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Motivic spectral sequence for relative homotopy K-theory

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Abstract. We construct a motivic spectral sequence for the relative homotopy invariant K-theory of a closed immersion of schemes $D \subset X$. The E_2-terms of this spectral sequence are the cdh-hypercohomology of a complex of equidimensional cycles.

Using this spectral sequence, we obtain a cycle class map from the relative motivic cohomology group of 0-cycles to the relative homotopy invariant K-theory. For a smooth scheme X and a divisor $D \subset X$, we construct a canonical homomorphism from the Chow groups with modulus $\mathrm{CH}^i(X|D)$ to the relative motivic cohomology groups $H^{2i}(X|D, \mathbb{Z}(i))$ appearing in the above spectral sequence. This map is shown to be an isomorphism when X is affine and $i = \dim(X)$.

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1. Introduction

In this text, we construct an Atiyah-Hirzebruch type spectral sequence for the relative algebraic K-theory of a closed immersion of smooth schemes, and relate the E_2-terms of this spectral sequence with the known Chow groups with modulus in various cases. This section provides the background of the problem, a summary of main results, their statements and outline of proofs.

1.1. The background

Since the advent of higher algebraic K-theory of rings and schemes by Quillen, the goal has been to search for tools for computing these higher K-groups. In his seminal work [4], Bloch introduced the theory of higher Chow groups of schemes. He showed that these higher Chow groups rationally coincide with the algebraic K-groups of schemes. It was later shown by Bloch-Lichtenbaum [7], Friedlander-Suslin [11] and Levine [26] that there exists an Atiyah-Hirzebruch type spectral

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sequence whose E_2-terms are Bloch's higher Chow groups and which abuts to the algebraic K-theory of a smooth scheme.

After the introduction of motivic homotopy theory by Voevodsky and his coauthors, it was subsequently observed by Voevodsky that the motivic T-spectra in the motivic stable homotopy category over a field (*e.g.*, the algebraic K-theory spectrum KGL) could be understood well via their slice filtration. Voevodsky [28, Chapter 19] also showed that the motivic cohomology groups appearing in the spectral sequence resulting from the slice filtration for the algebraic K-theory spectrum coincide with Bloch's higher Chow groups. Since then, Voevodsky's slice filtration has become a very powerful tool to compute algebraic K-theory of smooth schemes.

One immediate question that arises out of the works of Friedlander-Suslin and Voevodsky is if it is possible to construct a convergent spectral sequence which computes the relative algebraic K-theory $K(X, D)$ for a closed immersion of smooth schemes $D \subset X$, and which reduces to the earlier spectral sequence when $D = \emptyset$. The first problem that one faces in order to answer this question is to define a relative motivic cohomology theory which would constitute the E_2-terms of such a spectral sequence. Based on the theories of additive higher Chow groups by Bloch and Esnault [6] and Chow groups with modulus by Kerz and Saito [19], a theory of higher Chow groups with modulus was introduced by Binda and Saito in [3]. It is expected that these higher Chow groups with modulus (or some refined version of them) constitute the E_2-terms of a spectral sequence which would converge to the relative algebraic K-theory $K(X, D)$ whenever D is an effective Cartier divisor in a smooth scheme X over a field.

1.2. Summary of main results

Despite the introduction of higher Chow groups with modulus, connecting these groups to relative algebraic K-theory, and in particular, constructing the desired spectral sequence, remains one of the challenging current problems in algebraic K-theory. This paper is an attempt in this direction. Here, we do not construct a spectral sequence whose E_2-terms are the higher Chow groups with modulus. What we do instead is to expand and feed the machinery of Voevodsky's slice filtration into the setting of relative K-theory. What results is a strongly convergent spectral sequence abutting to the relative algebraic K-theory of a closed immersion of smooth schemes $D \subset X$. More generally, we show that such a spectral sequence exists for the relative homotopy invariant K-theory $KH(X, D)$ for any closed immersion of schemes $D \subset X$.

Given a closed immersion of a divisor D inside a smooth scheme X, we show that the E_2-terms of our spectral sequence can be described as the *cdh*-hypercohomology of a subcomplex of the complex of equidimensional cycles of Friedlander-Suslin-Voevodsky on a scheme S_X. This scheme is obtained from gluing two copies of X along D. If X is projective, these E_2-terms are shown to coincide with the motivic cohomology with compact support [12] of the complement of D in X.

Another aspect of our spectral sequence is its degeneration with rational coefficients. An important consequence of this is that it allows us to provide a complete description of the rational relative K-theory of projective schemes in terms of the motivic cohomology with compact support. Furthermore, it allows us to prove the Grothendieck Riemann-Roch theorem for relative K-theory of smooth schemes (see Theorem 5.6). Prior to this work, it was not known if the relative K-theory of a smooth pair of projective schemes could be described in terms of motivic cohomology with compact support.

Having this spectral sequence in hand, what remains to connect Chow groups with modulus with relative K-theory of a smooth divisor D inside a smooth scheme X is to show the agreement between the E_2-terms of our spectral sequence and the Chow groups with modulus. In an attempt in this direction, we construct a canonical homomorphism from the Chow groups with modulus to the E_2-terms of our spectral sequence. We then show that for 0-cycles, this map is in fact an isomorphism when X is affine. This provides some evidence that the spectral sequence constructed in this paper might be the answer to the question of relating Chow groups with modulus with relative algebraic K-theory of smooth pairs.

1.3. Statements of main results

The results we prove can be summarized as follows. The exact hypothesis of each statement, notation and terms used in these results will be explained and made precise at appropriate places in this text.

Let k be a perfect field and let Λ denote the ring $\mathbb{Z}$ if k admits resolution of singularities or, $\mathbb{Z}[\frac{1}{p}]$ if k has characteristic $p > 0$. For an Abelian group A, let $A_\Lambda = A \otimes_{\mathbb{Z}} \Lambda$. Given a morphism of schemes $f : D \to X$ over k, let $KH(X, D)$ denote the homotopy fiber of the map of Weibel's homotopy K-theory spectra $f^* : KH(X) \to KH(D)$. Note that $KH(X, D)$ coincides with the relative algebraic K-theory spectrum $K(X, D)$ if X and D are smooth.

Recall from [28, Lecture 16] that the presheaf of Abelian groups $z_{equi}(\mathbb{A}^q_k, 0)$ on the category of smooth schemes over k is defined by letting $z_{equi}(\mathbb{A}^q_k, 0)(U)$ be the free Abelian group generated by the closed and irreducible subschemes $Z \subsetneq U \times \mathbb{A}^q_k$ which are dominant and equidimensional of relative dimension zero over a component of U. Let $C_* z_{equi}(\mathbb{A}^q_k, 0)$ denote the chain complex of presheaves of Abelian groups associated, via the Dold-Kan correspondence, to the simplicial presheaf given by $C_n z_{equi}(\mathbb{A}^q_k, 0)(U) = z_{equi}(\mathbb{A}^q_k, 0)(U \times \Delta^n_k)$. Given a smooth scheme X over k and an effective Cartier divisor $D \subset X$, let S_X denote the scheme obtained by gluing two copies of X along D and let $\nabla : S_X \to X$ be the fold map. We let $\Lambda_{X|D}(q)[2q]$ denote the complex of sheaves on the cdh-site of X given by $\Lambda_{X|D}(q)[2q] = \mathrm{Ker}((\nabla_*(C_* z_{equi}(\mathbb{A}^q_k, 0)|_{(S_X)_{cdh}})) \to C_* z_{equi}(\mathbb{A}^q_k, 0)|_{X_{cdh}})$.

The relative motivic cohomology $H^a(X|D, \Lambda(b))$ where $D \subset X$ is a closed subscheme, is defined in the motivic stable homotopy category in terms of maps from the mapping cone of $D \to X$ into the Eilenberg-MacLane $H\Lambda$ spectrum representing motivic cohomology, see Subsection 4.2.1, Propositions 4.3, 4.7.

Theorem 1.1. *Let X be a separated scheme of finite type over k and let $D \subset X$ be a closed subscheme. Then the following hold:*

(1) *There exists a strongly convergent spectral sequence*

$$E_2^{a,b} = H^{a-b}(X|D, \Lambda(-b)) \Rightarrow KH_{-a-b}(X, D)_\Lambda.$$

This spectral sequence degenerates with rational coefficients;

(2) *The spectral sequence exists with integral coefficients if X and Y are regular;*

(3) *If X is regular and $D \subset X$ is a Cartier divisor, then $H^a(X|D, \Lambda(b)) \cong \mathbb{H}^{a-2b}_{cdh}(X, \Lambda_{X|D}(b)[2b])$;*

(4) *If X is projective over k, then $H^a(X|D, \Lambda(b)) = H^a_c(X \setminus D, \Lambda(b))$ is the Friedlander-Voevodsky motivic cohomology with compact support of $X \setminus D$.*

Theorem 1.2. *If X has dimension d, then:*

(1) *There exists a cycle class map*

$$cyc_i : H^{2d+i}(X|D, \Lambda(d+i)) \to KH_i(X, D)_\Lambda;$$

(2) *If k admits resolution of singularities, X is regular and $D \subset X$ is an effective Cartier divisor, there exist Chern class maps*

$$c_{X|D,a,b} : KH_a(X, D)_\Lambda \to H^{2b-a}(X|D, \Lambda(q))$$

which are functorial in the pair (X, D).

Theorem 1.3. *If X is regular, $D \subset X$ an effective Cartier divisor and $i \geq 0$ an integer, there exists a homomorphism*

$$\lambda_{X|D} : \mathrm{CH}^i(X|D)_\Lambda \to H^{2i}(X|D, \Lambda(i)).$$

If k is furthermore algebraically closed, X is affine of dimension d and D is regular, then there is an isomorphism $\lambda_{X|D} : \mathrm{CH}^d(X|D) \xrightarrow{\cong} H^{2d}(X|D, \mathbb{Z}(d))$.

1.4. Outline of proofs

We end this section with a brief outline of our proofs. The idea of the construction of the spectral sequence for relative K-theory came from our previous work [25], where such a spectral sequence was constructed for the KH-theory of singular schemes. Extending our techniques, we feed the machinery of the slice filtration into the relative setting. By mapping the mapping cone of a closed immersion of schemes into the slice filtration of KGL and generalizing some results of [25] to the relative case, we obtain the desired spectral sequence and its rational degeneration.

In order to get a tower for the relative K-theory spectrum leading to the spectral sequence, we need to use [26] which compares Voevodsky's slice filtration with

Levine's homotopy coniveau tower. This yields a tower for relative K-theory whose layers are identified with the relative motivic cohomology.

The remaining part of this text is devoted to showing a direct relation between the E_2-terms of the spectral sequence with the Chow groups with modulus. In Section 7, we construct a homomorphism from the higher Chow groups with modulus to the E_2-terms by again using the comparison between the slice and the homotopy coniveau tower for K-theory. We show in the final section that these maps are isomorphisms in the 0-cycle range for affine schemes. This critically uses the affine Roitman torsion theorem of [21] as the main input.

2. Review of motivic spaces and algebraic cycles

In this section we fix our notation and provide a limited recollection of some definitions and known results related to the stable homotopy category of smooth schemes over a base scheme. We recall the definitions of cycles with modulus on smooth schemes and Levine-Weibel Chow groups of singular schemes. This Chow group of singular schemes will play a crucial role in our comparison between Chow group of 0-cycles with modulus and relative motivic cohomology.

2.1. Definitions and notation

We will write k for a perfect field of exponential characteristic p (in some cases we will assume that the field k admits resolution of singularities [12, Definition 3.4]). Let $\mathbf{Sch}_k$ be the category of separated schemes of finite type over k and $\mathbf{Sm}_k$ be the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over k. If $X \in \mathbf{Sch}_k$, we will write $\mathbf{Sm}_X$ for the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over X. Let $(\mathbf{Sm}_k)_{Nis}$ (respectively $(\mathbf{Sm}_X)_{Nis}$, $(\mathbf{Sch}_k)_{cdh}$, $(\mathbf{Sch}_k)_{Nis}$) denote $\mathbf{Sm}_k$ equipped with the Nisnevich topology (respectively $\mathbf{Sm}_X$ equipped with the Nisnevich topology, $\mathbf{Sch}_k$ equipped with the cdh-topology, $\mathbf{Sch}_k$ equipped with the Nisnevich topology). To simplify the notation we will write $X \times Y$ for $X \times_{\mathrm{Spec}\,(k)} Y$.

Let $\mathcal{M}$ (respectively $\mathcal{M}_X$, $\mathcal{M}_{cdh}$) be the category of pointed simplicial presheaves on $\mathbf{Sm}_k$ (respectively $\mathbf{Sm}_X$, $\mathbf{Sch}_k$) equipped with the motivic model structure described in [16] considering the Nisnevich topology on $\mathbf{Sm}_k$ (respectively Nisnevich topology on $\mathbf{Sm}_X$, cdh-topology on $\mathbf{Sch}_k$) and the affine line $\mathbb{A}^1_k$ as an interval. A simplicial presheaf will often be called a *motivic space*.

Let T in $\mathcal{M}$ (respectively $\mathcal{M}_X$, $\mathcal{M}_{cdh}$) be the pointed simplicial presheaf represented by $S^1_s \wedge S^1_t$, where S^1_t is $\mathbb{A}^1_k \setminus \{0\}$ (respectively $\mathbb{A}^1_X \setminus \{0\}$, $\mathbb{A}^1_k \setminus \{0\}$) pointed by 1, and S^1_s denotes the simplicial circle. Given an arbitrary integer $r \geq 1$, let S^r_s (respectively S^r_t) denote the iterated smash product of S^1_s (respectively S^1_t) with r-factors: $S^1_s \wedge \cdots \wedge S^1_s$ (respectively $S^1_t \wedge \cdots \wedge S^1_t$); $S^0_s = S^0_t$ will be by definition equal to the pointed simplicial presheaf represented by the base scheme $\mathrm{Spec}\,(k)$ (respectively X, $\mathrm{Spec}\,(k)$).

Let $Spt(\mathcal{M})$ (respectively $Spt(\mathcal{M}_X)$, $Spt(\mathcal{M}_{cdh})$) denote the category of symmetric T-spectra on $\mathcal{M}$ (respectively $\mathcal{M}_X$, $\mathcal{M}_{cdh}$) equipped with the motivic model structure defined in [15, 8.7]. We will write $\mathcal{SH}$ (respectively $\mathcal{SH}_X$, $\mathcal{SH}_{cdh}$) for the homotopy category of $Spt(\mathcal{M})$ (respectively $Spt(\mathcal{M}_X)$, $Spt(\mathcal{M}_{cdh})$) which is a tensor triangulated category. For any two integers $m, n \in \mathbb{Z}$, let $\Sigma^{m,n}$ denote the automorphism $\Sigma_s^{m-n} \circ \Sigma_t^n : \mathcal{SH} \to \mathcal{SH}$ (this also makes sense in $\mathcal{SH}_X$ and $\mathcal{SH}_{cdh}$). We will write Σ_T^n for $\Sigma^{2n,n}$, and $E \wedge F$ for the smash product of E, $F \in \mathcal{SH}$ (respectively $\mathcal{SH}_X$, $\mathcal{SH}_{cdh}$).

Given a simplicial presheaf A, we will write A_+ for the pointed simplicial presheaf obtained by adding a disjoint base point (isomorphic to the base scheme) to A. For any $B \in \mathcal{M}$, let $\Sigma_T^\infty(B)$ denote the object $(B, T \wedge B, \cdots) \in Spt(\mathcal{M})$. This functor makes sense for objects in $\mathcal{M}_{cdh}$ and $\mathcal{M}_X$ as well.

If $F : \mathcal{A} \to \mathcal{B}$ is a functor with right adjoint $G : \mathcal{B} \to \mathcal{A}$, we shall say that $(F, G) : \mathcal{A} \to \mathcal{B}$ is an adjunction. We will use the following notation in all the categories under consideration: $*$ will denote the terminal object, and $\cong$ will denote that a map (respectively functor) is an isomorphism (respectively equivalence of categories).

Throughout this paper, Λ will denote the ring $\mathbb{Z}$ if k admits resolution of singularities, or the ring $\mathbb{Z}[\frac{1}{p}]$ otherwise. For any Abelian group M, we shall let $M_\Lambda = M \underset{\mathbb{Z}}{\otimes} \Lambda$.

2.2. Some known results in motivic homotopy theory

Let $X \in \mathbf{Sch}_k$ and let $v : X \to \operatorname{Spec}(k)$ denote the structure map. Let $\pi : (\mathbf{Sch}_k)_{cdh} \to (\mathbf{Sm}_k)_{Nis}$ be the canonical continuous map of sites. We will write $(\pi^*, \pi_*) : \mathcal{M} \to \mathcal{M}_{cdh}$, $(v^*, v_*) : \mathcal{M} \to \mathcal{M}_X$ for the adjunctions induced by π, v respectively. We will also consider the morphism of sites $\pi_X : (\mathbf{Sch}_k)_{cdh} \to (\mathbf{Sm}_X)_{Nis}$ and the corresponding adjunction $(\pi_X^*, \pi_{X*}) : \mathcal{M}_X \to \mathcal{M}_{cdh}$. The following result can be found in [25, 2.4 and 2.9].

Proposition 2.1. *The adjunctions* $(\pi^*, \pi_*) : \mathcal{M} \to \mathcal{M}_{cdh}$, $(v^*, v_*) : \mathcal{M} \to \mathcal{M}_X$, $(\pi_X^*, \pi_{X*}) : \mathcal{M}_X \to \mathcal{M}_{cdh}$ *are Quillen adjunctions.*

We further conclude from Proposition 2.1 and [15, Theorem 9.3] the following:

Proposition 2.2. *The pairs* (π^*, π_*), (v^*, v_*) *and* (π_X^*, π_{X*}) *are Quillen adjunctions between stable model categories.*

We deduce from Proposition 2.2 that there are pairs of adjoint functors $(\mathbf{L}\pi^*, \mathbf{R}\pi_*) : \mathcal{SH} \to \mathcal{SH}_{cdh}$, $(\mathbf{L}v^*, \mathbf{R}v_*) : \mathcal{SH} \to \mathcal{SH}_X$ and $(\mathbf{L}\pi_X^*, \mathbf{R}\pi_{X*}) : \mathcal{SH}_X \to \mathcal{SH}_{cdh}$ between the various stable homotopy categories of motivic T-spectra. We observe that for $a \geq b \geq 0$, the suspension functor $\Sigma^{a,b}$ in $\mathcal{SH}$ (respectively $\mathcal{SH}_X$, $\mathcal{SH}_{cdh}$) is the derived functor of the left Quillen functor $E \mapsto S_s^{a-b} \wedge S_t^b \wedge E$ in $Spt(\mathcal{M})$ (respectively $Spt(\mathcal{M}_X)$, $Spt(\mathcal{M}_{cdh})$). Since the functors π^*, v^*, π_X^* are simplicial and symmetric monoidal, we deduce that they commute with the suspension functors $\Sigma^{m,n}$, *i.e.*, for every $m, n \in \mathbb{Z}$: $\mathbf{L}\pi^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \mathbf{L}\pi^*(-)$, $\mathbf{L}v^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \mathbf{L}v^*(-)$ and $\mathbf{L}\pi_X^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \mathbf{L}\pi_X^*(-)$.

2.3. Higher Chow groups with modulus

For $n \geq 1$, let $\square^n$ denote the scheme $\mathbb{A}^n_k \cong (\mathbb{P}^1_k \setminus \{\infty\})^n$. Let $(y_1, \cdots, y_n)$ denote the coordinate of a point on $\square^n$. We shall denote the scheme $(\mathbb{P}^1_k)^n$ by $\overline{\square}^n$. For $1 \leq i \leq n$, let $F^\infty_{n,i}$ denote the closed subscheme of $\overline{\square}^n$ given by the equation $\{y_i = \infty\}$. We shall denote the divisor $\sum_{i=1}^{n} F^\infty_{n,i}$ by F^∞_n.

Let X be a smooth quasi-projective scheme of dimension $d \geq 0$ over k and let $D \subset X$ be an effective Cartier divisor. For $r \in \mathbb{Z}$ and $n \geq 0$, let $\underline{z}_r(X|D, n)$ be the free Abelian group on integral closed subschemes V of $X \times \square^n$ of dimension $r+n$ satisfying the following conditions.
(1) (Face condition) For each face F of $\square^n$, V intersects $X \times F$ properly:

$$\dim_k(V \cap (X \times F)) \leq r + \dim_k(F), \text{ and}$$

(2) (Modulus condition) V is a cycle with modulus D relative to F^∞_n:

$$\nu^*(D \times \overline{\square}^n) \leq \nu^*(X \times F^\infty_n),$$

where $\overline{V}$ is the closure of V in $X \times \overline{\square}^n$ and $\nu : \overline{V}^N \to \overline{V} \to X \times \overline{\square}^n$ is the composite map from the normalization of $\overline{V}$. We let $\underline{z}_r(X|D, n)_{\text{degn}}$ denote the subgroup of $\underline{z}_r(X|D, n)$ generated by cycles which are pull-back of some cycles under various projections $X \times \square^n \to X \times \square^m$ with $m < n$.

Definition 2.3. The *cycle complex with modulus* $(z_r(X|D, \bullet), d)$ of X in dimension r and with modulus D is the non-degenerate complex associated to the cubical Abelian group $\underline{n} \mapsto \underline{z}_r(X|D, n)$, *i.e.*,

$$z_r(X|D, n) := \frac{\underline{z}_r(X|D, n)}{\underline{z}_r(X|D, n)_{\text{degn}}}.$$

The homology $\mathrm{CH}_r(X|D, n) := H_n(z_r(X|D, \bullet))$ is called a *higher Chow group* of X with modulus D. Sometimes, we also write it as the Chow group of the *modulus pair* (X, D). If X has dimension d, we write $\mathrm{CH}^r(X|D, n) = \mathrm{CH}_{d-r}(X|D, n)$. We shall often write $\mathrm{CH}^r(X|D, 0)$ as $\mathrm{CH}^r(X|D)$. We refer to [24] for further details on this definition. The reader should note that $\mathrm{CH}_r(X|D, n)$ coincides with the usual higher Chow group of Bloch $\mathrm{CH}_r(X, n)$ if $D = \emptyset$.

2.4. Levine-Weibel Chow group of singular schemes

We recall the definition of the cohomological Chow group of 0-cycles for singular schemes from [2] and [27]. Let X be a reduced quasi-projective scheme of dimension $d \geq 1$ over k. Let X_{sing} and X_{reg} respectively denote the loci of the singular and the regular points of X. We let X^N denote the normalization of X. Given a nowhere dense closed subscheme $Y \subset X$ such that $X_{\text{sing}} \subseteq Y$ and no component of X is contained in Y, we let $\mathcal{Z}_0(X, Y)$ denote the free Abelian group on the closed points of $X \setminus Y$. We write $\mathcal{Z}_0(X, X_{\text{sing}})$ in short as $\mathcal{Z}_0(X)$.

Definition 2.4. Let C be a pure dimension one reduced scheme in $\mathbf{Sch}_k$. We shall say that a pair (C, Z) is *a good curve relative to* X if there exists a finite morphism $\nu \colon C \to X$ and a closed proper subscheme $Z \subsetneq C$ such that the following hold.

(1) No component of C is contained in Z;
(2) $\nu^{-1}(X_{\rm sing}) \cup C_{\rm sing} \subseteq Z$.
(3) ν is local complete intersection at every point $x \in C$ such that $\nu(x) \in X_{\rm sing}$.

Let (C, Z) be a good curve relative to X and let $\{\eta_1, \cdots, \eta_r\}$ be the set of generic points of C. Let $\mathcal{O}_{C,Z}$ denote the semilocal ring of C at $S = Z \cup \{\eta_1, \cdots, \eta_r\}$. Let $k(C)$ denote the ring of total quotients of C and write $\mathcal{O}^{\times}_{C,Z}$ for the group of units in $\mathcal{O}_{C,Z}$. Notice that $\mathcal{O}_{C,Z}$ coincides with $k(C)$ if $|Z| = \emptyset$. As C is Cohen-Macaulay, $\mathcal{O}^{\times}_{C,Z}$ is the subgroup of $k(C)^{\times}$ consisting of those f which are regular and invertible in the local rings $\mathcal{O}_{C,x}$ for every $x \in Z$.

Given any $f \in \mathcal{O}^{\times}_{C,Z} \hookrightarrow k(C)^{\times}$, we denote by $\operatorname{div}_C(f)$ (or $\operatorname{div}(f)$ in short) the divisor of zeros and poles of f on C, which is defined as follows. If $C_1, \dots, C_r$ are the irreducible components of C, and f_i is the factor of f in $k(C_i)$, we set $\operatorname{div}(f)$ to be the 0-cycle $\sum_{i=1}^r \operatorname{div}(f_i)$, where $\operatorname{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the classical sense. As f is an invertible regular function on C along Z, $\operatorname{div}(f) \in \mathcal{Z}_0(C, Z)$.

By definition, given any good curve (C, Z) relative to X, we have a push-forward map $\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X)$. We shall write $\mathcal{R}_0(C, Z, X)$ for the subgroup of $\mathcal{Z}_0(X)$ generated by the set $\{\nu_*(\operatorname{div}(f)) | f \in \mathcal{O}^{\times}_{C,Z}\}$. Let $\mathcal{R}_0(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the image of the map $\mathcal{R}_0(C, Z, X) \to \mathcal{Z}_0(X)$, where (C, Z) runs through all good curves relative to X. We let $\mathrm{CH}_0(X) = \frac{\mathcal{Z}_0(X)}{\mathcal{R}_0(X)}$.

If we let $\mathcal{R}^{LW}_0(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $\nu : C \to X$ is a closed immersion, then the resulting quotient group $\mathcal{Z}_0(X)/\mathcal{R}^{LW}_0(X)$ is denoted by $\mathrm{CH}^{LW}_0(X)$. Such curves on X are called the *Cartier curves*. There is a canonical surjection $\mathrm{CH}^{LW}_0(X) \twoheadrightarrow \mathrm{CH}_0(X)$. The Chow group $\mathrm{CH}^{LW}_0(X)$ was discovered by Levine and Weibel [27] in an attempt to describe the Grothendieck group of a singular scheme in terms of algebraic cycles. The modified version $\mathrm{CH}_0(X)$ was introduced in [2].

2.5. The double and its Chow group

Let X be a smooth quasi-projective scheme of dimension d over k and let $D \subset X$ be an effective Cartier divisor. Recall from [2, Section 2.1] that the double of X along D is a quasi-projective scheme $S(X, D) = X \amalg_D X$ so that

$$\begin{array}{ccc} D & \xrightarrow{\iota} & X \\ \iota\downarrow & & \downarrow\iota_+ \\ X & \xrightarrow[\iota_-]{} & S(X,D) \end{array} \tag{2.1}$$

is a co-Cartesian square in $\mathbf{Sch}_k$. In particular, the identity map of X induces a finite map $\nabla : S(X, D) \to X$ such that $\nabla \circ \iota_\pm = \mathrm{Id}_X$ and $\pi = \iota_+ \amalg \iota_- : X \amalg X \to S(X, D)$ is the normalization map. We let $X_\pm = \iota_\pm(X) \subset S(X, D)$ denote the two irreducible components of $S(X, D)$. We shall often write $S(X, D)$ as S_X when the divisor D is understood. S_X is a reduced quasi-projective scheme whose singular locus is $D_{\mathrm{red}} \subset S_X$. It is projective whenever X is so. It follows from [22, Lemma 2.2] that (2.1) is also a Cartesian square.

It is clear that the map $\mathcal{Z}_0(S_X, D) \xrightarrow{(\iota_+^*, \iota_-^*)} \mathcal{Z}_0(X_+, D) \oplus \mathcal{Z}_0(X_-, D)$ is an isomorphism. Notice also that there are push-forward inclusion maps $p_{\pm *}\colon \mathcal{Z}_0(X, D) \to \mathcal{Z}_0(S_X, D)$ such that $\iota_+^* \circ p_{+*} = \mathrm{Id}$ and $\iota_+^* \circ p_{-*} = 0$. The fundamental result that connects the 0-cycles with modulus on X and 0-cycles on S_X is the following.

Theorem 2.5 ([2, Theorem 1.12]). *Let X be a smooth quasi-projective scheme over k and let $D \subset X$ be an effective Cartier divisor. Then there is a split short exact sequence*

$$0 \to \mathrm{CH}_0(X|D) \xrightarrow{p_{+*}} \mathrm{CH}_0(S_X) \xrightarrow{\iota_-^*} \mathrm{CH}_0(X) \to 0.$$

3. Relative homotopy invariant K-theory

Let Spt_{S^1} be the category of Bousfield-Friedlander S^1-spectra equipped with the stable model structure [26, Section 1.2], and $\mathcal{H}(S^1)$ its homotopy category. We will write $Spt_{S^1}(\mathbf{Sch}_k)$ for the category of presheaves on $\mathbf{Sch}_k$ with values in Spt_{S^1}. We let $K \in Spt_{S^1}(\mathbf{Sch}_k)$ denote the Thomason-Trobaugh algebraic K^B-theory spectrum, and let $KH \in Spt_{S^1}(\mathbf{Sch}_k)$ denote Weibel's homotopy invariant K-theory spectrum [39], [8, 2.8]. Let $K_n(X)$, $n \in \mathbb{Z}$ be the n-th homotopy group of the spectrum $K(X)$. We use similar notation for the homotopy groups of $KH(X)$.

Definition 3.1. Let $f : Y \to X$ be a map in $\mathbf{Sch}_k$.

1. [(1)] We will write $K(f)$ (respectively $KH(f)$) for the homotopy fiber of the map $f^* : K(X) \to K(Y)$ (respectively $f^* : KH(X) \to KH(Y)$) induced by f in Spt_{S^1}. If f is a closed (respectively open) immersion, we will write $K(X, Y)$ (respectively $K^{X \setminus Y}(X)$) for $K(f)$. Analogous notation will be used for $KH(f)$ too. We will write $K_n(f)$ (respectively $KH_n(f)$) for the n-th homotopy group of the spectrum $K(f)$ (respectively $KH(f)$);
2. [(2)] We will write M_f for the mapping cone of f in $\mathcal{M}_{cdh}$. Namely, let X_+, Y_+ denote the simplicial presheaves represented by X, Y with a disjoint base point. We then factor f in $\mathcal{M}_{cdh}$:

$$\begin{array}{ccc} Y_+ & \xrightarrow{f} & X_+ \\ & \searrow^{c_f} & \uparrow{\scriptstyle w_f} \\ & & A_f \end{array}$$

where c_f (respectively w_f) is a cofibration (respectively trivial fibration) in $\mathcal{M}_{cdh}$ and let M_f be the pushout in $\mathcal{M}_{cdh}$ of the diagram: $* \leftarrow Y_+ \xrightarrow{c_f} A_f$. If f is a closed immersion, M_f is canonically identified with the quotient X/Y in $\mathcal{M}_{cdh}$.

Since the cofibrant replacement functor is functorial in $\mathcal{M}_{cdh}$, we deduce that every commutative diagram in $\mathbf{Sch}_k$:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array} \tag{3.1}$$

induces a commutative diagram in $\mathcal{M}_{cdh}$ which is natural in g and g' (3.1):

$$\begin{array}{ccccc} Y'_+ & \xrightarrow{c_{f'}} & A_{f'} & \longrightarrow & M_{f'} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle (g,g')_A} & & \downarrow{\scriptstyle (g,g')_M} \\ Y_+ & \xrightarrow{c_f} & A_f & \longrightarrow & M_f \end{array} \tag{3.2}$$

where the rows are cofiber sequences in $\mathcal{M}_{cdh}$.

3.1. Voevodsky's KGL spectrum

For any Noetherian separated scheme X of finite Krull dimension, the motivic T-spectrum $\mathrm{KGL}_X \in Spt(\mathcal{M}_X)$ was defined by Voevodsky (see [36, Section 6.2]). It represents algebraic K-theory of objects in $\mathbf{Sm}_X$ if X is regular. It was later shown by Cisinski [8] that for X not necessarily regular, KGL_X represents Weibel's homotopy invariant K-theory $KH_*(Y)$ for $Y \in \mathbf{Sm}_X$. We will write $\mathrm{KGL} \in Spt(\mathcal{M}_{cdh})$ for $\mathbf{L}\pi^*\mathrm{KGL}_k$.

We will write $Spt_{S^1}(\mathcal{M}_{cdh})$ for the category of symmetric S^1-spectra on $\mathcal{M}_{cdh}$ equipped with the motivic model structure defined in [15, 8.7], and $\mathcal{SH}^{S^1}_{cdh}$ for its homotopy category which is a tensor triangulated category. We shall denote the homotopy category of $Spt_{S^1}(\mathcal{M})$ by $\mathcal{SH}^{S^1}_{nis}$. There is a Quillen adjunction [35, §2]:

$$(\Sigma_t^\infty, \Omega_t^\infty) : Spt_{S^1}(\mathcal{M}_{cdh}) \to Spt(\mathcal{M}_{cdh}). \tag{3.3}$$

Consider the functor of global sections on $X \in \mathbf{Sch}_k$, $\Gamma_X : Spt_{S^1}(\mathcal{M}_{cdh}) \to Spt_{S^1}$, which admits a left adjoint $X_+ \otimes - : Spt_{S^1} \to Spt_{S^1}(\mathcal{M}_{cdh})$, $(E^0, E^1, \ldots) \mapsto (X_+ \wedge E^0, X_+ \wedge E^1, \ldots)$. Since $X_+ \otimes -$ is a left Quillen functor, we obtain a Quillen adjunction:

$$(X_+ \otimes -, \Gamma_X) : Spt_{S^1} \to Spt_{S^1}(\mathcal{M}_{cdh}). \tag{3.4}$$

We will write $\mathbf{R}\Gamma_X : \mathcal{SH}^{S^1}_{cdh} \to \mathcal{H}(S^1)$ (respectively $\mathbf{R}\Omega_t^\infty : \mathcal{SH}_{cdh} \to \mathcal{SH}^{S^1}_{cdh}$) for the right derived functor of Γ_X (respectively Ω_t^∞), $\mathbf{RHom}$ for the internal Hom-functor in $\mathcal{SH}^{S^1}_{cdh}$ and $\Sigma^\infty_{S^1} : \mathcal{M}_{cdh} \to Spt_{S^1}(\mathcal{M}_{cdh})$, $A \mapsto (A, S^1 \wedge A, \ldots)$ for the infinite suspension functor. When $X = \mathrm{Spec}\,(k)$, we will simply write $\mathbf{R}\Gamma_k$.

Remark 3.2. Since $X_+ \otimes -$ is naturally isomorphic to the composition $\Sigma^\infty_{S^1} X_+ \wedge (\mathrm{Spec}\,(k)_+ \otimes -)$, we deduce that $\mathbf{R}\Gamma_X$ is naturally isomorphic to

$$\mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} X_+, -).$$

By construction and Definition 3.1(2), there is a commutative diagram in $Spt_{S^1}(\mathcal{M}_{cdh})$:

$$\begin{array}{ccccc} \Sigma^\infty_{S^1} Y_+ & \longrightarrow & \Sigma^\infty_{S^1} A_f & \longrightarrow & \Sigma^\infty_{S^1} M_f \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle w_f} & & & \\ \Sigma^\infty_{S^1} X_+, & & & & \end{array}$$

where the top row is a cofiber sequence and w_f is a S^1-stable weak equivalence. So this induces a commutative diagram where the solid arrows form a fiber sequence in $\mathcal{SH}^{S^1}_{cdh}$ (recall that KGL is by definition $\mathbf{L}\pi^*\mathrm{KGL}_k$) and w_f^* is an isomorphism (see the last paragraph of [36, page 592]):

$$\begin{array}{ccc} \mathbf{RHom}(\Sigma^\infty_{S^1} M_f, \mathbf{R}\Omega^\infty_t \mathrm{KGL}) & \longrightarrow & \mathbf{RHom}(\Sigma^\infty_{S^1} A_f, \mathbf{R}\Omega^\infty_t \mathrm{KGL}) \\ & \overset{w_f^*}{\nearrow} & \downarrow \\ \mathbf{RHom}(\Sigma^\infty_{S^1} X_+, \mathbf{R}\Omega^\infty_t \mathrm{KGL}) & \underset{f^*}{\dashrightarrow} & \mathbf{RHom}(\Sigma^\infty_{S^1} Y_+, \mathbf{R}\Omega^\infty_t \mathrm{KGL}). \end{array}$$

Thus, applying $\mathbf{R}\Gamma_k$, we obtain the commutative diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} M_f, \mathbf{R}\Omega^\infty_t \mathrm{KGL}) & \longrightarrow & \mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} A_f, \mathbf{R}\Omega^\infty_t \mathrm{KGL}) \\ & \overset{w_f^*}{\nearrow} & \downarrow \\ \mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} X_+, \mathbf{R}\Omega^\infty_t \mathrm{KGL}) & \underset{f^*}{\dashrightarrow} & \mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} Y_+, \mathbf{R}\Omega^\infty_t \mathrm{KGL}). \end{array} \tag{3.5}$$

where the solid arrows form a fiber sequence and w_f^* is an isomorphism in $\mathcal{H}(S^1)$.

Lemma 3.3. *The composition of w_f^* with the right vertical arrow in* (3.5) *is canonically identified with the pull-back map $f^* : KH(X) \to KH(Y)$ in $\mathcal{H}(S^1)$.*

Proof. The lemma follows immediately by combining [8, Section 2.16] (see page 438, line 9) and [8, Proposition 2.19], which together imply that there is a canonical isomorphism $\mathbf{R}\Omega^\infty_t \mathrm{KGL} \cong KH$ in $\mathcal{SH}^{S^1}_{cdh}$. □

Corollary 3.4. $\mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} M_f, \mathbf{R}\Omega^\infty_t \mathrm{KGL}) \cong KH(f)$ *in $\mathcal{H}(S^1)$ and therefore* $KH_n(f) \cong \mathrm{Hom}_{\mathcal{SH}_{cdh}}(\Sigma^\infty_T M_f[n], \mathrm{KGL})$*. In particular, for a closed immersion $f : Y \hookrightarrow X$, we have*

$$KH_n(X, Y) \cong \mathrm{Hom}_{\mathcal{SH}_{cdh}}(\Sigma^\infty_T (X/Y)_+[n], \mathrm{KGL}). \tag{3.6}$$

Proof. The corollary follows directly from (3.5), Lemma 3.3, Quillen adjunctions (3.3) and (3.4), representability result [8, Théorème 2.20], and Definition 3.1(1). □

In this paper we shall write $KH(f)$ and $\mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} M_f, \mathbf{R}\Omega^\infty_t \mathrm{KGL})$ interchangeably hereafter. It follows from (3.5), Lemma 3.3 and Corollary 3.4 that the commutative diagram (3.2), associated to (3.1), induces $(g, g')^* : KH(f) \to KH(f')$ in $\mathcal{H}(S^1)$ which fits in a morphism of distinguished triangles in $\mathcal{H}(S^1)$:

$$\begin{array}{ccccc} KH(f) & \longrightarrow & KH(X) & \xrightarrow{f^*} & KH(Y) \\ {\scriptstyle (g,g')^*}\downarrow & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ KH(f') & \longrightarrow & KH(X') & \xrightarrow{f'^*} & KH(Y'). \end{array} \tag{3.7}$$

Remark 3.5. By construction, the diagram (3.7) is natural in (g, g') in (3.1).

4. Relative motivic cohomology

We continue to assume that k is a perfect field of exponential characteristic p. In this section, we define our relative motivic cohomology for a closed immersion of schemes $Y \subset X$ in $\mathbf{Sch}_k$. We shall then show that this relative motivic cohomology can be described as the cdh-hypercohomology of a presheaf of complexes of equidimensional cycles. These relative cohomology groups will later constitute the E_2-terms of our spectral sequence for relative KH-theory.

4.1. Motivic cohomology of singular schemes

Recall from [28, Lecture 16] that given $T \in \mathbf{Sch}_k$ and an integer $r \geq 0$, the presheaf $z_{equi}(T, r)$ on $\mathbf{Sm}_k$ is defined by letting $z_{equi}(T, r)(U)$ be the free Abelian group generated by the closed and irreducible subschemes $Z \subsetneq U \times T$ which are dominant and equidimensional of relative dimension r (any fiber is either empty or all its components have dimension r) over a component of U. It is known that $z_{equi}(T, r)$ is a sheaf on the big étale site of $\mathbf{Sm}_k$.

Let $C_* z_{equi}(T, r)$ denote the chain complex of presheaves of Abelian groups associated, via the Dold-Kan correspondence, to the simplicial presheaf on $\mathbf{Sm}_k$ given by $C_n z_{equi}(T, r)(U) = z_{equi}(T, r)(U \times \Delta^n_k)$. The simplicial structure on $C_* z_{equi}(T, r)$ is induced by the cosimplicial scheme $\Delta^\bullet_k$. Recall the following definition of motivic cohomology of singular schemes from [12, Definition 9.2].

Definition 4.1. The motivic cohomology groups of $X \in \mathbf{Sch}_k$ are defined as the hypercohomology

$$\begin{aligned} H^m(X, \mathbb{Z}(n)) &= \mathbb{H}^{m-2n}_{cdh}(X, \mathbf{L}\pi^*(C_* z_{equi}(\mathbb{A}^n_k, 0))) \\ &\cong \mathbb{H}^{m-2n}_{cdh}(X, C_* z_{equi}(\mathbb{A}^n_k, 0)_{cdh}). \end{aligned}$$

We will also need to consider $\mathbb{Z}[\frac{1}{p}]$-coefficients. In this case, we will write:

$$H^m(X, \mathbb{Z}[\tfrac{1}{p}](n)) = \mathbb{H}^{m-2n}_{cdh}(X, \mathbf{L}\pi^*(C_* z_{equi}(\mathbb{A}^n_k, 0)[\tfrac{1}{p}])).$$

For $n < 0$, we set $H^m(X, \mathbb{Z}(n)) = H^m(X, \mathbb{Z}[\frac{1}{p}](n)) = 0$.

4.2. Motivic cohomology via $\mathcal{SH}_{cdh}$

In order to represent the motivic cohomology of a singular scheme X in $\mathcal{SH}_{cdh}$, let us recall the Eilenberg-MacLane spectrum

$$H\mathbb{Z} = (K(0, 0), K(1, 2), \cdots, K(n, 2n), \cdots)$$

in $Spt(\mathcal{M})$, where $K(n, 2n)$ is the presheaf of simplicial Abelian groups on $\mathbf{Sm}_k$ associated to the presheaf of chain complexes $C_*\big(\frac{z_{equi}(\mathbb{P}^n_k, 0)}{z_{equi}(\mathbb{P}^{n-1}_k, 0)}\big)$ via the Dold-Kan correspondence. The assembly maps of this spectrum are induced by the canonical map $g : \mathbb{P}^1_k \to C_* z_{equi}(\mathbb{P}^1_k, 0)$. This map assigns to any map $U \to \mathbb{P}^1_k$ its graph in $U \times \mathbb{P}^1_k$. This in turn descends to maps $T \wedge K(n, 2n) \xrightarrow{\overline{g}\wedge \mathrm{Id}} K(1, 2) \wedge K(n, 2n) \xrightarrow{\times} K(n + 1, 2n + 2)$, where the latter is the obvious external product map. Using the localization theorem, it follows that $K(n, 2n)$ is weak equivalent to $C_* z_{equi}(\mathbb{A}^n_k, 0)$. We shall not distinguish between a simplicial Abelian group and the associated chain complex of Abelian groups from now on in this text and will use them interchangeably.

By [25, Theorem 3.10] motivic cohomology can be defined via $\mathcal{SH}_{cdh}$, so this leads naturally to the following definition of relative motivic cohomology. Recall our notation that Λ denotes the ring $\mathbb{Z}$ if k admits resolution of singularities, or the ring $\mathbb{Z}[\frac{1}{p}]$ otherwise.

4.2.1. *Relative motivic cohomology*

Let $f : Y \to X$ be any morphism in $\mathbf{Sch}_k$. For any commutative ring R, we define the relative motivic cohomology of the pair (X, Y) with coefficients in R by

$$H^m(X|Y, R(n)) = \mathrm{Hom}_{\mathcal{SH}_{cdh}}(\Sigma^\infty_T M_f, \Sigma^{m,n}\mathbf{L}\pi^* HR). \tag{4.1}$$

For $n < 0$, we set $H^m(X|Y, R(n)) = 0$. Notice that by [25, 3.10] this definition reduces to 4.1 when $Y = \emptyset$ and $R = \Lambda$.

Proposition 4.2. *Let k be a perfect field of exponential characteristic p. For any closed immersion $f : Y \hookrightarrow X$ in $\mathbf{Sm}_k$ and integers $m, n \in \mathbb{Z}$, there is a natural isomorphism*

$$\theta_{X|Y} : H^m(X|Y, \mathbb{Z}(n)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{SH}}(\Sigma^\infty_T M_f, \Sigma^{m,n} H\mathbb{Z}). \tag{4.2}$$

Proof. Follows from the definition of relative motivic cohomology (4.1) and [25, Corollary 2.17]. ∎

4.3. Motivic cohomology via DM_k

Let $\mathrm{MS}^{\mathrm{tr}}$, $\mathrm{MS}^{\mathrm{tr}}_{cdh}$ denote the category of presheaves with transfers on $\mathbf{Sm}_k$, $\mathbf{Sch}_k$, respectively; equipped with the model structure induced by the adjunction $(tr, U) : \mathcal{M} \to \mathrm{MS}^{\mathrm{tr}}$, $(tr, U) : \mathcal{M}_{cdh} \to \mathrm{MS}^{\mathrm{tr}}_{cdh}$, where U is the functor that forgets transfers, *i.e.*, a map f in $\mathrm{MS}^{\mathrm{tr}}$, $\mathrm{MS}^{\mathrm{tr}}_{cdh}$ is a weak equivalence (respectively a fibration) if and only if $U(f)$ is a weak equivalence (respectively a fibration) in $\mathcal{M}$ and $\mathcal{M}_{cdh}$.

We will write $Spt(\mathrm{MS}^{\mathrm{tr}})$, $Spt(\mathrm{MS}^{\mathrm{tr}}_{cdh})$ for the category of symmetric T-spectra on $\mathrm{MS}^{\mathrm{tr}}$, $\mathrm{MS}^{\mathrm{tr}}_{cdh}$, respectively; equipped with the stable model structure [15, 8.7], where T is identified with $C_*\left(\frac{z_{equi}(\mathbb{P}^1_K,0)}{z_{equi}(\mathbb{P}^0_K,0)}\right)$. Let DM_k (respectively $\mathrm{DM}_{\mathrm{cdh}}$) denote the homotopy category of $Spt(\mathrm{MS}^{\mathrm{tr}})$, $Spt(\mathrm{MS}^{\mathrm{tr}}_{cdh})$. Notice that DM_k is Voevodsky's triangulated category of motives.

There is a T-suspension functor $\Sigma^\infty_T : \mathrm{MS}^{\mathrm{tr}} \to Spt(\mathrm{MS}^{\mathrm{tr}})$, $\Sigma^\infty_T : \mathrm{MS}^{\mathrm{tr}}_{cdh} \to Spt(\mathrm{MS}^{\mathrm{tr}}_{cdh})$, and the forgetful functor induces levelwise Quillen adjunctions:

$$(tr, U) : Spt(\mathcal{M}) \to Spt(\mathrm{MS}^{\mathrm{tr}}), \quad (tr, U) : Spt(\mathcal{M}_{cdh}) \to Spt(\mathrm{MS}^{\mathrm{tr}}_{cdh}), \tag{4.3}$$

which fit in a commutative diagram of left Quillen functors:

$$\begin{array}{ccc} Spt(\mathcal{M}) & \xrightarrow{\pi^*} & Spt(\mathcal{M}_{cdh}) \\ {\scriptstyle tr}\downarrow & & \downarrow{\scriptstyle tr} \\ Spt(\mathrm{MS}^{\mathrm{tr}}) & \xrightarrow{\pi^*} & Spt(\mathrm{MS}^{\mathrm{tr}}_{cdh}). \end{array} \tag{4.4}$$

For $X \in \mathbf{Sch}_k$, we will write $M(X) \in \mathrm{DM}_{cdh}$ for $tr(\Sigma^\infty_T X)$ and $\mathbb{Z}(n)[m] \in \mathrm{DM}_{cdh}$, $m, n \in \mathbb{Z}$ for $tr(\Sigma^{m,n}\Sigma^\infty_T S^0_s)$. Given $E \in \mathrm{DM}_{cdh}$, we will write $E(n)$ for $E \otimes \mathbb{Z}(n)$.

Notice that by construction the spectrum $H\mathbb{Z} \in \mathcal{SH}_{cdh}$ representing motivic cohomology has transfers. Furthermore, for every $m, n \in \mathbb{Z}$ [9, 5.10-5.11]:

$$U(\Lambda(n)[m]) \cong \Sigma^{m,n}\mathbf{L}\pi^* H\Lambda. \tag{4.5}$$

Since $tr(\Sigma^\infty_T(X_+)) = M(X)$ for any $X \in \mathbf{Sch}_k$, it follows that $M(Y) \to M(X) \to tr(\Sigma^\infty_T(M_f))$ is a distinguished triangle in DM_{cdh} for any closed immersion $Y \hookrightarrow X$. We let $M(X/Y) := tr(\Sigma^\infty_T(M_f))$.

Let $\mathrm{DM}(k, \Lambda)$ and $\mathrm{DM}_{cdh}(k, \Lambda)$ denote Voevodsky's big categories of motives over k with coefficients in Λ with respect to the Nisnevich and cdh-topologies, respectively. Note that $\mathrm{DM}(k, \mathbb{Z}) = \mathrm{DM}_k$. It follows from [9, Proposition 8.1(c)] that the functor $\mathbf{L}\pi^* : \mathrm{DM}(k, \Lambda) \to \mathrm{DM}_{cdh}(k, \Lambda)$ is an equivalence of tensor triangulated categories. As a consequence of (4.3)-(4.5) and (4.1)-Proposition 4.2, we get:

Proposition 4.3. *Let k be a perfect field of exponential characteristic p. For any closed immersion $f : Y \hookrightarrow X$ in $\mathbf{Sch}_k$ and integers $m, n \in \mathbb{Z}$, there is a natural isomorphism*

$$H^m(X|Y, \Lambda(n)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M(X/Y), \Lambda(n)[m]).$$

If $X, Y \in \mathbf{Sm}_k$, there is a natural isomorphism

$$H^m(X|Y, \mathbb{Z}(n)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{DM}_k}(M(X/Y), \mathbb{Z}(n)[m]).$$

4.4. Motivic cohomology with (compact) supports

Let $X \in \mathbf{Sch}_k$, $j : U \to X$ an open immersion and $Z = X \backslash U$ its closed complement equipped with the reduced scheme structure. We will write $M^c(X) \in \mathrm{DM}_{cdh}$, $M_Z(X) \in \mathrm{DM}_{cdh}$ respectively for the motive of X with compact supports [37, Section 4.1] and the motive of X with supports on Z, where the latter is defined in terms of a distinguished triangle $M(U) \xrightarrow{j} M(X) \to M_Z(X)$ in DM_{cdh}.

Definition 4.4. With the notation of (4.5) and Subsection 4.4, let $m, n \in \mathbb{Z}$. The motivic cohomology of X with compact supports of degree m and weight n is given as:

$$H^m_c(X, \mathbb{Z}(n)) := \mathrm{Hom}_{\mathrm{DM}_{cdh}}(M^c(X), \mathbb{Z}(m)[n]).$$

Similarly, the motivic cohomology of X with supports on Z of degree m and weight n is given by $H^m_Z(X, \mathbb{Z}(n)) := \mathrm{Hom}_{\mathrm{DM}_{cdh}}(M_Z(X), \mathbb{Z}(n)[m])$. The motivic cohomology groups $H^m_c(X, \mathbb{Z}[\frac{1}{p}](n))$ and $H^m_Z(X, \mathbb{Z}[\frac{1}{p}](n))$ are defined analogously.

Proposition 4.5. *With the notation of Definition* 3.1, *assume that f is a closed (respectively open) immersion, and let $U = X \backslash Y$ (respectively $Z = X \backslash Y$). If f is closed, assume in addition that X is proper. Then there are natural isomorphisms:*

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{SH}_{cdh}}(\Sigma^\infty_T M_f, \Sigma^{m,n}\mathbf{L}\pi^* H\Lambda) \cong H^m_c(U, \Lambda(n)) \\ respectively\quad &\mathrm{Hom}_{\mathcal{SH}_{cdh}}(\Sigma^\infty_T M_f, \Sigma^{m,n}\mathbf{L}\pi^* H\Lambda) \cong H^m_Z(X, \Lambda(n)). \end{aligned}$$

If $X, Y \in \mathbf{Sm}_k$, these isomorphisms hold integrally.

Proof. We will give the argument for the first isomorphism since the other one is parallel. By construction (see Definition 3.1), there is a distinguished triangle $(cs) = \Sigma^\infty_T Y_+ \to \Sigma^\infty_T X_+ \to \Sigma^\infty_T M_f$ in $\mathcal{SH}_{cdh}$. Combining the Quillen adjunctions (4.3) with (4.5), we are reduced to show that $tr(\Sigma^\infty_T M_f) \cong M^c(U)$ in DM_{cdh} in general and in DM_k if $X, Y \in \mathbf{Sm}_k$. But this follows from [37, Proposition 4.1.5] and [9, Proposition 8.1(c)] since applying tr to (cs), we obtain a distinguished triangle $M(Y) \to M(X) \to tr(\Sigma^\infty_T M_f)$ in DM_{cdh} (in DM_k if $X, Y \in \mathbf{Sm}_k$). □

4.5. Relative motivic cohomology in terms of hypercohomology of sheaves of equidimensional cycles

Let $X \in \mathbf{Sm}_k$ and let $Y \subset X$ be an effective Cartier divisor. In this case, we can describe the relative motivic cohomology group in terms of the hypercohomology of a complex of equidimensional cycles. Let S_X denote the double of X along Y and let $\nabla : S_X \to X$ denote the fold map (see Section 2.5). We let ∇ also denote the morphism of sites $(S_X)_{cdh} \to X_{cdh}$. For any $Z \in \mathbf{Sch}_k$, let $Sh(Z)_{cdh}$ denote the category of sheaves of Abelian groups on the cdh-site Z_{cdh}.

Lemma 4.6. *The direct image functor* $\nabla_* : Sh(S_X)_{cdh} \to Sh(X)_{cdh}$ *is exact.*

Proof. It suffices to show that $\nabla : S_X \to X$ is a cdh-covering. But this follows immediately from [32, Lemma 5.8] because ∇ is finite and it admits a section (see Section 2.5). □

For any $Z \in \mathbf{Sch}_k$ and an integer $n \geq 0$, we let $\mathbb{Z}_Z(n)[2n]$ denote the complex of sheaves $C_* z_{equi}(\mathbb{A}^n_k, 0)_{Z_{cdh}}$ on Z_{cdh}. It follows from Lemma 4.6 that

$$H^m(S_X, \Lambda(n)) \cong \mathbb{H}^{m-2n}_{cdh}(X, \nabla_*(\Lambda_{S_X}(n)[2n])). \tag{4.6}$$

For any morphism of schemes $Y \to X$, we obtain maps $Y \xrightarrow{\iota_-} S_X \times_X Y \to Y$ whose composite is identity. Pulling back cycles along ι_-, we obtain a map $\iota_-^* : \nabla_*(\Lambda_{S_X}(n)[2n]) \to \Lambda_X(n)[2n]$ which admits a section $\nabla^* : \Lambda_X(n)[2n] \to \nabla_*(\Lambda_{S_X}(n)[2n])$. We let $\Lambda_{X|Y}(n)[2n] := \mathrm{Ker}(\iota_-^*)$ so that there is a split short exact sequence on X_{cdh}:

$$0 \to \Lambda_{X|Y}(n)[2n] \to \nabla_*(\Lambda_{S_X}(n)[2n]) \to \Lambda_X(n)[2n] \to 0. \tag{4.7}$$

Since (2.1) is a cdh-square (see Section 2.5), the map $\iota_+^* : H^m(S_X|X_-, \Lambda(n)) \to H^m(X|Y, \Lambda(n))$ is an isomorphism. We conclude from (4.6) and (4.7), the following description of the relative motivic cohomology of the pair (X, Y) in terms of equidimensional cycles.

Proposition 4.7. *For any integers* $m, n \in \mathbb{Z}$*, there is a natural isomorphism*

$$H^m(X|Y, \Lambda(n)) \cong \mathbb{H}^{m-2n}_{cdh}(X, \Lambda_{X|Y}(n)[2n]).$$

If X is furthermore projective, we have

$$H^m(X|Y, \Lambda(n)) \cong \mathbb{H}^{m-2n}_c(X \setminus Y, (C_* z_{equi}(\mathbb{A}^n_k, 0)_\Lambda)_{cdh}).$$

5. Slice spectral sequence for relative KH-theory

Let k be a perfect field of exponential characteristic p. Given $X \in \mathbf{Sch}_k$, recall that Voevodsky's slice filtration of $\mathcal{SH}_X$ is given as follows. For an integer $q \in \mathbb{Z}$, let $\Sigma^q_T \mathcal{SH}^{\mathrm{eff}}_X$ denote the smallest full triangulated subcategory of $\mathcal{SH}_X$ which contains C^q_{eff} and is closed under arbitrary coproducts, where

$$C^q_{\mathrm{eff}} = \{\Sigma^{m,n} \Sigma^\infty_T Y_+ : m \in \mathbb{Z}, n \geq q, Y \in \mathbf{Sm}_X\}. \tag{5.1}$$

In particular, $\mathcal{SH}^{\mathrm{eff}}_X$ is the smallest full triangulated subcategory of $\mathcal{SH}_X$ which is closed under infinite direct sums and contains all spectra of the type $\Sigma^\infty_T Y_+$ with

$Y \in \mathbf{Sm}_X$. The slice filtration of $\mathcal{SH}_X$ (see [38]) is the sequence of full triangulated subcategories

$$\cdots \subseteq \Sigma_T^{q+1}\mathcal{SH}_X^{\text{eff}} \subseteq \Sigma_T^{q}\mathcal{SH}_X^{\text{eff}} \subseteq \Sigma_T^{q-1}\mathcal{SH}_X^{\text{eff}} \subseteq \cdots$$

It is known [30] that the inclusion $i_q : \Sigma_T^q\mathcal{SH}_X^{\text{eff}} \to \mathcal{SH}_X$ admits a right adjoint $r_q : \mathcal{SH}_X \to \Sigma_T^q\mathcal{SH}_X^{\text{eff}}$ and that the functors $f_q, s_{<q}, s_q : \mathcal{SH}_X \to \mathcal{SH}_X$ are triangulated; where $r_q \circ i_q$ is the identity, $f_q = i_q \circ r_q$ and $s_{<q}, s_q$ are characterized by the existence of the following distinguished triangles in $\mathcal{SH}_X$:

$$f_qE \longrightarrow E \longrightarrow s_{<q}E \quad \text{and} \qquad f_{q+1}E \longrightarrow f_qE \longrightarrow s_qE \tag{5.2}$$

for every $E \in \mathcal{SH}_X$.

Definition 5.1. Let a, b, $n \in \mathbb{Z}$ and $Y \in \mathbf{Sm}_X$. Let $F^nE^{a,b}(Y)$ be the image of the map induced by $f_nE \to E$ (5.2): $\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b}f_nE) \to \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b}E)$. This determines a decreasing filtration $F^\bullet$ on $E^{a,b}(Y) = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b}E)$, and we will write $gr^nF^\bullet$ for the associated graded pieces $\{F^nE^{a,b}(Y)/F^{n+1}E^{a,b}(Y)\}$.

5.1. The slice spectral sequence

Let $Y \in \mathbf{Sm}_X$ be a smooth X-scheme and $G \in \mathcal{SH}_X$. Since $\mathcal{SH}_X$ is a triangulated category, the collection of distinguished triangles $\{f_{q+1}G \to f_qG \to s_qG\}_{q\in\mathbb{Z}}$ determines a (slice) spectral sequence of the form $E_1^{p,q} = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma_s^{p+q}s_pG)$ with $G^{m,n}(Y) = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{m,n}G)$ as its abutment and differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

In order to study the convergence of this spectral sequence, recall from [38, page 22] that $G \in \mathcal{SH}_X$ is called *bounded* with respect to the slice filtration if for every $m, n \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, there exists $q \in \mathbb{Z}$ such that:

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n}\Sigma_T^\infty Y_+, f_{q+i}G) = 0 \tag{5.3}$$

for every $i > 0$. Clearly the slice spectral sequence is strongly convergent when G is bounded.

5.2. The slice spectral sequence for KGL

Let k be a perfect field of exponential characteristic p, and let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$. Let $v : X \to \mathrm{Spec}\,(k)$ be the structure map. Recall that KGL is by definition $\mathbf{L}\pi^*\mathrm{KGL}_k$, and the map $\mathbf{L}v^*(\mathrm{KGL}_k) \to \mathrm{KGL}_X$ is an isomorphism by [8, Proposition 3.8].

By [31, 6.2.3.10, 5.3.18 and 5.3.10] we have the Chern character isomorphism $ch : (\mathrm{KGL}_k)_{\mathbb{Q}} \xrightarrow{\cong} \oplus_{q\in\mathbb{Z}}\Sigma_T^q H\mathbb{Q}$, and by [31, 5.3.17 and 5.3.10] $\Sigma_T^q H\mathbb{Q} \cong$

$(\mathrm{KGL}_k)^{(q)}_{\mathbb{Q}}$ where $(\mathrm{KGL}_k)^{(q)}_{\mathbb{Q}}$ is the qth Adams eigen-spectrum constructed in [31, 5.3.9]. In addition, one knows that $\phi_r : s_r\mathrm{KGL}_k \xrightarrow{\cong} \Sigma^r_T H\mathbb{Z}$ for $r \in \mathbb{Z}$ [26, Theorem 6.4.2], which implies $s_i(\Sigma^r_T H\mathbb{Z}) = 0$ for $i \neq r$ and $s_r(\Sigma^r_T H\mathbb{Z}) \cong \Sigma^r_T H\mathbb{Z}$. Now, since the effective categories $\Sigma^q_T \mathcal{SH}^{\mathrm{eff}}$ are closed under arbitrary coproducts it follows that the slices s_i commute with arbitrary coproducts. Hence, we obtain the following isomorphism:

$$s_i(ch) : s_i(\mathrm{KGL}_k)_{\mathbb{Q}} \xrightarrow{\cong} s_i(\oplus_{q\in\mathbb{Z}}\Sigma^q_T H\mathbb{Q}) \cong \oplus_{q\in\mathbb{Z}} s_i\,\Sigma^q_T H\mathbb{Q} \cong \Sigma^i_T H\mathbb{Q} \cong (\mathrm{KGL}_k)^{(i)}_{\mathbb{Q}}.$$

Thus, we conclude that the Chern character isomorphism splits the slice filtration:

$$\mathrm{ch} : (\mathrm{KGL}_k)_{\mathbb{Q}} \xrightarrow{\cong} \oplus_{q\in\mathbb{Z}} s_q(\mathrm{KGL}_k)_{\mathbb{Q}}. \tag{5.4}$$

in $\mathcal{SH}$. For $q \in \mathbb{Z}$, we let

$$\mathbb{Z}_{\mathrm{sl}}(M_f, q) := \mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} M_f, \mathbf{R}\Omega^\infty_t \Sigma^q_T H\mathbb{Z}) \in \mathcal{H}(S^1). \tag{5.5}$$

If f is a closed (respectively open) immersion, we write $\mathbb{Z}_{\mathrm{sl}}(M_f, q)$ as $\mathbb{Z}_{\mathrm{sl}}(X|Y, q)$ (respectively $\mathbb{Z}^Y_{\mathrm{sl}}(X, q)$).

Now, by construction (see Definition 3.1(2) and equation (3.2)), there is a commutative diagram in $Spt(\mathcal{M}_{cdh})$:

$$\begin{array}{ccccc} \Sigma^\infty_T Y_+ & \longrightarrow & \Sigma^\infty_T A_f & \longrightarrow & \Sigma^\infty_T M_f \\ f\big\downarrow & \swarrow_{w_f} & & & \\ \Sigma^\infty_T X_+ & & & & \end{array} \tag{5.6}$$

where the top row is a cofiber sequence. Since $w_f : A_f \to X_+$ is a weak equivalence of motivic spaces, the map $w_f : \Sigma^\infty_T A_f \to \Sigma^\infty_T X_+$ is a stable weak equivalence in $Spt(\mathcal{M}_{cdh})$ (*e.g.*, see [36, page 592]). Hence, mapping in $\mathcal{SH}_{cdh}$ (5.6) into the slice tower of KGL:

$$\cdots \to \mathbf{L}\pi^*(f_{q+1}\mathrm{KGL}_k) \to \mathbf{L}\pi^*(f_q\mathrm{KGL}_k) \to \cdots \to \mathbf{L}\pi^*(\mathrm{KGL}_k) = \mathrm{KGL}$$

and splicing together Lemma 3.3 and Corollary 3.4, we obtain the following result:

Theorem 5.2. *Let k be a perfect field, and let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$. Then there exists in $\mathcal{H}(S^1)$, a tower, natural in (X, Y):*

$$\begin{aligned} \cdots \to \phi_{q+1}KH(f) \to \phi_q KH(f) \to \cdots \to \phi_0 KH(f) \to \phi_{-1}KH(f) \\ \to \cdots \to KH(f) \end{aligned} \tag{5.7}$$

and an isomorphism for each $q \in \mathbb{Z}$:

$$\phi_q/\phi_{q+1}KH(f)_\Lambda \cong \Lambda_{\mathrm{sl}}(M_f, q). \tag{5.8}$$

where $\phi_q/\phi_{q+1}KH(f)$ is the cofiber in $\mathcal{H}(S^1)$ of the map $\phi_{q+1}KH(f) \to \phi_q KH(f)$.

Proof. If we let $\phi_q KH(f) = \mathbf{R}\Gamma_k \circ \mathbf{RHom}(\Sigma^\infty_{S^1} M_f, \mathbf{R}\Omega^\infty_t \mathbf{L}\pi^*(f_q \mathrm{KGL}_k))$, we only need to verify that the cofiber of the map $\phi_{q+1} KH(f)_\Lambda \to \phi_q KH(f)_\Lambda$ is isomorphic to $\Lambda_{\mathrm{sl}}(M_f, q)$. But this follows from the fact that there is an isomorphism $\Sigma^q_T H\mathbb{Z} \cong f_q \mathrm{KGL}_k / f_{q+1}\mathrm{KGL}_k$ in $\mathcal{SH}$ (*e.g.*, see [26, Section 11.3, Remark 11.3.4]). Moreover, the same holds for KGL in $\mathcal{SH}_{cdh}$ with Λ-coefficients in positive characteristic by [25, Lemma 4.8]. □

Using Theorem 5.2 together with the Quillen adjunctions of (3.3) and (3.4), we get our final result:

Theorem 5.3. *Let k be a perfect field and let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$. Then there is a commutative diagram of strongly convergent spectral sequences:*

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
E_2^{a,b} = H^{a-b}(M_f, \Lambda(-b)) & \Longrightarrow & KH_{-a-b}(f)_\Lambda \\
\downarrow & & \downarrow \\
E_2^{a,b} = H^{a-b}(X, \Lambda(-b)) & \Longrightarrow & KH_{-a-b}(X)_\Lambda \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
E_2^{a,b} = H^{a-b}(Y, \Lambda(-b)) & \Longrightarrow & KH_{-a-b}(Y)_\Lambda \\
\downarrow & & \downarrow \\
E_2^{a+1,b} = H^{a-b+1}(M_f, \Lambda(-b)) & \Longrightarrow & KH_{-a-b-1}(f)_\Lambda \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

with exact columns, where the differentials of the spectral sequence are given by $d_r : E_r^{a,b} \to E_r^{a+r,b-r+1}$; and for every $a, b \in \mathbb{Z}$, there exists $N > 0$ such that $E_r^{a,b} = E_\infty^{a,b}$ for $r \geq N$, where $E_\infty^{a,b}$ is the associated graded $gr^{-b}F^\bullet$ with respect to the descending filtration Definition 5.1 *on the groups in the right column. Furthermore, these spectral sequences degenerate with rational coefficients.*

Proof. Except for the strong convergence, all the claims follow from Theorem 5.2, (3.3) and (3.4) and an elementary reindexing to convert the resulting E_1-spectral sequence into an E_2-spectral sequence (see the proof of [26, Theorem 11.3.2]). Alternatively, one can use Lemma 3.3 and Corollary 3.4 and the method of [25, 4.27]) to prove the existence of the spectral sequences.

To prove the strong convergence, we recall from [25, 4.7 and 4.10] that $\mathrm{KGL}[\frac{1}{p}]$ is bounded with respect to the slice filtration (5.3). Let $m, n \in \mathbb{Z}$ be arbitrary integers, and $q_1, q_2 \in \mathbb{Z}$ such that the vanishing condition (5.3) holds for (m, n) and $(m+1, n)$ respectively. Then, if q is the maximum of q_1, q_2, we deduce by [25, Theorem 2.14] and (5.3) that for every $i > 0$, there is an exact sequence:

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{SH}_{cdh}}\left(\Sigma^{m+1,n}\Sigma_T^\infty Y_+,\, f_{q+i}\mathrm{KGL}\left[\tfrac{1}{p}\right]\right) \\ \to &\mathrm{Hom}_{\mathcal{SH}_{cdh}}\left(\Sigma^{m,n}\Sigma_T^\infty M_f,\, f_{q+i}\mathrm{KGL}\left[\tfrac{1}{p}\right]\right) \\ &\mathrm{Hom}_{\mathcal{SH}_{cdh}}\left(\Sigma^{m,n}\Sigma_T^\infty X_+,\, f_{q+i}\mathrm{KGL}\left[\tfrac{1}{p}\right]\right), \end{aligned}$$

where the terms on the top and bottom vanish, so the term in the middle also vanishes. This implies the desired convergence abutting to $KH_*(f)_\Lambda$. □

Remark 5.4. With the notation of Definition 3.1, we have the following:

(1) If f is an open immersion, then Corollary 3.4 and Proposition 4.5 together imply that the top row of Theorem 5.3 is same as (where $Z = X\backslash Y$):

$$E_2^{a,b} = H_Z^{a-b}(X, \Lambda(-b)) \Rightarrow KH^Z_{-a-b}(X)_\Lambda; \tag{5.9}$$

(2) If f is a closed immersion, then Corollary 3.4 and (4.1)-Proposition 4.2 together imply that the top row of Theorem 5.3 is the same as

$$E_2^{a,b} = H^{a-b}(X|Y, \Lambda(-b)) \Rightarrow KH_{-a-b}(X, Y)_\Lambda.$$

If in addition X is assumed to be proper, Corollary 3.4 and Proposition 4.5 together imply that the top row of Theorem 5.3 is the same as

$$E_2^{a,b} = H_c^{a-b}(X \setminus Y, \Lambda(-b)) \Rightarrow KH_{-a-b}(X, Y)_\Lambda. \tag{5.10}$$

As a combination of Theorem 5.3 and [34, Theorem 9.5, 9.6], we obtain the following result for the Thomason-Trobaugh relative algebraic K-theory $K(-)$ of singular schemes [34].

Corollary 5.5. *Let k be a perfect field of exponential characteristic p. Let $\ell \neq p$ be a prime and $m \geq 0$ any integer. Given any $f : Y \to X$ in $\mathbf{Sch}_k$, there exists a strongly convergent spectral sequence*

$$E_2^{a,b} = H^{a-b}(M_f, \mathbb{Z}/\ell^m(-b)) \Rightarrow K/\ell^m{}_{-a-b}(f). \tag{5.11}$$

If $p > 0$, there exists a strongly convergent spectral sequence

$$E_2^{a,b} = H^{a-b}\left(M_f, \mathbb{Z}\left[\tfrac{1}{p}\right](-b)\right) \Rightarrow K_{-a-b}(f)\left[\tfrac{1}{p}\right]. \tag{5.12}$$

5.3. Relative K-theory of smooth closed pair

Our main example of interest is the relative algebraic K-theory of the pair (X, Y) where $Y \subset X$ is a codimension one closed immersion of smooth schemes. We now show that Theorems 5.2 and Theorem 5.3 can be much simplified to give a more precise description of the tower of the relative K-theory and of the motivic cohomology in this case.

In any case, it follows from the definition of the tower (5.7) that there is a distinguished triangle in $\mathcal{H}(S^1)$:

$$\phi_q KH(f) \to \phi_q KH(X) \xrightarrow{f^*} \phi_q KH(Y) \to \phi_q KH(f)[1].$$

Let us now assume that f is a closed immersion of smooth schemes over k. In this case, we know that $K(f) \to KH(f)$ is a weak equivalence. On the other hand, it follows from [26, Theorem 9.0.3] that the slice tower $\phi_\bullet K(X)$ coincides with the homotopy coniveau tower $\psi_\bullet K(X)$, where $\psi_q K(X)$ is the diagonal of the simplicial spectrum $K^{(q)}(X, -)$, defined in [26, 2.1.2]. In particular, we get $\phi_q K(f) = \phi_q K(X, Y) = K(X, Y)$ for $q \le 0$. Furthermore, it follows from Theorem 5.2 and [26, Theorem 6.4.2] that $\mathbb{Z}_{\mathrm{sl}}(X, q) \cong \phi_q/\phi_{q+1} K(X) \cong \psi_q/\psi_{q+1} K(X) \cong \Sigma^\infty_{S^1} z^q(X, \bullet)$, where $z^q(X, \bullet)$ is Bloch's cycle complex of X [4]. The same holds for Y as well.

We let $z^q_{\mathcal{M}}(X|Y, \bullet) = \mathrm{Cone}(f^*)[-1]$, where $\mathrm{Cone}(f^*)$ denotes the cone of the restriction map of cycle complexes $f^* : z^q_Y(X, \bullet) \to z^q(Y, \bullet)$. Recall here that $z^q_Y(X, \bullet) \subset z^q(X, \bullet)$ is the subcomplex generated by irreducible cycles which intersect all faces of $Y \times \Delta^\bullet$ properly. By [23, Theorem 1.10] (which relies on Bloch's moving [5] lemma) this inclusion is a weak equivalence of simplicial Abelian groups. Letting $\mathrm{CH}^q_{\mathcal{M}}(X|Y, i) = H_i(z^q_{\mathcal{M}}(X|Y, \bullet))$ and $\mathrm{CH}^*_{\mathcal{M}}(X|Y, i) = \oplus_{q \in \mathbb{Z}} \mathrm{CH}^q_{\mathcal{M}}(X|Y, i)$, we conclude that Theorem 5.2 reduces to the following:

Theorem 5.6. *Let k be a perfect field and let $f : Y \hookrightarrow X$ be a closed immersion in $\mathbf{Sm}_k$. Let $(g, g') : (X', Y') \to (X, Y)$ be a projective morphism of closed pairs in $\mathbf{Sm}_k$ such that $Y' = Y \times_X X'$. Let $T_g \in K_0(Y')$ be the virtual relative tangent bundle for $g : X' \to X$. Then there is a tower in $\mathcal{H}(S^1)$:*

$$\cdots \to \phi_{q+1} K(X, Y) \to \phi_q K(X, Y) \to \cdots \to \phi_0 K(X, Y) = K(X, Y)$$

and isomorphisms for each $q, i \ge 0$:

(1) $\phi_q/\phi_{q+1} K(X, Y) \cong \Sigma^\infty_{S^1} z^q_{\mathcal{M}}(X|Y, \bullet)$;
(2) $\mathrm{CH}^q_{\mathcal{M}}(X|Y, i) \cong H^{2q-i}(X|Y, \mathbb{Z}(q))$;
(3) *There exists a strongly convergent spectral sequence*

$$E_2^{a,b} = \mathrm{CH}^{-b}_{\mathcal{M}}(X|Y, -a-b) \Rightarrow KH_{-a-b}(X, Y);$$

(4) *Grothendieck Riemann-Roch theorem: there is a commutative diagram*

$$\begin{array}{ccc} K_i(X', Y')_{\mathbb{Q}} & \xrightarrow{\mathrm{Td}(T_g)\mathrm{ch}} & \mathrm{CH}^*_{\mathcal{M}}(X'|Y', i)_{\mathbb{Q}} \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K_i(X, Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}} & \mathrm{CH}^*_{\mathcal{M}}(X|Y, i)_{\mathbb{Q}}, \end{array} \tag{5.13}$$

such that the horizontal arrows are isomorphisms.

Proof. The existence of the tower and (1) are already explained above. The item (3) follows from (1) and (2). For (2), we note from what is explained above and the Quillen adjunctions (3.3) and (3.4) that we do have natural maps

$$\mathrm{CH}^q_{\mathcal{M}}(X|Y, i) \to \mathbb{H}^{-i}(X, z^q_{\mathcal{M}}(-|Y, \bullet)|_{X_{zar}}) \to H^{2q-i}(X|Y, \mathbb{Z}(q)). \tag{5.14}$$

To show that the composite map is an isomorphism, we only have to observe that this is indeed the case if $Y = \emptyset$. We conclude using the 5-lemma. In the Riemann-Roch theorem (4), the commutativity of the diagram follows from the item (2) and [29, Theorem 3.15], if we note that a projective morphism of smooth schemes is a local complete intersection. Finally ch is an isomorphism by (5.4), which also implies that $\mathrm{Td}(T_g)\mathrm{ch}$ is an isomorphism since $Td(T_g)$ is a unit in $CH^*_{\mathcal{M}}(X', *)_{\mathbb{Q}}$ and $CH^*_{\mathcal{M}}(X'|Y', *)_{\mathbb{Q}}$ has the structure of $CH^*_{\mathcal{M}}(X', *)_{\mathbb{Q}}$-module. □

6. The cycle class and Chern class maps

As an application of Theorem 5.3, we shall now construct a cycle class map from the relative motivic cohomology in the 0-cycle range to relative KH-theory. We shall then apply the double construction to construct the Chern class maps.

6.1. The cycle class map

We continue with our assumption on the field k and the coefficient ring Λ. In order to construct the cycle class map, we shall use the connective version of the spectrum KGL.

Let $X \in \mathbf{Sch}_k$. Recall that the *connective K-theory* spectrum KGL^0_X is the motivic T-spectrum $f_0\mathrm{KGL}_X$ in $\mathcal{SH}_X$ (see (5.2)). In particular, there is a canonical map $u_X : \mathrm{KGL}^0_X \to \mathrm{KGL}_X$ which is universal for morphisms from objects of $\mathcal{SH}^{\mathrm{eff}}_X$ to KGL_X. With the notation of Definition 3.1, we let $CKH^{p,q}(M_f) = \mathrm{Hom}_{\mathcal{SH}_{cdh}}(\Sigma^\infty_T M_f, \Sigma^{p,q}\mathbf{L}\pi^*\mathrm{KGL}^0)$.

Apart from the connective cover of KGL, we also need the following vanishing result for the motivic cohomology with support.

Lemma 6.1. *Let $X \in \mathbf{Sch}_k$ be of dimension d and let $Y \subseteq X$ be a closed subscheme. Then $H^{2a-b}_Y(X, \Lambda(a)) = H^{2a-b}_c(X, \Lambda(a)) = 0$ whenever $a > d + b$.*

Proof. We first show that $H^{2a-b}_Y(X, \Lambda(a)) = 0$ whenever $a > d + b$. If (X, Y) is a smooth pair, then the purity theorem for motivic cohomology (see [32, Theorem 4.10]) implies that $H^{2a-b}_Y(X, \Lambda(a)) \cong H^{2(a-q)-b}(Y, \Lambda(a-q))$, where Y has codimension q in X. But it is shown in [25, Theorem 5.1] that the latter group is zero since $a - q > d - q + b = \dim(Y) + b$.

We shall now prove the lemma by induction on the dimensions of X and Y. We first keep our assumption that X is smooth but allow Y to be singular. If $\dim(Y) = 0$, then we can assume that Y is smooth and reduce to the previous case. We can therefore assume that $\dim(Y) \geq 1$. Let Z be the singular locus of Y with reduced induced closed subscheme structure. Since k is perfect, we have $\dim(Z) < \dim(Y)$. There is a commutative diagram

$$\begin{array}{ccccc} M(X \setminus Y) & \longrightarrow & M(X) & \to & M_Y(X) \\ \downarrow & & \| & & \downarrow \\ M(X \setminus Z) & \longrightarrow & M(X) & \to & M_Z(X) \\ \downarrow & & & & \\ M_{Y\setminus Z}(X \setminus Z) & & & & \end{array} \tag{6.1}$$

in DM_k so that we get a distinguished triangle $M_{Y\setminus Z}(X\setminus Z) \to M_Y(X) \to M_Z(X)$. This yields an exact sequence

$$H^{2a-b}_Z(X, \Lambda(a)) \to H^{2a-b}_Y(X, \Lambda(a)) \to H^{2a-b}_{Y\setminus Z}(X \setminus Z, \Lambda(a)).$$

The first term vanishes by induction on $\dim(Y)$ and the third term vanishes because $(X \setminus Z, Y \setminus Z)$ is a smooth pair. Hence the middle term vanishes.

We now allow X to be singular and work by induction on $\dim(X)$. If $\dim(X) = 0$, then we can assume X to be smooth. So we assume $\dim(X) \geq 1$. Let us first assume that there is a resolution of singularities $f : \widetilde{X} \to X$ and let $\widetilde{Y} = f^{-1}(Y)$. Let Z denote the singular locus of X and let $E \subset \widetilde{X}$ be the exceptional divisor. If we let $U = X \setminus Y$ and $\widetilde{U} = f^{-1}(U)$, we get a commutative diagram where the rows are distinguished triangles in DM_k:

$$\begin{array}{ccccc} M(U \cap Z) \oplus M(\widetilde{U}) & \to & M(U) & \to & M(E \cap \widetilde{U})[1] \\ \downarrow & & \downarrow & & \downarrow \\ M(Z) \oplus M(\widetilde{X}) & \longrightarrow & M(X) & \longrightarrow & M(E)[1] \end{array} \tag{6.2}$$

and this gives a distinguished triangle

$$M_{Y\cap Z}(Z) \oplus M_{\widetilde{Y}}(\widetilde{X}) \to M_Y(X) \to M_{\widetilde{Y}\cap E}(E)[1].$$

The associated long exact sequence of motivic cohomology groups is of the form

$$H^{2a-b-1}_{\widetilde{Y}\cap E}(E, \Lambda(a)) \to H^{2a-b}_Y(X, \Lambda(a)) \to H^{2a-b}_{Y\cap Z}(Z, \Lambda(a)) \oplus H^{2a-b}_{\widetilde{Y}}(\widetilde{X}, \Lambda(a)).$$

The end terms vanish either by induction on $\dim(X)$ or by the case of smooth ambient scheme. It follows that the middle term vanishes, as desired.

If X is not smooth and k has positive characteristic, we argue as follows. By a theorem of Gabber [18] and its strengthening by Temkin [33, Theorem 1.2.9], there exists $W \in \mathbf{Sm}_k$ and a surjective proper map $h : W \to X$, which is generically étale of degree $p^r, r \geq 1$. By a theorem of Raynaud-Gruson [13, Theorem 5.2.2], there exists a blow-up $f : \widetilde{X} \to X$ with nowhere dense center $Z \subset X$ such that the following diagram commutes, where h' is finite flat surjective of degree p^r and f' is the blow-up of W with center $h^{-1}(Z)$:

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{h'} & \widetilde{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[h]{} & X. \end{array} \tag{6.3}$$

Using the cdh-excision corresponding to the cdh-cover $\{\widetilde{X} \amalg Z \to X\}$ of X and arguing as above, we get an exact sequence:

$$H^{2a-b-1}_{\widetilde{Y}\cap E}(E, \Lambda(a)) \to H^{2a-b}_{Y}(X, \Lambda(a)) \to H^{2a-b}_{\widetilde{Y}}(\widetilde{X}, \Lambda(a)) \oplus H^{2a-b}_{Y\cap Z}(Z, \Lambda(a)),$$

where we set $E = X' \times_X Z$ and $\widetilde{Y} = f^{-1}(Y)$. Since $\dim(E)$ and $\dim(Z)$ are smaller than $\dim(X)$, it follows by induction on $\dim(X)$ that $H^{2a-b-1}_{\widetilde{Y}\cap E}(E, \Lambda(a)) = 0$ and $H^{2a-b}_{Y\cap Z}(Z, \Lambda(a)) = 0$, thus the map $f^* : H^{2a-b}_{Y}(X, \Lambda(a)) \to H^{2a-b}_{\widetilde{Y}}(\widetilde{X}, \Lambda(a))$ is injective.

We now let $Z' = h^{-1}(Y)$, $\widetilde{Z} = f'^{-1}(Z')$ and consider the commutative diagram resulting from (6.3):

$$\begin{array}{ccc} H^{2a-b}_{Y}(X, \Lambda(a)) & \xrightarrow{f^*} & H^{2a-b}_{\widetilde{Y}}(\widetilde{X}, \Lambda(a)) \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h'^*} \\ H^{2a-b}_{Z'}(W, \Lambda(a)) & \xrightarrow[f'^*]{} & H^{2a-b}_{\widetilde{Z}}(\widetilde{W}, \Lambda(a)). \end{array} \tag{6.4}$$

Since $W \in \mathbf{Sm}_k$, we know that $H^{2a-b}_{Z'}(W, \Lambda(a)) = 0$. Using (6.4), it suffices therefore to show that h'^* is injective. But this follows from Lemma 6.2. The proof of the first vanishing assertion is now complete.

To prove that $H^{2a-b}_c(X, \Lambda(a)) = 0$ whenever $a > d + b$, we choose an open immersion $j : X \hookrightarrow \overline{X}$ with dense image such that $\overline{X}$ is projective over k. Letting $Z = \overline{X} \setminus X$, it follows from [37] and [18, Chapter 5] that there is a distinguished triangle in $\mathrm{DM}(k, \Lambda)$:

$$M^c(Z) \to M^c(\overline{X}) \to M^c(X) \to M^c(Z)[1]. \tag{6.5}$$

In particular, there is an exact sequence

$$H^{2a-b-1}(Z, \Lambda(d)) \to H^{2a-b}_c(X, \Lambda(a)) \to H^{2a-b}(\overline{X}, \Lambda(a)),$$

where we have replaced the cohomology with compact support by the usual motivic cohomology on the two end terms because $\overline{X}$ and Z are projective over k. It follows from the first part of the lemma that the two end terms vanish. The desired assertion now follows. □

Lemma 6.2. *Let $f : W \to X$ be a finite and flat morphism of degree p^r in $\mathbf{Sch}_k$ with $\dim(X) = d$. Let $Y \subseteq X$ be a closed subscheme and $Z = Y \times_X W$. Then the pull-back map $f^* : H^{2a-b}_Y(X, \Lambda(a)) \to H^{2a-b}_Z(W, \Lambda(a))$ is injective whenever $a > d + b$.*

Proof. When Y (and hence Z) is empty, both sides are zero by [25, Theorem 5.1] and so the lemma holds. Otherwise, we let $U = X \setminus Y$ and $V = f^{-1}(U)$. Let $v : X \to \mathrm{Spec}\,(k)$ and $u : W \to \mathrm{Spec}\,(k)$ denote the structure maps.

By [18, Definition 4.3.1, Corollary 5.2.4], the spectrum $H\Lambda \in \mathcal{SH}$ has the structure of traces. In particular, for any $m, n \in \mathbb{Z}$, there exists a trace map $\mathrm{Tr}_f : \mathbf{R}f_*\mathbf{L}f^*\mathbf{L}v^*(\Sigma^{m,n}H\Lambda) \to \mathbf{L}v^*(\Sigma^{m,n}H\Lambda)$ in $\mathcal{SH}_X$ such that its composition with the unit of adjunction $(\mathbf{L}f^*, \mathbf{R}f_*)$:

$$\mathbf{L}v^*(\Sigma^{m,n}H\Lambda) \to \mathbf{R}f_*\mathbf{L}f^*\mathbf{L}v^*(\Sigma^{m,n}H\Lambda) \xrightarrow{\mathrm{Tr}_f} \mathbf{L}v^*(\Sigma^{m,n}H\Lambda)$$

is multiplication by p^r. In particular, the composite map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{SH}_X}(\mathbf{1}_X, \mathbf{L}v^*(\Sigma^{m,n}H\Lambda)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{SH}_X}(\mathbf{1}_X, \mathbf{R}f_*\mathbf{L}f^*\mathbf{L}v^*(\Sigma^{m,n}H\Lambda)) \\ & \searrow & \downarrow f_* \\ & & \mathrm{Hom}_{\mathcal{SH}_X}(\mathbf{1}_X, \mathbf{L}v^*(\Sigma^{m,n}H\Lambda)) \end{array} \tag{6.6}$$

is multiplication by p^r, where we let f_* denote the maps on the cohomology groups induced by Tr_f. It follows that this composite map is an isomorphism.

On the other hand, we have $\mathrm{Hom}_{\mathcal{SH}_X}(\mathbf{1}_X, \mathbf{L}v^*(\Sigma^{m,n}H\Lambda)) \cong H^m(X, \Lambda(n))$ by [25, Corollary 3.6]. Moreover,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{SH}_X}(\mathbf{1}_X, \mathbf{R}f_*\mathbf{L}f^*\mathbf{L}v^*(\Sigma^{m,n}H\Lambda)) &\cong^1 \mathrm{Hom}_{\mathcal{SH}_W}(\mathbf{L}f^*(\mathbf{1}_X), \mathbf{L}f^*\mathbf{L}v^*(\Sigma^{m,n}H\Lambda)) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_W}(\mathbf{1}_W, \mathbf{L}(v \circ f)^*(\Sigma^{m,n}H\Lambda)) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_W}(\mathbf{1}_W, \mathbf{L}u^*(\Sigma^{m,n}H\Lambda)) \\ &\cong^2 H^m(W, \Lambda(n)), \end{aligned}$$

where $\cong^1$ is a consequence of adjointness of the pair $(\mathbf{L}f^*, \mathbf{R}f_*)$ and $\cong^2$ follows from [25, Corollary 3.6]. Using these isomorphisms in (6.6), we get the maps

$$H^m(X, \Lambda(n)) \xrightarrow{f^*} H^m(W, \Lambda(n)) \xrightarrow{f_*} H^m(X, \Lambda(n))$$

whose composite is an isomorphism.

We now consider the commutative diagram

$$\begin{array}{ccccccc} H^{2a-b-1}(X, \Lambda(a)) & \to & H^{2a-b-1}(U, \Lambda(a)) & \to & H^{2a-b}_Y(X, \Lambda(a)) & \to & 0 \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \\ H^{2a-b-1}(W, \Lambda(a)) & \to & H^{2a-b-1}(V, \Lambda(a)) & \to & H^{2a-b}_Z(W, \Lambda(a)) & \to & 0 \\ \downarrow f_* & & \downarrow f_* & & \vdots & & \\ H^{2a-b-1}(X, \Lambda(a)) & \to & H^{2a-b-1}(U, \Lambda(a)) & \to & H^{2a-b}_Y(X, \Lambda(a)) & \to & 0, \end{array} \tag{6.7}$$

whose rows are exact because $H^{2a-b}(X, \Lambda(a)) = H^{2a-b}(W, \Lambda(a)) = 0$, as we saw in the beginning of the proof. It follows that there exists an arrow $f_* : H^{2a-b}_Z(W, \Lambda(a)) \to H^{2a-b}_Y(X, \Lambda(a))$ such that all squares in this diagram commute. Since the left and the middle composite vertical arrows are isomorphisms, as we just observed above, it follows that the right composite vertical arrow must also be an isomorphism. In particular, $f^* : H^{2a-b}_Y(X, \Lambda(a)) \to H^{2a-b}_Z(W, \Lambda(a))$ is injective. This finishes the proof. □

Theorem 6.3. *With the notation of Definition* 3.1*, assume that f is either a closed immersion such that* $\dim(Y) < \dim(X) = d$*, or an open immersion. Then the map* $\mathrm{KGL}^0_X[\frac{1}{p}] \to s_0\mathrm{KGL}_X[\frac{1}{p}] \cong H\mathbb{Z}[\frac{1}{p}]$ *induces for every integer $i \geq 0$, an isomorphism*

$$CKH^{2d+i,d+i}(M_f)_\Lambda \xrightarrow{\cong} H^{2d+i}(M_f, \Lambda(d+i)). \tag{6.8}$$

In particular, the canonical map $\mathrm{KGL}^0_X \to \mathrm{KGL}_X$ *induces a natural cycle class map*

$$cyc_i : H^{2d+i}(M_f, \Lambda(d+i)) \to KH_i(f)_\Lambda. \tag{6.9}$$

Proof. By [25, 5.12-5.13], the argument in Theorem 5.3 also applies to $\mathbf{L}\pi^*\mathrm{KGL}^0$. In particular, there is a strongly convergent spectral sequence:

$$E_2^{a,b} = H^{a-b}(M_f, \Lambda_{b\leq 0}(n-b)) \Rightarrow CKH^{a+b,n}(M_f)_\Lambda, \tag{6.10}$$

where $\Lambda_{b\leq 0} = \Lambda$ if $b \leq 0$ and is zero otherwise. Furthermore, this spectral sequence degenerates with rational coefficients.

Combining the spectral sequence (6.10) with Lemma 6.1, we conclude that the edge map $CKH^{2d+i,d+i}(M_f)_\Lambda \to H^{2d+i}(M_f, \Lambda(d+i))$ is an isomorphism for every $i \geq 0$. Finally, we compose the inverse of this isomorphism with the canonical map $CKH^{2d+i,d+i}(M_f)_\Lambda \to KH_i(M_f)_\Lambda$ to get the desired cycle class map. □

Note that the proofs of Lemma 6.1 and Theorem 6.3 show that they remain valid without inverting p and without assuming resolution of singularities if f is a closed immersion of smooth schemes. We thus get the following:

Corollary 6.4. *Let k be a perfect field and let $f : Y \hookrightarrow X$ be a closed immersion in $\mathbf{Sm}_k$. Then there exists a cycle class map for $i \geq 0$:*

$$cyc_i : H^{2d+i}(X|Y, \mathbb{Z}(d+i)) \to K_i(X, Y)$$

whose kernel is a torsion group.

6.2. The Chern class map to relative motivic cohomology

The Chern class maps from the homotopy invariant K-theory of singular schemes to their motivic cohomology was constructed in characteristic zero in [25, Section 6]. Using the recent result of Kerz, Strunk and Tamme [20] and the double construction of [2], we can generalize the construction of [25] to the relative setting and positive characteristic as follows.

Theorem 6.5. *Let k be a perfect field. Let X be a smooth scheme over k and let $f : Y \hookrightarrow X$ be the inclusion of an effective Cartier divisor. Assume that k admits resolution of singularities. Then there are Chern class maps*

$$c_{X,Y,b} : KH_a(X, Y) \to H^{2b-a}(X|Y, \mathbb{Z}(b))$$

which are functorial in the pair (X, Y).

Proof. To prove the theorem, we shall use the doubling trick of [2]. We let $S_X = S(X, Y)$ denote the double of X along Y (see Section 2.5). Using the cdh-descent for the motivic cohomology and KH-theory, it suffices to construct the Chern class map from $KH_a(S_X, X_-)$ to $H^{2b-a}(S_X|X_-, \mathbb{Z}(b))$ (see the proof of Lemma 4.6).

After this reduction, the proof is now identical to that of [25, Theorem 6.9] with very minor modifications that we explain. We follow the notation of [25, Section 6]. We only have to show that [25, Lemma 6.4] is valid in the present case too. But this is immediate from our assumption that k admits resolution of singularities and the recent result of [20] that the canonical map $KH(Z) \to \widetilde{\mathcal{K}}_{cdh}(Z)$ (induced by the cdh-descent for KH, shown in [8]) is a weak equivalence of spectra for any $Z \in \mathbf{Sch}_k$.

Following the rest of the argument of [25, Theorem 6.9] verbatim, we obtain a commutative diagram of Chern class maps

$$\begin{array}{ccccc} KH_a(S_X) & \xrightarrow{C_b} & \mathbb{H}^{-a}_{cdh}(S_X, C_* z_{equi}(\mathbb{A}^b_k, 0)_{cdh}) & \xrightarrow{\cong} & H^{2b-a}(S_X, \mathbb{Z}(b)) \\ \downarrow{\scriptstyle \iota_+^*} & & \downarrow{\scriptstyle \iota_+^*} & & \downarrow{\scriptstyle \iota_+^*} \\ KH_a(X_-) & \xrightarrow{C_b} & \mathbb{H}^{-a}_{cdh}(X_-, C_* z_{equi}(\mathbb{A}^b_k, 0)_{cdh}) & \xrightarrow{\cong} & H^{2b-a}(X_-, \mathbb{Z}(b)). \end{array} \tag{6.11}$$

Since $\iota_+ : X_- \hookrightarrow S_X$ is naturally split by $\nabla : S_X \to X$, we get the desired natural map

$$c_{X,Y,b} : KH_a(S_X, X_-) \to H^{2b-a}(S_X|X_-, \mathbb{Z}(b)). \tag{6.12}$$

The functoriality of $c_{X,Y,b}$ is clear from its construction. We refer to the proof of [25, Theorem 6.9] for further detail. □

7. Chow group with modulus and relative motivic cohomology

We keep the assumption on the ground field k and the coefficient ring Λ as before. In this section, we shall construct a natural map from the Chow groups with modulus to the relative motivic cohomology groups. We shall later show that this map is an isomorphism for 0-cycles on affine schemes.

7.1. Higher Chow groups and motivic cohomology with support

Consider a Cartesian square of quasi-projective schemes

$$\begin{array}{ccc} Z & \xrightarrow{u} & Z' \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle w} \\ X & \xrightarrow{v} & X' \end{array} \tag{7.1}$$

where (X', X) is a smooth pair with $\dim(X') = d \geq 1$, and X is a divisor. We assume that the vertical arrows are closed immersions of codimension $q \geq 1$ such that Z' is integral and is not contained in X. In particular, (7.1) is a transverse square. Let $\mathcal{L}$ be the line bundle on X' associated to X. For any locally closed subscheme $U' \subset X'$, let $z^i_X(U', \bullet)$ be the subcomplex of Bloch's cycle complex $z^i(U', \bullet)$ generated by integral cycles which intersect X properly. Given any open subset $U' \subset X'$ and $U = X \cap U'$, there is a commutative diagram

$$\begin{array}{ccccc} z^i_X(Z' \cap U', \bullet) & \to & z^{i+q}_X(U', \bullet) & \to & z^{i+q}_X(U' \setminus Z', \bullet) \\ \downarrow & & \downarrow & & \downarrow \\ z^i(Z' \cap U', \bullet) & \to & z^{i+q}(U', \bullet) & \to & z^{i+q}(U' \setminus Z', \bullet). \end{array} \tag{7.2}$$

The localization theorem for Bloch's complex says that the bottom row is a distinguished triangle in the derived category of chain complexes of Abelian groups. Since $X \cap (X' \setminus Z') \neq \emptyset$ and $X \subset X'$ is a divisor, the proof of Bloch's localization theorem shows easily that the top row is also a distinguished triangle. By [23, Theorem 1.10] it follow that the middle and the right vertical arrows are quasi-isomorphisms. So the left vertical arrow is also a quasi-isomorphism.

On the other hand, there is a commutative diagram of cycle complexes

$$\begin{array}{ccccc} z^i_X(Z' \cap U', \bullet) & \to & z^{i+q}_X(U', \bullet) & \to & z^{i+q}_X(U' \setminus Z', \bullet) \\ {\scriptstyle u^*}\downarrow & & \downarrow{\scriptstyle v^*} & & \downarrow{\scriptstyle v^*} \\ z^i(Z \cap U, \bullet) & \longrightarrow & z^{i+q}(U, \bullet) & \to & z^{i+q}(U \setminus Z, \bullet). \end{array} \tag{7.3}$$

In particular, there is a pull-back map $u^* : \mathrm{CH}^i(Z', j) \to \mathrm{CH}^i(Z, j)$ which is induced by capping with the first Chern class $c_1(\mathcal{L})$. Furthermore, it is immediate that (7.2) and (7.3) are compatible with respect to the inclusions of open subsets $U'_1 \subset U'_2$ in X'. By varying the open $U' \subset X'$, it follows that (7.2) and (7.3) form commutative diagrams of complexes of presheaves of Abelian groups on the small Zariski site of X'.

It is shown in [37, Proposition 4.2.9] that there is a monomorphism of chain complexes $\psi_{U'} : C_* z_{equi}(U', i) \hookrightarrow z^{d-i}(U', \bullet)$ which is functorial with respect to flat pull-back and proper push-forward. In particular, this induces a monomorphism of complexes of presheaves of Abelian groups on the small Zariski site of X'. Note here that by $C_* z_{equi}(U', i)$, we mean $C_* z_{equi}(U', i)(\mathrm{Spec}\,(k))$. For any complex $\mathcal{C}$ of presheaves of Abelian groups on the small Zariski site of X', we let $\mathcal{C}_{\mathrm{zar}}$ denote the sheafification of $\mathcal{C}$.

We get a sequence of morphisms of complexes of Zariski sheaves

$$\begin{array}{ccccc} C_* z_{equi}(-, i) & \longrightarrow & z^{d-i}(-, \bullet)_{\mathrm{zar}} & \longleftarrow & z^{d-i}_X(-, \bullet)_{\mathrm{zar}} \\ & & & & \downarrow \cap X \\ & & C_* z_{equi}(-, i-1)|_X & \to & z^{d-i}(-, \bullet)_{\mathrm{zar}}|_X . \end{array} \tag{7.4}$$

Note that $C_* z_{equi}(-, i)$ is already a complex of Zariski sheaves. All arrows except the vertical one in this diagram are quasi-isomorphisms of complexes of Zariski sheaves (see [37, Proposition 4.2.9]). It follows that there is a morphism $v^* : C_* z_{equi}(-, i) \to C_* z_{equi}(-, i-1)|_X$ in the derived category of complexes of Zariski sheaves and a commutative diagram

$$\begin{array}{ccccc} C_* z_{equi}(-, i)|_{Z'} & \longrightarrow & C_* z_{equi}(-, i)|_{X'} & \longrightarrow & C_* z_{equi}(-, i)|_{X' \setminus Z'} \\ & & \downarrow v^* & & \downarrow v^* \\ C_* z_{equi}(-, i-1)|_Z & \to & C_* z_{equi}(-, i-1)|_X & \to & C_* z_{equi}(-, i-1)|_{X \setminus Z} . \end{array} \tag{7.5}$$

It follows from (7.3) and (7.4) that the two rows in (7.5) are distinguished triangles in the derived category of complexes of Zariski sheaves and the right square commutes. We therefore obtain a morphism $u^* : C_* z_{equi}(-, i)|_{Z'} \to C_* z_{equi}(-, i-1)|_Z$ such that (7.5) commutes. Moreover, (7.4) implies that

$$\begin{array}{ccc} C_* z_{equi}(-, i)|_{Z'} & \longrightarrow & z^{d-i-q}(-, \bullet)_{\mathrm{zar}}|_{Z'} \\ u^* \downarrow & & \downarrow u^* \\ C_* z_{equi}(-, i-1)|_Z & \to & z^{d-i-q}(-, \bullet)_{\mathrm{zar}}|_Z \end{array} \tag{7.6}$$

commutes.

It follows from [37, Proposition 4.2.9, Theorem 4.3.7] (see also [28, Theorem 19.11]) that for any $W \in \mathbf{Sm}_k$, the Zariski hypercohomology of $C_* z_{equi}(-, i)|_W$ are the motivic cohomology groups of W. In particular, the commutative square of hypercohomology groups induced by the right square in (7.5) is isomorphic to the one induced on the motivic cohomology by the commutative diagram of motives

$$\begin{array}{ccc} M(X \setminus Z) & \longrightarrow & M(X) \\ v \downarrow & & \downarrow v \\ M(X' \setminus Z') & \to & M(X'). \end{array} \tag{7.7}$$

We conclude from (7.5) that the hypercohomology of $C_* z_{equi}(-, i)|_{Z'}$ and $C_* z_{equi}(-, i)|_Z$ are the motivic cohomology groups of $M_{Z'}(X')$ and $M_Z(X)$, respectively. Moreover, the left vertical arrow in (7.6) is the one induced by the canonical map $u : M_Z(X) \to M_{Z'}(X')$. We have thus shown the following.

Lemma 7.1. *Given the commutative diagram* (7.1)*, there are induced maps of motives* $v : M_Z(X) \to M_{Z'}(X')$ *and cycle complexes* $u^* : z^i(Z', \bullet) \to z^i(Z, \bullet)$*, and a commutative diagram of associated hypercohomology groups*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{DM}_k}(M_{Z'}(X'), \mathbb{Z}(i)[j]) & \xrightarrow{\alpha_{Z'}} & \mathrm{CH}^{i-q}(Z', 2i-j) \\ {\scriptstyle v^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathrm{Hom}_{\mathrm{DM}_k}(M_Z(X), \mathbb{Z}(i)[j]) & \xrightarrow{\alpha_Z} & \mathrm{CH}^{i-q}(Z, 2i-j) \end{array} \tag{7.8}$$

in which the horizontal arrows are isomorphisms.

7.2. Relation between Chow group and motivic cohomology

Let X be a smooth quasi-projective scheme of dimension $d \geq 0$ over k and let $D \subset X$ be an effective Cartier divisor. We let $U = X \setminus D$. We let S_X denote the double of X along D (see Section 2.5). We have the inclusions $U \cong U_\pm \subset X_\pm \subset S_X$.

Let $q \geq 1$ be an integer and let $Z \subset X$ be an integral cycle in $z^q(X|D, 0)$. The modulus condition implies that $Z \subset U$. We consider the embeddings $Z \xrightarrow{i} U_+ = S_X \setminus X_- \xrightarrow{j_+} S_X$, where the composite map is a closed immersion. We therefore have a sequence of maps

$$\begin{array}{ccc} \mathrm{CH}^0(Z)_\Lambda & & \\ {\scriptstyle \alpha_Z^{-1}}\downarrow\cong & & \\ \mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M_Z(U_+), \Lambda(q)[2q]) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M_Z(S_X), \Lambda(q)[2q]) \\ & & \downarrow \\ H^{2q}(S_X, \Lambda(q)) & \xleftarrow{\cong} & \mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M(S_X), \Lambda(q)[2q]). \end{array} \tag{7.9}$$

We let $\lambda_Z : \mathrm{CH}^0(Z)_\Lambda \to H^{2q}(S_X, \Lambda(q))$ denote the composite of all arrows in (7.9) and set $\lambda_{X|D}([Z]) = \Lambda_Z(1)$. We extend it linearly to define a group homomorphism $\lambda_{X|D} : z^q(X|D, 0) \to H^{2q}(S_X, \Lambda(q))$.

Lemma 7.2. *The map* $\lambda_{X|D}$ *induces a group homomorphism*

$$\lambda_{X|D} : \mathrm{CH}^q(X|D)_\Lambda \to H^{2q}(S_X, \Lambda(q)).$$

Proof. To prove the lemma, we consider the diagram

$$\begin{array}{ccccc} z^q(X|D, 1) & \hookrightarrow & z^q(\mathbb{A}^1_X|\mathbb{A}^1_D, 0) & \xrightarrow[\lambda_{\mathbb{A}^1_X|\mathbb{A}^1_D}]{} & H^{2q}(S_{\mathbb{A}^1_X}, \Lambda(q)) \\ {\scriptstyle \partial_1^*-\partial_0^*}\downarrow & & & & \downarrow{\scriptstyle \iota_1^*-\iota_0^*} \\ z^q(X|D, 0) & & \xrightarrow[\lambda_{X|D}]{} & & H^{2q}(S_X, \Lambda(q)). \end{array} \tag{7.10}$$

To show that $\Lambda_{X|D}$ kills the subgroup of cycles rationally equivalent to zero is equivalent to showing that $\lambda_{X|D}\circ(\partial_1^* - \partial_0^*) = 0$. It follows from [2, Proposition 2.3] that $S_{\mathbb{A}^1_X}$ is canonically isomorphic to $\mathbb{A}^1_{S_X}$ and the right vertical arrow in (7.10) is induced by the inclusion $\iota_t : S_X \hookrightarrow \mathbb{A}^1_{S_X}$ for $t = 0, 1$. It follows from the homotopy invariance of the motivic cohomology that the right vertical arrow in (7.10) is zero. Our assertion will therefore follow if we show that (7.10) commutes.

Let $W \in z^q(X|D, 1)$ be an integral cycle which intersects the faces of $\mathbb{A}^1_X$ properly and whose closure in $\mathbb{P}^1_X$ satisfies the modulus D. To show the commutativity of (7.10), it suffices to show that the diagram

$$\begin{array}{ccc}
 & & \bigoplus_{i=1}^{r_t} \mathrm{CH}^0(Z_i)_\Lambda \\
 & & \downarrow\cong \\
\mathrm{CH}^0(W)_\Lambda & \xrightarrow{\iota_t^*} & \mathrm{CH}^0(W_t)_\Lambda \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M_W(\mathbb{A}^1_{U_+}), \Lambda(q)[2q]) & \xrightarrow{\iota_t^*} & \mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M_{W_t}(U_+), \Lambda(q)[2q]) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M_W(\mathbb{A}^1_{S_X}), \Lambda(q)[2q]) & \xrightarrow{\iota_t^*} & \mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M_{W_t}(S_X), \Lambda(q)[2q]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M(\mathbb{A}^1_{S_X}), \Lambda(q)[2q]) & \xrightarrow{\iota_t^*} & \mathrm{Hom}_{\mathrm{DM}(k,\Lambda)}(M(S_X), \Lambda(q)[2q]) \\
\cong\downarrow & & \downarrow\cong \\
H^{2q}(\mathbb{A}^1_{S_X}, \Lambda(q)) & \xrightarrow{\iota_t^*} & H^{2q}(S_X, \Lambda(q))
\end{array} \tag{7.11}$$

commutes for $t = 0, 1$, where $\{Z_1, \cdots, Z_{r_t}\}$ are the irreducible components of $W_t = \iota_t^*(W)$. Note that $\iota_t^*(1)$ is the cycle class of $W_t = \partial_t^*([W])$ in $\mathrm{CH}^0(W_t)_\Lambda \cong \Lambda^{r_t}$.

The bottom three squares commute by the commutativity of the diagram

$$\begin{array}{ccccc}
M(S_X) & \to & M_{W_t}(S_X) & \xleftarrow{\cong} & M_{W_t}(U_+) \\
\iota_t\downarrow & & \downarrow\iota_t & & \downarrow\iota_t \\
M(\mathbb{A}^1_{S_X}) & \to & M_W(\mathbb{A}^1_{S_X}) & \xleftarrow{\cong} & M_W(\mathbb{A}^1_{U_+}).
\end{array} \tag{7.12}$$

The top square commutes by Lemma 7.1. We conclude that (7.11) commutes and this completes the proof. □

Our main result on the relation between cycles with modulus and relative motivic cohomology is the following.

Theorem 7.3. *Let k be a perfect field. Let X be a smooth quasi-projective scheme of dimension $d \geq 0$ over k and let $D \subset X$ be an effective Cartier divisor. Then the following hold for every integer $q \geq 0$.*

(1) *The map* $\lambda_{X|D} : z^q(X|D, 0) \to H^{2q}(S_X, \Lambda(q))$ *induces a map*

$$\lambda_{X|D} : \mathrm{CH}^q(X|D)_\Lambda \to H^{2q}(X|D, \Lambda(q)); \tag{7.13}$$

(2) *If* $D \in \mathbf{Sm}_k$, *then the inclusion* $z^q(X|D, \bullet) \hookrightarrow z^q(X, \bullet)$ *induces, for all* $i \geq 0$, *a map*

$$\lambda_{X|D} : \mathrm{CH}^q(X|D, i) \to H^{2q-i}(X|D, \mathbb{Z}(q)). \tag{7.14}$$

Proof. We have a commutative diagram with exact bottom row (see equation (4.7) and Proposition 4.7):

$$\begin{array}{ccccccccc} & & & & \mathrm{CH}^q(X|D)_\Lambda & & & & \\ & & & \swarrow & \downarrow{\scriptstyle \lambda_{X|D}} & & & & \\ 0 \to & H^{2q}(X|D, \Lambda(q)) & \xrightarrow{\nu_{+,*}} & & H^{2q}(S_X, \Lambda(q)) & \xrightarrow{\iota_-^*} & H^{2q}(X, \Lambda(q)) & \to 0. \end{array} \tag{7.15}$$

It is therefore enough to show that $\iota_-^* \circ \lambda_{X|D} = 0$. But this is clear from the construction of $\lambda_{X|D}$ in (7.9) because $M_Z(X_-) = 0$ and the diagram

$$\begin{array}{ccccc} M(X_-) & \xrightarrow{\iota_-} & M(S_X) & \longleftarrow & M(U_+) \\ \downarrow & & \downarrow & & \downarrow \\ M_Z(X_-) & \xrightarrow{\iota_-} & M_Z(S_X) & \xleftarrow{\cong} & M_Z(U_+) \end{array} \tag{7.16}$$

commutes for any integral cycle $Z \in z^q(X|D, 0)$. This proves (1).

We now prove (2). Using Theorem 5.6, it is enough to construct a map $\mathrm{CH}^q(X|D, i) \to \mathrm{CH}^q_{\mathcal{M}}(X|D, i)$. But this follows from the observation that

$$z^q_{\mathcal{M}}(X|D, \bullet) \to z^q_D(X, \bullet) \to z^q(D, \bullet) \to z^q_{\mathcal{M}}(X|D, \bullet)[1]$$

is a distinguished triangle in the derived category of Abelian groups and the composite map $z^q(X|D, \bullet) \to z^q_D(X, \bullet) \to z^q(D, \bullet)$ is zero. Hence, the inclusion $z^q(X|D, \bullet) \hookrightarrow z^q_D(X, \bullet)$ factors through a map $\lambda_{X|D} : z^q(X|D, \bullet) \to z^q_{\mathcal{M}}(X|D, \bullet)$ in the derived category. In particular, it induces the desired map between the homology groups. □

7.3. Cycle class map for Chow groups with modulus

If the relative K-theory is to be described by higher Chow groups with modulus, as conjectured, then there must exist a cycle class map from the higher 0-cycles with modulus to the relative higher K-groups. As a consequence of Corollary 6.4, Theorem 7.3, and the weak equivalence of spectra $K(X, D) \to KH(X, D)$, it follows immediately that this is indeed the case if $D \subset X$ is a smooth divisor.

Corollary 7.4. *Let* k *be a perfect field and let* $D \subset X$ *be an inclusion of a smooth divisor in* $\mathbf{Sm}_k$. *Then there exists a cycle class map for* $i \geq 0$:

$$cyc_i : \mathrm{CH}^{d+i}(X|D, i) \to K_i(X, D).$$

A cycle class map of the kind given in Corollary 7.4 was constructed in [1]. In that construction, Binda uses a different definition for the Chow groups with modulus compared to the ones described in Section 2.3. His modulus condition is stronger and this allows him to prove the result when D is a simple normal crossing divisor. A very general cycle class map for our definition of the Chow groups with modulus is constructed in [14] (where D is allowed to be any Cartier divisor). However, this construction exists only in the pro-setting where we need to consider the pro-Abelian groups $\{\mathrm{CH}^{d+i}(X|mD, i)\}_{m\geq 1}$. The main point of the new result Corollary 7.4 is that it shows that for smooth pairs, we do not need to go to the pro-setting.

8. The isomorphism theorem

The goal of this section is to show that the comparison map $\lambda_{X|D}$ of Theorem 7.3 is an isomorphism for 0-cycles on affine schemes. Our strategy for showing this is to use the doubling trick once again and combine this with Theorem 2.5. Other crucial ingredient is the Roitman torsion theorem of [21].

Throughout this section, we fix an algebraically closed field k. We also fix a smooth quasi-projective scheme X of dimension d over k. We let $D \subset X$ be a smooth divisor. Let $S_X = S(X, D)$ denote the double of X along D. We let $U = X \setminus D$ and $S_U = \nabla^{-1}(U) = U_+ \amalg U_-$, where $\nabla : S_X \to X$ is the fold map.

We first construct a comparison map for S_X. Let $x \in S_U$ be a closed point. We let $S = \mathrm{Spec}\,(k(x))$. Since $\iota_S : S \hookrightarrow S_X$ is a local complete intersection (lci), it follows from [29, Definition 2.32, Theorem 2.33] that there is a push-forward map $\iota_{S,*} : \mathbb{Z} \xrightarrow{\cong} H^0(S, \Lambda(0)) \to H^{2d}(S_X, \mathbb{Z}(d))$. Since (S_U, S) is a smooth pair, the map ι_* is induced by the maps of motives

$$M(S_X) \to M_S(S_X) \xleftarrow{\cong} M_S(S_U) \xrightarrow{\cong} M(S)(d)[2d]. \tag{8.1}$$

We let $\gamma_{S_X}([x]) = \iota_{S,*}(1) \in H^{2d}(S_X, \mathbb{Z}(d))$ and extend this construction linearly to get a map

$$\gamma_{S_X} : \mathcal{Z}_0(S_X) \to H^{2d}(S_X, \mathbb{Z}(d)). \tag{8.2}$$

Lemma 8.1. *The map γ_{S_X} descends to a group homomorphism $\gamma_{S_X} : \mathrm{CH}_0(S_X) \to H^{2d}(S_X, \mathbb{Z}(d))$ such that the diagram*

$$\begin{array}{ccc} \mathrm{CH}_0(S_X) & \xrightarrow{cyc_{S_X}} & K_0(S_X) \\ \gamma_{S_X}\downarrow & & \downarrow\kappa_{S_X} \\ H^{2d}(S_X, \mathbb{Z}(d)) & \xrightarrow{cyc_{S_X}} & KH_0(S_X) \end{array} \tag{8.3}$$

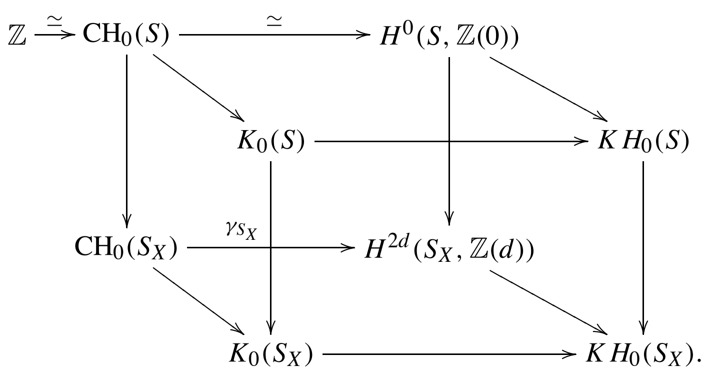
is commutative.

Proof. By the moving lemma, we can write $\mathrm{CH}_0(X) = \mathcal{Z}_0(U)/\mathcal{R}_0(X/U)$, where $\mathcal{R}_0(X/U) = \mathcal{Z}_0(U) \cap \mathcal{R}_0(X) \subset \mathcal{Z}_0(X)$. We then have a diagram of split exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{Z}_0(X|D) & \xrightarrow{p_{+*}} & \mathcal{Z}_0(S_X) & \xrightarrow{\iota_-^*} & \mathcal{Z}_0(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{CH}_0(X|D) & \xrightarrow{p_{+*}} & \mathrm{CH}_0(S_X) & \xrightarrow{\iota_-^*} & \mathrm{CH}_0(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_{X|D}} & & \downarrow{\scriptstyle \gamma_{S_X}} & & \downarrow{\scriptstyle \lambda_X} & & \\
0 & \to & H^{2d}(X|D, \mathbb{Z}(d)) & \xrightarrow{p_{+*}} & H^{2d}(S_X, \mathbb{Z}(d)) & \xrightarrow{\iota_-^*} & H^{2d}(X, \mathbb{Z}(d)) & \to & 0.
\end{array} \tag{8.4}$$

It is immediate from the definition of $\lambda_{X|D}$ in (7.9) and γ_{S_X} in (8.2) that all squares in the outer diagram (ignoring the middle row) in (8.4) commute. Let $\alpha \in \mathcal{Z}_0(S_X)$ be such that it dies in $\mathrm{CH}_0(S_X)$. We can uniquely write $\alpha = p_{+*}(\alpha_1) + \nabla^*(\alpha_2)$. Since $\iota_-^* \circ \nabla^*$ is identity, we must have that $\alpha_1 \in \mathcal{R}_0(X|D)$ and $\alpha_2 \in \mathcal{R}_0(X/U) \subset \mathcal{Z}_0(U)$. But then α_1 must die in $H^{2d}(X|D, \mathbb{Z}(d))$ by Theorem 7.3(2), and hence it must die in $H^{2d}(S_X, \mathbb{Z}(d))$. Similarly, α_2 must die in $H^{2d}(X, \mathbb{Z}(d))$. In particular, we must have $\gamma_{S_X}(\nabla^*(\alpha_2)) = 0$ in $H^{2d}(S_X, \mathbb{Z}(d))$. We conclude that $\gamma_{S_X}(\alpha) = \gamma_{S_X}(p_{+*}(\alpha_1)) + \gamma_{S_X}(\nabla^*(\alpha_2)) = 0$. This proves the first part.

To show that (8.3) commutes, we choose a closed point $x \in S_U$, set $S = \mathrm{Spec}\,(k(x))$ and consider the diagram



(8.5)

To show that the bottom face of this cube commutes, it suffices to show that its all other faces commute. Now, the top and front face clearly commute and the left face commutes by the definition of the cycle class map (see [2, Lemma 3.12]). The back face commutes by definition of γ_{S_X}. To show that the right face commutes, we can break it into a diagram

$$\begin{array}{ccccc} H^0(S,\mathbb{Z}(0)) & \to & H^{2d}_S(S_X,\mathbb{Z}(d)) & \to & H^{2d}(S_X,\mathbb{Z}(d)) \\ \downarrow & & \downarrow & & \downarrow \\ KH_0(S) & \longrightarrow & KH^S_0(S_X) & \longrightarrow & KH_0(S_X). \end{array} \tag{8.6}$$

The left square clearly commutes and the right square commutes by Theorem 5.2 and Theorem 5.6. The lemma is now proven. □

The final result of this paper is the following comparison theorem for 0-cycles.

Theorem 8.2. *Let X be a smooth affine scheme of dimension $d \geq 1$ over an algebraically closed field k and let $D \subset X$ be a smooth divisor. Then the map*

$$\lambda_{X|D} : \mathrm{CH}_0(X|D) \to H^{2d}(X|D, \mathbb{Z}(d))$$

is an isomorphism.

Proof. Since (X, D) is a smooth pair, the double S_X is a simple normal crossing variety in the sense of [10, Section 2.1]. In particular, it follows from [10, Proposition 6.4] that γ_{S_X} is surjective. We remark here that the surjectivity of γ_V is proven in the above cited work for an arbitrary simple normal crossing variety V if we work with Λ-coefficients (see the end of Section 2). However, it is an elementary checking that the proof yields this surjectivity with integral coefficients if we let $V = S_X$. We conclude from (8.4) that $\lambda_{X|D}$ is surjective.

To show that $\lambda_{X|D}$ is injective, it suffices to show using (8.4) that γ_{S_X} is injective. Using (8.3), it suffices to show that $\kappa_{S_X} \circ cyc_{S_X}$ is injective. By [21, Corollary 6.8], it suffices to show that κ_{S_X} is injective.

Since excision holds for the K-theory of affine schemes in degrees up to zero (see [2, Proposition 11.3]), and since it holds for KH-theory in all degrees [8], there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} K_1(S_X^N) & \longrightarrow & K_1(D) & \longrightarrow & K_0(S_X) & \longrightarrow & K_0(S_X^N) & \longrightarrow & K_0(D) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ KH_1(S_X^N) & \to & KH_1(D) & \to & KH_0(S_X) & \to & KH_0(S_X^N) & \to & KH_0(D). \end{array} \tag{8.7}$$

Since X and D are regular, all vertical arrows except possibly the middle one are isomorphisms. It follows that middle vertical arrow is also an isomorphism. In other words, κ_{S_X} is in fact an isomorphism. This finishes the proof. □

Remark 8.3. The proof of Theorem 8.2 shows that $\lambda_{X|D}$ is surjective even if X is not affine.

References

[1] F. BINDA, *A cycle class map from Chow groups with modulus to relative K-theory*, Doc. Math. **23** (2018), 407–444.

[2] F. BINDA and A. KRISHNA, *Zero cycles with modulus and zero cycles on singular varieties*, Compos. Math. **154** (2018), 120–187.

[3] F. BINDA and S. SAITO, *Relative cycles with moduli and regulator maps*, J. Math. Inst. Jussieu (2017), to appear, arXiv:1412.0315.

[4] S. BLOCH, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[5] S. BLOCH, *The moving lemma for higher Chow groups*, J. Algebraic Geom. **3** (1994), 537–568.

[6] S. BLOCH and H. ESNAULT, *The additive dilogarithm*, Doc. Math., Extra Vol.: Kazuya Kato's fiftieth birthday (2003), 131–155.

[7] S. BLOCH and S. LICHTENBAUM, *A spectral sequence for motivic cohomology*, unpublished.

[8] D. CISINSKI, *Descente par éclatements en K-théorie invariante par homotopie*, Ann. of Math. **177** (2013), 425–448.

[9] D.-C. CISINSKI and F. DÉGLISE, *Integral mixed motives in equal characteristic*, Doc. Math., Extra Vol.: Alexander S. Merkurjev's Sixtieth Birthday (2015), 145–194.

[10] H. ESNAULT, M. KERZ and O. WITTENBERG, *A restriction isomorphism for cycles of relative dimension zero*, Camb. J. Math. **4** (2016), 163–196.

[11] E. FRIEDLANDER and A. SUSLIN, *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Sci. École Norm. Sup. **35** (2002), 773–875.

[12] E. FRIEDLANDER and V. VOEVODSKY, *Bivariant cycle cohomology*, In: "Cycles, Transfers and Motivic Homology Theories", Ann. of Math. Stud., Vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, 138–187.

[13] L. GRUSON and M. RAYNAUD, *Critères de platitude et de projectivité*, Invent. Math. **13** (1971), 1–89.

[14] R. GUPTA and A. KRISHNA, *Zero cycles with modulus and relative K-theory*, Ann. K-Theory, to appear, 2020.

[15] M. HOVEY, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra **165** (2001), 63–127.

[16] D. ISAKSEN, *Flasque model structures for simplicial presheaves*, K-Theory **36** (2005), 371–395.

[17] K. KATO and S. SAITO, *Global class field theory of arithmetic schemes*, Applications of algebraic K-theory to algebraic geometry and number theory, Contemp. Math., Vol. 55, Amer. Math. Soc., Providence, RI, 1986, 255–331.

[18] S. KELLY, *Triangulated categories of motives in positive characteristic*, arXiv:1305.5349, (2013).

[19] M. KERZ and S. SAITO, *Chow group of* 0*-cycles with modulus and higher-dimensional class field theory*, Duke Math. J. **165** (2016), 2811–2897.

[20] M. KERZ, F. STRUNK and G. TAMME, *Algebraic K-theory and descent for blow-ups*, Invent. Math. **211** (2018), 523–577.

[21] A. KRISHNA, *Murthy's conjecture on* 0*-cycles*, Invent. Math. **217** (2019), 549–602.

[22] A. KRISHNA, *Torsion in the* 0*-cycle group with modulus*, Algebra Number Theory **12** (2018), 1431–1469.

[23] A. KRISHNA and M. LEVINE, *Additive higher Chow groups of schemes*, J. Reine Angew. Math. **619** (2008), 75–140.

[24] A. KRISHNA and J. PARK, *A module structure and a vanishing theorem for cycles with modulus*, Math. Res. Lett. **24** (2017), 1149–1178.

[25] A. KRISHNA and P. PELAEZ, *Slice spectral sequence for singular schemes and applications* Ann. K-Theory **3** (2018), 657–708.

[26] M. LEVINE, *The homotopy coniveau tower*, J. Topol. **1** (2008). 217–267.

[27] M. LEVINE and C. WEIBEL, *Zero cycles and complete intersections on singular varieties*, J. Reine Angew. Math. **359** (1985), 106–120.
[28] C. MAZZA, V. VOEVODSKY and C. WEIBEL, "Lecture Notes on Motivic Cohomology", Clay Mathematics Monographs, Vol. 2, American Mathematical Society, Providence, 2006.
[29] A. NAVARRO, *Riemann-Roch for homotopy invariant K-theory and Gysin morphisms* (2016), arXiv:1605.00980.
[30] P. PELAEZ, *On the functoriality of the slice filtration*, J. K-Theory **11** (2013), 55–71.
[31] J. RIOU, *Algebraic K-theory,* $\mathbf{A}^1$*-homotopy and Riemann-Roch theorems*, J. Topol. **3** (2010), 229–264.
[32] A. SUSLIN and V. VOEVODSKY, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, In: "The Arithmetic and Geometry of Algebraic Cycles", NATO Sci. Ser. C Math. Phys. Sci., Vol. 548, Kluwer Acad, Publ., Dordrecht, 2000, 117–189.
[33] M. TEMKIN, *Tame destillation and desingularization by p-alterations*, Ann. of Math. **186** (2017), 1–30.
[34] R. THOMASON and T. TROBAUGH, "Higher Algebraic K-Theory of Schemes and of Derived Categories", The Grothendieck Festschrift III, Progress in Math., Vol. 88, Birkhäuser.
[35] V. VOEVODSKY, *A possible new approach to the motivic spectral sequence for algebraic K-theory*, In: "Recent Progress in Homotopy Theory", Contemp. Math., Vol. 293, Amer. Math. Soc., Providence RI, 2002, 371–379.
[36] V. VOEVODSKY, $\mathbb{A}^1$*-homotopy theory*, In: "Proceedings of the International Congress of Mathematicians", Vol. I, (Berlin, 1998), Doc. Math., Extra Vol. I, 1998, 579–604.
[37] V. VOEVODSKY, *Triangulated categories of motives over a field*, In: "Cycles, Transfers, and Motivic Homology Theories", Ann. of Math. Stud., Vol. 143, Princeton Univ. Press, Princeton NJ, 2000, 188–238.
[38] V. VOEVODSKY, *Cohomological theory of presheaves*, In: "Cycles, Transfers, and Motivic Homology Theories", Ann. of Math. Stud., Vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, 87–137.
[39] C. WEIBEL, *Homotopy algebraic K-theory*, In: "Algebraic K-Theory and Algebraic Number Theory", (Honolulu, HI, 1987), Contemp. Math., Vol. 83, Amer. Math. Soc., Providence, RI, 1989, 461–488.

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