\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\neq 0$, $(1+|y|)\sim R$, $y=\frac{x-\xi}{\mu_0}$. As for the second term $\tilde{\phi}(\Delta-\partial_t)\eta_R$, by direct computations, we have

$$\begin{aligned}
\left|\tilde{\phi}(\Delta-\partial_t)\eta_R\right| \lesssim& \frac{\left|\Delta\eta\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\right|}{R^2\mu_0^2}\mu_0^{-\frac{n-2}{2}}|\phi| \\
&+\left|\eta'\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\left(\frac{|x-\xi|}{R\mu_0^2}\dot{\mu}_0+\frac{1}{R\mu_0}\dot{\xi}\right)\right|\mu_0^{-\frac{n-2}{2}}|\phi|.
\end{aligned} \tag{7.5}$$

From the definition of $\tilde{\phi}$, we have the following estimate for the first term in the right-hand side of (7.5),

$$\begin{aligned}
\frac{\left|\Delta\eta\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\right|}{R^2\mu_0^2}\mu_0^{-\frac{n-2}{2}}|\phi| &\lesssim \frac{\left|\Delta\eta\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\right|}{R^2\mu_0^2}\frac{\mu_0^{\frac{n-2}{2}+\sigma}}{(1+|y|^{\alpha})}\|\phi\|_{n-2s+\sigma,\alpha} \\
&\lesssim \frac{1}{R^{a-2}}\|\phi\|_{n-2+\sigma,\alpha}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha}},
\end{aligned} \tag{7.6}$$

here the fact that $\left|\Delta\eta\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\right|\sim\frac{1}{1+|\frac{y}{R}|^2}$ was used. From (4.9), we estimate the second term in the right-hand side of (7.5) as

$$\begin{aligned}
&\left|\eta'\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\left(\frac{|x-\xi|\dot{\mu}_0+\mu_0\dot{\xi}}{R\mu_0^2}\right)\right|\mu_0^{-\frac{n-2}{2}}|\phi| \\
&\qquad\lesssim \frac{\left|\eta'\left(\left|\frac{x-\xi}{R\mu_0}\right|\right)\right|}{R^2\mu_0^2}(\mu_0^{n-2}R^2+\mu_0^{n-2+\sigma}R)\mu_0^{-\frac{n-2}{2}}|\phi| \\
&\qquad\lesssim \frac{1}{R^{\alpha-2}}\|\phi\|_{n-2+\sigma,\alpha}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha}}.
\end{aligned} \tag{7.7}$$

From (7.4)-(7.7), we obtain (4.21). □

Proof of (4.22). Since $p-2\geq 0$ when $n\leq 6$, we have the following

$$\tilde{N}_A(\psi+\psi_1+\eta_R\tilde{\phi}) \lesssim \begin{cases} |u_A^*|^{p-2}\left[|\psi|^2+|\psi_1|^2+|\eta_R\tilde{\phi}|^2\right] & \text{when } 6\geq n \\ |\psi|^p+|\psi_1|^p+|\eta_R\tilde{\phi}|^p & \text{when } 6<n. \end{cases} \tag{7.8}$$

When $6\geq n$, there hold

$$\left|(u_A^*)^{p-2}(\eta_R\tilde{\phi})^2\right| \lesssim \frac{\mu_0^{\frac{3n}{2}-5+2\sigma}}{1+|y|^{2\alpha}}\|\phi\|^2_{n-2+\sigma,\alpha} \lesssim \mu_0^{n-2+\sigma}R^{\alpha-2}\|\phi\|^2_{n-2+\sigma,\alpha}\frac{1}{R^{\alpha-2}}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}}{1+|y|^{\alpha}}$$

and

$$\left|(u_A^*)^{p-2}\psi^2\right| \lesssim \mu_0^{-\frac{6-n}{2}}\frac{t^{-2\beta}}{1+|y|^{2(\alpha-2)}}\|\psi\|^2_{**,\beta,\alpha} \lesssim R^{\alpha-2}\mu_0^{n-4+\sigma+\alpha-2}\|\psi\|^2_{**,\beta,\alpha}\frac{1}{R^{\alpha-2}}\frac{\mu_j^{-2}t^{-\beta}}{1+|y|^{\alpha}}.$$

When $6<n$, one has

$$\left|\eta_R\tilde{\phi}\right|^p \lesssim \frac{\mu_0^{(\frac{n-2}{2}+\sigma)p}}{1+|y|^{\alpha p}}\|\phi\|^p_{n-2+\sigma,\alpha} \lesssim \mu_0^{2+(p-1)\sigma}R^{\alpha-2}\mu_0^2\|\phi\|^p_{n-2+\sigma,\alpha}\frac{1}{R^{\alpha-2}}\frac{\mu^{-2}\mu_0^{\frac{n-2}{2}+\sigma}}{1+|y|^{\alpha}}$$

and

$$|\psi|^p \lesssim \frac{t^{-p\beta}}{1+|y|^{p(\alpha-2)}}\|\psi\|^p_{**,\beta,\alpha} \lesssim \mu^{4(1+\frac{\sigma}{n-2})+p(\alpha-2)-\alpha}R^{\alpha-2}\|\psi\|^p_{**,\beta,\alpha}\frac{1}{R^{\alpha-2}}\frac{\mu_j^{-2}\mu_0^{\frac{n-2}{2}+\sigma}}{1+|y|^{\alpha}}.$$

The estimate for ψ_1 is similar. This proves (4.22). □

8. Stability result in dimension 5 and 6

In dimension 5 and 6, we have $p-1=\frac{4}{n-2}\geq 1$. In this case, all the equations can be solved by the Contraction Mapping Theorem since the operators $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are Lipschitz continuous with respect to the parameter functions. Therefore, Theorem 1.1 can be proved by the Contraction Mapping Theorem arguments in dimension 5 and 6, moreover, we have the following stability result.

Theorem 8.1. *Assume k_0 is a sufficiently large integer, $n = 5, 6$ and q is a point in Ω, then the conclusion of Theorem 1.1 holds when $k \geq k_0$. Furthermore, there exists a sub-manifold $\mathcal{M}$ with codimension K in $C^1(\overline{\Omega})$ containing $u_q(x, 0)$ such that, if $u_0 \in \mathcal{M}$ and is sufficiently close to $u_q(x, 0)$, the solution $u(x, t)$ to (1.1) still has the form*

$$u(x,t) = \tilde{\lambda}(t)^{-\frac{n-2}{2}} \left(Q_k \left(\frac{x - \tilde{\xi}(t)}{\tilde{\lambda}(t)} \right) + \tilde{\varphi}(x,t) \right),$$

where $\tilde{q} = \lim_{t \to +\infty} \tilde{\xi}(t)$ is close to q.

Recalling that K is the dimension of the space $V := \{f \in \dot{H}^1(\mathbb{R}^n) | \langle Lf, f \rangle < 0\}$ and L is defined in (1.10). The proof is similar to [2] and [32], so we give a sketch here. We divide the whole process into three steps.

Step 1. Solving the outer problem (4.15)

Proposition 8.2. *Assume $\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}$ and $\dot{\theta}$ satisfy (4.9) and (4.10), ϕ satisfies (4.13), $\psi_0 \in C^2(\overline{\Omega})$ and*

$$\|\psi_0\|_{L^\infty(\overline{\Omega})} + \|\nabla \psi_0\|_{L^\infty(\overline{\Omega})} \leq \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}.$$

Then (4.15) has a unique solution $\psi = \Psi[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi]$, for $y = \frac{x-\xi}{\mu_0}$, there exist small constants $\sigma > 0$ and $\varepsilon > 0$ such that

$$|\psi(x,t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \frac{\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha-2}} + e^{-\delta(t-t_0)} \|\psi_0\|_{L^\infty(\overline{\Omega})}$$

$$|\nabla \psi(x,t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \frac{\mu^{-1}\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha-1}} \text{ for } |y| \leq R$$

hold. Here R is defined in (3.1).

Proposition 8.2 is a direct consequence of Proposition 4.3, Proposition 4.4 and the Contraction Mapping Theorem, whose proof we omit here. This result indicates that for any small initial datum ψ_0, (4.15) has a solution ψ. Moreover, the following proposition clarifies the dependence of $\Psi[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi]$ on the parameter functions $\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi$ which is proved by estimating, for instance,

$$\partial_\phi \Psi[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi][\bar{\phi}] = \partial_s \Psi[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi + s\bar{\phi}]|_{s=0}$$

as a bounded linear operator between weighted parameter spaces. For simplicity, the above operator is denoted by $\partial_\phi \Psi[\bar{\phi}]$. Similarly, we define $\partial_\lambda \Psi[\bar{\lambda}]$, $\partial_\xi \Psi[\bar{\xi}]$, $\partial_a \Psi[\bar{a}]$, $\partial_\theta \Psi[\bar{\theta}]$, $\partial_{\dot{\lambda}} \Psi[\dot{\bar{\lambda}}]$, $\partial_{\dot{\xi}} \Psi[\dot{\bar{\xi}}]$, $\partial_{\dot{a}} \Psi[\dot{\bar{a}}]$ and $\partial_{\dot{\theta}} \Psi[\dot{\bar{\theta}}]$.

Proposition 8.3. *Under the assumptions of Proposition* 8.2, Ψ *depends smoothly on the parameter functions* λ, ξ, a, θ, $\dot{\lambda}$, $\dot{\xi}$, $\dot{a}$, $\dot{\theta}$, ϕ *and, for* $y = \frac{x-\xi}{\mu_0}$, *the following hold:*

$$
\begin{aligned}
\left|\partial_\lambda \Psi[\bar{\lambda}]\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\bar{\lambda}\|_{1+\sigma} \frac{\mu_0^{\frac{n-2}{2}-1}(t)}{1+|y|^{\alpha-2}},\\
\left|\partial_\xi \Psi[\bar{\xi}]\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \left(\|\bar{\xi}\|_{1+\sigma} \frac{\mu_0^{\frac{n-2}{2}-1}(t)}{1+|y|^{\alpha-2}} \right),\\
\left|\partial_a \Psi[\bar{a}]\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \left(\|\bar{a}\|_{\sigma} \frac{\mu_0^{\frac{n-2}{2}-2}(t)}{1+|y|^{\alpha-2}} \right),\\
\left|\partial_\theta \Psi[\bar{\theta}]\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \left(\|\bar{\theta}\|_{\sigma} \frac{\mu_0^{\frac{n-2}{2}-2}(t)}{1+|y|^{\alpha-2}} \right),\\
\left|\partial_{\dot{\xi}} \Psi[\dot{\bar{\xi}}](x,t)\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\dot{\bar{\xi}}(t)\|_{n-3+\sigma} \left(\frac{\mu_0^{-\frac{n-6}{2}-1+\sigma}(t)}{1+|y|^{\alpha-2}} \right),\\
\left|\partial_{\dot{\lambda}} \Psi[\dot{\bar{\lambda}}](x,t)\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\dot{\bar{\lambda}}(t)\|_{n-3+\sigma} \left(\frac{\mu_0^{-\frac{n-6}{2}-1+\sigma}(t)}{1+|y|^{\alpha-2}} \right),\\
\left|\partial_{\dot{a}} \Psi[\dot{\bar{a}}](x,t)\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\dot{\bar{a}}(t)\|_{n-4+\sigma} \left(\frac{\mu_0^{-\frac{n-6}{2}-2+\sigma}(t)}{1+|y|^{\alpha-2}} \right),\\
\left|\partial_{\dot{\theta}} \Psi[\dot{\bar{\theta}}](x,t)\right| &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\dot{\bar{\theta}}(t)\|_{n-4+\sigma} \left(\frac{\mu_0^{-\frac{n-6}{2}-2+\sigma}(t)}{1+|y|^{\alpha-2}} \right),\\
\left|\partial_\phi \Psi[\bar{\phi}](x,t)\right| &\lesssim \frac{1}{R^{\alpha-2}} \|\bar{\phi}(t)\|_{n-2+\sigma,\alpha} \left(\frac{\mu_0^{\frac{n-2}{2}+\sigma}(t)}{1+|y|^{\alpha-2}} \right).
\end{aligned} \tag{8.1}
$$

Proof. We prove (8.1). Decompose the term $\partial_\lambda \Psi[\bar{\lambda}](x,t) = Z_1 + Z$ with $Z_1 = \mathcal{T}_2(0, -\partial_\lambda u_A^*[\bar{\lambda}], 0)$, where $\mathcal{T}_2$ is defined by Proposition 4.3. Then Z is a solution of the following problem

$$
\begin{cases}
\partial_t Z = \Delta Z + V_A Z + \partial_\lambda V_A[\bar{\lambda}]\psi \\
\qquad\qquad + \partial_\lambda \tilde{N}_A\left(\psi + \phi^{in}\right)[\bar{\lambda}] + \partial_\lambda S_{\mathrm{out}}[\bar{\lambda}] & \text{in } \Omega \times (t_0, \infty) \\
Z = 0 & \text{in } \partial\Omega \times (t_0, \infty) \\
Z(\cdot, t_0) = 0 & \text{in } \Omega.
\end{cases} \tag{8.2}
$$

For any $x \in \partial\Omega$,

$$\begin{aligned} \left|\partial_\lambda u_A^*[\bar{\lambda}](x,t)\right| &\lesssim \mu_0^{\frac{n}{2}-1+\sigma}(t)|\bar{\lambda}(t)| \\ &\lesssim \mu_0^{\frac{n}{2}+2\sigma}(t)\|\bar{\lambda}(t)\|_{1+\sigma}. \end{aligned} \tag{8.3}$$

From (8.3) and Proposition 4.3, we obtain

$$|Z_1(x,t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\left(\|\bar{\lambda}(t)\|_{1+\sigma}\frac{\mu_0^{\frac{n-2}{2}-1}(t)}{1+|y|^{\alpha-2}}\right).$$

To prove the estimation for Z, which can be viewed as a fixed point for the operator

$$\mathcal{A}(Z) = \mathcal{T}_2\,(g,0,0) \tag{8.4}$$

with

$$g = \partial_\lambda V_A[\bar{\lambda}]\psi + \partial_\lambda \tilde{N}_A\left(\psi + \phi^{in}\right)[\bar{\lambda}] + \partial_\lambda S_{\mathrm{out}}[\bar{\lambda}],$$

we estimate $\partial_\lambda S_{\mathrm{out}}[\bar{\lambda}]$ first. In the region $|x-q| > \delta$, from (2.40), (4.9) and (4.10), we have

$$\begin{aligned} \left|\partial_\lambda S_{\mathrm{out}}[\bar{\lambda}](x,t)\right| &\lesssim \mu_0^{\frac{n-2}{2}-1} f(x,\mu_0^{-1}\mu,\xi,a,\theta)|\bar{\lambda}(t)| \\ &\lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\left(\|\bar{\lambda}(t)\|_{1+\sigma}\frac{\mu_0^{\frac{n-2}{2}-1}(t)}{1+|y|^{\alpha-2}}\right), \end{aligned}$$

where the function f is smooth and bounded depending on $(x,\mu_0^{-1}\mu,\xi,a,\theta)$. In the region $|x-q| \le \delta$, from (2.42), we have

$$\partial_\lambda S(u_A^*)[\bar{\lambda}](x,t) = \partial_\lambda S(u_A)[\bar{\lambda}](x,t)(1+\mu_0 f(x,\mu_0^{-1}\mu,\xi,a,\theta)),$$

where the function f is smooth and bounded depending on $(x,\mu_0^{-1}\mu,\xi,a,\theta)$. Differentiating (2.19) with respect to λ, easy but long computations yield that

$$\left|\partial_\lambda S(u_A)[\bar{\lambda}]\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\left(\|\bar{\lambda}(t)\|_{1+\sigma}\frac{\mu_0^{\frac{n-2}{2}-1}(t)}{1+|y|^{\alpha-2}}\right). \tag{8.5}$$

By the definition of S_{out} together with (8.5), we obtain

$$\left|\partial_\lambda S_{\mathrm{out}}[\bar{\lambda}](x,t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\left(\|\bar{\lambda}(t)\|_{1+\sigma}\frac{\mu_0^{\frac{n-2}{2}-1}(t)}{1+|y|^{\alpha-2}}\right).$$

Now we estimate the other terms of g. When $n = 5, 6$, we have

$$\begin{aligned}\partial_\lambda V_A[\bar{\lambda}](x,t) =& p(p-1)\Big[|u_A^*|^{p-3} u_A^* \partial_\lambda u_A^*[\bar{\lambda}] \\ & - \eta_R \left| \mu^{-\frac{n-2}{2}} Q(y) \right|^{p-3} \mu^{-\frac{n-2}{2}} Q(y) \partial_\lambda \big(\mu^{-\frac{n-2}{2}} Q(y) \big)[\bar{\lambda}] \Big].\end{aligned}$$

Since $\left|\partial_\lambda \big(\mu^{-\frac{n-2}{2}} Q(y)\big)\right| \lesssim \mu_0^{-1} \left|\mu^{-\frac{n-2}{2}} Q(y)\right|$ and $\beta = \frac{n-2}{2(n-4)} + \frac{\sigma}{n-4}$, we obtain

$$\left|\partial_\lambda V_A[\bar{\lambda}]\psi(x,t)\right| \lesssim \|\psi\|_{**,\beta,\alpha} \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\bar{\lambda}(t)\|_{1+\sigma} \frac{\mu^{-2}\mu_0^{\frac{n-2}{2}-1+\sigma}}{1+|y|^\alpha}.$$

Similarly, we estimate the term $p(p-1)|u_A^*|^{p-3}u_A^*(\psi + \phi^{in})\partial_\lambda u_A^*[\bar{\lambda}]$ as

$$\left| p(p-1)|u_A^*|^{p-3}u_A^*(\psi + \phi^{in})\partial_\lambda u_A^*[\bar{\lambda}] \right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\bar{\lambda}\|_{1+\sigma} \frac{\mu^{-2}\mu_0^{\frac{n-2}{2}-1+\sigma}}{1+|y|^\alpha}$$

when $n = 5, 6$. The last term $p\left[\left|u_A^* + \psi + \phi^{in}\right|^{p-1} u_A^* - \left|u_A^*\right|^{p-1} u_A^* \right]$ can be estimated analogously.

In the set of functions satisfying

$$|Z(x,t)| \le M \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \|\bar{\lambda}\|_{1+\sigma} \frac{\mu_0^{\frac{n-2}{2}-1}}{1+|y|^{\alpha-2}}$$

for a fixed large constant M, the operator $\mathcal{A}$ defined in (8.4) has a fixed point. Indeed, $\mathcal{A}$ is a contraction map when R is large in terms of t_0. Hence (8.1) holds. The proof of the other estimates are similar, we omit them. □

Substituting the solution $\psi = \Psi[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi]$ of (4.15) given by Proposition 8.2 into (3.7), the full problem becomes

$$\begin{aligned}\mu_0^2 \partial_t \phi =& \Delta_y \phi + p|Q|^{p-1}(y)\phi \\ & + H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi](y,t),\ y \in B_{2R}(0).\end{aligned} \tag{8.6}$$

Similar to Section 4.1, using change of variables

$$t = t(\tau), \quad \frac{dt}{d\tau} = \mu_0^2(t),$$

(8.6) reduces to

$$\partial_\tau \phi = \Delta_y \phi + p|Q|^{p-1}(y)\phi + H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi](y, t(\tau))$$

for $y \in B_{2R}(0)$, $\tau \geq \tau_0$, τ_0 is the unique positive number such that $t(\tau_0) = t_0$. We try to find a solution ϕ to the equation

$$\begin{cases} \partial_\tau \phi = \Delta_y \phi + p|Q|^{p-1}(y)\phi \\ \qquad\qquad + H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi](y, t(\tau)) & y \in B_{2R}(0), \quad \tau \geq \tau_0 \\ \phi(y, \tau_0) = \sum_{l=1}^{K} e_{0l} Z_l(y) & y \in B_{2R}(0), \end{cases} \tag{8.7}$$

for some suitable constants e_{0l}, $l = 1, \cdots, K$. To apply the linear theory Proposition 4.1, the parameter functions λ, ξ, a, θ need to satisfy the following orthogonality conditions

$$\int_{B_{2R}} H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi](y, t(\tau)) z_l(y) dy = 0, \quad l = 0, 1, \cdots, 3n-1. \tag{8.8}$$

Step 2. Choosing the parameter functions

By the Lipschitz properties for $\Psi = \Psi[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi]$ given by Proposition 8.3, Proposition 4.2 can be strengthened as

Proposition 8.4. (8.8) *is equivalent to*

$$\begin{cases} \dot{\lambda} + \frac{1+(n-4)}{(n-4)t}\lambda = \Pi_0[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{\xi}_l = \Pi_l[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), & l = 1, \cdots, n, \\ \dot{\theta}_{12} = \mu_0^{-1} \Pi_{n+1}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{a}_1 = \mu_0^{-1} \Pi_{n+2}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{a}_2 = \mu_0^{-1} \Pi_{n+3}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), \\ \dot{\theta}_{1l} = \mu_0^{-1} \Pi_{n+l+1}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), & l = 3, \cdots, n, \\ \dot{\theta}_{2l} = \mu_0^{-1} \Pi_{2n+l-1}[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t), & l = 3, \cdots, n. \end{cases} \tag{8.9}$$

The terms in the right-hand side of the above system can be expressed as

$$\Pi_0[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t) = \frac{t_0^{-\varepsilon}}{R^{\alpha-2}} \mu_0^{n-3+\sigma}(t) f_0(t) + \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}$$
$$\times \Theta_0\left[\dot{\lambda}, \dot{\xi}, \mu_0 \dot{a}, \mu_0 \dot{\theta}, \mu_0^{n-4}(t)\lambda, \mu_0^{n-4}(\xi - q), \mu_0^{n-3} a, \mu_0^{n-3}\theta, \mu_0^{n-3+\sigma}\phi, \mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t)$$

and for $j = 1, \cdots, 3n-1$,

$$\Pi_j[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi, \psi](t)$$
$$= \mu_0^{n-2} c_j \left[b^{n-2} \nabla H(q, q)\right] + \mu_0^{n-2+\sigma}(t) f_j(t) + \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}$$
$$\times \Theta_j\left[\dot{\lambda}, \dot{\xi}, \mu_0 \dot{a}, \mu_0 \dot{\theta}, \mu_0^{n-4}(t)\lambda, \mu_0^{n-4}(\xi - q), \mu_0^{n-3} a, \mu_0^{n-3}\theta, \mu_0^{n-3+\sigma}\phi, \mu_0^{\frac{n-2}{2}+\sigma}\psi\right](t),$$

where $f_j(t)$ *and* $\Theta_j[\cdots](t)$ *(* $j = 0, \cdots, 3n-1$ *) are bounded smooth functions for* $t \in [t_0, \infty)$*,* c_j *(* $j = 0, \cdots, 3n-1$ *) are suitable constants. Moreover, we have*

$$\left|\Theta_j[\dot\lambda_1](t) - \Theta_j[\dot\lambda_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}|\dot\lambda_1(t) - \dot\lambda_2(t)|$$

$$\left|\Theta_j[\dot\xi_1](t) - \Theta_j[\dot\xi_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}|\dot\xi_1(t) - \dot\xi_2(t)|,$$

$$\left|\Theta_j[\mu_0\dot a_1^{(1)}](t) - \Theta_j[\mu_0\dot a_1^{(2)}](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0|\dot a_1^{(1)}(t) - \dot a_1^{(2)}(t)|,$$

$$\left|\Theta_j[\mu_0\dot a_2^{(1)}](t) - \Theta_j[\mu_0\dot a_2^{(2)}](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0|\dot a_2^{(1)}(t) - \dot a_2^{(2)}(t)|,$$

$$\left|\Theta_j[\mu_0\dot\theta_1](t) - \Theta_j[\mu_0\dot\theta_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0|\dot\theta_1(t) - \dot\theta_2(t)|,$$

$$\left|\Theta_j[\mu_0^{n-4}\lambda_1](t) - \Theta_j[\mu_0^{n-4}\lambda_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}|\lambda_1(t) - \lambda_2(t)|,$$

$$\left|\Theta_j[\mu_0^{n-4}(\xi_1 - q)](t) - \Theta_j[\mu_0^{n-4}(\xi_2 - q)](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}|\xi_1(t) - \xi_2(t)|,$$

$$\left|\Theta_j[\mu_0^{n-3}a_1^{(1)}](t) - \Theta_j[\mu_0^{n-3}a_1^{(2)}](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0|a_1^{(1)}(t) - a_1^{(2)}(t)|,$$

$$\left|\Theta_j[\mu_0^{n-3}a_2^{(1)}](t) - \Theta_j[\mu_0^{n-3}a_2^{(2)}](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0|a_2^{(1)}(t) - a_2^{(2)}(t)|,$$

$$\left|\Theta_j[\mu_0^{n-3}\theta_1](t) - \Theta_j[\mu_0^{n-3}\theta_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\mu_0|\theta_1(t) - \theta_2(t)|,$$

$$\left|\Theta[\mu_0^{n-3+\sigma}\phi_1](t) - \Theta[\mu_0^{n-3+\sigma}\phi_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\|\phi_1(t) - \phi_2(t)\|_{n-2+\sigma,\alpha}.$$

System (8.9) is solvable for λ, ξ, a, θ satisfying (4.9) and (4.10). Indeed, we have:

Proposition 8.5. (8.9) *has a solution* $\lambda = \lambda[\phi](t)$*,* $\xi = \xi[\phi](t)$*,* $a = a[\phi](t)$ *and* $\theta = \theta[\phi](t)$ *satisfying estimates* (4.9) *and* (4.10)*. Moreover, for* $t \in (t_0, \infty)$*, there hold*

$$\mu_0^{-(1+\sigma)}(t)\left|\lambda[\phi_1](t) - \lambda[\phi_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\|\phi_1 - \phi_2\|_{n-2+\sigma,\alpha},$$

$$\mu_0^{-(1+\sigma)}(t)\left|\xi[\phi_1](t) - \xi[\phi_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\|\phi_1 - \phi_2\|_{n-2+\sigma,\alpha},$$

$$\mu_0^{-\sigma}(t)\left|a[\phi_1](t) - a[\phi_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\|\phi_1 - \phi_2\|_{n-2+\sigma,\alpha},$$

$$\mu_0^{-\sigma}(t)\left|\theta[\phi_1](t) - \theta[\phi_2](t)\right| \lesssim \frac{t_0^{-\varepsilon}}{R^{\alpha-2}}\|\phi_1 - \phi_2\|_{n-2+\sigma,\alpha}.$$

Using Proposition 8.3, the proof of Proposition 8.4 and 8.5 is similar to that of [2] and [32], we omit it.

Step 3. Gluing: the inner problem
After choosing parameter functions $\lambda = \lambda[\phi](t)$, $\xi = \xi[\phi](t)$, $a = a[\phi](t)$ and $\theta = \theta[\phi](t)$ such that (8.8) hold, we solve problem (8.7) in the class of functions with $\|\phi\|_{n-2+\sigma,\alpha}$ bounded. Problem (8.7) is a fixed point of

$$\phi = \mathcal{A}_1(\phi) := \mathcal{T}_2(H[\lambda, \xi, a, \theta, \dot{\lambda}, \dot{\xi}, \dot{a}, \dot{\theta}, \phi]).$$

It is easy to see that

$$\left|H[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t)\right| \lesssim t_0^{-\varepsilon} \frac{\mu_0^{n-2+\sigma}}{1 + |y|^{2+\alpha}} \tag{8.10}$$

and

$$\left|H[\phi^{(1)}] - H[\phi^{(2)}]\right|(y, t) \lesssim t_0^{-\varepsilon} \|\phi^{(1)} - \phi^{(2)}\|_{n-2+\sigma,\alpha} \tag{8.11}$$

hold. From (8.10) and (8.11), $\mathcal{A}_1$ has a fixed point ϕ in the set of functions $\|\phi\|_{n-2s+\sigma,\alpha} \le c t_0^{-\varepsilon}$ for suitable large constant $c > 0$. From the Contraction Mapping Theorem, we obtain a solution to (2.11). Then the rest argument to the stability part of Theorem 8.1 is the same as [2], we omit it.

9. Appendix

9.1. Proof of Lemma 6.1

Let us recall from [9] and [33] that

$$Q_k(x) = U(x) - \sum_{j=1}^{k} U_j(x) + \tilde{\phi}(x) \quad \text{with} \quad U(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{n-2}{2}}$$

and

$$U_j(x) = \zeta_k^{-\frac{n-2}{2}} U(\zeta_k^{-1}(x - \xi_j)).$$

Here ζ_k is a positive constant satisfying $\zeta_k \sim k^{-2}$, $\xi_j = \sqrt{1 - \zeta_k^2}(\mathbf{n}_j, 0)$, $\mathbf{n}_j = (\cos\theta_j, \sin\theta_j, 0), \theta_j = \frac{2\pi}{k}(j-1)$ and $\tilde{\phi}$ is a small term than $U(x) - \sum_{j=1}^{k} U_j(x)$. Let us introduce the functions

$$Z_0(x) = \frac{n-2}{2} U(x) + \nabla U(x) \cdot x, \qquad \pi_0(x) = \frac{n-2}{2} \tilde{\phi}(x) + \nabla \tilde{\phi}(x) \cdot x$$

and

$$Z_\alpha(x) = \frac{\partial}{\partial x_\alpha} U(x), \quad \pi_\alpha(x) = \frac{\partial}{\partial x_\alpha} \tilde{\phi}(x) \quad \text{for} \quad \alpha = 1, \ldots, n.$$

For $l = 1, \ldots, k$, define

$$Z_{0l}(x) = \frac{n-2}{2} U_l(x) + \nabla U_l(x) \cdot (x - \xi_l).$$

From (1.11) and (1.12),

$$z_0(x) = Z_0(x) - \sum_{l=1}^{k} \left[Z_{0l}(x) + \sqrt{1 - \zeta_k^2} \cos\theta_l \frac{\partial}{\partial x_1} U_l(x) + \sqrt{1 - \zeta_k^2} \sin\theta_l \frac{\partial}{\partial x_2} U_l(x)\right] + \pi_0(x).$$

For $l = 1, \dots, k$, define

$$Z_{1l}(x) = \sqrt{1-\zeta_k^2}\left[\cos\theta_l \frac{\partial}{\partial x_1}U_l(x) + \sin\theta_l \frac{\partial}{\partial x_2}U_l(x)\right],$$
$$Z_{2l}(x) = \sqrt{1-\zeta_k^2}\left[-\sin\theta_l \frac{\partial}{\partial x_1}U_l(x) + \cos\theta_l \frac{\partial}{\partial x_2}U_l(x)\right],$$
$$Z_{\alpha l}(x) = \frac{\partial}{\partial x_\alpha}U_l(x), \quad \text{for} \quad \alpha = 3, \dots, n.$$

Then we have

$$z_0(x) = Z_0(x) - \sum_{l=1}^{k}[Z_{0l}(x) + Z_{1l}(x)] + \pi_0(x), \tag{9.1}$$

$$\begin{aligned} z_1(x) &= Z_1(x) - \sum_{l=1}^{k}\frac{\partial}{\partial x_1}U_l(x) + \pi_1(x) \\ &= Z_1(x) - \sum_{l=1}^{k}\frac{[\cos\theta_l Z_{1l}(x) - \sin\theta_l Z_{2l}(x)]}{\sqrt{1-\zeta_k^2}} + \pi_1(x), \end{aligned} \tag{9.2}$$

$$\begin{aligned} z_2(x) &= Z_2(x) - \sum_{l=1}^{k}\frac{\partial}{\partial x_2}U_2(x) + \pi_2(x) \\ &= Z_2(x) - \sum_{l=1}^{k}\frac{[\sin\theta_l Z_{1l}(x) + \cos\theta_l Z_{2l}(x)]}{\sqrt{1-\zeta_k^2}} + \pi_2(x), \end{aligned} \tag{9.3}$$

$$z_\alpha(x) = Z_\alpha(x) - \sum_{l=1}^{k} Z_{\alpha l} + \pi_\alpha(x) \text{ for } \alpha = 3, \cdots, n. \tag{9.4}$$

Moreover, the following identities hold,

$$z_{n+1}(x) = \sum_{l=1}^{k} Z_{2l}(x) + x_2\pi_1(x) - x_1\pi_2(x), \tag{9.5}$$

$$\begin{aligned} z_{n+2}(x) = &\sum_{l=1}^{k}\sqrt{1-\zeta_k^2}\cos\theta_l Z_{0l}(x) - \sum_{l=1}^{k}\sqrt{1-\zeta_k^2}\cos\theta_l Z_{1l}(x) \\ &- 2x_1\pi_0(x) + |x|^2\pi_1(x), \end{aligned} \tag{9.6}$$

$$\begin{aligned} z_{n+3}(x) = &\sum_{l=1}^{k}\sqrt{1-\zeta_k^2}\sin\theta_l Z_{0l}(x) - \sum_{l=1}^{k}\sqrt{1-\zeta_k^2}\sin\theta_l Z_{1l}(x) \\ &- 2x_2\pi_0(x) + |x|^2\pi_2(x), \end{aligned} \tag{9.7}$$

$$z_{n+\alpha+1}(x) = \sqrt{1-\zeta_k^2}\sum_{l=1}^{k}\cos\theta_l Z_{\alpha l}(x) + x_1\pi_\alpha(x), \text{ for } \alpha = 3, \dots, n, \tag{9.8}$$

$$z_{2n+\alpha-1}(x) = \sqrt{1-\zeta_k^2}\sum_{l=1}^{k}\sin\theta_l Z_{\alpha l}(x) + x_2\pi_\alpha(x), \text{ for } \alpha = 3, \dots, n. \tag{9.9}$$

Then we have the following estimations:

Lemma 9.1.

$$\int_{\mathbb{R}^n} Z_{\alpha l}(x) Z_0(x) dx \tag{9.10}$$
$$= \begin{cases} \int_{\mathbb{R}^n} Z_0^2(x) dx + O(k^{-1}) & \text{if } \alpha = 0,\ l = 0 \\ O(k^{-1}) & \text{otherwise}, \end{cases}$$

$$\int_{\mathbb{R}^n} Z_{\alpha l}(x) Z_\beta(x) dx \tag{9.11}$$
$$= \begin{cases} \int_{\mathbb{R}^n} Z_1^2(x) dx + O(k^{-1}) & \text{if } \alpha = \beta \in \{1, \cdots, n\},\ l = 0 \\ O(k^{-1}) & \text{otherwise}, \end{cases}$$

$$\int_{\mathbb{R}^n} Z_{\alpha l}(x) Z_{0j}(x) dx \tag{9.12}$$
$$= \begin{cases} \int_{\mathbb{R}^n} Z_0^2(x) dx + O(k^{-1}) & \text{if } \alpha = 0,\ l = j \\ O(k^{-1}) & \text{otherwise}, \end{cases}$$

$$\int_{\mathbb{R}^n} Z_{\alpha l}(x) Z_{\beta j}(x) dx \tag{9.13}$$
$$= \begin{cases} \int_{\mathbb{R}^n} Z_1^2(x) dx + O(k^{-1}) & \text{if } \alpha = \beta \in \{1, \cdots, n\},\ l = j \\ O(k^{-1}) & \text{otherwise}, \end{cases}$$

$$\int_{\mathbb{R}^n} \frac{|x|^2 - 2}{\left(1 + |x|^2\right)^{\frac{n-2}{2}+1}} Z_{\beta j}(x) dx \tag{9.14}$$
$$= \begin{cases} \int_{\mathbb{R}^n} \frac{(|x|^2 - 2)}{\left(1 + |x|^2\right)^{\frac{n-2}{2}+1}} Z_0(x) dx + O(k^{-1}) & \text{if } \beta = 0,\ \ j = 0 \\ O(k^{-1}) & \text{otherwise}, \end{cases}$$

$$\int_{\mathbb{R}^n} \frac{x_i}{\left(1 + |x|^2\right)^{\frac{n-2}{2}+1}} Z_{\beta j}(x) dx \tag{9.15}$$
$$= \begin{cases} \int_{\mathbb{R}^n} \frac{x_i}{\left(1 + |x|^2\right)^{\frac{n-2}{2}+1}} Z_i(x) dx + O(k^{-1}) & \text{if } \ \beta = 0,\ \ j = i \in \{1, \cdots, n\} \\ O(k^{-1}) & \text{otherwise}. \end{cases}$$

Proof. We prove (9.12). Let $\eta > 0$ be a small fixed real number independent of k. Then

$$\int_{\mathbb{R}^n} Z_{\alpha l}(x) Z_{0j}(x)dx = \int_{B(\xi_l, \frac{\eta}{k})} Z_{\alpha l}(x) Z_{0l}(x)dx + \int_{\mathbb{R}^n \setminus B(\xi_l, \frac{\eta}{k})} Z_{\alpha l}(x) Z_{0j}(x)dx.$$
$$:= i_1 + i_2.$$

Changing the variable via $x = \xi_l + \zeta_k y$, we obtain

$$i_1 = \int_{B(0, \frac{\eta}{k\zeta_k})} Z_\alpha(x) Z_0(x)dx = \begin{cases} \left(\int_{\mathbb{R}^n} Z_0^2(x)dx + O((\zeta_k k)^n)\right) & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha \neq 0. \end{cases}$$

As for the term i_2, decompose

$$i_2 = \int_{\mathbb{R}^n \setminus \cup_{j=1}^k B(\xi_j, \frac{\eta}{k})} Z_{\alpha l}(x) Z_{0j}(x)dx + \sum_{j \neq l} \int_{B(\xi_j, \frac{\eta}{k})} Z_{\alpha l}(x) Z_{0j}(x)dx = i_{21} + i_{22}.$$

i_{21} can be estimated as

$$\begin{aligned} |i_{21}| &\leq C\zeta_k^{n-2} \int_{\{|x| \geq \frac{\eta}{k}\}}^\infty \frac{1}{|x|^{2n-4}} dx = C\zeta_k^{n-2} \int_{\frac{\eta}{k}}^\infty \frac{r^{n-1}}{r^{2n-4}} dr \\ &\leq C\zeta_k^{n-2} k^{n-4} = O\left(\frac{1}{k}\right). \end{aligned}$$

And

$$\begin{aligned} |i_{22}| \leq C \sum_{j \neq l} \int_{B(\xi_j, \frac{\eta}{k})} \frac{\zeta_k^{\frac{n-2}{2}}}{|x - \xi_l|^{n-2}} Z_{0j}(x)dx &\leq C\zeta_k^{\frac{n-2}{2}} k^{n-2} \int_0^{\frac{\eta}{k}} \frac{r^{n-1}}{r^{n-2}} dr \\ &\leq C\zeta_k^{\frac{n-2}{2}} k^{n-4} \leq C\zeta_k = O\left(\frac{1}{k}\right), \end{aligned}$$

where C are generic positive constants independent of k. Hence we have (9.12). The proofs of (9.10), (9.11), (9.13), (9.14) and (9.15) are similar, we omit them. This concludes the proof. □

Then Lemma 6.1 follows from Lemma 9.1, (9.1)-(9.9) and Proposition 2.1 of [33] by long but easy estimates.

9.2. Proof of (2.30)

First, we claim that

$$\int_{\mathbb{R}^n} |Q|^{p-1}(y) Z_0(y)dy = \int_{\mathbb{R}^n} U^{p-1}(y) Z_0(y)dy + O\left(\frac{1}{k^s}\right) \tag{9.16}$$

for some small $s > 0$. Indeed, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |Q|^{p-1}(y) Z_0(y) dy \\
&= \int_{\mathbb{R}^n} \left| U(y) - \sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y) \right|^{p-1} Z_0(y) dy \\
&= \int_{\mathbb{R}^n} U^{p-1}(y) Z_0(y) dy + (p-1) \int_{\mathbb{R}^n} U^{p-2}(y) \left| -\sum_{j=1}^{k} U_j(y) + \tilde{\phi}(y) \right| Z_0(y) dy \\
&\quad + O\left(\sum_{j=l}^{k} \int_{\mathbb{R}^n} |U_j|^{p-1}(y) Z_0(y) dy \right) + O\left(\int_{\mathbb{R}^n} |\tilde{\phi}(y)|^{p-1}(y) Z_0(y) dy \right) \\
&= \int_{\mathbb{R}^n} U^{p-1}(y) Z_0(y) dy + O\left(\sum_{j=1}^{k} \int_{\mathbb{R}^n} U^{p-2}(y) U_j(y) Z_0(y) dy \right) \\
&\quad + O\left(\sum_{j=l}^{k} \int_{\mathbb{R}^n} |U_j|^{p-1}(y) Z_0(y) dy \right) + O\left(\int_{\mathbb{R}^n} |\tilde{\phi}(y)|^{p-1}(y) Z_0(y) dy \right) \\
&\quad + O\left(\int_{\mathbb{R}^n} U^{p-2}(y) \left| \tilde{\phi}(y) \right| Z_0(y) dy \right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |U_j|^{p-1}(y) Z_0(y) dy = 4 \int_{\mathbb{R}^n} \frac{\zeta_k^2}{(\zeta_k^2 + |y - \xi_j|^2)^2} Z_0(y) dy \\
&= 4\zeta_k^{n-2} \int_{\mathbb{R}^n} \frac{1}{(1+|z|^2)^2} Z_0(\zeta_k z + \xi_j) dy \\
&\le C\zeta_k^{n-2} \int_{\mathbb{R}^n} \frac{1}{(1+|z|^2)^2} \frac{1}{\left|\zeta_k z + \xi_j\right|^{n-2}} dy \\
&= C\zeta_k^{n-2} \int_{|z| \le \frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^2} \frac{1}{\left|\zeta_k z + \xi_j\right|^{n-2}} dy \\
&\quad + C\zeta_k^{n-2} \int_{|z| \ge \frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^2} \frac{1}{\left|\zeta_k z + \xi_j\right|^{n-2}} dy \\
&= C\zeta_k^{n-2} \int_{|z| \le \frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^2} \frac{1}{\left|\xi_j\right|^{n-2}} \left(1 + O\left(\frac{\zeta_k z}{|\xi_j|} \right) \right) dy \\
&\quad + C\zeta_k^{n-2} \int_{|z| \ge \frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^2} \frac{1}{|\zeta_k z|^{n-2}} \left(1 + O\left(\frac{|\xi_j|}{\zeta_k z} \right) \right) dy \\
&= O\left(\zeta_k^2 \right) = O\left(\frac{1}{k^4} \right),
\end{aligned}
$$

$$\int_{\mathbb{R}^n} U^{p-2}(y)U_j(y)Z_0(y)dy \le C\int_{\mathbb{R}^n} \frac{\zeta_k^{\frac{n-2}{2}}}{(\zeta_k^2+|y-\xi_j|^2)^{\frac{n-2}{2}}}\frac{1}{(1+|y|)^4}dy$$

$$= C\zeta_k^{\frac{n}{2}+1}\int_{\mathbb{R}^n} \frac{1}{(1+|z|^2)^{\frac{n-2}{2}}}\frac{1}{(1+\left|\zeta_k z+\xi_j\right|)^4}dy$$

$$\le C\zeta_k^{\frac{n}{2}+1}\int_{\mathbb{R}^n} \frac{1}{(1+|z|^2)^{\frac{n-2}{2}}}\frac{1}{\left|\zeta_k z+\xi_j\right|^4}dy$$

$$= C\zeta_k^{\frac{n}{2}+1}\int_{|z|\le\frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^{\frac{n-2}{2}}}\frac{1}{\left|\zeta_k z+\xi_j\right|^4}dy$$

$$+ C\zeta_k^{\frac{n}{2}+1}\int_{|z|\ge\frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^{\frac{n-2}{2}}}\frac{1}{\left|\zeta_k z+\xi_j\right|^4}dy$$

$$= C\zeta_k^{\frac{n}{2}+1}\int_{|z|\le\frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^{\frac{n-2}{2}}}\frac{1}{\left|\xi_j\right|^4}\left(1+O\left(\frac{\zeta_k z}{|\xi_j|}\right)\right)dy$$

$$+ C\zeta_k^{\frac{n}{2}+1}\int_{|z|\ge\frac{1}{2\zeta_k}} \frac{1}{(1+|z|^2)^{\frac{n-2}{2}}}\frac{1}{|\zeta_k z|^4}\left(1+O\left(\frac{|\xi_j|}{\zeta_k z}\right)\right)dy$$

$$= O\left(\zeta_k^{\frac{n}{2}-1}\right),$$

$$\int_{\mathbb{R}^n} |\tilde\phi(y)|^{p-1}(y)Z_0(y)dy = O\left(k^{-\frac{n}{q}\frac{4}{n-2}}\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+2}}dy\right) = O\left(k^{-\frac{n}{q}\frac{4}{n-2}}\right),$$

$$\int_{\mathbb{R}^n} U^{p-2}(y)\left|\tilde\phi(y)\right|Z_0(y)dy = O\left(k^{-\frac{n}{q}}\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+2}}dy\right) = O\left(k^{-\frac{n}{q}}\right)$$

hold. Now (9.16) follows from the above estimates. Similarly, we have

$$\int_{\mathbb{R}^n} |Q|^{p-1}(y)Z_{0l}(y)dy = \int_{\mathbb{R}^n} |U_l|^{p-1}(y)Z_{0l}(y)dy + O\left(\frac{1}{k^{1+s}}\right)$$

and

$$\int_{\mathbb{R}^n} |Q|^{p-1}(y)Z_{1l}(y)dy = \int_{\mathbb{R}^n} |U_l|^{p-1}(y)Z_{1l}(y)dy + O\left(\frac{1}{k^{1+s}}\right).$$

Moreover,

$$-p\int_{\mathbb{R}^n} U^{p-1}(y)Z_0(y)dy = \frac{n-2}{2}\int_{\mathbb{R}^n} U^p(y)dy > 0,$$

$$-p\int_{\mathbb{R}^n} |U_l|^{p-1}(y)Z_{0l}(y)dy = \zeta_k^{\frac{n}{2}-1}\left(-p\int_{\mathbb{R}^n} U^{p-1}(y)Z_0(y)dy\right)$$

$$= \zeta_k^{\frac{n}{2}-1}\frac{n-2}{2}\int_{\mathbb{R}^n} U^p(y)dy,$$

$$\int_{\mathbb{R}^n} |U_l|^{p-1}(y)Z_{1l}(y)dy = 0.$$

Then from (9.1),

$$\begin{aligned}
&- p\int_{\mathbb{R}^n} |Q|^{p-1}(y)z_0(y)dy\\
&= -p\int_{\mathbb{R}^n} |Q|^{p-1}(y)Z_0(y)dy + p\sum_{l=1}^{k}\int_{\mathbb{R}^n} |Q|^{p-1}(y)Z_{0l}(y)dy\\
&\quad + p\sum_{l=1}^{k}\int_{\mathbb{R}^n} |Q|^{p-1}(y)Z_{1l}(y)dy - p\int_{\mathbb{R}^n}\pi_0(y)Z_{0l}(y)dy\\
&= -p\int_{\mathbb{R}^n} U^{p-1}(y)Z_0(y)dy + p\sum_{l=1}^{k}\int_{\mathbb{R}^n} |U_l|^{p-1}(y)Z_{0l}(y)dy\\
&\quad + p\sum_{l=1}^{k}\int_{\mathbb{R}^n} |U_l|^{p-1}(y)Z_{1l}(y)dy - p\int_{\mathbb{R}^n}\pi_0(y)Z_{0l}(y)dy + O\left(\frac{1}{k^s}\right)\\
&= (1+k\zeta_k^{\frac{n}{2}-1})\frac{n-2}{2}\int_{\mathbb{R}^n} U^p(y)dy + O\left(\frac{1}{k^s}\right)
\end{aligned}$$

which is positive when k is large. This proves $c_1 > 0$ when k_0 is large enough.

Finally, we prove $c_2 > 0$. From (9.1), $z_0(y) - \frac{D_{n,k}(2-|y|^2)}{(1+|y|^2)^{\frac{n}{2}}}$ can be written as

$$\begin{aligned}
&z_0(y) - \frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\\
&= \left(Z_0(y) - \frac{n-2}{2}\frac{\alpha_n(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)\\
&\quad - \sum_{l=1}^{k}\left(Z_{0l}(y) - \zeta_k^{\frac{n-2}{2}} f_n\frac{(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)\\
&\quad - \sum_{l=1}^{k} Z_{1l}(y) + \pi_0(y) - o(1)h_n\frac{(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}, \quad (k\to+\infty).
\end{aligned}$$

A direct computation yields that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(Z_0(y) - \frac{n-2}{2}\frac{\alpha_n(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)z_0(y)dy\\
&= \int_{\mathbb{R}^n}\left(Z_0(y) - \frac{n-2}{2}\frac{\alpha_n(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)Z_0(y)dy + O\left(\frac{1}{k}\right)\\
&= \alpha_n\frac{n-2}{2}\frac{\sqrt{\pi}2^{-n}\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n+1}{2}\right)} + O\left(\frac{1}{k}\right),
\end{aligned}$$

$$\sum_{l=1}^{k}\int_{\mathbb{R}^n}\left(Z_{0l}(y)-\zeta_k^{\frac{n-2}{2}}f_n\frac{(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)z_0(y)dy=O\left(\frac{1}{k}\right),$$

$$\sum_{l=1}^{k}\int_{\mathbb{R}^n}Z_{1l}(y)z_0(y)dy=O\left(\frac{1}{k}\right),$$

$$\sum_{l=1}^{k}\int_{\mathbb{R}^n}\left(\pi_0(y)-o(1)h_n\frac{(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)z_0(y)dy=O\left(\frac{1}{k}\right).$$

Therefore,

$$\int_{\mathbb{R}^n}\left(z_0(y)-\frac{D_{n,k}(2-|y|^2)}{\left(1+|y|^2\right)^{\frac{n}{2}}}\right)z_0(y)dy=\alpha_n\frac{n-2}{2}\frac{\sqrt{\pi}2^{-n}\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n+1}{2}\right)}+O\left(\frac{1}{k}\right)$$

which is positive when k is large enough. Hence $c_2>0$ if k_0 is sufficiently large.

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