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The maximal operator of a normal Ornstein-Uhlenbeck semigroup is of weak type (1, 1)

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Abstract. Consider a normal Ornstein-Uhlenbeck semigroup in Euclidean space, whose covariance is given by a positive definite matrix. The drift matrix is assumed to have eigenvalues only in the left half-plane. We prove that the associated maximal operator is of weak type (1, 1) with respect to the invariant measure. This extends earlier work by G. Mauceri and L. Noselli. The proof goes via the special case where the matrix defining the covariance is I and the drift matrix is diagonal.

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1. Introduction

Let Q be a real, symmetric and positive definite $n \times n$ matrix, and B a real $n \times n$ matrix whose eigenvalues have negative real parts; here $n \geq 1$. One defines the covariance matrices

$$Q_t = \int_0^t e^{sB}\, Q\, e^{sB^*} ds\,, \qquad t \in (0, +\infty],$$

and the family of Gaussian measures in $\mathbb{R}^n$

$$d\gamma_t(x) = (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_t^{-1} x, x\rangle} dx\,, \qquad t \in (0, +\infty].$$

Here $d\gamma_\infty$ is the unique invariant measure.

On the space $\mathcal{C}_b(\mathbb{R}^n)$ of bounded continuous functions, we consider the Ornstein-Uhlenbeck semigroup $\big(\mathcal{H}_t^{Q,B}\big)_{t>0}$, explicitly given by the Kolmogorov formula

$$\mathcal{H}_t^{Q,B} f(x) = \int f\big(e^{tB}x - y\big) d\gamma_t(y)\,, \quad x \in \mathbb{R}^n\,,$$

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(see [7]). Its infinitesimal generator is given by

$$\mathcal{L}^{Q,B} f = \frac{1}{2} \operatorname{tr}\left(Q\nabla^2 f\right) + \langle Bx, \nabla f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

and $\mathcal{S}(\mathbb{R}^n)$ is a core of $\mathcal{L}^{Q,B}$. Here $Q\nabla^2 f$ denotes the product of Q and the Hessian matrix of f.

The relevance of this semigroup is also due to the fact that $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ is the transition semigroup of the Ornstein-Uhlenbeck process

$$\mathcal{X}(t,x) = e^{tB}x + \int_0^t e^{(t-s)B} dW(s)$$

on $\mathbb{R}^n$, where W denotes an n-dimensional Brownian motion with covariance matrix Q. This process describes the random motion of a particle subject to friction; *cf.* [15] or [4].

Among its various properties, we only recall here that $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ is strongly continuous in $\mathcal{C}_0(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$ for all $1 \le p < \infty$ [2,3,8], while strong continuity fails to hold in the space of bounded, uniformly continuous functions in $\mathbb{R}^n$ endowed with the supremum norm [3, Lemma 3.2], [19]. For some relevant results about differentiability and analyticity of $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ in the L^p spaces, we refer the reader to [2,12].

We consider the maximal operator

$$\mathcal{H}_*^{Q,B} f(x) = \sup_{t>0} \left|\mathcal{H}_t^{Q,B} f(x)\right|, \qquad t > 0, \tag{1.1}$$

which is an essential tool in the study of the almost everywhere convergence of $\mathcal{H}_t^{Q,B} f$ as $t \to 0$ for $f \in L^p(\gamma_\infty)$, $1 \le p < \infty$.

The boundedness properties of $\mathcal{H}_*^{Q,B}$ are essentially known when $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ is *symmetric*, *i.e.*, when $\mathcal{H}_t^{Q,B}$ is self-adjoint on $L^2(\gamma_\infty)$ for all $t > 0$. Indeed, for $1 < p \le \infty$, the boundedness of $\mathcal{H}_*^{Q,B}$ on $L^p(\gamma_\infty)$ then follows from the general Littlewood-Paley-Stein theory for symmetric semigroups of contractions on Lebesgue spaces [18].

G. Mauceri and L. Noselli [9] addressed the nonsymmetric case, assuming only that $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ is *normal*, *i.e.*, that $\mathcal{H}_t^{Q,B}$ is for each $t > 0$ a normal operator on $L^2(\gamma_\infty)$. Then, by generalizing Stein's results to a semigroup of normal contractions whose infinitesimal generator is a sectorial operator of angle less than $\pi/2$, they were able to prove that $\mathcal{H}_*^{Q,B}$ is bounded on $L^p(\gamma_\infty)$, for all $1 < p \le \infty$.

Since the operator $\mathcal{H}_*^{Q,B}$ is always unbounded on $L^1(\gamma_\infty)$, one is led to analyze the weak type (1, 1) of the maximal operator. This means seeking an estimate of the form

$$\gamma_\infty\left\{x \in \mathbb{R}^n : \mathcal{H}_*^{Q,B} f(x) > \alpha\right\} \lesssim \frac{\|f\|_1}{\alpha},$$

holding for all $\alpha > 0$ and all $f \in L^1(\gamma_\infty)$. In the special case $Q = I$ and $B = -I$, which is symmetric, this was proved by B. Muckenhoupt in the one-dimensional case [14] and by the third author in higher dimension [17]; the proof in [17] was then simplified by T. Menárguez, S. Pérez and F. Soria [11] (see also [10, 16]). Another simple argument is given in [6]. For a nice discussion of the different techniques we refer the reader to [1].

In [9] Mauceri and Noselli applied a factorization known from [13], saying that an arbitrary normal Ornstein-Uhlenbeck semigroup $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ can be written as the product of more elementary semigroups, called building blocks. Each building block is an Ornstein-Uhlenbeck semigroup with $Q = I$ and $B = \lambda(R - I)$, for some positive λ and a real skew-adjoint matrix R. Mauceri and Noselli were able to prove that for such a building block the truncated maximal operator, defined by taking the supremum in (1.1) only over $0 < t \leq T < \infty$, is of weak type (1, 1). If, in addition, R generates a periodic group, they proved that the full maximal operator $H_*^{Q,B}$ is of weak type (1, 1). The case when the semigroup involves several building blocks seems not to have been considered as yet. Indeed, Mauceri and Noselli write “already the case where B is a diagonal matrix with at least two different eigenvalues seems to require new ideas”.

In this paper, we give the complete solution of the problem studied in [9], as follows.

Theorem 1.1. *The maximal operator* $\mathcal{H}_*^{Q,B}$ *of an arbitrary normal Ornstein-Uhlenbeck semigroup* $\left(\mathcal{H}_t^{Q,B}\right)_{t>0}$ *is of weak type* (1, 1) *with respect to the invariant measure* $d\gamma_\infty$.

We first consider the special case when $Q = I$ and $B = \mathrm{diag}(-\lambda_1, -\lambda_2, \dots, -\lambda_n)$, with $\lambda_j > 0$ for $j = 1, \dots, n$, and state in Theorem 2.1 the weak type (1, 1) of $H_*^{Q,B}$. The proof of this result involves some geometry and occupies most of this paper. Theorem 2.1 already extends the results in [9], and forms the basis of the proof of Theorem 1.1.

The paper is organized as follows. In Section 2 we introduce the notation, in particular for the relevant Mehler kernel $K_t(x, u)$, and state the intermediate result Theorem 2.1. Sections 3, 4, 5, and 6 are devoted to the proof of Theorem 2.1. More precisely, in Section 3 we introduce a localization procedure for those coordinates in which the variables x and u are close to each other. In Section 4, we consider the remaining variables, and reduce the problem to an ellipsoidal annulus. A system of polar-like coordinates is also introduced. Then we prove in Section 5 the weak type (1, 1) for that part of the maximal operator given by large t. Section 6 is devoted to the more delicate part corresponding to small t. Finally, in Section 7 we consider the building blocks of an arbitrary normal Ornstein-Uhlenbeck semigroup, and deduce Theorem 1.1 from Theorem 6.3, which is a slight generalization of Theorem 2.1.

In the following, we shall use the symbols c and C with $0 < c, C < \infty$ to denote constants which are not necessarily equal at different occurrences. They depend only on the dimension and the parameters of the semigroup considered. The symbol $\simeq$ between two positive expressions means that their ratio is bounded above and below by such constants. For two positive quantities a and b, we write $a \lesssim b$ instead of $a \le Cb$ and $a \gtrsim b$ for $b \lesssim a$. The symbol $|E|$ will denote the Lebesgue measure of a measurable set E. By $\mathbb{N}$ we mean the set of all nonnegative integers. Finally, we write $\lfloor x \rfloor$ to denote the greatest integer smaller than or equal to $x \in \mathbb{R}$.

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2. Restriction to a special case

In this and the following four sections, we consider the case when $Q = I$ and

$$B = \operatorname{diag}\big(-\lambda_1, -\lambda_2, \dots, -\lambda_n \big), \tag{2.1}$$

with $\lambda_j > 0$ for $j = 1, \dots, n$. We set $\lambda_{\max} = \max \lambda_j$ and $\lambda_{\min} = \min \lambda_j$.

Then the covariance matrices and the Gaussian measures are given by

$$Q_t = \operatorname{diag}\left(\frac{1}{2\lambda_1}\big(1 - e^{-2\lambda_1 t}\big), \frac{1}{2\lambda_2}\big(1 - e^{-2\lambda_2 t}\big), \dots, \frac{1}{2\lambda_n}\big(1 - e^{-2\lambda_n t}\big) \right)$$

and

$$d\gamma_t(x) = \pi^{-\frac{n}{2}} \frac{\sqrt{\Pi_{j=1}^n \lambda_j}}{\sqrt{\Pi_{j=1}^n (1 - e^{-2\lambda_j t})}} \exp\left(- \sum_{j=1}^n \frac{\lambda_j}{1 - e^{-2\lambda_j t}}\, x_j^2 \right) dx_1 \dots dx_n.$$

The invariant measure is

$$d\gamma_\infty(x) = \pi^{-\frac{n}{2}} \sqrt{\Pi_{j=1}^n \lambda_j}\, \exp\left(- \sum_{j=1}^n \lambda_j x_j^2 \right) dx_1 \dots dx_n. \tag{2.2}$$

We denote the Ornstein-Uhlenbeck semigroup simply by $\mathcal{H}_t$, suppressing the indices Q, B. It may be written as

$$\mathcal{H}_t f(x) = \pi^{-\frac{n}{2}} \frac{\sqrt{\Pi_{j=1}^n \lambda_j}}{\sqrt{\Pi_{j=1}^n (1 - e^{-2\lambda_j t})}} \int f\big(e^{-t\lambda_1} x_1 - y_1, \dots, e^{-t\lambda_n} x_n - y_n\big)$$
$$\times \exp\left(- \sum_{j=1}^n \frac{\lambda_j}{1 - e^{-2\lambda_j t}} y_j^2 \right) dy_1 \dots dy_n.$$

A straightforward computation leads to

$$\mathcal{H}_t f(x) = \frac{\exp\left(\sum_{j=1}^{n} \lambda_j x_j^2\right)}{\sqrt{\Pi_{j=1}^{n}(1 - e^{-2\lambda_j t})}} \times \int f(u_1, \dots, u_n) \exp\left(-\sum_{j=1}^{n} \frac{\lambda_j}{1 - e^{-2\lambda_j t}} \left(x_j - e^{-\lambda_j t} u_j\right)^2\right) \times d\gamma_\infty(u_1, \dots, u_n).$$

We write this as

$$\mathcal{H}_t f(x) = \int K_t(x, u)\, f(u)\, d\gamma_\infty(u)\,,$$

where K_t denotes the Mehler kernel, given by

$$K_t(x, u) = \frac{\exp\left(\sum_{j=1}^{n} \lambda_j x_j^2\right)}{\sqrt{\Pi_{j=1}^{n}(1 - e^{-2\lambda_j t})}} \exp\left(-\sum_{j=1}^{n} \frac{\lambda_j}{1 - e^{-2\lambda_j t}} \left(x_j - e^{-\lambda_j t} u_j\right)^2\right)$$

for $x, u \in \mathbb{R}^n$. It is clearly the tensor product of the one-dimensional kernels

$$K_{t,j}(x_j, u_j) = \frac{\exp(\lambda_j x_j^2)}{\sqrt{1 - e^{-2\lambda_j t}}} \exp\left(-\frac{\lambda_j}{1 - e^{-2\lambda_j t}} \left(x_j - e^{-\lambda_j t} u_j\right)^2\right). \qquad (2.3)$$

The maximal operator is

$$\mathcal{H}_* f(x) = \sup_{t>0} \left|\mathcal{H}_t f(x)\right|.$$

We will prove the following special case of Theorem 1.1.

Theorem 2.1. *If $Q = I$ and B is diagonal and given by* (2.1)*, then $\mathcal{H}_* = \mathcal{H}_*^{I,B}$ is of weak type* $(1, 1)$ *with respect to the invariant measure $d\gamma_\infty$.*

In the proof of this theorem, we distinguish between global and local variables. For $k \in \{0, \dots, n\}$ we define

$$M_k = \left\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |x_j - u_j| > \frac{1}{1 + |x_j|},\ j = 1, \dots, k\,, \right. \\ \left. \text{and} \quad |x_j - u_j| \le \frac{1}{1 + |x_j|},\ j = k + 1, \dots, n \right\}.$$

If $k = 0$ or $k = n$, this means that the second or the first inequality, respectively, applies to all j. We call the inequalities $|x_j - u_j| > \frac{1}{1+|x_j|}$ and $|x_j - u_j| \le \frac{1}{1+|x_j|}$ the global and the local condition, respectively. If $(x, u) \in M_k$ for some $k \in \{0, \dots, n\}$, we write

$$x = (\xi, x_{\mathrm{loc}}), \qquad \text{with} \qquad \xi = (x_1, \dots, x_k) \qquad \text{and} \qquad x_{\mathrm{loc}} = (x_{k+1}, \dots, x_n).$$

Thus $x = x_{\mathrm{loc}}$ for $k = 0$ and $x = \xi$ for $k = n$. We use similar notation for u and write

$$u = (\eta, u_{\mathrm{loc}}), \qquad \text{with} \qquad \eta = (u_1, \dots, u_k) \qquad \text{and} \qquad u_{\mathrm{loc}} = (u_{k+1}, \dots, u_n).$$

Then let

$$\mathcal{H}_*^k f(x) = \sup_{t>0} \left| \int K_t(x, u)\, \chi_{M_k}(x, u)\, f(u)\, d\gamma_\infty(u) \right|,$$

where $k \in \{0, \dots, n\}$.

Observe that $\mathcal{H}_*^0$ is the local part of $\mathcal{H}_*$. To prove Theorem 2.1, it is for obvious symmetry reasons enough to show that each $\mathcal{H}_*^k$, $k = 0, \dots, n$, is of weak type $(1, 1)$ with respect to the invariant measure $d\gamma_\infty$. The proof is quite long and will be divided in several steps.

3. The localization procedure

We start by proving a simple estimate for the local coordinates.

Lemma 3.1. *If for some $j \in \{1, \dots, n\}$ the point $(x_j, u_j) \in \mathbb{R} \times \mathbb{R}$ satisfies the local condition $|x_j - u_j| \le 1/(1 + |x_j|)$, then*

$$\left|K_{t,j}\big(x_j, u_j\big)\right| \lesssim \frac{\exp\left(\lambda_j x_j^2\right)}{\big(\min(1, t)\big)^{1/2}} \exp\left(-c\, \frac{(x_j - u_j)^2}{\min(1, t)}\right), \qquad t > 0.$$

Proof. The following argument is well known, see, *e.g.*, [9, proof of Lemma 5.3]. We have

$$\begin{aligned}
\frac{(x_j - e^{-\lambda_j t} u_j)^2}{1 - e^{-2\lambda_j t}} &= \frac{(x_j - u_j + u_j - e^{-\lambda_j t} u_j)^2}{1 - e^{-2\lambda_j t}} \\
&\ge \frac{(x_j - u_j)^2 - 2|u_j|\,|x_j - u_j|(1 - e^{-\lambda_j t})}{1 - e^{-2\lambda_j t}} \\
&\ge \frac{(x_j - u_j)^2}{1 - e^{-2\lambda_j t}} - \frac{2|x_j|\,|x_j - u_j|}{1 + e^{-\lambda_j t}} - \frac{2(u_j - x_j)^2}{1 + e^{-\lambda_j t}} \qquad (3.1) \\
&\ge \frac{(x_j - u_j)^2}{1 - e^{-2\lambda_j t}} - \frac{2|x_j|}{1 + |x_j|} - \frac{2}{(1 + |x_j|)^2} \\
&\ge \frac{(x_j - u_j)^2}{1 - e^{-2\lambda_j t}} - 4.
\end{aligned}$$

Inserting this in (2.3), one obtains the desired conclusion. □

Next, we simplify the problem by means of a localization process for the local variables, covering $\mathbb{R}^{n-k}$ with suitable rectangles. Assume $0 \le k < n$. First we split the real line into pairwise disjoint intervals of the type

$$I_s = \left(s - \frac{1}{1+|s|},\ s + \frac{1}{1+|s|} \right].$$

Clearly, this can be done with values of s in an increasing sequence $\left(s^{(\nu)}\right)_{\nu\in\mathbb{Z}}$. We claim that for each s

$$s' \in I_s, \quad |s'' - s'| \le \frac{1}{1+|s'|} \qquad \Rightarrow \qquad s'' \in 3I_s, \tag{3.2}$$

where $3I_s$ denotes the concentric scaling of I_s by a factor 3. Indeed, since $|s'-s| \le 1/(1+|s|)$,

$$1 + |s| \le 1 + |s'| + \frac{1}{1+|s|} \le 2(1+|s'|)\,,$$

and it follows that

$$|s'' - s| \le |s'' - s'| + |s' - s| \le \frac{1}{1+|s'|} + \frac{1}{1+|s|} \le \frac{3}{1+|s|}.$$

Observe also that the scaled intervals $3I_{s^{(\nu)}}$, $\nu \in \mathbb{Z}$, have bounded overlap. A similar splitting was used in [5].

Next, we apply this in each variable in $\mathbb{R}^{n-k}$, assuming $k < n$. Denoting by $\nu = (\nu_{k+1}, ..., \nu_n) \in \mathbb{Z}^{n-k}$ a multiindex, we split $\mathbb{R}^{n-k}$ into closed rectangles

$$\mathcal{C}_\nu = \prod_{j=k+1}^{n} \left[s^{(\nu_j)} - \frac{1}{1+|s^{(\nu_j)}|},\ s^{(\nu_j)} + \frac{1}{1+|s^{(\nu_j)}|} \right], \qquad \nu \in \mathbb{Z}^{n-k},$$

with centers $s^\nu = (s^{(\nu_j)})_{j=k+1}^n$. A consequence of (3.2) is that

$$(x,u) \in M_k, \quad x_{\text{loc}} \in \mathcal{C}_\nu \qquad \Rightarrow \qquad u_{\text{loc}} \in \tilde{\mathcal{C}}_\nu,$$

where $\tilde{\mathcal{C}}_\nu = 3\mathcal{C}_\nu$ is the concentric scaling. This implication assures that the values of $\mathcal{H}_*^k f$ in $\mathbb{R}^k \times \mathcal{C}_\nu$ only depend on the restriction of f to $\mathbb{R}^k \times \tilde{\mathcal{C}}_\nu$. Further, the rectangles $\mathcal{C}_\nu$ are pairwise disjoint except for boundaries, and the $\tilde{\mathcal{C}}_\nu$ have bounded overlap.

In each set $\mathbb{R}^k \times \tilde{\mathcal{C}}_\nu$ the Gaussian density varies little with the local coordinates, in the following way.

Lemma 3.2. *Let $\nu \in \mathbb{Z}^{n-k}$, $k \in \{0, \dots, n-1\}$. Then for any $u_{loc} \in \tilde{\mathcal{C}}_\nu$,*

$$\exp\left(\sum_{j=k+1}^{n} \lambda_j u_j^2\right) \sim \exp(D_\nu),$$

where $D_\nu = \sum_{j=k+1}^{n} \lambda_j (s^{(\nu_j)})^2$.

Proof. This is a well-known and simple fact (see, for example, [17, page 74]). □

To prove Theorem 2.1, it suffices to show for each $k \in \{0, 1, \dots, n\}$ and each $\nu \in \mathbb{Z}^{n-k}$ that $\mathcal{H}_*^k$ maps $L^1(\mathbb{R}^k \times \tilde{\mathcal{C}}_\nu;\ d\gamma_\infty)$ boundedly into $L^{1,\infty}(\mathbb{R}^k \times \mathcal{C}_\nu;\ d\gamma_\infty)$, uniformly in ν. Indeed, the bounded overlap of the $\tilde{\mathcal{C}}_\nu$ will then allow summing in ν. In the case $k = n$, there is no need for the $\mathcal{C}_\nu$ and $\tilde{\mathcal{C}}_\nu$.

With ν fixed, Lemma 3.2 then makes it natural to replace $d\gamma_\infty$ by the measure

$$d\gamma_\infty^k(x) = \pi^{-\frac{k}{2}} \sqrt{\Pi_{j=1}^k \lambda_j}\ \exp\left(-\sum_{j=1}^{k} \lambda_j x_j^2\right) dx_1 \dots dx_k\, dx_{\mathrm{loc}},$$

where $dx_{\mathrm{loc}} = dx_{k+1} \dots dx_n$. Observe that $d\gamma_\infty^n = d\gamma_\infty$.

We are now led to the kernel

$$\begin{aligned} K_t^{k,\nu}(x,u) = & \frac{\exp\left(\sum_{j=1}^{k} \lambda_j x_j^2\right)}{\sqrt{\Pi_{j=1}^n (1 - e^{-2\lambda_j t})}} \\ & \times \exp\left(-\sum_{j=1}^{n} \frac{\lambda_j}{1-e^{-2\lambda_j t}} \left(x_j - e^{-\lambda_j t} u_j\right)^2\right) \chi_{M_k}(x,u)\, \chi_{\mathcal{C}_\nu}(x_{\mathrm{loc}}), \end{aligned} \tag{3.3}$$

which vanishes for $u_{\mathrm{loc}} \notin \tilde{\mathcal{C}}_\nu$, and to the operator

$$\mathcal{H}_*^{k,\nu} f(x) = \sup_{t>0} \left| \int K_t^{k,\nu}(x,u)\, f(u)\, d\gamma_\infty^k(u) \right|. \tag{3.4}$$

As easily verified by means of a small computation, Theorem 2.1 can be rephrased as follows.

Theorem 3.3. *Let $k \in \{0, \dots, n\}$. For all functions $f \in L^1(\gamma_\infty^k)$*

$$\gamma_\infty^k \big\{ x : \mathcal{H}_*^{k,\nu} f(x) > \alpha \big\} \lesssim \frac{1}{\alpha} \|f\|_{L^1(\gamma_\infty^k)}, \qquad \alpha > 0, \tag{3.5}$$

uniformly in $\nu \in \mathbb{Z}^{n-k}$.

We first show that Theorem 3.3 holds in the (entirely local) case $k = 0$.

Proposition 3.4. *The maximal operator* $\mathcal{H}_*^{0,\nu}$ *is of weak type* $(1,1)$*, uniformly in* ν.

Proof. Lemma 3.1 implies that for $(x,u) \in M_0$, $x \in \mathcal{C}_\nu$ and $u \in \tilde{\mathcal{C}}_\nu$

$$\left|K_t^{0,\nu}(x,u)\right| \lesssim \frac{1}{\big(\min(1,t)\big)^{n/2}} \exp\left(-c\,\frac{|x-u|^2}{\min(1,t)}\right), \qquad t>0.$$

Standard methods now allow us to estimate $\mathcal{H}_*^{0,\nu} f$ in $L^{1,\infty}(\mathcal{C}_\nu)$ in terms of the norm of f in $L^1(\tilde{\mathcal{C}}_\nu)$. For further details, see for example [6, Section 3]. □

When proving Theorem 3.3 for $k>0$, we can assume that f is nonnegative, supported in $\mathbb{R}^k \times \tilde{\mathcal{C}}_\nu$ and normalized in the sense that

$$\|f\|_{L^1(\gamma_\infty^k)} = 1.$$

The level set in (3.5) is contained in $\mathbb{R}^k \times \mathcal{C}_\nu$, and $\gamma_\infty^k(\mathbb{R}^k \times \mathcal{C}_\nu) \lesssim 1$. We may assume that α is large, since (3.5) is trivial in the opposite case. The meaning of "large" here will be specified later and will depend only on the dimension and the parameters of the semigroup.

4. Some elliptic geometry

4.1. Reduction to an ellipsoidal annulus

We simplify the proof of Theorem 3.3 by restricting the global variables to an ellipsoidal annulus, defined in terms of the quadratic form

$$R(\xi) = \sum_{j=1}^{k} \lambda_j x_j^2, \tag{4.1}$$

where $\xi = (x_1, \dots, x_k)$. Fixing a large α, we shall see that it is not restrictive to assume that $x = (\xi, x_{\text{loc}})$ in (3.5) is such that ξ is in the set

$$\mathcal{E} = \left\{\xi \in \mathbb{R}^k : \frac{1}{2}\log\alpha \le R(\xi) \le 2\log\alpha\right\}. \tag{4.2}$$

We first consider the set of points not verifying the inequality $R(\xi) \le 2\log\alpha$, which satisfies

$$\begin{aligned} &\gamma_\infty^k\big\{(\xi, x_{\text{loc}}) \in \mathbb{R}^k \times \mathcal{C}_\nu : R(\xi) > 2\log\alpha\big\} \\ &\lesssim |\mathcal{C}_\nu| \int_{R(\xi)>2\log\alpha} \exp(-R(\xi))\,d\xi \\ &\lesssim (2\log\alpha)^{(k-2)/2} \exp(-2\log\alpha) \\ &\lesssim \frac{1}{\alpha}; \end{aligned} \tag{4.3}$$

to get the second inequality here, one uses polar coordinates after the change of variables $x'_j = x_j\sqrt{\lambda_j}$.

Further, we claim that for any $(x, u) \in M_k$,

$$R(\xi) < \frac{1}{2}\log\alpha \qquad \Rightarrow \qquad K_t^{k,\nu}(x,u) \lesssim \alpha. \tag{4.4}$$

This requires a lemma which will also be useful later; recall that $x = (\xi, x_{\mathrm{loc}})$.

Lemma 4.1. *If* $(x, u) \in M_k$ *and* $0 < t \le 1$, *then*

$$\frac{1}{(1+|\xi|)^2} \lesssim t^2|\xi|^2 + \sum_1^k \bigl(x_j - e^{-\lambda_j t}u_j\bigr)^2.$$

Proof. From the definition of M_k we have

$$\begin{aligned}\frac{1}{1+|\xi|} &\le \sum_1^k |x_j - u_j| = \sum_1^k \Bigl|\bigl(1-e^{\lambda_j t}\bigr)x_j + e^{\lambda_j t}x_j - u_j\Bigr| \\ &\lesssim t\sum_1^k |x_j| + \sum_1^k e^{\lambda_j t}\Bigl|x_j - e^{-\lambda_j t}u_j\Bigr| \lesssim t\Bigl|\xi| + \sum_1^k |x_j - e^{-\lambda_j t}u_j\Bigr|.\end{aligned}$$

The lemma follows. □

To verify (4.4), we first assume that $t > 1$. Then because of (3.3)

$$K_t^{k,\nu}(x,u) \lesssim e^{R(\xi)} < \sqrt{\alpha} \le \alpha,$$

since α is large. In the case when $t \le 1$, we have

$$K_t^{k,\nu}(x,u) \lesssim \frac{e^{R(\xi)}}{t^{n/2}} \exp\left(-c\sum_{j=1}^k \frac{(x_j - e^{-\lambda_j t}u_j)^2}{t}\right).$$

It follows from Lemma 4.1 that

$$t^2 \gtrsim \frac{1}{(1+|\xi|)^4} \qquad \text{or} \qquad \sum_{j=1}^k \frac{(x_j - e^{-\lambda_j t}u_j)^2}{t} \gtrsim \frac{1}{(1+|\xi|)^2 t}.$$

The first inequality here implies that

$$K_t^{k,\nu}(x,u) \lesssim e^{R(\xi)}\,(1+|\xi|)^n \lesssim e^{2R(\xi)} < \alpha.$$

If the second inequality holds, we have

$$K_t^{k,\nu}(x,u) \lesssim \frac{e^{R(\xi)}}{t^{n/2}} \exp\left(-\frac{c}{(1+|\xi|)^2 t}\right) \lesssim e^{R(\xi)}\,(1+|\xi|)^n,$$

and the same estimate follows. Thus (4.4) is verified.

Replacing α by $C\alpha$ for some C, we see from (4.3) and (4.4) that we can assume $\xi \in \mathcal{E}$ in the proof of Theorem 3.3.

4.2. Polar-like coordinates in $\mathbb{R}^k$

Fix $\beta > 0$ and consider the ellipsoid

$$E_\beta = \{\xi \in \mathbb{R}^k : R(\xi) = \beta\}.$$

We introduce the anisotropic dilations

$$e^{\lambda s}\,\xi = \left(e^{\lambda_j s}\,x_j\right)_{j=1}^k.$$

Then each $\xi \in \mathbb{R}^k \setminus \{0\}$ may be written in a unique way as $\xi = e^{\lambda s}\,\tilde{\xi}$ with $s \in \mathbb{R}$ and $\tilde{\xi} = (\tilde{\xi}_j)_{j=1}^k \in E_\beta$. Thus $x = (\xi, x_{\mathrm{loc}}) \in \mathbb{R}^n$ is given by

$$x = \left(e^{\lambda s}\tilde{\xi}, x_{\mathrm{loc}}\right). \tag{4.5}$$

The Lebesgue measure $d\xi$ in $\mathbb{R}^k$ satisfies

$$d\xi \simeq \left|e^{\lambda s}\tilde{\xi}\right|\,ds\,dS(\tilde{\xi}), \tag{4.6}$$

where dS is the area measure of the ellipsoid E_β. Indeed, we will see in the next subsection that the curve $s \mapsto e^{\lambda s}\tilde{\xi}$ is transverse to the family of ellipsoids defined by $R(\xi)$.

In the following result, we estimate the distance between two points in terms of the coordinates $s, \tilde{\xi}$.

Lemma 4.2. *Let $\xi^{(0)},\ \xi^{(1)} \in \mathbb{R}^k \setminus \{0\}$ and assume $R(\xi^{(0)}) > \beta/2$. Write $\xi^{(0)} = e^{\lambda s^{(0)}}\,\tilde{\xi}^{(0)}$ and $\xi^{(1)} = e^{\lambda s^{(1)}}\,\tilde{\xi}^{(1)}$ with $s^{(0)}, s^{(1)} \in \mathbb{R}$ and $\tilde{\xi}^{(0)},\ \tilde{\xi}^{(1)} \in E_\beta$.*

(a) *Then*

$$\left|\xi^{(0)} - \xi^{(1)}\right| \geq c\left|\tilde{\xi}^{(0)} - \tilde{\xi}^{(1)}\right|; \tag{4.7}$$

(b) *If also $s^{(1)} \geq 0$, then*

$$\left|\xi^{(0)} - \xi^{(1)}\right| \geq c\sqrt{\beta}\left|s^{(0)} - s^{(1)}\right|. \tag{4.8}$$

Proof. Let $\Gamma : [0, 1] \to \mathbb{R}^k$ be a differentiable curve with $\Gamma(0) = \xi^{(0)}$ and $\Gamma(1) = \xi^{(1)}$. It is clearly enough to bound the length of any such curve from below by the right-hand sides of (4.7) and (4.8).

For each $\tau \in [0, 1]$, we write $\Gamma(\tau) = e^{\lambda s(\tau)}\,\tilde{\xi}(\tau)$ with $\tilde{\xi}(\tau) = (\tilde{\xi}_j(\tau))_1^k \in E_\beta$, so that $s(0) = s^{(0)}$ and $s(1) = s^{(1)}$. The tangent vector is

$$\Gamma'(\tau) = \left(s'(\tau)\,\lambda_j\,e^{\lambda_j s(\tau)}\,\tilde{\xi}_j(\tau) + e^{\lambda_j s(\tau)}\,\tilde{\xi}_j'(\tau)\right)_{j=1}^k,$$

and

$$|\Gamma'(\tau)|^2 = \sum_1^k e^{2\lambda_j s(\tau)} \Big(s'(\tau)\,\lambda_j\,\tilde\xi_j(\tau) + \tilde\xi_j'(\tau)\Big)^2$$
$$\geq \min_j e^{2\lambda_j s(\tau)} \left| s'(\tau)\,\lambda\,\tilde\xi(\tau) + \tilde\xi'(\tau)\right|^2,$$

where $\lambda\,\tilde\xi(\tau)$ denotes the vector $(\lambda_j\,\tilde\xi_j(\tau))_{j=1}^k$. This vector is normal to E_β at $\tilde\xi(\tau)$ and so orthogonal to the tangent vector $\tilde\xi'(\tau)$, and we conclude that

$$|\Gamma'(\tau)|^2 \geq \min_j e^{2\lambda_j s(\tau)} \Big(s'(\tau)^2 \left|\lambda\,\tilde\xi(\tau)\right|^2 + \left|\tilde\xi'(\tau)\right|^2 \Big). \tag{4.9}$$

We need a lower estimate of $s(0)$. If $s(0) < 0$, the assumption $R(\xi^{(0)}) > \beta/2$ implies that

$$\beta/2 < \sum_j \lambda_j e^{2\lambda_j s(0)} \Big(\tilde\xi_j^{(0)}\Big)^2 \leq e^{2\lambda_{\min} s(0)}\, R\Big(\tilde\xi^{(0)}\Big) = e^{2\lambda_{\min} s(0)}\,\beta\,.$$

Thus we always have

$$s(0) > -\tilde s,$$

where $\tilde s = \log 2/(2\lambda_{\min})$.

Assume now that $s(\tau) > -2\tilde s$ for all $\tau \in [0,1]$. Then the minimum in (4.9) stays away from 0 and we get

$$|\Gamma'(\tau)| \gtrsim |s'(\tau)| \left|\lambda\,\tilde\xi(\tau)\right| \gtrsim \sqrt{\beta}\,|s'(\tau)|$$

and

$$|\Gamma'(\tau)| \gtrsim \left|\tilde\xi'(\tau)\right|.$$

Integrating each of these two estimates with respect to τ in $[0,1]$, we see that the length of Γ is bounded below by the right-hand sides of (4.8) and (4.7).

If instead $s(\tau) \leq -2\tilde s$ for some $\tau \in [0,1]$, the image $s([0,1])$ contains the interval $[-2\tilde s, \max(s(0), s(1))]$. Then we can find a closed subinterval $I \subset [0,1]$ such that for $\tau \in I$

$$-2\tilde s \leq s(\tau) \leq \max(s(0), s(1))$$

and, moreover, equality holds in the left-hand inequality here at one endpoint of I and in the right-hand inequality at the other endpoint. For the length of Γ, we now have, in view of (4.9),

$$\int_0^1 |\Gamma'(\tau)|\,d\tau \geq \int_I |\Gamma'(\tau)|\,d\tau \gtrsim \sqrt{\beta} \int_I |s'(\tau)|\,d\tau \geq \sqrt{\beta}\,\big(\max(s(0), s(1)) + 2\tilde s\big).$$

Since $s(0) > -\tilde s$, the last quantity here is larger than $\sqrt{\beta}\,|\tilde s| \gtrsim \sqrt{\beta} \sim \operatorname{diam} E_\beta$. Thus the length of the curve is bounded below by the right-hand side of (4.7). If we also assume $s^{(1)} \geq 0$, the same is true with (4.7) replaced by (4.8), since then

$$\max(s(0), s(1)) + 2\tilde s \geq |s(0) - s(1)|.$$

The proof of the lemma is complete. □

4.3. The Gaussian measure of a tube

We will need a geometric, k-dimensional lemma. In $\mathbb{R}^k$ we write points as $\xi = (x_j)_{j=1}^k$ and use the measure

$$d\mu_R(\xi) = e^{-R(\xi)}\, d\xi\,,$$

where $R(\xi)$ was defined in (4.1). Recall that $e^{\lambda t}\,\xi = (e^{\lambda_j t}\,x_j)_{j=1}^k$ and that $\alpha > 0$ is large.

We fix β with $\frac{1}{2}\log\alpha \le \beta \le 2\log\alpha$ and consider a spherical cap of the ellipsoid E_β, centered at some point $\xi^{(1)} \in E_\beta$. Explicitly, we define

$$\Omega = \left\{\xi \in \mathbb{R}^k : R(\xi) = \beta,\ \left|\xi - \xi^{(1)}\right| < a\right\}$$

with $a > 0$. Observe that $|\xi| \simeq \sqrt{\beta}$ for $\xi \in \Omega$. Then we define the tube

$$Z = \left\{e^{\lambda s}\,\xi : s > 0,\ \xi \in \Omega\right\}. \tag{4.10}$$

Lemma 4.3. *The $d\mu_R$ measure of Z satisfies*

$$\mu_R(Z) \lesssim \frac{a^{k-1}}{\sqrt{\beta}}\, e^{-\beta}.$$

Proof. For $s \ge 0$ the set

$$\Omega_s = \left\{e^{\lambda s}\,\xi : \xi \in \Omega\right\}$$

is a slice of Z. The selfadjoint linear map

$$F_s : \xi \mapsto e^{\lambda s}\xi$$

is a bijection between Ω and Ω_s. To estimate $\mu_R(Z)$, we need an estimate of the area $\left|\Omega_s\right|$ of the $(k-1)$-dimensional surface Ω_s.

A normal vector to $\Omega_0 = \Omega$ at the point $\xi \in \Omega$ is $v = (\lambda_j x_j)_{j=1}^k$, and the tangent hyperplane at ξ is $v^\perp$. For $s > 0$ the tangent hyperplane of Ω_s at the point $F_s(\xi)$ is $F_s(v^\perp)$. Thus a normal to Ω_s at the same point is $w = F_s^{-1}(v) = (e^{-\lambda_j s}\lambda_j x_j)_{j=1}^k$. The angle $\psi(s,\xi)$ between w and $F_s(v) = (e^{\lambda_j s}\lambda_j x_j)_{j=1}^k$ is given by

$$\cos\psi(s,\xi) = \frac{w \cdot F_s(v)}{\|w\|\,\|F_s(v)\|} = \frac{\sum_1^k \lambda_j^2 x_j^2}{\sqrt{\sum_1^k e^{-2\lambda_j s}\lambda_j^2 x_j^2}\,\sqrt{\sum_1^k e^{2\lambda_j s}\lambda_j^2 x_j^2}}.$$

We remark that this shows that $\cos\psi(s,\xi)$ stays away from zero; this yields the transversality mentioned in the preceding subsection.

Since $F_s(v) = \partial F_s(\xi)/\partial s$, the distance from a point $F_s(\xi) \in \Omega_s$ to Ω_{s+h} in the normal direction is, for small $h > 0$, essentially

$$h|F_s(v)| \cos \psi(s, \xi).$$

Thus the Lebesgue measure in Z is given by $|F_s(v)| \cos \psi(s, \xi)\, dS_s\, ds$, where dS_s denotes the $(k-1)$-dimensional area measure of Ω_s. It follows that

$$\mu_R(Z) = \int_0^\infty \int_{\Omega_s} |F_s(v)| \cos \psi(s, \xi)\, e^{-R(e^{\lambda s}\xi)}\, dS_s\, ds. \tag{4.11}$$

To evaluate this, we must first estimate the area $|\Omega_s|$. The area of Ω can be approximated by that of a union of small $(k-1)$-dimensional simplices, *i.e.* small convex k-gons, tangent to Ω. Similarly, that of Ω_s is approximated by the images under F_s of these simplices. Let S be such a simplex, situated in the tangent hyperplane of Ω at the point $\xi \in \Omega$ and containing ξ. We shall compare its area $|S|$ with the area $|F_s(S)|$ of its image. With v as before and $\varepsilon > 0$, the convex hull of S and the point $\xi + \varepsilon v$ is a k-dimensional simplex S_ε. Its volume is $|S_\varepsilon| = k^{-1}\,\varepsilon |S|\,|v|$. Its image $F_s(S_\varepsilon)$ is spanned by $F_s(S)$ and $F_s(\xi) + \varepsilon F_s(v)$, and so has volume $|F_s(S_\varepsilon)| = k^{-1}\,\varepsilon |F_s(S)|\,|F_s(v)| \cos \psi(s, \xi)$.

On the other hand, the quotient $|F_s(S_\varepsilon)|/|S_\varepsilon|$ equals the Jacobian of F_s, which is $\exp(\sum_1^k \lambda_\nu s)$. Combining, one finds that

$$\frac{|F_s(S)|}{|S|} = \frac{\exp\left(\sum\limits_1^k \lambda_\nu s\right)\, |v|}{|F_s(v)| \cos \psi(s, \xi)} = \exp\left(\sum_1^k \lambda_\nu s\right) \frac{\sqrt{\sum\limits_1^k e^{-2\lambda_j s} \lambda_j^2 x_j^2}}{\sqrt{\sum\limits_1^k \lambda_j^2 x_j^2}}$$

$$= \frac{\sqrt{\sum_{j=1}^k \exp\left[2\left(\sum\limits_{\nu=1}^k \lambda_\nu - \lambda_j\right) s\right] \lambda_j^2 x_j^2}}{\sqrt{\sum\limits_1^k \lambda_j^2 x_j^2}}.$$

It follows that

$$1 \le \frac{|F_s(S)|}{|S|} \le e^{(k-1)\lambda_{\max} s}.$$

Summing over small simplices, we conclude that also

$$1 \le \frac{|\Omega_s|}{|\Omega|} \le e^{(k-1)\lambda_{\max} s}, \tag{4.12}$$

for any $s > 0$.

Next, we estimate the factors in (4.11), still assuming $s > 0$. First, $|F_s(v)| \le e^{\lambda_{\max} s}|v|$ and $|v| \simeq |\xi| \simeq \sqrt{\beta}$, so that

$$|F_s(v)| \lesssim e^{\lambda_{\max} s}\sqrt{\beta}.$$

Further,

$$\begin{aligned} R(e^{\lambda s}\xi) &= \sum_j \lambda_j e^{2\lambda_j s}\, x_j^2 \ge \sum_j \lambda_j (1 + 2\lambda_{\min} s)\, x_j^2 \\ &= (1 + 2\lambda_{\min} s)\, R(\xi) = (1 + 2\lambda_{\min} s)\beta, \end{aligned}$$

since $R(\xi) = \beta$.

Inserted in (4.11), these two estimates lead to

$$\mu_R(Z) \lesssim \sqrt{\beta}\, e^{-\beta} \int_0^\infty e^{\lambda_{\max} s - 2\lambda_{\min} \beta s} \int_{\Omega_s} dS_s\, ds.$$

The inner integral here is $|\Omega_s|$, so we can use (4.12) and observe that $|\Omega| \lesssim a^{k-1}$, to get

$$\mu_R(Z) \lesssim \sqrt{\beta}\, e^{-\beta}\, a^{k-1} \int_0^\infty e^{(k\lambda_{\max} - 2\lambda_{\min} \beta)s}\ ds.$$

We can assume that α is so large that $\lambda_{\min}\beta > k\lambda_{\max}$, and then the last integral will be less than $1/(\lambda_{\min}\beta) \sim 1/\beta$, which proves the assertion. □

5. The case of large *t*

We prove part of Theorem 3.3, considering the supremum in (3.4) taken only over $t > 1$.

Proposition 5.1. *Let $k \in \{1, \dots, n\}$. Then the maximal operator*

$$\sup_{t>1} \left| \int_{\mathbb{R}^n} K_t^{k,\nu}(x, u)\, f(u)\, d\gamma_\infty^k(u) \right|$$

is of weak type $(1, 1)$ *with respect to the invariant measure $d\gamma_\infty^k$, uniformly in $\nu \in \mathbb{Z}^{n-k}$.*

Proof. As before, f is nonnegative, supported in $\mathbb{R}^k \times \tilde{\mathcal{C}}_\nu$ and normalized in $L^1(\gamma_\infty^k)$. We need only consider points $x = (\xi, x_{\mathrm{loc}}) \in \mathcal{E} \times \mathcal{C}_\nu$ and $u = (\eta, u_{\mathrm{loc}}) \in \mathbb{R}^k \times \tilde{\mathcal{C}}_\nu$. Moreover, we shall use for both x and u the coordinates introduced in (4.5) with $\beta = \log\alpha$, that is,

$$\xi = e^{\lambda s}\tilde{\xi}, \qquad \eta = e^{\lambda s'}\tilde{\eta},$$

where $\tilde{\xi}, \tilde{\eta} \in E_{\log\alpha}$ and $s, s' \in \mathbb{R}$. Observe here that $|s| < C$, since $\xi \in \mathcal{E}$. Then (3.3) and the fact that $t > 1$ imply

$$K_t^{k,\nu}(x,u) \lesssim \exp(R(\xi)) \exp\left(-\sum_{j=1}^{k} \lambda_j \big(x_j - e^{-\lambda_j t} u_j\big)^2\right).$$

Since $\xi \in \mathcal{E}$ and $e^{-\lambda t}\eta = e^{\lambda(s'-t)}\tilde{\eta}$, we can apply Lemma 4.2 (a) getting

$$\sum_{j=1}^{k} \lambda_j \big(x_j - e^{-\lambda_j t} u_j\big)^2 \geq \lambda_{\min} \big|\xi - e^{-\lambda t}\eta\big|^2 \gtrsim \big|\tilde{\xi} - \tilde{\eta}\big|^2,$$

so that

$$K_t^{k,\nu}(x,u) \lesssim \exp(R(\xi)) \exp\Big(-c\,\big|\tilde{\xi} - \tilde{\eta}\big|^2\Big).$$

By integrating we obtain

$$\int K_t^{k,\nu}(x,u) f(u)\, d\gamma_\infty^k(u) \lesssim \exp\Big(R\big(e^{\lambda s}\,\tilde{\xi}\big)\Big) \int \exp\Big(-c\,\big|\tilde{\xi} - \tilde{\eta}\big|^2\Big) f(u)\, d\gamma_\infty^k(u).$$

The right-hand side here is increasing in s, and therefore the inequality

$$\exp\Big(R\big(e^{\lambda s}\,\tilde{\xi}\big)\Big) \int \exp\Big(-c\,\big|\tilde{\xi} - \tilde{\eta}\big|^2\Big) f(u)\, d\gamma_\infty^k(u) > \alpha \tag{5.1}$$

holds if and only if $s > s_\alpha(\tilde{\xi})$ for some $s_\alpha(\tilde{\xi})$, with equality for $s = s_\alpha(\tilde{\xi})$. Since $\alpha > 1$ and the last integral is less than $\|f\|_{L^1(\gamma_\infty^k)} = 1$, it follows that $s_\alpha(\tilde{\xi}) > 0$.

We see that the set of x where the supremum in the statement of Proposition 5.1 is larger than $C\alpha$ for some C is contained in the set $\mathcal{A}^{k,\nu}(\alpha)$ of points $(\xi, x_{\mathrm{loc}}) \in \mathcal{E} \times \mathcal{C}_\nu$ satisfying (5.1).

Applying (4.6), where now $\big|e^{\lambda s}\tilde{\xi}\big| \simeq \sqrt{\log\alpha}$ and $\beta = \log\alpha$, and observing that $\big|\tilde{\mathcal{C}_\nu}\big| \lesssim 1$, we conclude that

$$\gamma_\infty^k\big(\mathcal{A}^{k,\nu}(\alpha)\big) \lesssim \sqrt{\log\alpha} \int_{E_{\log\alpha}} \int_{s_\alpha(\tilde{\xi})<s<C} \exp\left(-\sum_{j=1}^{k} \lambda_j e^{2\lambda_j s} \tilde{\xi}_j^2\right) ds\, dS(\tilde{\xi}).$$

To estimate the integrand here, we observe that for $s_\alpha(\tilde{\xi}) < s < C$ the inequality

$$e^{2\lambda_j s} = e^{2\lambda_j s_\alpha(\tilde{\xi})}\, e^{2\lambda_j (s - s_\alpha(\tilde{\xi}))} \geq e^{2\lambda_j s_\alpha(\tilde{\xi})}\Big(1 + 2\lambda_j\big(s - s_\alpha\big(\tilde{\xi}\big)\big)\Big)$$

implies that

$$\begin{aligned}
&\exp\left(-\sum_{j=1}^{k}\lambda_j e^{2\lambda_j s}\tilde{\xi}_j^2\right)\\
&\le \exp\left(-\sum_{j=1}^{k}\lambda_j e^{2\lambda_j s_\alpha(\tilde{\xi})}\tilde{\xi}_j^2\right)\exp\left(-2\sum_{j=1}^{k}\lambda_j^2 e^{2\lambda_j s_\alpha(\tilde{\xi})}\big(s-s_\alpha(\tilde{\xi})\big)\tilde{\xi}_j^2\right)\\
&\le \exp\Big(-R\big(e^{\lambda s_\alpha(\tilde{\xi})}\tilde{\xi}\big)\Big)\exp\Big(-2\lambda_{\min}\big(s-s_\alpha(\tilde{\xi})\big)R\big(e^{\lambda s_\alpha(\tilde{\xi})}\tilde{\xi}\big)\Big)\\
&\le \exp\Big(-R\big(e^{\lambda s_\alpha(\tilde{\xi})}\tilde{\xi}\big)\Big)\exp\Big(-c\,\big(s-s_\alpha(\tilde{\xi})\big)\log\alpha\Big),
\end{aligned}$$

because $R(e^{\lambda s_\alpha(\tilde{\xi})}\tilde{\xi}) \simeq \log\alpha$ here. Thus

$$\begin{aligned}
&\gamma_\infty^k\big(\mathcal{A}^{k,\nu}(\alpha)\big)\\
&\lesssim \sqrt{\log\alpha}\int_{E_{\log\alpha}}\int_{s>s_\alpha(\tilde{\xi})}\exp\Big(-R\big(e^{\lambda s_\alpha(\tilde{\xi})}\tilde{\xi}\big)\Big)\exp\Big(-c\big(s-s_\alpha(\tilde{\xi})\big)\log\alpha\Big)\,ds\,dS\big(\tilde{\xi}\big)\\
&\lesssim \frac{1}{\sqrt{\log\alpha}}\int_{E_{\log\alpha}}\exp\Big(-R\big(e^{\lambda s_\alpha(\tilde{\xi})}\tilde{\xi}\big)\Big)\,dS\big(\tilde{\xi}\big).
\end{aligned}$$

Next we combine this estimate with the case of equality in (5.1). Changing then the order of integration, we finally get

$$\begin{aligned}
\gamma_\infty^k\big(\mathcal{A}^{k,\nu}(\alpha)\big) &\lesssim \frac{1}{\alpha\sqrt{\log\alpha}}\int_{E_{\log\alpha}}\int\exp\Big(-c\,\big|\tilde{\xi}-\tilde{\eta}\big|^2\Big)\,f(u)\,d\gamma_\infty^k(u)dS\big(\tilde{\xi}\big)\\
&\lesssim \frac{1}{\alpha\sqrt{\log\alpha}}\int\int_{E_{\log\alpha}}\exp\Big(-c\,\big|\tilde{\xi}-\tilde{\eta}\big|^2\Big)dS\big(\tilde{\xi}\big)\,f(u)\,d\gamma_\infty^k(u)\\
&\lesssim \frac{1}{\alpha\sqrt{\log\alpha}}\int f(u)\,d\gamma_\infty^k(u)\,,
\end{aligned}$$

proving Proposition 5.1. □

6. The case of small t

The following proposition, combined with Proposition 5.1, will complete the proof of Theorem 3.3.

Proposition 6.1. *Let $k \in \{1, \dots, n\}$. Then the maximal operator*

$$\sup_{t\le 1}\left|\int_{\mathbb{R}^n} K_t^{k,\nu}(x,u)\,f(u)\,d\gamma_\infty^k(u)\right|$$

is of weak type $(1,1)$ with respect to the invariant measure $d\gamma_\infty^k$, uniformly in $\nu \in \mathbb{Z}^{n-k}$.

Proof. We fix the multiindex $\nu \in \mathbb{Z}^{n-k}$. As before, $f \in L^1(\gamma_\infty^k)$ is nonnegative, supported in $\mathbb{R}^k \times \tilde{\mathcal{C}}_\nu$ and normalized, and we write $\eta = (u_j)_{j=1}^k$ and $e^{-\lambda t}\eta = \big(e^{-\lambda_j t}u_j\big)_{j=1}^k$. For $m_1, m_2 \in \mathbb{N}$ and $0 < t \le 1$, we introduce regions $\mathcal{S}_t^{m_1,m_2}$, depending also on ν. If $m_1,\ m_2 > 0$, let

$$\mathcal{S}_t^{m_1,m_2} = \Big\{(x,u) \in M_k :\ 2^{m_1-1}\sqrt{t} < |\xi - e^{-\lambda t}\eta| \le 2^{m_1}\sqrt{t}\,, \\ 2^{m_2-1}\sqrt{t} < |x_{\text{loc}} - u_{\text{loc}}| \le 2^{m_2}\sqrt{t}\,,\ x_{\text{loc}} \in \mathcal{C}_\nu\,,\ u_{\text{loc}} \in \tilde{\mathcal{C}}_\nu\Big\}.$$

If $m_1 = 0$, we replace the condition $2^{m_1-1}\sqrt{t} < |\xi - e^{-\lambda t}\eta| \le 2^{m_1}\sqrt{t}$ by $|\xi - e^{-\lambda t}\eta| \le \sqrt{t}$. Analogously, if $m_2 = 0$, the inequalities $2^{m_2-1}\sqrt{t} < |x_{\text{loc}} - u_{\text{loc}}| \le 2^{m_2}\sqrt{t}$ are replaced by $|x_{\text{loc}} - u_{\text{loc}}| \le \sqrt{t}$. Observe that for any fixed t these sets form a partition of $(\mathbb{R}^k \times \mathcal{C}_\nu) \times (\mathbb{R}^k \times \tilde{\mathcal{C}}_\nu) \cap M_k$.

In the set $\mathcal{S}_t^{m_1,m_2}$ we can apply (3.3), and also (3.1) for the local coordinates, to get

$$K_t^{k,\nu}(x,u) \lesssim \frac{\exp(R(\xi))}{t^{n/2}} \exp\Big(-c2^{2m_1} - c2^{2m_2}\Big).$$

Thus for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t > 0$,

$$K_t^{k,\nu}(x,u) \lesssim \sum_{m_1,m_2} \mathcal{K}_t^{m_1,m_2}(x,u)\,,$$

where we define

$$\mathcal{K}_t^{m_1,m_2}(x,u) = \frac{\exp(R(\xi))}{t^{n/2}} \exp\Big(-c2^{2m_1} - c2^{2m_2}\Big)\, \chi_{\mathcal{S}_t^{m_1,m_2}}(x,u), \tag{6.1}$$

omitting the indices ν and k.

Therefore, we need only show that

$$\gamma_\infty^k \left\{ x \in \mathbb{R}^n : \sup_{t\le 1} \int \mathcal{K}_t^{m_1,m_2}(x,u) f(u)\, d\gamma_\infty^k(u) > \alpha \right\} \\ \lesssim \frac{1}{\alpha} \exp\Big(-c2^{2m_1} - c2^{2m_2}\Big), \tag{6.2}$$

since this will allow summing in m_1, m_2 in the space $L^{1,\infty}$.

Observe that $\mathcal{K}_t^{m_1,m_2}(x,u) \ne 0$ implies $(x,u) \in M_k$ and $|\xi - e^{-\lambda t}\eta| \le 2^{m_1}\sqrt{t}$, and then Lemma 4.1 yields

$$1 \lesssim (1+|\xi|)^4 t^2 + (1+|\xi|)^2 2^{2m_1} t \le \big((1+|\xi|)^2 2^{2m_1} t\big)^2 + (1+|\xi|)^2 2^{2m_1} t.$$

From this it follows that

$$(1+|\xi|)^2\, 2^{2m_1}\, t \gtrsim 1 \tag{6.3}$$

as soon as there exists a point u with $\mathcal{K}_t^{m_1,m_2}(x,u) \neq 0$. Then $t \geq \varepsilon$ for some $\varepsilon > 0$ which may depend on $m_1,\ m_2$ and α. We conclude that the supremum in (6.2) can as well be taken over $\varepsilon \leq t \leq 1$, and that this supremum is a continuous function of $x \in \mathcal{E} \times \mathcal{C}_\nu$.

To verify (6.2), our idea is to construct a finite sequence of pairwise disjoint sets $\big(\mathcal{B}^{(\ell)}\big)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$ and a sequence of sets $\big(\mathcal{Z}^{(\ell)}\big)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$, called forbidden zones, which will contain the level set in (6.2). We will show that

$$\left\{x = (\xi, x_{\mathrm{loc}}) \in \mathcal{E} \times \mathcal{C}_\nu : \sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^{m_1,m_2}(x,u)\, f(u)\, d\gamma_\infty^k(u) \geq \alpha\right\} \subset \bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}, \quad (6.4)$$

and that for each ℓ

$$\gamma_\infty^k(\mathcal{Z}^{(\ell)}) \lesssim \frac{1}{\alpha} \exp\left(-c2^{2m_1} - c2^{2m_2}\right) \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty^k(u). \quad (6.5)$$

Since the $\mathcal{B}^{(\ell)}$ will be pairwise disjoint, we will then be able to conclude

$$\begin{aligned} \gamma_\infty^k\left(\bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}\right) &\lesssim \frac{1}{\alpha} \exp\left(-c2^{2m_1} - c2^{2m_2}\right) \sum_{\ell=1}^{\ell_0} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty^k(u) \\ &\lesssim \frac{1}{\alpha} \exp\left(-c2^{2m_1} - c2^{2m_2}\right) \|f\|_{L^1(\gamma_\infty^k)}. \end{aligned}$$

This will imply (6.2) and finish the proof of Proposition 6.1.

The sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ will be defined recursively, by means of points $x^{(\ell)}$, $\ell = 1, \dots, \ell_0$. To find the first point $x^{(1)}$, we consider the minimum of the quadratic form $R(\xi)$ in the compact set

$$\mathcal{A}_0 = \left\{x \in \mathcal{E} \times \mathcal{C}_\nu : \sup_{\varepsilon \leq t \leq 1} \int \mathcal{K}_t^{m_1,m_2}(x,u)\, f(u)\, d\gamma_\infty^k \geq \alpha\right\}.$$

(Should this set be empty, (6.2) is immediate.)

By continuity this minimum is attained at some point $x^{(1)} = \big(\xi^{(1)}, x_{\mathrm{loc}}^{(1)}\big)$ of $\mathcal{A}_0$. Moreover, there is some t, called t_1, in $[\varepsilon, 1]$ for which the supremum is attained, so that

$$\int \mathcal{K}_{t_1}^{m_1,m_2}\Big(x^{(1)}, u\Big)\, f(u)\, d\gamma_\infty^k(u) \geq \alpha.$$

Because of the expression (6.1) for the kernel $\mathcal{K}_t^{m_1,m_2}$ and the definition of $\mathcal{S}_t^{m_1,m_2}$, this implies

$$\alpha \leq R\big(\xi^{(1)}\big)\, t_1^{-n/2} \exp\left(-c2^{2m_1} - c2^{2m_2}\right) \int_{\mathcal{B}^{(1)}} f(u)\, d\gamma_\infty^k(u), \quad (6.6)$$

where the set $\mathcal{B}^{(1)}$ is defined by

$$\mathcal{B}^{(1)}=\Big\{\big(\eta,u_{\rm loc}\big)\in\mathbb{R}^k\times\tilde{\mathcal{C}}_\nu:\Big|\xi^{(1)}-e^{-\lambda t_1}\eta\Big|\le 2^{m_1}\sqrt{t}_1\,,\ \Big|x^{(1)}_{\rm loc}-u_{\rm loc}\Big|\le 2^{m_2}\sqrt{t}_1\Big\}.$$

Next we introduce the first *forbidden zone* (the terminology is taken from [17])

$$\mathcal{Z}^{(1)}=\Big\{\big(e^{\lambda s}\eta,u_{\rm loc}\big)\in\mathbb{R}^k\times\tilde{\mathcal{C}}_\nu:\ s\ge 0,\ R(\eta)=R\big(\xi^{(1)}\big),\\ \big|\eta-\xi^{(1)}\big|<A2^{3m_1}\sqrt{t_1}\,,\Big|u_{\rm loc}-x^{(1)}_{\rm loc}\Big|<B2^{2m_1+m_2}\sqrt{t_1}\Big\},$$

for some $A,B>0$ to be determined, depending only on the dimension and the parameters of the semigroup.

The construction now proceeds by recursion. Assume that we have selected $x^{(h)},\mathcal{B}^{(h)}$ and $\mathcal{Z}^{(h)}$ for $h=1,\dots,\ell-1$. The definition of the point $x^{(\ell)}$ is analogous to that of $x^{(1)}$ above, except that the forbidden zones $\mathcal{Z}^{(h)}$, $h=1,\dots,\ell-1$, are now excluded. More precisely, if the set

$$\mathcal{A}_\ell=\left\{x\in(\mathcal{E}\times\mathcal{C}_\nu)\setminus\bigcup_{h=1}^{\ell-1}\mathcal{Z}^{(h)}:\ \sup_{\varepsilon\le t\le 1}\int\mathcal{K}^{m_1,m_2}_t(x,u)\,f(u)\,d\gamma^k_\infty(u)\ge\alpha\right\}\tag{6.7}$$

is nonempty, we choose $x^{(\ell)}=\big(\xi^{(\ell)},x^{(\ell)}_{\rm loc}\big)$ as a point minimizing $R(\xi)$ in $\mathcal{A}_\ell$. But if $\mathcal{A}_\ell=\emptyset$, the process stops at $\ell_0=\ell-1$. We shall soon see that this actually occurs for some finite ℓ_0, which will depend on $m_1,\ m_2$ and α.

Assume now that $\mathcal{A}_\ell\neq\emptyset$. We verify below that $\mathcal{A}_\ell$ is compact, so that $x^{(\ell)}$ can be chosen. Then there is some $t_\ell\in[\varepsilon,1]$ for which

$$\int\mathcal{K}^{m_1,m_2}_{t_\ell}\big(x^{(\ell)},u\big)\,f(u)\,d\gamma^k_\infty(u)\ge\alpha.$$

We observe that (6.3) applies to t_ℓ and $x^{(\ell)}$, so that

$$\big(1+\big|\xi^{(\ell)}\big|\big)^2\,2^{2m_1}\,t_\ell\gtrsim 1.\tag{6.8}$$

Further, we define

$$\mathcal{B}^{(\ell)}=\Big\{\big(\eta,u_{\rm loc}\big)\in\mathbb{R}^k\times\tilde{\mathcal{C}}_\nu:\big|\xi^{(\ell)}-e^{-\lambda t_\ell}\eta\big|\le 2^{m_1}\sqrt{t_\ell}\,,\ \big|x^{(\ell)}_{\rm loc}-u_{\rm loc}\big|\le 2^{m_2}\sqrt{t_\ell}\Big\},$$

and the associated forbidden region is

$$\mathcal{Z}^{(\ell)}=\Big\{\big(e^{\lambda s}\eta,u_{\rm loc}\big)\in\mathbb{R}^k\times\tilde{\mathcal{C}}_\nu:\ s\ge 0,\ R(\eta)=R\big(\xi^{(\ell)}\big),\ \big|\eta-\xi^{(\ell)}\big|\\ <A2^{3m_1}\sqrt{t_\ell},\ \ \big|u_{\rm loc}-x^{(\ell)}_{\rm loc}\big|<B2^{2m_1+m_2}\sqrt{t_\ell}\Big\}.$$

To see that $\mathcal{A}_\ell$ is closed and thus compact, observe that for $1 \le h \le \ell - 1$ the minimum property of $x^{(h)}$ implies that $\mathcal{A}_\ell \subset \mathcal{A}_h \subset \{x = (\xi, x_{\mathrm{loc}}) : R(\xi) \ge R(\xi^{(h)})\}$. Thus

$$\begin{aligned}\mathcal{A}_\ell &= \mathcal{A}_\ell \cap \left\{x = (\xi, x_{\mathrm{loc}}) : R(\xi) \ge R(\xi^{(h)}),\ 1 \le h \le \ell - 1\right\} \\ &= \bigcap_{h=1}^{\ell-1} \Bigg\{x \in (\mathcal{E} \times \mathcal{C}_\nu) \setminus \mathcal{Z}^{(h)} : R(\xi) \ge R\big(\xi^{(h)}\big), \\ &\qquad\qquad \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^{m_1,m_2}(x,u)\, f(u)\, d\gamma_\infty^k(u) \ge \alpha\Bigg\}.\end{aligned}$$

The sets in this intersection are all closed because of the definition of $\mathcal{Z}^{(h)}$, and so $\mathcal{A}_\ell$ is closed. This completes the description of the recursive procedure.

In analogy with (6.6) we have

$$\alpha \le \exp\Big(R\big(\xi^{(\ell)}\big)\Big)\, t_\ell^{-n/2} \exp\left(-c2^{2m_1} - c2^{2m_2}\right) \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty^k(u). \tag{6.9}$$

We now verify that the sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ have the required properties.

Lemma 6.2. *The collection of sets $\mathcal{B}^{(\ell)}$ is pairwise disjoint.*

Proof. We prove that any two sets $\mathcal{B}^{(\ell)}$ and $\mathcal{B}^{(\ell')}$ with $\ell < \ell'$ are disjoint. Since

$$\left|\xi^{(\ell)} - e^{-\lambda t_\ell}\eta\right| = \left|e^{-\lambda t_\ell}\big(e^{\lambda t_\ell}\xi^{(\ell)} - \eta\big)\right| \ge e^{-\lambda_{\max} t_\ell}\left|e^{\lambda t_\ell}\xi^{(\ell)} - \eta\right|$$

for $t \le 1$, the projection of $\mathcal{B}^{(\ell)}$ in $\mathbb{R}^k$ is contained in a ball with center $e^{\lambda t_\ell}\xi^{(\ell)}$ and radius $2^{m_1} e^{\lambda_{\max}}\sqrt{t_\ell}$. Moreover, the projection of $\mathcal{B}^{(\ell)}$ in $\mathbb{R}^{n-k}$ is contained in a ball with center $x_{\mathrm{loc}}^{(\ell)}$ and radius $2^{m_2}\sqrt{t_\ell}$. The projections of $\mathcal{B}^{(\ell')}$ have analogous properties.

Thus it is enough to prove that the centers of these balls in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ are far from each other; more precisely, that

$$\left|e^{\lambda t_\ell}\xi^{(\ell)} - e^{\lambda t_{\ell'}}\xi^{(\ell')}\right| \ge 2^{m_1} e^{\lambda_{\max}}\big(\sqrt{t_\ell} + \sqrt{t_{\ell'}}\big), \tag{6.10}$$

or

$$\left|x_{\mathrm{loc}}^{(\ell)} - x_{\mathrm{loc}}^{(\ell')}\right| \ge 2^{m_2}\big(\sqrt{t_\ell} + \sqrt{t_{\ell'}}\big). \tag{6.11}$$

Using the coordinates from Subsection 4.2 with $\beta = R(\xi^{(\ell)})$, we write

$$\xi^{(\ell')} = e^{\lambda s}\tilde{\xi}^{(\ell')}$$

for some $\tilde{\xi}^{(\ell')}$ with $R(\tilde{\xi}^{(\ell')}) = R(\xi^{(\ell)})$ and some $s \in \mathbb{R}$. Here $s \ge 0$, because $R(\xi^{(\ell')}) \ge R(\xi^{(\ell)})$. Since $x^{(\ell')}$ is not in the forbidden zone $\mathcal{Z}^{(\ell)}$, we must have

$$\left|\tilde{\xi}^{(\ell')} - \xi^{(\ell)}\right| \ge A2^{3m_1}\sqrt{t_\ell} \tag{6.12}$$

or

$$\left|x_{\mathrm{loc}}^{(\ell')} - x_{\mathrm{loc}}^{(\ell)}\right| \geq B2^{2m_1+m_2}\sqrt{t_\ell}. \tag{6.13}$$

Assume first that $t_{\ell'} \geq M2^{4m_1}t_\ell$, for some $M \geq 2$ to be chosen. Together with Lemma 4.2 (b), this assumption implies

$$\left|e^{\lambda t_\ell}\xi^{(\ell)} - e^{\lambda t_{\ell'}}\xi^{(\ell')}\right| = \left|e^{\lambda t_\ell}\xi^{(\ell)} - e^{\lambda(t_{\ell'}+s)}\tilde{\xi}^{(\ell')}\right| \gtrsim \left|\xi^{(\ell)}\right|(t_{\ell'}+s-t_\ell) \gtrsim \left|\xi^{(\ell)}\right| t_{\ell'}.$$

We now apply the assumption again and then (6.8), observing that $|\xi^{(\ell)}| \simeq \log\alpha > 1$ because $\xi^{(\ell)} \in \mathcal{E}$. This gives

$$\begin{aligned}\left|e^{\lambda t_\ell}\xi^{(\ell)} - e^{\lambda t_{\ell'}}\xi^{(\ell')}\right| &\gtrsim \left|\xi^{(\ell)}\right|\sqrt{M}\,2^{2m_1}\sqrt{t_\ell}\,\sqrt{t_{\ell'}} \\ &\gtrsim \sqrt{M}\,2^{m_1}\sqrt{t_{\ell'}} \\ &\gtrsim \sqrt{M}\,2^{m_1}\left(\sqrt{t_{\ell'}}+\sqrt{t_\ell}\right).\end{aligned}$$

Fixing M conveniently, depending on the implicit constants, we obtain (6.10).

In the remaining case $t_{\ell'} < M2^{4m_1}t_\ell$, we have

$$\sqrt{t_\ell} > \frac{2^{-2m_1-1}}{\sqrt{M}}\left(\sqrt{t_{\ell'}}+\sqrt{t_\ell}\right).$$

Applying this to (6.12) or (6.13), we arrive at (6.10) or (6.11) by choosing $A = 2e^{\lambda_{\max}}\sqrt{M}$ and $B = 2\sqrt{M}$. □

We next verify that the sequence $(x^{(\ell)})$ is finite. For $\ell < \ell'$, we have as in the preceding proof (6.12) or (6.13). In the case of (6.12), Lemma 4.2 (a) implies

$$\left|\xi^{(\ell')} - \xi^{(\ell)}\right| \gtrsim A2^{3m_1}\sqrt{t_\ell}.$$

Since $t_\ell \geq \varepsilon$, we see that in both cases the distance $\left|x^{(\ell')} - x^{(\ell)}\right|$ is bounded below by a positive constant. But all the $x^{(\ell)}$ are contained in the bounded set $\mathcal{E}\times\mathcal{C}_\nu$, so they are finite in number. Thus the set considered in (6.7) must be empty for some $\ell - 1 = \ell_0$. This implies (6.4).

We now prove (6.5) . Observe that the global component of the forbidden zone $\mathcal{Z}^{(\ell)}$ corresponds to some region Z, as defined in (4.10), where $a = A2^{3m_1}\sqrt{t_\ell}$ and $\beta = R(\xi^{(\ell)})$. By applying Lemma 4.3 and taking also the local component into account, we get

$$\begin{aligned}\gamma_\infty^k(\mathcal{Z}^{(\ell)}) &\lesssim \frac{\left(A2^{3m_1}\sqrt{t_\ell}\right)^{k-1}}{\sqrt{R(\xi^{(\ell)})}}\ \exp\left(-R\left(\xi^{(\ell)}\right)\right)\left(B2^{2m_1+m_2}\sqrt{t_\ell}\right)^{n-k} \\ &\lesssim \frac{1}{\sqrt{\log\alpha}}\left(A2^{3m_1}\right)^{k-1}\left(B2^{2m_1+m_2}\right)^{n-k} t_\ell^{(n-1)/2}\ \exp\left(-R\left(\xi^{(\ell)}\right)\right),\end{aligned}$$

since $|\xi^{(\ell)}| \simeq \sqrt{\log \alpha}$. Estimating the exponential here by means of (6.9), we obtain

$$\gamma_\infty^k\big(\mathcal{Z}^{(\ell)}\big) \lesssim \frac{1}{\alpha\sqrt{t_\ell \log \alpha}} \big(A 2^{3m_1}\big)^{k-1} \big(B 2^{2m_1+m_2}\big)^{n-k} e^{-c2^{2m_1}-c2^{2m_2}} \int_{\mathcal{B}^{(\ell)}} f(u) d\gamma_\infty^k(u).$$

Applying also (6.8), we finally conclude

$$\begin{aligned} &\gamma_\infty^k(\mathcal{Z}^{(\ell)}) \\ &\lesssim \frac{2^{m_1}}{\alpha} \big(A 2^{3m_1}\big)^{k-1} \big(B 2^{2m_1+m_2}\big)^{n-k} \exp\big(-c2^{2m_1} - c2^{2m_2}\big) \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty^k(u) \\ &\lesssim \frac{1}{\alpha} \exp\big(-c2^{2m_1} - c2^{2m_2}\big) \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty^k(u). \end{aligned}$$

This proves (6.5) and ends the proof of Proposition 6.1. □

Finally, combining Proposition 3.4, Proposition 5.1, and Proposition 6.1, we complete the proof of Theorem 3.3, and therefore also that of Theorem 2.1.

In the next section, we will need a variant of Theorem 2.1, where the Mehler kernel is slightly modified. The proof of Theorem 2.1 also yields the following result.

Theorem 6.3. *Let $\kappa > 0$. The maximal operator associated with the kernel*

$$\frac{\exp\left(\sum_{j=1}^{n} \lambda_j x_j^2\right)}{\sqrt{\Pi_{j=1}^n (1 - e^{-2\lambda_j t})}} \exp\left(-\kappa \sum_{j=1}^{n} \frac{\lambda_j}{1 - e^{-2\lambda_j t}} \big(x_j - e^{-\lambda_j t} u_j\big)^2\right), \quad t > 0, \quad (6.14)$$

is of weak type $(1, 1)$ *with respect to the invariant measure* $d\gamma_\infty$.

7. The general case

We go back to the setting of Section 1 and prove Theorem 1.1. Thus we assume that the semigroup $\big(\mathcal{H}_t^{Q,B}\big)_{t>0}$ is normal.

Metafune, Prüss, Rhandi and Schnaubelt found in [13] a decomposition of $\mathbb{R}^n$ into subspaces invariant under $\mathcal{H}_t$ called building blocks. The restriction of $\mathcal{H}_t$ to each building block has covariance $Q = I$ and drift $B = \lambda(R - I)$, where $\lambda > 0$ and R is a real skew-symmetric matrix. In [9] Mauceri and Noselli then decomposed each building block into invariant subspaces of dimensions 1 and 2, in which the kernel of $\mathcal{H}_t$ has an explicit and rather simple form.

Combining the decompositions in [13] and [9], the result is that after a change of coordinates we will have covariance matrix $Q = I$ and a drift matrix of the form

$$B = \operatorname{diag}\big(B_2, B_4, \dots, B_{2m}, -\lambda_{2m+1}, \dots, -\lambda_n\big).$$

Here B_{2j}, $j = 1, \dots, m$, is a 2×2 block matrix of the form

$$B_{2j} = \begin{pmatrix} -\lambda_{2j} & q_j \\ -q_j & -\lambda_{2j} \end{pmatrix},$$

with $\lambda_{2j} > 0$ and $q_j \in \mathbb{R} \setminus \{0\}$. Also $\lambda_i > 0$ for $2m < i \le n$.

With Q and B of this form, we will determine the kernel of $\mathcal{H}_t$; as before the integration is with respect to the invariant measure $d\gamma_\infty$. To begin with, we consider the semigroup in $\mathbb{R}^2$ with covariance matrix I whose drift matrix is B_{2j}. The corresponding invariant measure is independent of q_j and has density $\pi^{-1}\lambda_{2j} \exp\left(-\lambda_{2j}|x|^2\right)$. For this see [9, page 185], where our λ_{2j} corresponds to $1/(2\alpha)$. As verified in [9, (3.6) and (3.7)], the kernel of this two-dimensional semigroup is

$$\begin{aligned} K_t^{2j}(x,u) = {} & \frac{\exp\left(\lambda_{2j}|x|^2\right)}{1 - e^{-2\lambda_{2j}t}} \exp\left(- \frac{\lambda_{2j}}{1 - e^{-2\lambda_{2j}t}} \left|x - e^{-\lambda_{2j}t}u\right|^2\right) \\ & \times \exp\left[-\lambda_{2j} \frac{e^{-\lambda_{2j}t}}{1 - e^{-2\lambda_{2j}t}} \Big((1 - \cos(q_j t))\langle x, u\rangle + \sin(q_j t) x \wedge u\Big)\right], \end{aligned} \tag{7.1}$$

where $x, u \in \mathbb{R}^2$ and $x \wedge u = x_1 u_2 - x_2 u_1$. In [9], $q_j = \theta$ and $\lambda_{2j} = 1$; the simple transformation needed to pass to any $\lambda_{2j} > 0$ is indicated in [9, page 185]. We shall use the following estimate of K_t^{2j}; notice that the bound is independent of q_j.

Proposition 7.1. *For $x, u \in \mathbb{R}^2$ and $t > 0$, one has*

$$K_t^{2j}(x,u) \le \frac{\exp\left(\lambda_{2j}|x|^2\right)}{1 - e^{-2\lambda_{2j}t}} \exp\left(-\frac{1}{2} \frac{\lambda_{2j}}{1 - e^{-2\lambda_{2j}t}} \left|x - e^{-\lambda_{2j}t}u\right|^2\right).$$

Proof. Let $z = x - e^{-\lambda_{2j}t}u$, so that x can be replaced by $z + e^{-\lambda_{2j}t}u$. We then rewrite (7.1) as

$$K_t^{2j}(x,u) = \frac{\exp\left(\lambda_{2j}|x|^2\right)}{1 - e^{-2\lambda_{2j}t}} \exp\left(- \frac{\lambda_{2j}}{1 - e^{-2\lambda_{2j}t}} F\right), \tag{7.2}$$

with

$$F = |z|^2 + e^{-\lambda_{2j}t}\left[(1 - \cos(q_j t))(e^{-\lambda_{2j}t}|u|^2 + \langle z, u\rangle) + \sin(q_j t)\, z \wedge u\right].$$

Let $\beta \in (-\pi, \pi]$ be the angle between the vectors z and u, with the sign chosen so that $z \wedge u = |z||u| \sin\beta$. Then

$$\begin{aligned} F = {} & |z|^2 + e^{-2\lambda_{2j}t}(1 - \cos(q_j t))|u|^2 \\ & + e^{-\lambda_{2j}t}|z||u|\left[(1 - \cos(q_j t))\cos\beta + \sin(q_j t)\sin\beta\right]. \end{aligned}$$

But

$$\begin{aligned}(1-\cos(q_j t))\cos\beta+\sin(q_j t)\sin\beta &= \cos\beta-\cos(q_j t+\beta)\\ &= 2\sin(q_j t/2)\sin(\beta+q_j t/2).\end{aligned}$$

Thus

$$F \geq |z|^2 + e^{-2\lambda_{2j}t}(1-\cos(q_j t))|u|^2 - 2e^{-\lambda_{2j}t}|z||u|\,|\sin(q_j t/2)|.$$

Applying the inequality between the geometric and arithmetic means to the last term here, we conclude

$$F \geq |z|^2 + e^{-2\lambda_{2j}t}(1-\cos(q_j t))|u|^2 - 2e^{-2\lambda_{2j}t}|u|^2\sin^2(q_j t/2) - |z|^2/2 = |z|^2/2.$$

Because of (7.2), this implies the proposition. □

Consider now the semigroup $\mathcal{H}_t$. The block diagonal structure of the drift matrix B implies that $\mathcal{H}_t$ is the product of commuting semigroups acting in $\mathbb{R}^2$ and $\mathbb{R}$. Those in $\mathbb{R}^2$ are as just described, and those in $\mathbb{R}$ are like the ones considered in Section 2, with kernels given by (2.3). This implies a tensor product structure both for the invariant measure and for the kernel of $\mathcal{H}_t$. Let $\lambda_{2j-1} = \lambda_{2j}$ for $j = 1, \dots, m$. Then the invariant measure of $\mathcal{H}_t$ will be given by the expression (2.2). Further, Proposition 7.1 implies that the kernel of $\mathcal{H}_t$ satisfies

$$K_t(x,u) \leq \frac{\exp\left(\sum_{i=1}^{n} \lambda_i |x_i|^2\right)}{\sqrt{\Pi_{i=1}^{n}(1-e^{-2\lambda_i t})}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{\lambda_i}{1-e^{-2\lambda_i t}}\left|x_i - e^{-\lambda_i t}u_i\right|^2\right),$$

for all $t > 0$ and $x, u \in \mathbb{R}^n$. Observing now that the last expression coincides with the kernel given by (6.14) with $\kappa = 1/2$, we conclude the proof of Theorem 1.1 using Theorem 6.3.

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