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Defect measures on graded Lie groups

CLOTILDE FERMANIAN-KAMMERER AND VÉRONIQUE FISCHER

Abstract. In this article, we define a generalisation of microlocal defect measures (also known as H-measures) to the setting of graded nilpotent Lie groups. This requires to develop the notions of homogeneous symbols and classical pseudo-differential calculus adapted to this setting and defined via the representations of the groups. Our method relies on the study of the C^*-algebra of 0-homogeneous symbols. Then, we compute microlocal defect measures for concentrating and oscillating sequences, which also requires to investigate the notion of oscillating sequences in graded Lie groups. Finally, we discuss compensated compactness approaches in the context of graded nilpotent Lie groups.

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Contents

1. Introduction

The aim of this article is to develop a new approach for analysing the lack of compactness of bounded square integrable families on nilpotent Lie groups. The idea is to generalise the notions of microlocal defect measures (MDM) which were originally defined and studied in the Euclidean setting by Luc Tartar and Patrick Gérard independently in the 90's, see [48] and [32] respectively; the original definition of [32] is recalled in the next paragraph. These notions have given a new insight on compensated compactness theorems introduced by François Murat [44] and

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Luc Tartar [47] and on averaging theorems discovered by François Golse, Benoît Perthame and Rémi Sentis [36] and further developed by Ronald Di Perna and Pierre-Louis Lions [19]. Such theorems allow one to pass to the limit on quadratic quantities appearing in mechanics for example and of the form (Au_k, v_k) for weakly converging subsequences (u_k) and (v_k) provided their MDM's and the operator A satisfy convenient assumptions. Such descriptions were already possible in some cases thanks to the *Div-Curl* Lemma [44, 47]. The analysis of MDM's extends the range of applications of the ideas which are behind this lemma (see [29,30] for examples). The reader can refer to the book [49] for a presentation of MDMs in the context of the theory of homogenization, especially to Chapters 28 to 33 therein. We also want to point out that the MDM and their semi-classical counterpart (also called semi-classical or Wigner measures, see [33,34,41] or the ex-post review paper [35]) have also proved useful for the analysis of pde-s in different context, from quantum chemistry to theory of chaos and analysis of quantum ergodicity, including control theory [39,40]. And questions not so far to those of the latter references are now addressed in the context of sub-laplacians (see [15]). The Div-Curl lemma has recently been studied in the context of Lie groups: see the article [6] in the context of the Heisenberg group and [7] for Carnot groups. This motivates the investigation of MDM's and of compensated compactness questions on Lie groups.

Before discussing the setting of nilpotent Lie groups in more details, let us recall briefly the definition of a MDM in the Euclidean case. On an open subset Ω of $\mathbb{R}^n$, a MDM of a sequence $(u_k)_{k\in\mathbb{N}}$ of functions converging weakly to a function u in $L^2(\Omega, \text{loc})$ is a positive measure γ on $\Omega \times \mathbb{S}^{n-1}$ such that, up to extraction of a subsequence, we have the convergence

$$(A(u_{k_j} - u), u_{k_j} - u)_{L^2} \underset{j\to\infty}{\longrightarrow} \int_{\Omega\times\mathbb{S}^{n-1}} a_0(x, \xi)\, d\gamma(x, \xi), \tag{1.1}$$

for any test pseudodifferential operator A of order 0 with principal symbol a_0; by test pseudo-differential operators we mean for instance operators in the classical Hörmander calculus, properly supported, defined through inverse Fourier transform by

$$Au(x) = \int_{\mathbb{R}^n} a_0(x, \xi) e^{ix\cdot\xi} \widehat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \tag{1.2}$$

(for simplicity, in the formula above, we have assumed that the symbol of A is exactly a_0 and have no term of lower order). This extends easily to closed manifolds by replacing $\mathbb{S}^{n-1} \times \Omega$ with the spherical co-tangent bundle, and also to vector-valued functions by taking suitable traces. Note that in the approach of Luc Tartar [48] test operators are the ones which are tensor products of multiplication operators with Fourier multipliers, which is enough to construct the measure γ.

As we see from the paragraph above, the notion of (Euclidean) MDM relies on microlocal analysis and the theory of pseudodifferential operators which has been developed since the 70's in the Euclidean setting (see [37, 51], or the review books [20, 22]). The development of a pseudo-differential theory on nilpotent Lie groups has been the purpose of works by several authors, see, *e.g.*, [8, 9, 16, 17, 31, 50]. The recent contribution of the second author with her collaborators in [5] for the Heisenberg group has been followed by the monograph [25] of the first author and her collaborator, where they have defined a pseudodifferential calculus on graded nilpotent Lie groups. As in the Euclidean context, they are defined thanks to inverse Fourier transform with the major difference that the Fourier transform of a function at a (unitary irreducible) representation is an operator on the space of the representation. Consequently the symbols of pseudo-differential operators introduced in [25] are measurable fields of operators on $G \times \widehat{G}$ where $\widehat{G}$ is the unitary dual, *i.e.* the set of unitary irreducible representations of G modulo equivalence; as is customary, we will use the same notation for a unitary irreducible representation $\pi(x)$, $x \in G$ and its equivalence class in $\widehat{G}$. Then, the operator A whose symbol is the field of operator $\sigma(x,\pi)$ satisfies

$$Au(x) = \int_{\widehat{G}} \operatorname{tr}\left(\pi(x)\sigma(x,\pi)\widehat{u}(\pi)\right) d\mu(\pi), \quad u \in \mathcal{S}(G), \quad x \in G,$$

which is the analogue of (1.2) (precise definitions are given in Sections 2 and 3). It is on this latter result that relies the construction of MDM's developed hereafter. However, we shall need to extend the theory and we develop in Section 4 the classes of homogeneous symbols and of classical symbols, together with the notion of principal symbol.

We will see that the MDM's on a graded nilpotent Lie group G defined in this paper are non commutative objects, and this is not surprising since the Fourier transform is operator-valued. More precisely, a MDM on G consists of a positive measure γ on $G \times \Sigma_1$ and a γ-integrable field Γ of trace-class operators on $G \times \Sigma_1$, where Σ_1 is the quotient set $(\widehat{G}\backslash\{1\})/\mathbb{R}^+$ defined by use of dilations; the class of $\pi \in \widehat{G}$ will be denote by $\dot{\pi}$, see Section 2.3. Then, the analogue of formula (1.1) which is proved in Section 6 writes

$$\left(A(u_{k_j} - u), u_{k_j} - u\right)_{L^2} \underset{j\to\infty}{\longrightarrow} \int_{G\times\Sigma_1} \operatorname{tr}\left(\sigma_0(x,\dot{\pi})\Gamma(x,\dot{\pi})\right) d\gamma(x,\dot{\pi}), \qquad (1.3)$$

where σ_0 is the principal symbol of the operator A, as defined in Section 4. Note that operator-valued measures have been introduced in semi-classical settings since the 90's [23,24,43,45] and, more recently, in the context of quantum ergodicity [2–4, 42]. As in the Euclidean case, one can develop applications to compensated compactness as discussed in Section 7.

An important difference with the Euclidean context is the lack of Gårding inequality, and this prevents us to adapt the main steps of the proof of existence of Euclidean MDM's given in [32]. This is overcome by the use of C^*-algebra formalism and the notion of state. More precisely, we prove that convergent limits of quantities similar to the left-hand side of (1.3) in the nilpotent setting define positive linear functionals on the algebra of homogeneous symbols of order 0, and that these linear functionals extend to states on certain C^*-algebras. Understanding the spectrum of these C^*-algebras and decomposing these states yield the main result of the paper. This type of argument was suggested to the authors by Vladimir Georgescu as an alternative (albeit too sophisticated) proof of existence of MDM in the case of the abelian group $\mathbb{R}^n$ with arguments which were fitting for generalisations. Note also that, although this paper belongs to the fields of micro-local analysis and non-commutative analysis, many of its tools and techniques relies on the progress of the last four decades in harmonic analysis on Lie groups: for instance, in understanding the properties of spectral multipliers in sub-laplacians on nilpotent Lie groups (or more generally positive Rockland operators), or in describing homogeneous convolution operators in terms of their kernels.

Finally, we want to emphasize that the nilpotent Lie groups considered in this paper and in [25] are graded, but this is a natural restriction. Indeed, the class of graded nilpotent Lie groups contains the class of stratified Lie groups (also called Carnot groups in more geometric contexts), the prime example being the Heisenberg group. Graded or even stratified groups are the groups usually appearing in applications of analysis on nilpotent Lie groups, for instance in the study of operators sums of squares of vector fields, as in [15]. It is likely that many aspects of the results in this paper, especially the description of the algebras of operators in terms of their kernels, could also be done in the more general context of homogeneous nilpotent Lie groups and would then coincide with the seminal paper by Michael Christ, Daryl Geller, Pawel Głowacki, and Larry Polin [14]; however, this would require a more sophisticated presentation than the one performed here, rendering it more remote from the ideas of Euclidean micro-local analysis.

Our article is organised as follows. Section 2 is devoted to definitions on graded Lie groups and results in analysis in this setting that we shall use. Then we recall in Section 3 the definition of pseudodifferential operators on graded Lie groups and we introduce in Section 4 the notion of homogeneous symbols and of principal symbols. In Section 5, we analyse the C^*-algebras formed by 0-homogeneous symbols. The core of the paper consists in Section 6 where we prove the existence of MDM and analyse the fundamental examples of concentrating and oscillating sequences. Then, in Section 7, we link our results with the compensated compactness theory and the definition of *Curl* operators on Lie groups.

Convention: In the paper, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ denotes the Banach space of bounded linear mappings from $\mathcal{X}$ to $\mathcal{Y}$. If a linear operator A is densely defined in a Banach space $\mathcal{X}$ and valued in a Banach space $\mathcal{Y}$, then writing $A \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$ means that A extends to a bounded operator $\mathcal{X} \to \mathcal{Y}$ and that we identify the operator A with its bounded extension on $\mathcal{X}$ (which is unique). If $\mathcal{X} = \mathcal{Y}$, we write $\mathscr{L}(\mathcal{X}, \mathcal{X}) = \mathscr{L}(\mathcal{X})$. If $\mathcal{X}$ is a separable Hilbert space, we define by $\mathscr{L}^1(\mathcal{X})$ the trace-class operators on $\mathcal{X}$ and we set $\|A\|_{\mathscr{L}^1(\mathcal{X})} = \operatorname{tr}|A|$.

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2. Preliminaries: graded Lie groups

In this section, after defining graded Lie groups, we recall their homogeneous structure, the definition of the Fourier tranfrom and results on the dual. A complete description of the notions of graded and homogeneous nilpotent Lie groups may be found in [27, Chapter 1] and [25, Chapter 3].

2.1. Graded Lie groups

We will be concerned with graded Lie groups G which means that G is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits an $\mathbb{N}$-gradation $\mathfrak{g} = \oplus_{\ell=1}^{\infty}\mathfrak{g}_\ell$ where the $\mathfrak{g}_\ell$, $\ell = 1, 2, \ldots$, are vector subspaces of $\mathfrak{g}$, almost all equal to $\{0\}$, and satisfying $[\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subset \mathfrak{g}_{\ell+\ell'}$ for any $\ell, \ell' \in \mathbb{N}$. This implies that the group G is nilpotent. Examples of such groups are the Heisenberg group and, more generally, all stratified groups (which by definition correspond to the case $\mathfrak{g}_1$ generating the full Lie algebra $\mathfrak{g}$).

We construct a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ adapted to the gradation, by choosing a basis $\{X_1, \ldots X_{n_1}\}$ of $\mathfrak{g}_1$ (this basis is possibly reduced to $\emptyset$), then $\{X_{n_1+1}, \ldots, X_{n_1+n_2}\}$ a basis of $\mathfrak{g}_2$ (possibly $\{0\}$ as well as the others) and so on. The exponential mapping $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism from $\mathfrak{g}$ onto G, and we may identify the points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with the points

$$x = \exp_G(x_1X_1 + \cdots + x_nX_n) \in G.$$

Consequently we allow ourselves to denote by $C(G)$, $\mathcal{D}(G)$ and $\mathcal{S}(G)$ etc, the spaces of continuous functions, of smooth and compactly supported functions or of Schwartz functions on G identified with $\mathbb{R}^n$, and similarly for distributions with the duality notation $\langle\cdot,\cdot\rangle$.

This basis also leads to a corresponding Lebesgue measure on $\mathfrak{g}$ and a Haar measure dx on the group G – which we will fix once and for all – hence $L^p(G) \cong L^p(\mathbb{R}^n)$. The group convolution of two functions f_1 and f_2, for instance square integrable, is defined via

$$(f_1 * f_2)(x) := \int_G f_1(y)f_2\big(y^{-1}x\big)dy.$$

The convolution is not commutative: in general, $f_1 * f_2 \neq f_2 * f_1$.

The coordinate function $x = (x_1, \ldots, x_n) \in G \mapsto x_j \in \mathbb{R}$ is denoted by x_j. More generally we define for every multi-index $\alpha \in \mathbb{N}_0^n$, $x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, as a function on G. Similarly we set $X^\alpha = X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}$ in the (complex) universal enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. Let us recall that a vector of $\mathfrak{g}$ defines a left-invariant vector field on G and, more generally, that the universal enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic with the left-invariant differential operators; we keep the same notation for the vectors and the corresponding operators. However if $X \in \mathfrak{g}$, then $\tilde{X}$ denotes the corresponding right invariant vector field. More generally, if $T \in \mathfrak{U}(\mathfrak{g})$, we denote by $\tilde{T}$ the right-invariant differential operator.

For any $r > 0$, we define the linear mapping $D_r : \mathfrak{g} \to \mathfrak{g}$ by $D_r X = r^\ell X$ for every $X \in \mathfrak{g}_\ell$, $\ell \in \mathbb{N}$. Then the Lie algebra $\mathfrak{g}$ is endowed with the family of dilations $\{D_r, r > 0\}$ and becomes a homogeneous Lie algebra in the sense of [27]. We re-write the set of integers $\ell \in \mathbb{N}$ such that $\mathfrak{g}_\ell \neq \{0\}$ into the increasing sequence of positive integers $\upsilon_1, \dots, \upsilon_n$ counted with multiplicity, the multiplicity of $\mathfrak{g}_\ell$ being its dimension. In this way, the integers $\upsilon_1, \dots, \upsilon_n$ become the weights of the dilations and we have $D_r X_j = r^{\upsilon_j} X_j$, $j = 1, \dots, n$, on the chosen basis of $\mathfrak{g}$. The associated group dilations are defined by

$$D_r(x) = rx := \left(r^{\upsilon_1} x_1, r^{\upsilon_2} x_2, \dots, r^{\upsilon_n} x_n\right), \quad x = (x_1, \dots, x_n) \in G, \ r > 0.$$

In a canonical way, this leads to the notions of homogeneity for functions and operators. For instance the degree of homogeneity of x^α and X^α, viewed respectively as a function and a differential operator on G, is $[\alpha] = \sum_j \upsilon_j \alpha_j$. This also leads to the notion of homogeneous distribution: the Haar measure is Q-homogeneous:

$$r^Q \int_G f(rx)dx = \int_G f(y)dy, \tag{2.1}$$

where

$$Q := \sum_{\ell \in \mathbb{N}} \ell \dim \mathfrak{g}_\ell = \upsilon_1 + \dots + \upsilon_n,$$

is called the *homogeneous dimension* of G.

Recall that a *homogeneous quasi-norm* on G is a continuous function $|\cdot| : G \to [0, +\infty)$ homogeneous of degree 1 on G which vanishes only at 0. This often replaces the Euclidean norm in the analysis on homogeneous Lie groups. Any homogeneous quasi-norm $|\cdot|$ on G satisfies a triangle inequality up to a constant:

$$\exists C \geq 1, \quad \forall x, y \in G, \quad |xy| \leq C(|x| + |y|).$$

Any two homogeneous quasi-norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent in the sense that

$$\exists C > 0, \quad \forall x \in G, \quad C^{-1}|x|_2 \leq |x|_1 \leq C|x|_2.$$

There is an analogue of polar coordinates on G:

Proposition 2.1. *Let $|\cdot|$ be a fixed homogeneous quasi-norm on G. Then there is a (unique) positive Borel measure σ on the unit sphere $\mathfrak{S} := \{x \in G : |x| = 1\}$, such that for all $f \in L^1(G)$, we have*

$$\int_G f(x)dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr. \tag{2.2}$$

There is also an analogue of the mean value theorem:

Lemma 2.2. *We fix $|\cdot|$ a homogeneous quasi-norm on G. Then there exist constants $C > 0$ and $\eta > 1$ such that for any $f \in C^1(G)$, $x \in G$, we have*

$$|f(x) - f(0)| \leq C \sum_{j=0}^{n} |x|^{\upsilon_j} \sup_{|y| \leq \eta |x|} |X_j f(y)|.$$

2.2. The dual of *G* and the Plancherel theorem

Here we set some notations and recall some properties regarding the representations of the group G (especially the Plancherel theorem) and its enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$.

In this paper, we always assume that the representations of the group G are strongly continuous and acting on separable Hilbert spaces.Unless otherwise stated, the representations of G will also be assumed unitary. For a representation π of G, we keep the same notation for the corresponding infinitesimal representation which acts on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra of the group. It is characterised by its action on $\mathfrak{g}$:

$$\pi(X) = \partial_{t=0} \pi\left(e^{tX}\right), \quad X \in \mathfrak{g}. \tag{2.3}$$

The infinitesimal action acts on the space $\mathcal{H}_\pi^\infty$ of smooth vectors, that is, the space of vectors $v \in \mathcal{H}_\pi$ such that the mapping $G \ni x \mapsto \pi(x)v \in \mathcal{H}_\pi$ is smooth.

Example 2.3. Vectors of the form $\pi(\phi)v$ where $\phi \in \mathcal{D}(G)$ or $\mathcal{S}(G)$ and $v \in \mathcal{H}_\pi$ are smooth.

Here we have used the usual notation for the *group Fourier transform* of a function $f \in L^1(G)$ at π:

$$\pi(f) \equiv \widehat{f}(\pi) \equiv \mathcal{F}_G(f)(\pi) = \int_G f(x)\pi(x)^* dx \in \mathscr{L}(\mathcal{H}_\pi).$$

We denote by $\widehat{G}$ the unitary dual of G, that is, the set of unitary irreducible representations of G modulo equivalence, and identify a unitary irreducible representation with its class in $\widehat{G}$. The set $\widehat{G}$ is naturally equipped with a structure of standard Borel space.

The Plancherel measure is the unique positive Borel measure μ on $\widehat{G}$ such that for any $f \in C_c(G)$, we have:

$$\int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\mathcal{F}_G(f)(\pi)\|^2_{HS(\mathcal{H}_\pi)} d\mu(\pi). \tag{2.4}$$

Here $\|\cdot\|_{HS(\mathcal{H}_\pi)}$ denotes the Hilbert-Schmidt norm on $\mathcal{H}_\pi$. This implies that the group Fourier transform extends unitarily from $L^1(G) \cap L^2(G)$ to $L^2(G)$ onto $L^2(\widehat{G}) := \int_{\widehat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* d\mu(\pi)$ which we identify with the space of μ-square integrable fields on $\widehat{G}$. Consequently, (2.4) holds for any $f \in L^2(G)$; this formula is called the Plancherel formula. Furthermore, for any $\phi_1, \phi_2 \in L^2(G)$, the quantity $\int_{\widehat{G}} \mathrm{tr}\, |\widehat{\phi}_1(\pi)\widehat{\phi}_2(\pi)| d\mu(\pi)$ is finite and we have the Parseval formula

$$\int_G \phi_1(x)\bar{\phi}_2(x)\, dx = \int_{\widehat{G}} \mathrm{tr}\, \left(\widehat{\phi}_1(\pi)\widehat{\phi}_2(\pi)^*\right) d\mu(\pi). \tag{2.5}$$

The orbit method furnishes an expression for the Plancherel measure μ, see [18, Section 4.3]. However we will not need this here.

The general theory on locally compact unimodular group of type I applies (see [21]): let $\mathscr{L}(L^2(G))$ be the space of bounded linear operators on $L^2(G)$ and let $\mathscr{L}_L(L^2(G))$ be the subspace of those operators $T \in \mathscr{L}(L^2(G))$ which are left-invariant, that is, commute with the left translation:

$$T(f(g\cdot))(g_1) = (Tf)(gg_1), \quad f \in L^2(G),\ g, g_1 \in G.$$

Then there exists a field of bounded operators $\widehat{T}(\pi) \in \mathscr{L}(\mathcal{H}_\pi), \pi \in \widehat{G}$, such that

$$\forall f \in L^2(G), \quad \mathcal{F}_G(Tf)(\pi) = \widehat{T}(\pi)\ \widehat{f}(\pi) \quad \text{for } \mu - \text{almost all } \pi \in \widehat{G}.$$

Moreover the operator norm of T is equal to

$$\|T\|_{\mathscr{L}(L^2(G))} = \sup_{\pi \in \widehat{G}} \left\|\widehat{T}(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)}.$$

The supremum here has to be understood as the essential supremum with respect to the Plancherel measure μ. We denote by $L^\infty(\widehat{G})$ the space of measurable fields of operators $\sigma_\pi \in \mathscr{L}(\mathcal{H}_\pi), \pi \in \widehat{G}$, with

$$\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\sigma_\pi\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty,$$

modulo equivalence under the Plancherel measure μ. Conversely, any field in $L^\infty(\widehat{G})$ naturally yields a left-invariant bounded operator on $L^2(G)$.

By the Schwartz kernel theorem, any operator $T \in \mathscr{L}_L(L^2(G))$ is a convolution operator and we denote by $T\delta_0 \in \mathcal{S}'(G)$ its convolution kernel: $Tf = f * (T\delta_0)$, $f \in \mathcal{S}(G)$. We denote by $\mathcal{K}(G)$ the space of convolution kernels of operators in $\mathscr{L}_L(L^2(G))$ and we define the group Fourier transform of $T\delta_0$ as

$$\mathcal{F}_G(T\delta_0) \equiv \widehat{T}.$$

We may call $T\delta_0$ the kernel of T or of $\widehat{T}$. This extends the previous definition of the group Fourier transforms from $L^1(G) \cap \mathcal{K}(G)$ or $L^2(G) \cap \mathcal{K}(G)$ to $\mathcal{K}(G)$ onto $L^\infty(\widehat{G})$. The group Fourier transform also extends for instance to the space of convolution kernels $\mathcal{K}_{a,b}(G)$ of operators in $\mathscr{L}_L(L^2_a(G), L^2_b(G))$ where $L^2_a(G)$ denotes the Sobolev spaces on G, see [25, Section 4] and Section 2.5 below.

2.3. Dilations on $\widehat{G}$

Since the group G is a (connected, simply connected) nilpotent Lie group, one can use the orbit method to construct unitary irreducible representations of G (see, *e.g.*, [18]): with this method, to any linear functional $\varphi \in \mathfrak{g}^*$, one associates a class $\pi_\varphi \in \widehat{G}$ of equivalent unitary irreducible representations. Any element of $\widehat{G}$ may be realised in this way and two such classes in $\widehat{G}$ coincide when the linear functionals are on the same orbit for the co-adjoint action of G on $\mathfrak{g}^*$. In other words, one obtains a bijection $\mathfrak{g}^*/G \longleftrightarrow \widehat{G}$, known as Kirillov's map.

The dilations of G provide an action of $\mathbb{R}^+ = (0, \infty)$ on the Lie algebra $\mathfrak{g}$, hence on $\mathfrak{g}^*$ by duality, and one easily checks that quotienting by the co-adjoint action of G and by the $\mathbb{R}^+$-action commutes. Hence one obtains an action of $\mathbb{R}^+$ on $\mathfrak{g}^*/G$. The dilations also provide an action of $\mathbb{R}^+$ on the group G thus on its dual via

$$r \cdot \pi(x) = \pi(rx), \quad x \in G,\ \pi \in \widehat{G},\ r > 0. \tag{2.6}$$

One checks that Kirillov's map $\mathfrak{g}^*/G \longleftrightarrow \widehat{G}$ is $\mathbb{R}^+$-equivariant.

As usual, $\widehat{G}$ is equipped with the hull-kernel topology and $\mathfrak{g}^*/G$ with the quotient Euclidean topology. It is known [11] that Kirillov's map is a homeomorphism. One easily checks that, quotienting by the $\mathbb{R}^+$-actions, the map $(\mathfrak{g}^*/G)/\mathbb{R}^+ \longleftrightarrow \widehat{G}/\mathbb{R}^+$ is a homeomorphism.

If $X \in \mathfrak{g}$ is of degree d, then it follows from (2.3) that we have

$$\begin{aligned}(r \cdot \pi)(X) &= \partial_{t=0}(r \cdot \pi)\big(e^{tX}\big) = \partial_{t=0}\pi\left(re^{tX}\right)\\ &= \partial_{t=0}\pi\left(e^{tD_r(X)}\right) = \partial_{t=0}\pi\left(e^{tr^dX}\right) = r^d\pi(X).\end{aligned}$$

More generally for any $\alpha \in \mathbb{N}_0^n$, we have:

$$(r \cdot \pi)\big(X^\alpha\big) = r^{[\alpha]}\pi\big(X^\alpha\big), \quad r > 0. \tag{2.7}$$

If $f \in L^1(G)$, then so does $f \circ D_{r^{-1}}$ and using (2.1), we have

$$\begin{aligned}(r \cdot \pi)(f) &= \int_G f(x) r \cdot \pi(x)^* dx = \int_G f(x)\pi(rx)^* dx\\ &= \int_G f \circ D_{r^{-1}}(x)\pi(x)^* r^{-Q} dx = r^{-Q}\pi\left(f \circ D_{r^{-1}}\right).\end{aligned}$$

More generally, using the properties of the group Fourier transforms, we obtain

$$(r \cdot \pi)(f) = \pi\big(f_{(r)}\big) \quad \text{where } f_{(r)} = r^{-Q} f \circ D_{r^{-1}}, \quad r > 0, \tag{2.8}$$

for any f in $L^1(G)$, $L^2(G)$ or $\mathcal{K}_{a,b}(G)$.

Formula (2.8) and the Plancherel measure being unique, easily imply that for any positive measurable or integrable function F on $\widehat{G}$ and any $r > 0$, we have

$$\int_{\widehat{G}} F(r \cdot \pi) d\mu(\pi) = r^{-Q} \int_{\widehat{G}} F(\pi) d\mu(\pi). \tag{2.9}$$

Let us fix a quasi-norm $|\cdot|$ on G. This yields a map on $\mathfrak{g}^*$ for which we keep the same notation. We set

$$|[\varphi]| := \inf\left\{|\varphi'|, \varphi' \in [\varphi]\right\} = \min\left\{|\varphi'|, \varphi \in [\varphi]\right\},$$

where $[\varphi]$ denotes the co-adjoint class of $\varphi \in \mathfrak{g}^*$. Naturally, the map $[\varphi] \mapsto |[\varphi]|$ is continuous $\mathfrak{g}^*/\mathbb{R}^+ \to [0, \infty)$. We set for each $\pi \in \widehat{G}$,

$$|\pi| := \inf\left\{|\varphi|, \varphi \in \mathfrak{g}^* \text{ so that } \pi \equiv \pi_\varphi\right\} = \min\left\{|\varphi|, \varphi \in \mathfrak{g}^* \text{ so that } \pi \equiv \pi_\varphi\right\}, \quad (2.10)$$

where π_φ is the class of unitary irreducible representations of G corresponding to the co-adjoint orbit containing φ. This mapping is nothing else than the map $[\varphi] \mapsto |[\varphi]|$ transported by the Kirillov mapping. There the function $\pi \mapsto |\pi|$ is continuous from $\widehat{G}$ onto $[0, \infty)$.

One easily checks that the map $[\varphi] \mapsto |[\varphi]|$, and therefore the map $\pi \mapsto |\pi|$ respect the dilations in the following way:

$$\left|\left[\varphi\left(r^{-1} \cdot\right)\right]\right| = r|[\varphi]| \quad \text{and} \quad |r \cdot \pi| = r|\pi|\,, \quad r > 0,\ \pi \in \widehat{G},\ \varphi \in \mathfrak{g}^*. \quad (2.11)$$

Furthermore

$$|[\varphi]| = 0 \implies \varphi = 0\,, \qquad |\pi| = 0 \implies \pi = 1\,,$$

where 1 denotes the trivial representation of G.

This induces a continuous surjection from the 'sphere' in $\widehat{G}$

$$\Sigma_{1,|\cdot|} := \left\{\pi \in \widehat{G},\ |\pi| = 1\right\} \quad \text{onto} \quad \Sigma_1 := \left(\widehat{G}/\mathbb{R}^+\right)\backslash\{1\} = \left(\widehat{G}\backslash\{1\}\right)/\mathbb{R}^+. \quad (2.12)$$

This shows the following property:

Lemma 2.4. *The $\mathbb{R}^+$-quotient of Kirillov's map is a homeomorphism between the compact spaces $(\mathfrak{g}^*/G)/\mathbb{R}^+$ and $\widehat{G}/\mathbb{R}^+$. Moreover, the set Σ_1 is a compact subset of $\widehat{G}/\mathbb{R}^+$.*

Remark 2.5. In the case of the $2n+1$-dimensional Heisenberg group, this sphere Σ_1 is the union of two points with the horizontal sphere (that is, the unit sphere of $\mathbb{R}^{2n}$).

Having defined a unit sphere on $\widehat{G}$, we can state a polar decomposition:

Lemma 2.6. *Let $|\cdot|$ be a quasi-norm on G and let $|\cdot|$ be the associated mapping on $\widehat{G}$ and the sphere $\Sigma_{1,|\cdot|}$ defined in* (2.12) *above.*

(1) *The linear mapping $f \longmapsto \int_{1\leq|\pi|\leq e} f(|\pi|^{-1}\cdot\pi)|\pi|^{-Q}d\mu(\pi)$, defines a continuous positivity-preserving linear functional on the Banach space $C(\Sigma_{1,|\cdot|})$ of continuous function on the compact space $\Sigma_{1,|\cdot|}$. We denote by $\varsigma_{|\cdot|}$ the corresponding Radon measure;*

(2) *For any measurable function* $F : \widehat{G} \to [0, \infty)$, *we have:*

$$\int_G F(\pi) d\mu(\pi) = \int_{\Sigma_{1,|\cdot|} \times (0,\infty)} F(r \cdot \pi) d\varsigma_{|\cdot|}(\pi) r^{Q-1} dr;$$

(3) *In particular, if* $u \in L^2(G)$, *then*

$$\int_{\Sigma_{1,|\cdot|} \times (0,\infty)} \|\hat{u}(r \cdot \pi)\|_{HS}^2 d\varsigma_{|\cdot|}(\pi) r^{Q-1} dr = \|u\|_{L^2(G)}^2;$$

(4) *If* $F \in L^1(\widehat{G} \backslash \{1\}, \text{loc})$ *is* $(-Q)$*-homogeneous, that is,* $F(r \cdot \pi) = r^{-Q} F(\pi)$, *then we have*

$$\forall r > 0, \qquad \int_{1 \le |\pi| \le r} F(\pi)\, d\mu(\pi) = |\ln r| \int_{1 \le |\pi| \le e} F(\pi)\, d\mu(\pi).$$

Furthermore the quantity $\int_{1 \le |\pi| \le e} F(\pi)\, d\mu(\pi)$ *is independent of* $|\cdot|$.

Proof. One easily checks Part (1). As the Plancherel measure is the unique measure such that the Plancherel formula holds, it suffices to show Part (3) which follows from simple manipulations and (2.9). Part (4) is obtained easily by adapting the ideas of the proof of the polar decomposition on a homogeneous Lie group (see, *e.g.*, [25, Section 3.1.7]). □

Part (4) may shed some light on our choice of definition for $\varsigma_{|\cdot|}$ and enables us to define the measure ς on Σ_1 in the following way:

$$\int_{\Sigma_1} F(\dot{\pi}) d\varsigma(\dot{\pi}) = \int_{\Sigma_{1,|\cdot|}} F(\dot{\pi}) d\varsigma_{|\cdot|}(\pi), \tag{2.13}$$

for any quasi-norm $|\cdot|$ on G and any measurable function $F : \Sigma_1 \to [0, \infty)$ which we also identify with a (0-homogeneous) measurable function $F : \widehat{G} \to [0,\infty)$. Part (2) then yields for any integrable function $f : \widehat{G} \to [0, \infty)$,

$$\int_{\widehat{G}} f(\pi) d\mu(\pi) = \int_{\Sigma_1} F(\dot{\pi}) d\varsigma(\dot{\pi}) \qquad \text{where} \qquad F(\dot{\pi}) = \int_{r=0}^{+\infty} f(r \cdot \pi) r^Q \frac{dr}{r}.$$

2.4. Rockland operators

Here we recall the definition of Rockland operators and their main properties. See [25, Chapter 4] for proofs and references.

Definition 2.7. A *Rockland operator* $\mathcal{R}$ on G is a left-invariant differential operator which is homogeneous of positive degree and satisfies the Rockland condition: (R) for each unitary irreducible representation π on G, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, that is, for all $v \in \mathcal{H}_\pi^\infty$, $\pi(\mathcal{R}) v = 0 \implies v = 0$.

Example 2.8. In the stratified case, one can check easily that any (left-invariant negative) *sub-Laplacian*, that is

$$\begin{aligned} &\mathcal{L} = Z_1^2 + \ldots + Z_{n'}^2 \\ &\text{with } Z_1, \ldots, Z_{n'} \text{ forming any basis of the first stratum } \mathfrak{g}_1, \end{aligned} \tag{2.14}$$

is a Rockland operator.

Example 2.9. On any graded group G, it is not difficult to see that the operator

$$\sum_{j=1}^{n} (-1)^{\frac{\nu_o}{\upsilon_j}} c_j X_j^{2\frac{\nu_o}{\upsilon_j}} \quad \text{with} \quad c_j > 0, \tag{2.15}$$

is a Rockland operator of homogeneous degree $2\nu_o$ if ν_o is any common multiple of $\upsilon_1, \ldots, \upsilon_n$.

Hence Rockland operators do exist on any graded Lie group (not necessarily stratified).

One easily checks that if $\mathcal{R}$ is a Rockland operator then so are $\mathcal{R}^t$ and $\bar{\mathcal{R}}$ defined as elements of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$; recall that if $A \in \mathfrak{U}(\mathfrak{g})$ then it is written uniquely as a finite linear combination of X^α, $\alpha \in \mathbb{N}_0^n$, *i.e.* $A = \sum_{(\alpha)} c_\alpha X^\alpha$, and that A^t and $\bar{A}$ are defined via

$$A^t = \sum_{(\alpha)} c_\alpha (X^\alpha)^t, \qquad \bar{A} = \sum_{(\alpha)} \bar{c}_\alpha X^\alpha, \tag{2.16}$$

where $(X^\alpha)^t = (-1)^{|\alpha|} X_n^{\alpha_n} \ldots X_1^{\alpha_1}$. Consequently if $\mathcal{R}$ is a Rockland operator then so is $\mathcal{R}^* = \bar{\mathcal{R}}^t$. (defined as element in $\mathfrak{U}(\mathfrak{g})$).

If the Rockland operator $\mathcal{R}$ is formally self-adjoint, that is, $\mathcal{R}^* = \mathcal{R}$ as elements in $\mathfrak{U}(\mathfrak{g})$, then $\mathcal{R}$ and $\pi(\mathcal{R})$ admit self-adjoint extensions on $L^2(G)$ and $\mathcal{H}_\pi$ respectively. We keep the same notation for their self-adjoint extension. We denote by E and E_π their spectral measure:

$$\mathcal{R} = \int_{\mathbb{R}} \lambda dE(\lambda) \quad \text{and} \quad \pi(\mathcal{R}) = \int_{\mathbb{R}} \lambda dE_\pi(\lambda).$$

Example of formally self-adjoint Rockland operators are the positive Rockland operators, that is, Rockland operators $\mathcal{R}$ that satisfy

$$\forall f \in \mathcal{S}(G), \qquad \int_G \mathcal{R} f(x)\, \overline{f(x)}\, dx \geq 0.$$

One easily checks that the operator in (2.15) is positive. This shows that positive Rockland operators always exist on any graded Lie group. Note that if G is stratified and $\mathcal{L}$ is a (left-invariant negative) sub-Laplacian, then it is customary to privilege $-\mathcal{L}$ as a positive Rockland operator.

The next lemma says that the point 0 can be neglected in the spectrum of a positive Rockland operator and its group Fourier transform.

Lemma 2.10. *Let $\mathcal{R}$ be a positive Rockland operator with spectral measure E.*

(1) *Then for any $f \in L^2(G)$,*

$$\|E[0,\epsilon)f\|_2 \searrow 0 \quad \textit{and} \quad \|E(\epsilon,+\infty)f\|_{L^2(G)} \nearrow \|f\|_{L^2(G)} \quad \textit{as} \quad \epsilon \searrow 0;$$

(2) *If π is a non-trivial unitary irreducible representation of G, then the spectrum of $\pi(\mathcal{R})$ is a discrete subset of $(0,\infty)$;*
(3) *Let $\psi \in C^\infty(\mathbb{R})$ be a scalar valued function satisfying $\psi \equiv 1$ on (Λ,∞) for some $\Lambda > 0$. Then $\psi(t\mathcal{R})$ and $\psi(t\pi(\mathcal{R}))$ converges to the identity mapping of $L^2(G)$ and $\mathcal{H}_\pi$ for the strong operator topology (SOT) as $t \to 0^+$. Moreover we have*

$$\forall \pi \in \widehat{G}, \quad \exists r = r_\pi > 0, \quad \forall r > r_\pi, \qquad \psi(r\cdot\pi(\mathcal{R})) = \psi(r^\nu \pi(\mathcal{R})) = \mathrm{I}_{\mathcal{H}_\pi}. \quad (2.17)$$

Sketch of the proof. Let us recall that the heat kernel h_t of $\mathcal{R}$ is by definition the right convolution kernel of $e^{-t\mathcal{R}}$ and that it satisfies $h_t = t^{-\frac{Q}{\nu}} h_1 \circ D_{t^{-\frac{1}{\nu}}}$ with $h_1 \in \mathcal{S}(G)$. This has the two following consequences. Firstly, it yields classically

$$\left\|e^{-t\mathcal{R}} f\right\|_2 = \|f * h_t\| \underset{t\to 0^+}{\longrightarrow} 0, \qquad f \in L^2(G),$$

which implies Part (1). Secondly, it implies that the operators $\pi(h_t)$, $t > 0$, are compact and form a continuous semi-group. One easily checks that $\pi(\mathcal{R})$ is its infinitesimal generator, and this yields Part (2). Part (3) follows easily from spectral properties and Parts (1) and (2). □

Remark 2.11. We can give a value for r_π in (2.17):

$$r_\pi = \frac{\lambda_{\min}(\pi)}{\Lambda} \in (0,\infty),$$

where $\lambda_{\min}(\pi)$ is the minimum eigenvalue of $\pi(\mathcal{R})$, see Part 2 of Lemma 2.10. In this case, r_π is ν-homogeneous in π:

$$\forall t > 0,\ \pi \in \widehat{G}, \qquad r_{t\cdot\pi} = \frac{\lambda_{\min}(t\cdot\pi)}{\Lambda} = \frac{t^\nu \lambda_{\min}(\pi)}{\Lambda}.$$

Hence the range of r_π as π runs over $\widehat{G}$ is $(0,\infty)$.

The properties of the functional calculus of $\mathcal{R}$ and of the group Fourier transform imply the following lemma.

Lemma 2.12. *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν and $f : \mathbb{R}^+ \to \mathbb{C}$ be a measurable function. We assume that the domain of the operator $f(\mathcal{R}) = \int_{\mathbb{R}} f(\lambda) dE(\lambda)$ contains $\mathcal{S}(G)$. Then for any $r > 0$*

$$\left(f(r^\nu\mathcal{R})\phi\right) \circ D_r = f(\mathcal{R})\,(\phi \circ D_r), \quad \phi \in \mathcal{S}(G),$$

and, denoting by $f(\mathcal{R})\delta_0$ the right convolution kernel of $f(\mathcal{R})$,

$$f\big(r^{\nu}\mathcal{R}\big)\delta_0(x) = r^{-Q} f(\mathcal{R})\delta_0\big(r^{-1}x\big), \quad x \in G. \tag{2.18}$$

Besides, if π in an irreducible unitary representation, the domain of the operator $f(\pi(\mathcal{R})) = \int_{\mathbb{R}} f(\lambda)dE_\pi(\lambda)$ contains $\mathcal{H}_\pi^\infty$ and we have

$$\mathcal{F}\{f(\mathcal{R})\phi\}(\pi) = f(\pi(\mathcal{R}))\,\widehat{\phi}(\pi), \quad \phi \in \mathcal{S}(G). \tag{2.19}$$

2.5. Sobolev spaces

The (inhomogeneous) *Sobolev spaces* $L^2_a(G)$, respectively the *homogeneous Sobolev spaces* $\dot{L}^2_a(G)$, $a \in \mathbb{R}$, as the completion of the domain $\mathrm{Dom}(\mathrm{I}+\mathcal{R})^{\frac{a}{\nu}}$ of $(\mathrm{I}+\mathcal{R})^{\frac{a}{\nu}}$, respectively the domain $\mathrm{Dom}(\mathcal{R}^{\frac{a}{\nu}})$ of $\mathcal{R}^{\frac{a}{\nu}}$, for the Sobolev norm

$$\|f\|_{L^2_a(G),\mathcal{R}} := \big\|(\mathrm{I}+\mathcal{R})^{\frac{a}{\nu}} f\big\|_{L^2(G)}, \quad \text{respectively} \quad \|f\|_{\dot{L}^2_a(G),\mathcal{R}} := \big\|\mathcal{R}^{\frac{a}{\nu}} f\big\|_{L^2(G)}.$$

We realise the elements of $L^2_a(G)$ as tempered distributions and we have

$$\mathcal{S}(G) \subset \mathrm{Dom}(\mathrm{I}+\mathcal{R})^{\frac{a}{\nu}} \subset L^2_a(G) \subset \mathcal{S}'(G).$$

We realise the elements of $\dot{L}^2_a(G)$ as the linear functionals f on $\mathrm{Dom}(\bar{\mathcal{R}}^{-\frac{a}{\nu}})$ satisfying

$$\exists C > 0, \qquad \forall \phi \in \mathrm{Dom}\big(\bar{\mathcal{R}}^{-\frac{a}{\nu}}\big), \qquad |f(\phi)| \le C\big\|\bar{\mathcal{R}}^{-\frac{a}{\nu}}\phi\big\|_{L^2(G)},$$

where the Rockland operator $\bar{\mathcal{R}}$ is defined via (2.16). Each $f \in \dot{L}^2_a(G)$ defines a unique function $\mathcal{R}^{\frac{a}{\nu}} f \in L^2(G)$ via the continuous linear functional $\psi \mapsto f(\bar{\mathcal{R}}^{\frac{a}{\nu}}\psi)$.

The following lemma implies that the Sobolev spaces defined above do not depend on the choice of Rockland operators:

Lemma 2.13. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two positive Rockland operators of homogeneous degree ν_1 and ν_2 respectively. Then for any $a \in \mathbb{R}$, the operators $(\mathrm{I}+\mathcal{R}_1)^{\frac{a}{\nu_1}}(\mathrm{I}+\mathcal{R}_2)^{-\frac{a}{\nu_2}}$ and $(\mathcal{R}_1)^{\frac{a}{\nu_1}}(\mathcal{R}_2)^{-\frac{a}{\nu_2}}$ extends to bounded operators on $L^2(G)$.*

The Sobolev spaces defined above satisfy the following natural properties:

Theorem 2.14.

(1) *The spaces $L^2_a(G)$ and $\dot{L}^2_a(G)$ are Banach spaces. Different choices of positive Rockland operators yield equivalent* (*homogeneous*) *Sobolev norms;*
(2) (*Sobolev embeddings*) *We have the continuous inclusions*

$$L^2_a(G) \subset C(G), \ a > Q/2,$$

where $C(G)$ denotes the Banach space of continuous and bounded functions on G;

(3) *If* $a = 0$ *then* $L^2_0(G) = L^2(G)$. *If* $a > 0$ *then* $\mathrm{Dom}(\mathcal{R}^{\frac{a}{\nu}}) = \mathrm{Dom}(\mathrm{I} + \mathcal{R})^{\frac{a}{\nu}} \supset \mathcal{S}(G)$, *and* $L^2_a(G) = L^2(G) \cap \dot{L}^2_a(G)$ *with*

$$\|f\|_{L^2_a(G)} \asymp \|f\|_{L^2(G)} + \|f\|_{\dot{L}^2_a(G)},$$

after a choice of positive Rockland operators to realise the Sobolev norms;

(4) *For any* $\alpha \in \mathbb{N}_0^n$, X^α *maps continuously* $L^2_s(G)$ *to* $L^2_{s-[\alpha]}(G)$ *and* $\dot{L}^2_s(G)$ *to* $\dot{L}^2_{s-[\alpha]}(G)$, *for any* $s \in \mathbb{R}$;

(5) *Let* $\mathcal{R}$ *be a positive Rockland operator of degree* ν. *Let also* $a, s \in \mathbb{R}$. *Then the operator* $(\mathrm{I} + \mathcal{R})^{\frac{a}{\nu}}$ *maps continuously* $L^2_s(G)$ *to* $L^2_{s-a}(G)$ *and the operator* $\mathcal{R}^{\frac{a}{\nu}}$ *maps continuously* $\dot{L}^2_s(G)$ *to* $\dot{L}^2_{s-a}(G)$;

(6) *For any* $s \in \mathbb{R}$, *the Banach spaces* $L^2_{-s}(G)$ *and* $\dot{L}^2_{-s}(G)$ *are the duals of* $L^2_s(G)$ *and* $\dot{L}^2_s(G)$ *respectively via the dualities*

$$\langle f, g\rangle_{L^2_s \times L^2_{-s}} = \left\langle (\mathrm{I} + \mathcal{R})^{\frac{s}{\nu}} f, (\mathrm{I} + \mathcal{R})^{-\frac{s}{\nu}} g\right\rangle_{L^2 \times L^2},$$
$$\langle f, g\rangle_{\dot{L}^2_s \times \dot{L}^2_{-s}} = \left\langle \mathcal{R}^{\frac{s}{\nu}} f, \mathcal{R}^{-\frac{s}{\nu}} g\right\rangle_{L^2 \times L^2};$$

(7) *The Banach spaces* $L^2_a(G)$ *and* $\dot{L}^2_a(G)$ *satisfy the properties of interpolation* (*in the sense of* [25, Theorem 4.4.9 and Proposition 4.4.15]).

In order to distinguish the Sobolev spaces $L^2_s(G)$ on the graded group G and the usual Sobolev spaces on the underlying $\mathbb{R}^n$, we denote by H^s the Euclidean Sobolev spaces on $\mathbb{R}^n$. The spaces H^s and $L^2_s(G)$ are not comparable globally (we assume that G is not abelian), but they are locally:

Proposition 2.15. *For any* $s \in \mathbb{R}$ *and any* $\chi \in \mathcal{D}(\mathbb{R}^n)$, *the mapping* $\mathcal{S}(G) \ni f \mapsto \chi f$ *extends* (*uniquely*) *to a continuous operator of* $H^s \to L^2_{s\upsilon_1}(G)$ *and to a continuous operator of* $L^2_s(G) \to H^{s/\upsilon_n}$ (*where* $\upsilon_1 \leq \ldots \leq \upsilon_n$ *are the dilation's weights in increasing order*).

2.6. Bessel potential and Fourier Inversion Formula

Classical considerations on Bessel potentials in this context imply that the convolution kernel of $(\mathrm{I} + \mathcal{R})^{\frac{s_1}{\nu}}$ is square integrable when $s_1 < -Q/2$, *i.e.* $(\mathrm{I} + \mathcal{R})^{\frac{s_1}{\nu}} \delta_0 \in L^2(G)$, see [25, Section 4.3.3]. The Plancherel formula (2.4) then yields

$$\int_{\widehat{G}} \left\| \pi(\mathrm{I} + \mathcal{R})^{\frac{s_1}{\nu}} \right\|^2_{HS(\mathcal{H}_\pi)} d\mu(\pi) < \infty \quad \text{for } s_1 < -Q/2, \tag{2.20}$$

and consequently,

$$\int_{\widehat{G}} \mathrm{tr} \left| \pi(\mathrm{I} + \mathcal{R})^{\frac{s_1}{\nu}} \right| d\mu(\pi) < \infty \quad \text{for } s_1 < -Q. \tag{2.21}$$

Naturally, the Plancherel theorem (*cf.* Section 2.2) implies a Fourier Inverse Formula (FIF), at least formally.

Proposition 2.16 (Fourier Inversion Formula). *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $\sigma = \{\sigma(\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ be a field of operators on $\widehat{G}$ defined (at least) on $\int_{\widehat{G}} \mathcal{H}_\pi^\infty d\mu(\pi)$. Let $s > Q$. We assume that one of the quantity*

$$S_l := \sup_{\pi \in \widehat{G}} \left\| \pi(\mathrm{I} + \mathcal{R})^{\frac{s}{\nu}} \sigma(\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \quad \text{or} \quad S_r := \sup_{\pi \in \widehat{G}} \left\| \sigma(\pi) \pi(\mathrm{I} + \mathcal{R})^{\frac{s}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)}$$

is finite. Then $\sigma \in L^2(\widehat{G})$ and $\kappa = \mathcal{F}^{-1}\sigma$ coincides with a continuous and bounded function on G. Moreover,

$$\int_{\widehat{G}} \mathrm{tr}\, |\sigma(\pi)| d\mu(\pi) < \infty \qquad \text{and} \qquad \kappa(0) = \int_{\widehat{G}} \mathrm{tr}\, \sigma(\pi)\, d\mu(\pi).$$

In the statement, since $\{\pi(\mathrm{I} + \mathcal{R})^{\frac{s}{\nu}} : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, \pi \in \widehat{G}\}$ acts on $\int_{\widehat{G}} \mathcal{H}_\pi^\infty d\mu(\pi)$ [25, Lemma 5.1.2], the fields of operators $\{\sigma(\pi)\pi(\mathrm{I} + \mathcal{R})^{\frac{s}{\nu}}, \pi \in \widehat{G}\}$ and $\{\pi(\mathrm{I} + \mathcal{R})^{\frac{s}{\nu}} \sigma(\pi), \pi \in \widehat{G}\}$ are well defined on $\int_{\widehat{G}} \mathcal{H}_\pi^\infty d\mu(\pi)$ and the hypotheses make sense. Here we use the convention that if A and B are two unbounded operators on a Hilbert space $\mathcal{H}$, then the composition AB is the operator given by $ABv = A(Bv)$ where $v \in \mathcal{H}$ is in the domain of B and such that Bv is in the domain of A.

Since we shall make extensive use of the Fourier Inversion Formula and for the sake of completeness, we give a proof of this result. We will use the following classical properties for approximations of δ_0:

Lemma 2.17. *Let $\psi_1 \in \mathcal{S}(G)$. For $\epsilon > 0$, we set $\psi_\epsilon = (\psi_1)_{(\epsilon)}$, that is, $\psi_\epsilon(x) = \epsilon^{-Q}\psi_1(\epsilon^{-1}x)$. We also denote $c := \int_G \psi_1 = \int_G \psi_\epsilon$.*

(1) *As $\epsilon \to 0$, we have $\psi_\epsilon \longrightarrow c\delta_0$ in $\mathcal{S}'(G)$, and if $\kappa \in \mathcal{S}'(G)$ is continuous and bounded then $\int_G \kappa \psi_\epsilon \longrightarrow c\kappa(0)$;*
(2) *If π is a continuous unitary representation of G, then $(\widehat{\psi_\epsilon}(\pi))_{\epsilon>0}$ converges to $c\mathrm{I}_{\mathcal{H}_\pi}$ in the strong operator topology (SOT) on $\mathcal{H}_\pi$.*

Proof of Lemma 2.17. Part 1 is classical, see, *e.g.*, [25, Section 3.1.10]. For Part 2, we write

$$\widehat{\psi_\epsilon}(\pi)^* - c\mathrm{I}_{\mathcal{H}_\pi} = \int_G \psi_\epsilon(x)\pi(x)dx - c\mathrm{I}_{\mathcal{H}_\pi} = \int_G \psi_1(x) \left(\pi(\epsilon x) - \mathrm{I}_{\mathcal{H}_\pi}\right) dx.$$

Thus, applying a vector $v \in \mathcal{H}_\pi$ and for any $R > 0$ decomposing the integral as $\int_G = \int_{|x|<R} + \int_{|x|\geq R}$, we obtain:

$$\left\| \left(\widehat{\psi_\epsilon}(\pi)^* - c\mathrm{I}_{\mathcal{H}_\pi}\right) v \right\|_{\mathcal{H}_\pi} \leq \sup_{|x'| \leq \epsilon R} \left\| \pi(x')v - v \right\|_{\mathcal{H}_\pi} \int_G |\psi_1| + 2\|v\|_{\mathcal{H}_\pi} \int_{|x|>R} |\psi_1(x)|.$$

And the conclusion follows easily from the continuity of $x' \mapsto \pi(x')v$ at 0 and the integrability of $\psi_1 \in \mathcal{S}(G)$. □

Proof of Proposition 2.16. Let us assume S_l finite. The membership of σ in $L^2(\widehat{G})$ follows from $\|\sigma\|_{L^2(\widehat{G})} \leq \|\pi(\mathrm{I}+\mathcal{R})^{-\frac{s}{\nu}}\|_{L^2(\widehat{G})} S_l$, since the first term of the right-hand side is square integrable by (2.20). We also have

$$\int_{\widehat{G}} \operatorname{tr}|\sigma(\pi)| d\mu(\pi) \leq S_l \int_{\widehat{G}} \operatorname{tr}\left|\pi(\mathrm{I}+\mathcal{R})^{-\frac{s}{\nu}}\right| d\mu(\pi)$$

and the last integral is finite by (2.21). Hence $\int_{\widehat{G}} \operatorname{tr}|\sigma(\pi)| d\mu(\pi)$ is finite.

Let $\kappa = \mathcal{F}^{-1}\sigma$. As $\sigma \in L^2(\widehat{G})$, $\kappa \in L^2(G)$. Moreover

$$\begin{aligned}\left\|(\mathrm{I}+\mathcal{R})^{\frac{s}{2\nu}}\kappa\right\|_{L^2(G)} &= \left\|\pi(\mathrm{I}+\mathcal{R})^{\frac{s}{2\nu}}\sigma\right\|_{L^2(\widehat{G})} \\ &\leq \left\|\pi(\mathrm{I}+\mathcal{R})^{-\frac{s}{2\nu}}\right\|_{L^2(\widehat{G})} \sup_{\pi\in\widehat{G}} \left\|\pi(\mathrm{I}+\mathcal{R})^{\frac{s}{\nu}}\sigma(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)}.\end{aligned}$$

Hence $\kappa \in L^2_{s/2}$. The Sobolev embedding, see Theorem 2.14, implies that κ is continuous and bounded on G.

Let $\psi_1 \in \mathcal{S}(G)$ with $\int_G \psi_1 = 1$. We construct the δ_0-approximate $(\psi_\epsilon)_{\epsilon>0} \subset \mathcal{S}(G)$ as in Lemma 2.17. By the Parseval formula, see (2.5), we have:

$$\int_G \kappa(x)\bar{\psi}_\epsilon(x)dx = \int_{\widehat{G}} \operatorname{tr}\left(\sigma(\pi)\widehat{\psi}_\epsilon(\pi)^*\right) d\mu(\pi). \tag{2.22}$$

By Lemma 2.17, the left-hand side of (2.22) tends to $\kappa(0)$ as $\epsilon \to 0$. Note that the right-hand side of (2.22) is integrable since:

$$\left|\operatorname{tr}\left(\sigma(\pi)\widehat{\psi}_\epsilon(\pi)^*\right)\right| \leq \left\|\widehat{\psi}_\epsilon(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)} \operatorname{tr}|\sigma| \leq \|\psi_1\|_{L^1(G)} \operatorname{tr}|\sigma|.$$

Lemma 2.17 and the Lebesgue Dominated Convergence Theorem imply that the right-hand side of (2.22) converges to $\int_{\widehat{G}} \operatorname{tr}\sigma(\pi)d\mu(\pi)$ as $\epsilon \to 0$. Taking the limit in both sides of (2.22) as $\epsilon \to 0$ concludes the proof of Proposition 2.16 under the condition that S_l is finite.

The case of S_r finite may be obtained by taking the adjoint or by using right Sobolev spaces instead of the left ones $L^2_{s/2}$. □

Remark 2.18. In fact, the proof above shows that if $\psi_1 \in \mathcal{S}(G)$ and $\psi_\epsilon = (\psi_1)_{(\epsilon)}$ as in Lemma 2.17 and if σ and κ are as in Proposition 2.16, then

$$\int_{\widehat{G}}\int_{\widehat{G}} \operatorname{tr}\left|\sigma(\pi)\widehat{\psi}_\epsilon^*(\pi)\right| d\mu(\pi) < \infty, \text{ and } \lim_{\epsilon\to 0}\int_{\widehat{G}} \operatorname{tr}\left(\sigma(\pi)\widehat{\psi}_\epsilon^*(\pi)\right) d\mu(\pi) = c\,\kappa(0),$$

where $c = \int_G \psi_1$.

Corollary 2.19. *Let $\sigma \in L^\infty(\widehat{G})$. Then for any $\phi \in \mathcal{S}(G)$, we have*

$$\int_{\widehat{G}} \text{tr} \left|\sigma(\pi)\mathcal{F}_G(\check{\phi})(\pi)\right| d\mu(\pi) < \infty$$

and denoting by $\kappa \in \mathcal{S}'(G)$ the kernel of σ, i.e. *$\sigma = \widehat{\kappa}$, we have:*

$$\langle \kappa, \phi \rangle = \int_{\widehat{G}} \text{tr} \left(\sigma(\pi)\mathcal{F}_G(\check{\phi})(\pi)\right) d\mu(\pi), \tag{2.23}$$

where $\check{\phi}(x) = \phi(x^{-1})$.

Proof. If $\phi \in \mathcal{S}(G)$, then $\widehat{\phi}$ satisfies the hypotheses of Proposition 2.16 since $(\text{I} + \mathcal{R})^N\phi \in \mathcal{S}(G)$ is integrable for any $N \in \mathbb{N}$. For the same reason $\sigma\mathcal{F}_G(\check{\phi})$ satisfies the hypotheses of Proposition 2.16. We conclude with $\langle \kappa, \phi \rangle = \check{\phi} * \kappa(0)$ and $\mathcal{F}_G(\check{\phi} * \kappa) = \sigma\mathcal{F}_G(\check{\phi})$. □

Remark 2.20. Corollary 2.19 implies that if $\sigma \in L^\infty(\widehat{G})$ and $\kappa \in \mathcal{S}'(G)$ are such that (2.23) holds for any $\phi \in \mathcal{S}(G)$ or $\mathcal{D}(G)$ then κ is the kernel of σ, *i.e.* $\sigma = \widehat{\kappa}$.

We will also need the following inversion formula:

Proposition 2.21. *Let κ be a compactly supported distribution on G. Then for each unitary representation π of G and $v, w \in \mathcal{H}_\pi$, we can define*

$$(\widehat{\kappa}(\pi)v, w)_{\mathcal{H}_\pi} = \int_G \kappa(x)(\pi(x)^*v, w)_{\mathcal{H}_\pi} dx,$$

since $x \mapsto (\pi(x)^*v, w)_{\mathcal{H}_\pi}$ is smooth and bounded on G.

For any smooth and bounded function ϕ on G, we have

$$\int_{\widehat{G}} \text{tr} \left(\widehat{\kappa}(\pi)\, \widehat{\phi}(\pi)\right) d\mu(\pi) = \langle \kappa, \phi \rangle,$$

interpreting the left-hand side as the limits (in this order) of the absolutely convergent double integral:

$$\lim_{R\to\infty} \lim_{N\to+\infty} \int_{N\cdot\mathcal{C}} \int_G \text{tr}_N \left(\widehat{\kappa}(\pi)\pi(x)\right) \phi(x)\chi_R(x)\, dx d\mu(\pi),$$

where $\chi \in \mathcal{D}(G)$ with $\chi \equiv 1$ on a neighbourhood of 0 and $\chi_R(x) := \chi(R^{-1}x)$, $\mathcal{C}$ a compact neighbourhood of $1 \in \widehat{G}$ such that $\cup_{N\in\mathbb{N}} N \cdot \mathcal{C} = \widehat{G}$, and tr_N denotes the trace of the operators projected on the subspace spanned by the first N vectors, having fixed a fundamental sequence of vector fields.

For instance, having fixed a quasinorm, we can choose $\mathcal{C} := \{|\pi| \leq 1\}$, see Section 2.3. The definition of a fundamental sequence of vector fields may be found in [21, A93].

Proof. Corollary 2.19 implies the result when $\kappa \in \mathcal{D}(G)$. Bedsides, if κ is a compactly supported distribution, we consider $\psi_1 \in \mathcal{D}(G)$ satisfying $\psi(0) = 1$, and $\psi_\epsilon(x) := \epsilon^{-Q}\psi(\epsilon^{-1}x)$. Then $\kappa_\epsilon = \kappa * \psi_\epsilon$ is in $\mathcal{D}(G)$ and we conclude the proof by passing carefully to the limit using Lemma 2.17. □

2.7. Operators of type ν

The properties of kernels or operators of type ν extend from the Euclidean setting to the case of homogeneous Lie groups, so in particular to graded Lie groups (see, *e.g.*, [27, Chapter 6 A] or [25, Section 3.2]):

Definition 2.22. A distribution $\kappa \in \mathcal{D}'(G)$ which is smooth away from the origin and homogeneous of degree $\nu - Q$ is called a *kernel of type* $\nu \in \mathbb{C}$ on G. The corresponding convolution operator $f \in \mathcal{D}(G) \mapsto f * \kappa$ is called an *operator of type* ν.

Example 2.23. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. For any $a \in \mathbb{C}$, $\Re a \in [0, Q)$, the operator $\mathcal{R}^{-\frac{a}{\nu}}$ is of type a. See [25, Section 4.3].

The next statement summarises the properties of the operators of type ν used in the paper:

Proposition 2.24. *Let G be a graded group.*

(1) *An operator of type ν with $\nu \in [0, Q)$ is $(-\nu)$-homogeneous and extends to a bounded operator from $L^p(G)$ to $L^q(G)$ whenever $p, q \in (1, \infty)$ satisfy $\frac{1}{p} - \frac{1}{q} = \frac{\Re\nu}{Q}$;*
(2) *Let κ be a smooth function away from the origin, homogeneous of degree ν with $\Re\nu = -Q$. Then κ is a kernel of type ν, if and only if its mean value is zero, that is, when $\int_{\mathfrak{S}} \kappa \, d\sigma = 0$ where σ is the measure on the unit sphere of a homogeneous quasi-norm given by the polar change of coordinates, see Proposition* 2.1*; (This condition is independent of the choice of a homogeneous quasi-norm.)*
(3) *Let κ be a kernel of type $s \in [0, Q)$. Let T be a homogeneous left differential operator of degree ν_T. If $s - \nu_T \in [0, Q)$, then $T\kappa$ is a kernel of type $s - \nu_T$;*
(4) *Suppose κ_1 is a kernel of type $\nu_1 \in \mathbb{C}$ with $\Re\nu_1 > 0$ and κ_2 is a kernel of type $\nu_2 \in \mathbb{C}$ with $\Re\nu_2 \geq 0$. We assume $\Re(\nu_1 + \nu_2) < Q$. Then $\kappa_1 * \kappa_2$ is well defined as a kernel of type $\nu_1 + \nu_2$. Moreover if $f \in L^p(G)$ where $1 < p < Q/(\Re(\nu_1 + \nu_2))$ then $(f * \kappa_1) * \kappa_2$ and $f * (\kappa_1 * \kappa_2)$ belong to $L^q(G)$, $\frac{1}{q} = \frac{1}{p} - \frac{\Re(\nu_1+\nu_2)}{Q}$, and they are equal.*

The L^2-boundedness of operators of type 0 (see Part (1) in the case $\nu = 0$) and the characterisation of Part (2) are proved using the classical construction of a principal value distribution and quasi-orthogonality. The next lemma summarises the result in more detail with the vocabulary of this paper:

Lemma 2.25.

(1) *Let $\kappa \in C^1(G\backslash\{0\})$ be $(-Q)$-homogeneous and with vanishing mean value. Then κ extended to a distribution on G which is the kernel of a convolution operator bounded on $L^2(G)$. We fix a homogeneous quasi-norm $|\cdot|$. For each $j \in \mathbb{Z}$, we define the integrable function κ_j via $\kappa_j(x) := \kappa(x) 1_{2^j \leq |x| \leq 2^{j+1}}$, $x \in G$. Then for each $\pi \in \widehat{G}$, and each $v \in \mathcal{H}_\pi$, the limit $\sum_{j=-M_1}^{M_2} \widehat{\kappa}_j(\pi) v$*

converges in $\mathcal{H}_\pi$ as $M_1, M_2 \to \infty$. This defines a field of operators $\sum_{j\in\mathbb{Z}} \widehat{\kappa}_j$ wich is 0-homogeneous and satisfies:

$$\sup_{\pi\in\widehat{G}} \left\| \sum_{j\in\mathbb{Z}} \widehat{\kappa}_j(\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C \sup_{|z|=1, |\alpha|\leq 1} |X^\alpha \kappa(z)|, \tag{2.24}$$

where C is a constant which depends on the structural constants of the group G and of the choice of a homogeneous quasi-norm $|\cdot|$, but not on κ;

(2) *Let $\sigma = \{\sigma(\pi) \in \mathscr{L}(\mathcal{H}_\pi), \pi \in \widehat{G}\}$ be a measurable field of operators such that:*

- *σ is 0-homogeneous,* i.e. *$\sigma(r\pi) = \sigma(\pi)$ for (almost) all $\pi \in \widehat{G}$ and all $r > 0$;*
- *σ is bounded,* i.e. $\sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty$*;*
- *The kernel associated with σ,* i.e. *$\kappa \in \mathcal{S}'(G)$ such that $\widehat{\kappa} = \sigma$, coincides with a C^1 function on $G\backslash\{0\}$.*

Then the mean value of κ vanishes. Using the notation of Part 1, the sum $\sum_j \kappa_j$ converges in $\mathcal{S}'(G)$ and defines a tempered distribution which coincides with κ on $G\backslash\{0\}$. We have

$$\kappa = \sum_j \kappa_j + c_\sigma \delta_0,$$

where $c_\sigma = \int_G \kappa(z)\chi(z)dz$ where $\chi \in \mathcal{D}(G)$ is such that $\chi(0) = 1$ and $\chi(z) = \chi_1(|z|)$ for some $\chi_1 \in \mathcal{D}(\mathbb{R})$. The constant c_σ does not depend on χ or $|\cdot|$.

As a representative of the measurable field σ, we may choose the one given by

$$\sigma(\pi) = \sum_{j=-\infty}^{+\infty} \widehat{\kappa}_j(\pi) + c_\sigma \mathrm{I}_{\mathcal{H}_\pi}, \qquad \pi \in \widehat{G}.$$

Sketch of the proof of Lemma 2.25. See, *e.g.*, [25, Section 3.2.5] for the proof of Part (1). For Part (2), let $\tilde{\kappa}$ be the kernel associated with the symbol $\sum_{j\in\mathbb{Z}} \widehat{\kappa}_j$. Then $\tilde{\kappa} = \sum_j \kappa_j$ is a $(-Q)$-homogeneous tempered distribution. For any $\phi \in \mathcal{D}(G)$, the sum $\sum_j \langle \kappa_j, \phi\rangle$ is absolutely convergent and its sum is $\langle \tilde{\kappa}, \phi\rangle$. Hence $\tilde{\kappa}$ coincides with κ on $G\backslash\{0\}$ so the distribution $\kappa - \tilde{\kappa}$ being $-Q$-homogeneous and supported at the origin must be a multiple of δ_0. If there exists $\phi_1 \in C(\mathbb{R})$ such that $\phi(z) = \phi_1(|z|)$ then

$$\langle \kappa_j, \phi\rangle = \int_{2^j\leq|z|\leq 2^{j+1}} \kappa(z)\phi_1(|z|)dz = 0$$

as the mean value of κ is zero and $\langle \tilde{\kappa}, \phi\rangle = 0$. This together with $\kappa = \tilde{\kappa} + c_\sigma\delta_0$ with $c_\sigma \in \mathbb{C}$ implies the rest of the statement. ☐

3. Pseudo-differential calculus

Here we outline the pseudo-differentical calculus developed in [25].

3.1. Quantisation

A *symbol* is a measurable field of operators $\sigma(x,\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty$, parametrised by $x \in G$ and $\pi \in \widehat{G}$. We formally associate to σ the operator $\mathrm{Op}(\sigma)$ as follows

$$\mathrm{Op}(\sigma) f(x) := \int_{G} \mathrm{tr}\,\big(\pi(x)\sigma(x,\pi)\widehat{f}(\pi)\big)\, d\mu(\pi),$$

where $f \in \mathcal{S}(G)$ and $x \in G$.

Regarding symbols, when no confusion is possible, we will allow ourselves some notational shortcuts, for instance writing $\sigma(x,\pi)$ when considering the field of operators $\{\sigma(x,\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, (x,\pi) \in G \times \widehat{G}\}$ with the usual identifications for $\pi \in \widehat{G}$ and μ-measurability.

This quantisation has already been observed in [5,25,50] for instance. It can be viewed as an analogue of the Kohn-Nirenberg quantisation since the inverse formula can be written as

$$f(x) := \int_{G} \mathrm{tr}\,\big(\pi(x)\widehat{f}(\pi)\big)\, d\mu(\pi), \quad f \in \mathcal{S}(G),\ x \in G.$$

This also shows that the operator associated with the symbol $\mathrm{I} = \{\mathrm{I}_{\mathcal{H}_\pi}, (x,\pi) \in G \times \widehat{G}\}$ is the identity operator $\mathrm{Op}(\mathrm{I}) = \mathrm{I}$.

Note that (formally or whenever it makes sense), if we denote the (right convolution) kernel of $\mathrm{Op}(\sigma)$ by κ_x, that is,

$$\mathrm{Op}(\sigma)\phi(x) = \phi * \kappa_x, \quad x \in G,\ \phi \in \mathcal{S}(G),$$

then it is given by

$$\pi(\kappa_x) = \sigma(x,\pi).$$

Moreover the integral kernel of $\mathrm{Op}(\sigma)$ is

$$K(x,y) = \kappa_x\big(y^{-1}x\big), \quad \text{where} \quad \mathrm{Op}(\sigma)\phi(x) = \int_G K(x,y)\phi(y)dy.$$

We shall abuse the vocabulary and call κ_x the kernel of σ, and K its integral kernel.

3.2. Difference operators

The difference operators are aimed at replacing the derivatives with respect to the Fourier variable in the Euclidean case. For each $\alpha \in \mathbb{N}_0^n$, the difference operator Δ^α is defined via

$$\Delta^\alpha \widehat{f}(\pi) = \mathcal{F}_G\big(x^\alpha f\big)(\pi), \quad \pi \in \widehat{G}.$$

Here f is in a distributional space on which the group Fourier transform has been defined, *i.e.* $L^1(G)$, $L^2(G)$ or $\mathcal{K}_{a,b}(G)$ etc...

The difference operators satisfy the Leibniz rule:

$$\Delta^\alpha(\sigma_1\sigma_2) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\Delta^{\alpha_1}\sigma_1\ \Delta^{\alpha_2}\sigma_2, \tag{3.1}$$

where c_{α_1,α_2} are universal constants. By "universal constants", we mean that they depend only on G and the choice of the basis $\{X_j\}_{j=1}^n$. This comes from the fact that for any $\alpha \in \mathbb{N}_0^n$, with the same constants c_{α_1,α_2} as above, we have

$$(xy)^\alpha = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2} x^{\alpha_1} y^{\alpha_2}. \tag{3.2}$$

Note that $c_{0,\alpha_2'} = \delta_{\alpha=\alpha_2'}$ and $c_{\alpha_1',0} = \delta_{\alpha_1'=\alpha}$.

From (3.2), one also deduces that if $\phi \in \mathcal{S}(G)$ and $\kappa \in \mathcal{S}'(G)$

$$(x^\alpha\phi) * \kappa = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}(-1)^{\alpha_2} x^{\alpha_1}\phi * (x^{\alpha_2}\kappa).$$

Taking the Fourier transform, the following property holds for any $\phi \in \mathcal{S}(G)$ and $\sigma \in L^\infty(\widehat{G})$ satisfying $\Delta^{\alpha'}\sigma \in L^\infty(\widehat{G})$ for any $\alpha' \in \mathbb{N}_0^n$ with $[\alpha'] \leq [\alpha]$:

$$\sigma\Delta^\alpha\phi = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}(-1)^{\alpha_2}\Delta^{\alpha_1}\left\{\Delta^{\alpha_2}\sigma\ \widehat{\phi}\right\}. \tag{3.3}$$

Example 3.1. One can prove easily

$$\Delta^\alpha\pi(X)^\beta = \begin{cases} 0 & \text{if } [\alpha] > [\beta], \\ \sum\limits_{[\alpha']=[\alpha]-[\beta]} c'_{\alpha',\alpha,\beta}\pi(X)^{\alpha'} & \text{if } [\alpha] \leq [\beta], \end{cases}$$

where $c'_{\alpha',\alpha,\beta}$ are universal constants.

Using (2.8), if f and $x^\alpha f$ are integrable, then

$$(r\cdot\pi)(x^\alpha f) = \pi(x^\alpha f)_{(r)} = r^{-[\alpha]}\pi\left(x^\alpha f_{(r)}\right) = r^{-[\alpha]}\Delta^\alpha \widehat{f_{(r)}}.$$

Hence denoting $\sigma = \widehat{f}$ and $\sigma_{r\,\cdot} = \{\sigma_{r\cdot\pi}, \pi \in \widehat{G}\}$, we have $\sigma_{r\,\cdot} = \widehat{f_{(r)}}$ by (2.8) and we have obtained:

$$\Delta^\alpha\left(\sigma_{r\,\cdot}\right)(\pi) = r^{[\alpha]}\left(\Delta^\alpha\sigma\right)(r\cdot\pi), \quad r > 0,\ \pi \in \widehat{G}. \tag{3.4}$$

One easily checks that (3.4) holds as long as it makes sense.

We also have the following integration by parts:

$$\int_{\widehat{G}} \operatorname{tr}\left(\Delta^\alpha\sigma_1\ \ \sigma_2\right) d\mu = (-1)^{|\alpha|}\int_{\widehat{G}} \operatorname{tr}\left(\sigma_1\ \Delta^\alpha\sigma_2\right) d\mu,$$

if $\sigma_1, \sigma_2 \in \mathcal{F}_G\mathcal{S}(G)$ and $\alpha \in \mathbb{N}_0^n$. Indeed in this case, using the FIF, see Proposition 2.16, both sides are equal to $\int_G \mathcal{F}_G^{-1}\sigma_1(x) x^\alpha \mathcal{F}_G^{-1}\sigma_2(x^{-1})dx$. Along the same idea, we have:

Lemma 3.2. *Let σ be a symbol such that at least one of the two quantities*

$$\sup_{\pi\in\widehat{G}} \left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{s}{\nu}}\sigma(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)}, \qquad \sup_{\pi\in\widehat{G}} \left\| \sigma(\pi)\pi(\mathrm{I}+\mathcal{R})^{\frac{s}{\nu}}\right\|_{\mathscr{L}(\mathcal{H}_\pi)},$$

is finite for some $s > Q$. Then for any $\alpha \in \mathbb{N}_0^n\backslash\{0\}$, we have:

$$\int_G \mathrm{tr}\left(\Delta^\alpha \sigma(\pi)\right) d\mu(\pi) = 0,$$

in the sense that if $(\psi_\epsilon)_{\epsilon>0}$ is any δ_0-approximate as in Lemma 2.17*, then each quantity*

$$\int_G \mathrm{tr}\left|\sigma(\pi)\ \Delta^\alpha \widehat{\psi}_\epsilon(\pi)\right| d\mu(\pi), \quad \epsilon > 0,$$

is finite and the following limit exists and is zero:

$$\lim_{\epsilon\to 0}\int_G \mathrm{tr}\left(\sigma(\pi)\ \Delta^\alpha \widehat{\psi}_\epsilon(\pi)\right) d\mu(\pi) = 0.$$

By Remark 2.18, in the case $\alpha = 0$, the limit above is $c\kappa(0)$ where $c = \int_G \psi_1$ and κ is the kernel of σ.

Proof of Lemma 3.2. We set $\phi_1(x) = x^\alpha \psi_1(x)$ and $\phi_\epsilon(x) = \epsilon^{-Q}\phi_1(\epsilon^{-1}x)$. The statement follows from $\widehat{\phi}_\epsilon = \epsilon^{-[\alpha]}\Delta^\alpha \widehat{\psi}_\epsilon$ by (3.4) and by Remark 2.18,

$$\int_{\widehat{G}} \mathrm{tr}\left(\sigma(\pi)\ \widehat{\phi}_\epsilon(\pi)\right) d\mu(\pi) \underset{\epsilon\to 0}{\longrightarrow} c\,\mathcal{F}^{-1}\sigma(0), \quad \text{with} \quad c = \int_G \phi_1. \qquad \square$$

3.3. The symbol classes $S^m(G)$ and the calculus

In this section, we recall the definition and properties of the symbolic pseudo-differential calculus defined on graded Lie groups in [25, Section 5].

Definition 3.3. A symbol $\sigma = \{\sigma(x,\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, (x,\pi) \in G \times \widehat{G}\}$ is called a *symbol of order m* whenever, for each $\alpha, \beta \in \mathbb{N}_0^n$ and $\gamma \in \mathbb{R}$ we have

$$\sup_{x\in G,\pi\in\widehat{G}} \|\pi(\mathrm{I}+\mathcal{R})^{\frac{[\alpha]-m+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\pi(\mathrm{I}+\mathcal{R})^{-\frac{\gamma}{\nu}}\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty, \tag{3.5}$$

where $\mathcal{R}$ is a (fixed) positive Rockland operator of homogeneous degree ν. The *symbol class $S^m = S^m(G)$* is the set of symbols of order m.

By Lemma 2.13, each symbol class S^m is independent of $\mathcal{R}$. For a chosen positive Rockland operator $\mathcal{R}$ of homogeneous degree ν, the seminorms

$$\|\sigma\|_{S^m,a,b,c} := \max_{\substack{[\alpha]\le a\\ [\beta]\le b, |\gamma|\le c}} \sup_{\substack{x\in G\\ \pi\in\widehat{G}}} \left\|\pi(\mathrm{I}+\mathcal{R})^{\frac{[\alpha]-m+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\pi(\mathrm{I}+\mathcal{R})^{-\frac{\gamma}{\nu}}\right\|_{\mathscr{L}(\mathcal{H}_\pi)},$$

yield a structure of Fréchet spaces on each S^m, $m \in \mathbb{R}$. One checks that $S^{m_1} \subset S^{m_2}$ if $m_1 \leq m_2$. We also define the space of smoothing symbols

$$S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m,$$

which is endowed with the topology of projective limit.

The corresponding spaces of operators

$$\Psi^m \equiv \Psi^m(G) := \mathrm{Op}(S^m), \quad m \in \mathbb{R} \cup \{-\infty\},$$

yield a calculus:

Theorem 3.4.

(1) *The set of operators* $\cup_{m \in \mathbb{R}} S^m$ *is an algebra of symbols in the sense that product, taking the adjoint and applying spacial and dual derivatives*

$$\begin{cases} S^{m_1} \times S^{m_2} \longrightarrow S^{m_1+m_2} \\ (\sigma_1, \sigma_2) \longmapsto \sigma_1 \sigma_2, \end{cases} \qquad \begin{cases} S^m \longrightarrow S^m \\ \sigma \longmapsto \sigma^* \end{cases} \quad \text{and} \quad \begin{cases} S^m \longrightarrow S^{m-[\alpha]} \\ \sigma \longmapsto X^\beta \Delta^\alpha \sigma \end{cases}$$

(*for any* $m, m_1, m_2 \in \mathbb{R}$ *and* $\alpha, \beta \in \mathbb{N}_0^n$) *are continuous operations;*

(2) *Furthermore* $\sigma \in S^m$ *if and only if it satisfies* (3.5) *with* $\gamma = 0$. *The seminorms* $\| \cdot \|_{S^m, a, b, 0}$ *yield an equivalent family of seminorms for the Fréchet topology of* S^m*;*

(3) *The set of operators* $\cup_{m \in \mathbb{R}} \Psi^m$ *is a calculus in the sense that product and taking the adjoint*

$$\begin{cases} \Psi^{m_1} \times \Psi^{m_2} \longrightarrow \Psi^{m_1+m_2} \\ (T_1, T_2) \longmapsto T_1 T_2 \end{cases} \quad \text{and} \quad \begin{cases} \Psi^m \longrightarrow \Psi^m \\ T \longmapsto T^* \end{cases}$$

(*for any* $m, m_1, m_2 \in \mathbb{R}$) *are continuous operations, and that any operator in* Ψ^m *maps continuously* $L^2_s(G)$ *to* $L^2_{s-m}(G)$*;*

(4) *We have the asymptotic expansions:*

$$\mathrm{Op}^{-1}\left(\mathrm{Op}(\sigma_1)\mathrm{Op}(\sigma_2)\right) \sim \sum_{[\alpha]} c_\alpha \Delta^\alpha \sigma_1 X_x^\alpha \sigma_2,$$
$$\mathrm{Op}^{-1}\left(\mathrm{Op}(\sigma)^*\right) \sim \sum_{[\alpha]} c'_\alpha X_x^\alpha \Delta^\alpha \sigma^*,$$

where the constants c_α *and* c'_α, $\alpha \in \mathbb{N}_0^n$, *are universal with* $c_0 = c'_0 = 1$.

In the statement above we use asymptotic expansions of the form

$$\sigma \sim \sum_{\ell=0}^{\infty} \sigma_\ell, \quad \sigma_\ell \in S^{m_\ell}, \quad \text{with } m_\ell \text{ strictly decreasing to } -\infty. \tag{3.6}$$

This means that for any $M \in \mathbb{N}$,

$$\sigma - \sum_{\ell \leq M} \sigma_\ell \in S^{m_{M+1}}.$$

More precisely in Theorem 3.4, this was used with

$$\sigma_\ell = \sum_{[\alpha]=w_\ell} c_\alpha \Delta^\alpha \sigma_1 X_x^\alpha \sigma_2 \in S^{m_1+m_2-w_\ell} \quad \text{and} \quad \sigma_\ell = \sum_{[\alpha]=w_\ell} c'_\alpha X_x^\alpha \Delta^\alpha \sigma^* \in S^{m-w_\ell}.$$

Note that any formal asymptotic yields a symbol modulo a smoothing operator:

Theorem 3.5. *Let $\{\sigma_\ell\}_{\ell \in \mathbb{N}_0}$ be a sequence of symbols such that $\sigma_\ell \in S^{m_\ell}$ with m_ℓ strictly decreasing to $-\infty$. Then there exists $\sigma \in S^{m_0}$, unique modulo $S^{-\infty}$, such that $\sigma \sim \sum_\ell \sigma_\ell$.*

Naturally the calculus $\cup_{m\in\mathbb{R}} \Psi^m$ contains the left-invariant calculus since we have:

Example 3.6.

(1) For any $\alpha \in \mathbb{N}_0^n$, $X^\alpha = \mathrm{Op}(\pi(X^\alpha)) \in \Psi^{[\alpha]}$;
(2) The set Ψ^0 contains any smooth function $f : G \to \mathbb{C}$ with bounded left-derivatives, that is,

$$\forall \beta \in \mathbb{N}_0^n, \qquad \sup_{x\in G} \left| X^\beta f(x) \right| < \infty. \tag{3.7}$$

Another important class of symbols in the calculus is given by multipliers in Rockland operators. The precise class of multipliers that we consider is the following. Let $\mathcal{M}_m$ be the space of functions $f \in C^\infty(\mathbb{R}^+)$ such that the following quantities for all $\ell \in \mathbb{N}_0$ are finite:

$$\|f\|_{\mathcal{M}_{m,\ell}} := \sup_{\lambda>0,\, \ell'=0,\ldots,\ell} (1+\lambda)^{-m+\ell'} \left| \partial_\lambda^{\ell'} f(\lambda) \right|.$$

In other words, the class of functions f that appears in the definition above are the functions which are smooth on $\mathbb{R}^+$ and have the symbolic behaviour at infinity of the Hörmander class $S^m_{1,0}(\mathbb{R})$ on the real line. For instance, for any $m \in \mathbb{R}$, the function $\lambda \mapsto (1+\lambda)^m$ is in $\mathcal{M}_m$.

Proposition 3.7. *Let $m \in \mathbb{R}$ and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. If $f \in \mathcal{M}_{\frac{m}{\nu}}$, then $f(\mathcal{R})$ is in Ψ^m and its symbol $\{f(\pi(\mathcal{R})), \pi \in \widehat{G}\}$ satisfies*

$$\forall a,b,c \in \mathbb{N}_0, \qquad \exists \ell \in \mathbb{N}, \qquad \exists C > 0, \qquad \|f(\pi(\mathcal{R}))\|_{a,b,c} \leq C\|f\|_{\mathcal{M}_{\frac{m}{\nu}},\ell},$$

with $\ell, a, b, c \in \mathbb{N}_0$ and C independent of f.

Corollary 3.8. *If $\chi \in C^\infty(\mathbb{R})$ is such that* $(\text{supp}\,\chi) \cap [0,+\infty)$ *is compact, then the symbol $\{\chi(\pi(\mathcal{R})), \pi \in \widehat{G}\}$ is in $S^{-\infty}$. Moreover its kernel $\chi(\mathcal{R})\delta_0$ is Schwartz, $\{\chi(\pi(\mathcal{R})), \pi \in \widehat{G}\} \in L^2(\widehat{G})$ and*

$$\int_{\widehat{G}} \text{tr}\,|\chi(\pi(\mathcal{R}))| d\mu(\pi) < \infty.$$

Note that the membership of $\chi(\mathcal{R})\delta_0$ in $\mathcal{S}(G)$ was already proved in [38]. It is sometimes called the Hulanicki theorem and is used to show Proposition 3.7.

We conclude this section with recalling the following properties of the kernel associated with a symbol in the calculus:

Proposition 3.9. *Let $\sigma \in S^m$ and let κ its associated kernel. This means that for each $x \in G$, $\kappa_x \in \mathcal{S}'(G)$ and $\sigma(x,\pi) = \widehat{\kappa_x}(\pi)$. Furthermore $x \mapsto \kappa_x \in \mathcal{S}'(G)$ is smooth on G, and for each $x \in G$, $\kappa_x \in C^\infty(G\backslash\{0\})$. Furthermore, for any $N \in \mathbb{N}_0$ and any $\alpha, \beta \in \mathbb{N}_0^n$, there exist a constant $C > 0$ and a seminorm $\|\cdot\|_{S^m,a,b,c}$ such that*

$$\forall x, z \in G, \quad z \neq 0, \qquad \left|X_z^\alpha X_x^\beta \kappa_x(z)\right| \leq C_s \|\sigma\|_{S^m,a,b,c}|z|^{-N}.$$

The constant C and the seminorm $\|\cdot\|_{S^m,a,b,c}$ may depend on N, α, β but are independent of σ.

4. Homogeneous and principal symbols, classical calculus

In this section, we define the notions of homogeneous symbols, classical symbol classes and principal symbol in a way analogous to the Euclidean case.

4.1. Homogeneous symbol classes $\dot{S}^m$

Definition 4.1. A symbol $\sigma = \{\sigma(x,\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, (x,\pi) \in G \times \widehat{G}\}$ is said to be *homogeneous* of degree $m \in \mathbb{R}$ or m-homogeneous when

$$\sigma(x, r\cdot\pi) = r^m \sigma(x,\pi),$$

for all $x \in G$ and μ-almost all $\pi \in \widehat{G}$ and dr-almost all $r > 0$.

A m-homogeneous symbol $\sigma = \{\sigma(x,\pi)\}$ is *regular* if for any $\alpha, \beta \in \mathbb{N}_0^n$, $\gamma \in \mathbb{R}$:

$$\sup_{\substack{\pi\in\widehat{G}\\ x\in G}} \left\| \pi(\mathcal{R})^{\frac{[\alpha]-m+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathcal{L}(\mathcal{H}_\pi)} < \infty \tag{4.1}$$

where $\mathcal{R}$ is a positive Rockland operator of degree ν.

We denote by $\dot{S}^m$ the space of regular m-homogeneous symbols.

Remark 4.2.

(1) As in the inhomogeneous case, Lemma 2.13 implies that each symbol class $\dot{S}^m$ is independent of $\mathcal{R}$;
(2) The property of interpolation of Sobolev spaces (*cf.* Theorem 2.14) also implies that it suffices to have (4.1) only for a sequence $(\gamma_\ell)_{\ell\in\mathbb{Z}}$ with $\gamma_\ell \to_{\ell\to\pm\infty} \pm\infty$.

Before giving some concrete examples and an equivalent description for symbols in $\dot{S}^m$, let us mention some routine properties regarding classes of symbols. Each $\dot{S}^m$, $m \in \mathbb{R}$, is a Fréchet vector space when equipped with the seminorms

$$\|\sigma\|_{\dot{S}^m,a,b,c} := \max_{\substack{[\alpha]\le a\\ [\beta]\le b, |\gamma|\le c}} \sup_{\substack{x\in G\\ \pi\in\widehat{G}}} \left\| \pi(\mathcal{R})^{\frac{[\alpha]-m+\gamma}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)}, \quad a,b,c \in \mathbb{N}_0,$$

where $\mathcal{R}$ is a positive Rockland operator of homogeneous degree ν. This Fréchet structure is independent of the chosen positive Rockland operator and we will see later in Corollary 4.12 that we may assume $c = 0$. Furthermore taking the product and the adjoint and applying spacial and dual derivatives

$$\begin{cases} \dot{S}^{m_1} \times \dot{S}^{m_2} \longrightarrow \dot{S}^{m_1+m_2} \\ (\sigma_1,\sigma_2) \longmapsto \sigma_1\sigma_2 \end{cases}, \quad \begin{cases} \dot{S}^{m} \longrightarrow \dot{S}^{m} \\ \sigma \longmapsto \sigma^* \end{cases}, \quad \text{and} \quad \begin{cases} \dot{S}^{m} \longrightarrow \dot{S}^{m-[\alpha]} \\ \sigma \longmapsto X^\beta \Delta^\alpha \sigma \end{cases},$$

(for any $m, m_1, m_2 \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{N}_0^n$) are continuous operations for this topology.

Example 4.3. The symbol $\pi(X)^\alpha$ is homogeneous of degree $[\alpha]$ and regular. (See (2.7) and Example 3.1)

Example 4.4. The symbol given by a function $\sigma(x)$ independent of $\widehat{G}$ is homogeneous of degree 0. It is regular if the function is smooth with bounded left invariant derivatives, see (3.7).

Example 4.5. If $\mathcal{R}$ is a positive Rockland operator of degree ν and if $m \in \mathbb{R}$, then the symbol $\pi(\mathcal{R})^{\frac{m}{\nu}}$ (defined spectrally) is regular and homogeneous of degree m.

Proof for Example 4.5. The homogeneity may be obtained from the properties of the Rockland operator as in Lemma 2.12. The regularity will be a direct consequence of Proposition 4.6 below. □

We now give equivalent properties characterising a symbol in $\dot{S}^m$. In the abelian case, the statement boils down to the fact that a regular homogeneous symbol yields a (non-homogeneous) symbol in $S^m(\mathbb{R}^n)$ once the low frequencies have been cut off.

Proposition 4.6. *Let $\sigma = \{\sigma(x,\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, (x,\pi) \in G \times \widehat{G}\}$ be a homogeneous symbol of degree $m \geq 0$. The following properties are equivalent:*

(1) *σ is in $\dot{S}^m$;*
(2) *There exist a positive Rockland operator $\mathcal{R}$ and a real-valued function $\psi \in C^\infty(\mathbb{R})$ satisfying $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on (Λ, ∞) for some $\Lambda > 0$ such that the two symbols*

$$\left\{\psi(\pi(\mathcal{R}))\sigma(x,\pi),(x,\pi)\in G\times\widehat{G}\right\} \text{ and } \left\{\sigma(x,\pi)\psi(\pi(\mathcal{R})),(x,\pi)\in G\times\widehat{G}\right\},$$

are in S^m;
(3) *Property (2) holds for any such $\mathcal{R}$ and ψ.*

Moreover the mapping

$$\begin{cases} \dot{S}^m \longrightarrow S^m \times S^m \\ \sigma \longmapsto (\psi(\pi(\mathcal{R}))\sigma, \sigma\psi(\pi(\mathcal{R}))), \end{cases}$$

is continuous, injective and open (i.e. *with continuous inverse*) *onto its image.*

The proof is given in the next subsection, but first let us notice that using $\psi(\pi(\mathcal{R}))\sigma$ or $\sigma\psi(\pi(\mathcal{R}))$ is essentially equivalent as we have:

Corollary 4.7. *If $\sigma \in \dot{S}^m$, $\mathcal{R}$ is a positive Rockland operator and $\psi \in C^\infty(\mathbb{R})$ is a scalar valued function such that $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on $[\Lambda, \infty)$ for some $\Lambda > 0$, then $\psi(\pi(\mathcal{R}))\sigma - \sigma\psi(\pi(\mathcal{R}))$ is in $S^{-\infty}$.*

Proof of Corollary 4.7. We keep the hypotheses of the corollary and denote $\Psi(\pi) := \psi(\pi(\mathcal{R}))$. As $\Psi(\pi) = (1-\psi)(\pi(\mathcal{R}))$, by Corollary 3.8, $\mathrm{I} - \Psi$ is smoothing. Since $\Psi\sigma$ and $\sigma\Psi$ are in S^m by Proposition 4.6, the symbol $\Psi\sigma - \sigma\Psi = \Psi\sigma(\mathrm{I}-\Psi) - (\mathrm{I}-\Psi)\sigma\Psi$, is smoothing. □

4.2. Proof of Proposition 4.6

The underlying idea is to find a replacement for the following property in the Euclidean case: in the case of $\mathbb{R}^n$, if a cut-off function $\psi(\xi)$ on the Fourier side is constant for $|\xi| > \Lambda$ (Λ large enough), then its derivatives are $\partial_\xi^\alpha \psi(\xi) = 0$ if $|\xi| > \Lambda$. In our case, we cannot say anything in general about vanishing derivatives. However, we can show that these derivatives are smoothing and behaves well enough in the following way:

Lemma 4.8. *Let $\psi \in C^\infty(\mathbb{R})$ be a real valued function satisfying $\psi_{|[\Lambda,+\infty)} = 1$ for some $\Lambda > 0$. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Then for any $\alpha \in \mathbb{N}_0^n\backslash\{0\}$, the symbol given by $\Delta^\alpha\psi(\pi(\mathcal{R}))$ is smoothing,* i.e. *is in $S^{-\infty}$. Furthermore for each $a, b \in \mathbb{R}$, the fields of operators given by*

$$\pi(\mathcal{R})^{\frac{b}{\nu}}\Delta^\alpha\psi(\pi(\mathcal{R}))\pi(\mathcal{R})^{\frac{a}{\nu}}, \quad \pi(\mathrm{I}+\mathcal{R})^{\frac{b}{\nu}}\Delta^\alpha\psi(\pi(\mathcal{R}))\pi(\mathcal{R})^{\frac{a}{\nu}},$$
$$\pi(\mathcal{R})^{\frac{b}{\nu}}\Delta^\alpha\psi(\pi(\mathcal{R}))\pi(\mathrm{I}+\mathcal{R})^{\frac{a}{\nu}},$$

are in $L^\infty(\widehat{G})$.

Proof of Lemma 4.8. By Example 3.1, $\Delta^\alpha 1 = 0$ thus $\Delta^\alpha\psi(\pi(\mathcal{R})) = -\Delta^\alpha(1-\psi)(\pi(\mathcal{R}))$. Corollary 3.8 then implies the first part. If $a, b \in \nu\mathbb{N}_0$, the given fields of operators are bounded since $\Delta^\alpha\psi(\pi(\mathcal{R})) \in S^{-\infty}$ while $\pi(\mathcal{R})^{\frac{m}{\nu}}$ and $\pi(\mathrm{I}+\mathcal{R})^{\frac{m}{\nu}}$ are in Ψ^m for any $m \in \nu\mathbb{N}_0$. Hence this is also the case for $a, b \in \nu(-\mathbb{N}_0)$ by duality (see Theorem 2.14 (2.14)), and then for any $a, b \in \mathbb{R}$ by interpolation (see Theorem 2.14 (2.14)); indeed the adjoint of $(\mathrm{Op}\,(\Delta^\alpha\psi(\pi(\mathcal{R}))))^*$ is a linear combination of $\left(\mathrm{Op}\left(\Delta^\beta\psi(\pi(\mathcal{R}))\right)\right)$, $[\beta] = [\alpha]$. ☐

We will also need the following technical lemma.

Lemma 4.9. *Let $\sigma = \{\sigma(\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, \pi \in \widehat{G}\}$ be a measurable field of operators which is 0-homogeneous in the sense that $\sigma(r\cdot\pi) = \sigma(\pi)$ for any $r > 0$ and $\pi \in \widehat{G}$.*

(1) *If there exist a positive Rockland operator $\mathcal{R}$ and a scalar valued function $\psi \in C^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on (Λ, ∞) for some $\Lambda > 0$, and $\{\psi(\pi(\mathcal{R}))\sigma(\pi)\} \in L^\infty(\widehat{G})$ then $\sigma \in L^\infty(\widehat{G})$.*
Conversely, if $\sigma \in L^\infty(\widehat{G})$, then $\{\psi(\pi(\mathcal{R}))\sigma(\pi)\} \in L^\infty(\widehat{G})$ for any positive Rockland operator $\mathcal{R}$ and any scalar valued function $\psi \in C^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on (Λ, ∞) for some $\Lambda > 0$ and we have:

$$\|\sigma\|_{L^\infty(\widehat{G})} \leq \sup_{\pi\in\widehat{G}} \|\psi(\pi(\mathcal{R}))\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq \sup_{\lambda\geq 0}|\psi(\lambda)|\ \|\sigma\|_{L^\infty(\widehat{G})}.$$

We have the same property with $\{\sigma(\pi)\psi(\pi(\mathcal{R}))\}$;

(2) *We assume that $\sigma \in L^\infty(\widehat{G})$ and that there exist a real-valued function $\psi \in C^\infty(\mathbb{R})$ satisfying $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on $(\Lambda, +\infty)$ for some $\Lambda > 0$, a positive Rockland operator $\mathcal{R}$ of homogeneous degree ν and a number $m' \in \mathbb{R}$*

$$\sup_{\pi\in\widehat{G}}\left\|\pi(\mathcal{R})^{\frac{m'}{\nu}}\psi(\pi(\mathcal{R}))\sigma(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)} \quad or \quad \sup_{\pi\in\widehat{G}}\left\|\sigma(\pi)\psi(\pi(\mathcal{R}))\pi(\mathcal{R})^{\frac{m'}{\nu}}\right\|_{\mathscr{L}(\mathcal{H}_\pi)},$$

is finite. If $m' > 0$ then $\sigma = 0$.

Proof of Lemma 4.9. Let $\sigma = \{\sigma(\pi) \in \mathscr{L}(\mathcal{H}_\pi)\}$ be a 0-homogeneous symbol. Let us assume that the quantity

$$\sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} = \sup_{\substack{\pi\in\widehat{G}\\ u\in\mathcal{H}_\pi, \|u\|_{\mathcal{H}_\pi}=1}} \|\sigma(\pi)u\|_{\mathcal{H}_\pi}$$

is finite. Then, for any $\epsilon > 0$, there exists $\pi_0 \in \widehat{G}$ and $u_0 \in \mathcal{H}_{\pi_0}$, $\|u_0\|_{\mathcal{H}_{\pi_0}} = 1$ such that

$$\sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le \|\sigma(\pi_0)u_0\|_{\mathcal{H}_{\pi_0}} + \epsilon.$$

By (2.17), for any $r > r_{\pi_0}$, we have $\psi(r\cdot\pi_0(\mathcal{R})) = \mathrm{I}_{\mathcal{H}_{\pi_0}}$ thus

$$\psi(r\cdot\pi_0(\mathcal{R}))\sigma(\pi_0)u_0 = \sigma(\pi_0)u_0.$$

As σ is 0-homogeneous, we have $\sigma(\pi_0)u_0 = \sigma(r\cdot\pi_0)u_0$. Hence

$$\begin{aligned}\sup_{\pi\in\widehat{G}} \|\psi(\pi(\mathcal{R}))\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} &\ge \|\psi(r\cdot\pi_0(\mathcal{R}))\sigma(\pi_0)u_0\|_{\mathscr{L}(\mathcal{H}_{\pi_0})}\\ &\ge \sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} - \epsilon.\end{aligned}$$

This is true for any $\epsilon > 0$ and this shows

$$\sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le \sup_{\pi\in\widehat{G}} \|\psi(\pi(\mathcal{R}))\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}.$$

The rest of Part (1) follows from similar considerations and properties of the functional calculus of $\pi(\mathcal{R})$.

For Part (2), let us assume that the first quantity is finite, that is,

$$\sup_{\pi\in\widehat{G}} \left\|\pi(\mathcal{R})^{\frac{m'}{\nu}}\psi(\pi(\mathcal{R}))\sigma(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty.$$

For each $r > 0$, we use the change of index $\pi \mapsto r\cdot\pi$ and this quantity becomes by homogeneity

$$\begin{aligned}&\sup_{\pi\in\widehat{G}} \left\|r\cdot\pi(\mathcal{R})^{\frac{m'}{\nu}}\psi(r\cdot\pi(\mathcal{R}))\sigma(r\cdot\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)}\\ &= r^{m'}\sup_{\pi\in\widehat{G}} \left\|\pi(\mathcal{R})^{\frac{m'}{\nu}}\psi(r^\nu\pi(\mathcal{R}))\sigma(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)}\\ &\ge r^{m'}\sup_{\substack{\pi\in\widehat{G}\\ r>r_\pi}} \left\|\pi(\mathcal{R})^{\frac{m'}{\nu}}\sigma(\pi)\right\|_{\mathscr{L}(\mathcal{H}_\pi)},\end{aligned} \tag{4.2}$$

having used (2.17). Note that the limit

$$\lim_{r\to\infty} \sup_{\substack{\pi\in\widehat{G}\\ r>r_\pi}} \left\| \pi(\mathcal{R})^{\frac{m'}{\nu}} \sigma(\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} = \sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{m'}{\nu}} \sigma(\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)}, \tag{4.3}$$

is infinite unless either $m' = 0$ or $\sigma = 0$. Indeed, for the same reason as above, we have for any $r_1 > 0$:

$$\begin{aligned} \sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{m'}{\nu}} \sigma(\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} &= \sup_{\pi\in\widehat{G}} \left\| r_1 \cdot \pi(\mathcal{R})^{\frac{m'}{\nu}} \sigma(r_1 \cdot \pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\ &= r_1^{m'} \sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{m'}{\nu}} \sigma(\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)}. \end{aligned}$$

If the limit in (4.3) is infinite, as each side of (4.2) must be finite as $r \to \infty$, we then must have $r^{m'} \to 0$ as $r \to \infty$, that is, $m' < 0$. This concludes the proof of Part (2). □

We can now prove Proposition 4.6.

Proof of Proposition 4.6. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $\psi \in C^\infty(\mathbb{R})$ be a real valued function satisfying $\psi \equiv 0$ on $(-\infty, \epsilon_o)$ and $\psi \equiv 1$ on (Λ, ∞) for some $0 < \epsilon_o < \Lambda$.

Let us assume Property (1), that is, let $\sigma \in \dot{S}^m$. We will prove that for any $\alpha, \beta \in \mathbb{N}_0^n$,

$$\sup_{x\in G, \pi\in\widehat{G}} \left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{[\alpha]-m}{\nu}} X_x^\beta \Delta^\alpha \{\psi(\pi(\mathcal{R}))\sigma(x,\pi)\} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty. \tag{4.4}$$

The case $\alpha = 0$ of (4.4) follows from

$$\begin{aligned} &\left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{-m}{\nu}} X^\beta \psi(\pi(\mathcal{R}))\sigma(x,\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\ &\le \left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{-m}{\nu}} \psi(\pi(\mathcal{R}))\pi(\mathcal{R})^{-\frac{-m}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \left\| \pi(\mathcal{R})^{\frac{-m}{\nu}} X^\beta \sigma(x,\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\ &\le \sup_{\lambda>0} \left| (1+\lambda)^{\frac{-m}{\nu}} \psi(\lambda)\lambda^{-\frac{-m}{\nu}} \right| \left\| \pi(\mathcal{R})^{\frac{-m}{\nu}} X^\beta \sigma(x,\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty. \end{aligned}$$

Let $\alpha \in \mathbb{N}_0^n$ be such that $|\alpha| = 1$. Using the Leibniz rule (3.1), we obtain

$$\begin{aligned} &\left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{[\alpha]-m}{\nu}} X^\beta \Delta^\alpha \{\psi(\pi(\mathcal{R}))\sigma(x,\pi)\} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\ &\lesssim \left\| (\mathrm{I}+\pi(\mathcal{R}))^{\frac{[\alpha]-m}{\nu}} \left(\Delta^\alpha \psi(\pi(\mathcal{R}))\right) X^\beta \sigma(x,\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\ &\quad + \left\| (\mathrm{I}+\pi(\mathcal{R}))^{\frac{[\alpha]-m}{\nu}} \psi(\pi(\mathcal{R})) \left(\Delta^\alpha X^\beta \sigma(x,\pi)\right) \right\|_{\mathscr{L}(\mathcal{H}_\pi)}. \end{aligned}$$

For the second term on the right hand-side, we proceed as above. We modify the argument for the first term using Lemma 4.8. Recursively we prove (4.4) with the same arguments. The case of $\sigma(x,\pi)\psi(\pi(\mathcal{R}))$ are handled in a similar way, the details are left to the reader. We have proved (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3).

Since (3) $\Longrightarrow$ (2), it only remains to prove that (2) $\Longrightarrow$ (1). We assume that $\{\psi(\pi(\mathcal{R}))\sigma(x,\pi)\}$ and $\{\sigma(x,\pi)\psi(\pi(\mathcal{R}))\}$ are in S^m and we want to prove that

$$\sup_{x\in G,\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{[\alpha]-m+\gamma}{\nu}} X_x^\beta \Delta^\alpha \{\psi(\pi(\mathcal{R}))\sigma(x,\pi)\}\pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty,$$

for any $\alpha,\beta\in\mathbb{N}_0^n$ and for a sequence $(\gamma_\ell)_{\ell\in\mathbb{Z}}$ with $\gamma_\ell \to_{\ell\to\pm\infty} \pm\infty$ (see Remark 4.2 (2)), and similarly for $\sigma\psi(\pi(\mathcal{R}))$. Clearly it suffices to show it for $\beta=0$. Using recursively the Leibniz rule (see (3.1)) and Lemma 4.8, it suffices to show

$$\sup_{x\in G,\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{[\alpha]-m+\gamma}{\nu}} \psi(\pi(\mathcal{R}))\Delta^\alpha \{\sigma(x,\pi)\}\pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty,$$

for any $\alpha\in\mathbb{N}_0^n$ and for a sequence $(\gamma_\ell)_{\ell\in\mathbb{Z}}$ with $\gamma_\ell \to_{\ell\to\pm\infty} \pm\infty$ (see Remark 4.2 (2)), and similarly for $\Delta^\alpha\{\sigma(x,\pi)\}\ \psi(\pi(\mathcal{R}))$. By homogeneity of the operator Δ^α (see (3.4)), it suffices to prove the case $\alpha=0$ which we now do.

The field of operators $\{\pi(\mathcal{R})^{\frac{-m+\gamma}{\nu}}\sigma(x,\pi)\pi(\mathcal{R})^{-\frac{\gamma}{\nu}}, \pi\in\widehat{G}\}$ is 0-homogeneous. Thus by (1) of Lemma 4.9 and functional analysis

$$\begin{aligned}
&\sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{-m+\gamma}{\nu}}\sigma(x,\pi)\pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\
&\le \sup_{\pi\in\widehat{G}} \left\| \psi(\pi(\mathcal{R}))\pi(\mathcal{R})^{\frac{-m+\gamma}{\nu}}\sigma(x,\pi)\pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\
&\le C \sup_{\pi\in\widehat{G}} \left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{-m+\gamma}{\nu}}\{\psi(\pi(\mathcal{R}))\sigma(x,\pi)\}\pi(\mathrm{I}+\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)},
\end{aligned}$$

with a constant

$$C = C_{m,\gamma,\nu} = \sup_{\lambda_1>\epsilon_o} \left(\frac{1+\lambda_1}{\lambda_1}\right)^{\frac{m-\gamma}{\nu}} \sup_{\lambda_2>0} \left(\frac{\lambda_2}{1+\lambda_2}\right)^{-\frac{\gamma}{\nu}},$$

finite for $\gamma\ge 0$. We apply the same argument to σ^* and obtain

$$\begin{aligned}
&\sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{-\frac{\gamma}{\nu}} X^\beta\sigma(x,\pi)\pi(\mathcal{R})^{\frac{-m+\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\
&= \sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{-m+\gamma}{\nu}} X^\beta\sigma(x,\pi)^*\pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\
&\le C \sup_{\pi\in\widehat{G}} \left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{-m+\gamma}{\nu}} X^\beta\big\{\psi(\pi(\mathcal{R}))\sigma(x,\pi)^*\big\}\pi(\mathrm{I}+\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} \\
&= C \sup_{\pi\in\widehat{G}} \left\| \pi(\mathrm{I}+\mathcal{R})^{\frac{-m+\gamma}{\nu}} X^\beta\{\sigma(x,\pi)\psi(\pi(\mathcal{R}))\}\pi(\mathrm{I}+\mathcal{R})^{-\frac{\gamma}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)}.
\end{aligned}$$

This concludes the proof of (2) $\Longrightarrow$ (1). The rest of the proof follows easily. □

4.3. Consequence of Proposition 4.6 and of its proof

The proof of Proposition 4.6, especially the implication (1)$\Longrightarrow$(2), together with Proposition 3.7 imply

Corollary 4.10. *Let $\mathcal{R}$ be a positive Rockland operator. Then for any seminorm $\|\cdot\|_{S^m,a,b,c}$ of S^m, there exist a seminorm $\|\cdot\|_{\dot{S}^m,a',b',c'}$ of $\dot{S}^m$, $C>0$, and $k\in\mathbb{N}$ such that for any $\sigma\in\dot{S}^m$ and for any real-valued function $\psi\in C^\infty(\mathbb{R})$ satisfying $\psi\equiv 0$ on a neighbourhood of 0 and $\psi\equiv 1$ on (Λ,∞) for some $\Lambda>0$, we have:*

$$\max(\|\psi(\pi(\mathcal{R}))\sigma\|_{S^m,a,b,c},\ \|\sigma\psi(\pi(\mathcal{R}))\|_{S^m,a,b,c})$$
$$\leq C \sup_{\substack{\lambda>0\\ \ell=0,\dots,k}} (1+\lambda)^\ell \left|\partial_\lambda^\ell \psi(\lambda)\right| \ \|\sigma\|_{\dot{S}^m,a',b',c'}.$$

Note that the constant C and the integer k are independent of σ or ψ.

In Section 5, we will analyse more precisely the homogeneous symbols of degree 0. From Proposition 4.6 and Corollary 4.10, we can already prove the following regularity of their kernel. If $\sigma\in\dot{S}^0$, then, for each $x\in G$, $\sigma(x,\cdot)\in L^\infty(\widehat{G})$ has a kernel $\kappa_x\in\mathcal{S}'(G)$ such that $\sigma(x,\pi)=\widehat{\kappa}_x$, see Section 2.2. The regularity of the symbol implies that this distribution coincides with a smooth function:

Proposition 4.11. *Let $\sigma\in\dot{S}^0$ and let $\kappa_x\in\mathcal{S}'(G)$ be its kernel,* i.e. $\pi(\kappa_x)=\sigma(x,\pi)$.

Then for each $x\in G$, the distribution κ_x is $(-Q)$-homogeneous:

$$\forall r>0, \qquad \kappa_x(ry)=r^{-Q}\kappa_x(y).$$

For each $x\in G$, the distribution κ_x coincides with a smooth function away from the origin and the function $(x,z)\mapsto\kappa_x(z)$ is smooth on $G\times(G\backslash\{0\})$. Furthermore, for any compact subset S of $G\backslash\{0\}$ and any $\alpha,\beta\in\mathbb{N}_0^n$, there exist a constant $C>0$ and a seminorm $\|\cdot\|_{\dot{S}^0,a,b,c}$ such that

$$\sup_{\substack{x\in G\\ z\in S}} \left|X_z^\alpha X_x^\beta \kappa_x(z)\right| \leq C_s\|\sigma\|_{\dot{S}^0,a,b,c}.$$

The constant C and the seminorm $\|\cdot\|_{\dot{S}^0,a,b,c}$ may depend on S and α,β but are independent of σ.

Proof of Proposition 4.11. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $\psi\in C^\infty(\mathbb{R})$ be a real-valued function satisfying $\psi\equiv 0$ on a neighbourhood of 0 and $\psi\equiv 1$ on (Λ,∞) for some $\Lambda>0$.

Let $\sigma\in\dot{S}^0$ and let κ_x be its associated kernel. For each $t>0$, we set $\sigma_{(t)}(x,\pi)=\sigma(x,\pi)\psi(t\pi(\mathcal{R}))$. By Proposition 4.6, this defines a symbol $\sigma_{(t)}$ in S^0 and we denote by $\kappa_{(t)}$ its kernel. Lemma 2.10 (3) and the L^2-boundedness of $\text{Op}(S^0)$ imply that for each $x\in G$, $\text{Op}(\sigma_{(t)}(x,\cdot))=\text{Op}(\sigma(x,\cdot))\psi(t\mathcal{R})$ converges to $\mathrm{Op}(\sigma(x,\cdot))$ as $t \to 0$ for the strong operator topology of $L^2(G)$. This implies that $\kappa_{(t),x}$ converges to κ_x in $\mathcal{S}'(G)$ for each $x \in G$ as $t \to 0$. More generally, for each $x \in G$ and each $\beta \in \mathbb{N}_0^n$, $X_x^\beta \kappa_{(t),x}$ converges to $X_x^\beta \kappa_x$ in $\mathcal{S}'(G)$ as $t \to 0$. The statement now follows from the convergence in distribution, Proposition 3.9 and Corollary 4.10. □

Another consequence of Proposition 4.6 and its proof is that as in the inhomogeneous case (see Theorem 3.4, Part (2)), we can simplify the conditions on the regularity of the symbol:

Corollary 4.12. *Let σ be a homogeneous symbol of degree $m \in \mathbb{R}$. Then σ is in $\dot{S}^m$ if and only if*

$$\sup_{\substack{\pi\in\widehat{G}\\ x\in G}} \left\| \pi(\mathcal{R})^{\frac{[\alpha]-m}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)}, \quad \sup_{\substack{\pi\in\widehat{G}\\ x\in G}} \left\| X_x^\beta \Delta^\alpha \sigma(x,\pi) \pi(\mathcal{R})^{\frac{[\alpha]-m}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)},$$

are finite for all $\alpha, \beta \in \mathbb{N}_0^n$. Here $\mathcal{R}$ is a fixed positive Rockland operator of degree ν. Furthermore, for a fixed positive Rockland operator, these quantities yield an equivalent family of seminorms for the Fréchet topology of $\dot{S}^m$.

Finally, we observe that Proposition 4.6 implies the following property:

Corollary 4.13. *Let $\chi \in \mathcal{D}(\mathbb{R})$ with support in $(0,\infty)$. Let $\mathcal{R}$ be a positive Rockland operator. Let $\sigma \in \dot{S}^m$. Then $\sigma\, \chi(\pi(\mathcal{R}))$ and $\chi(\pi(\mathcal{R}))\,\sigma$ are smoothing,* i.e. *in $S^{-\infty}$. Consequently if σ does not depend on x then their kernels are Schwartz.*

Proof. The first part follows from Proposition 4.6 and Corollary 3.8. The consequence follows from [25, Theorem 5.4.9]. □

4.4. Homogeneous asymptotic and principal part

In this subsection, we give a meaning to a homogeneous asymptotic sum

$$\sigma \sim \sum_{\ell=0}^{\infty} \sigma_\ell, \quad \sigma_\ell \in \dot{S}^{m_\ell}, \quad \text{with } m_\ell \text{ strictly decreasing to } -\infty, \tag{4.5}$$

which is different to the (inhomogeneous) asymptotic sum in (3.6). This will enable us to define the principal part σ_0 of such an expansion. In order to give a meaning to (4.5), we show:

Proposition 4.14. *Let $\{\sigma_\ell\}_{\ell\in\mathbb{N}_0}$ be a sequence of homogeneous symbols such that $\sigma_\ell \in \dot{S}^{m_\ell}_{\rho,\delta}$ with m_ℓ strictly decreasing to $-\infty$. If $\mathcal{R}$ is any positive Rockland operator and $\psi \in C^\infty(\mathbb{R})$ is any real-valued function satisfying $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on (Λ,∞) for some $\Lambda > 0$, then the two sums*

$$\sum_\ell \sigma_\ell \psi(\pi(\mathcal{R})) \quad \text{and} \quad \sum_\ell \psi(\pi(\mathcal{R}))\sigma_\ell,$$

define the same symbol in S^{m_0} modulo $S^{-\infty}$.

Moreover, this symbol modulo $S^{-\infty}$ does not depend on the choice of $\mathcal{R}$ and ψ. And, if this symbol is in S^m with $m < m_0$, then the first term in the homogeneous expansion is $\sigma_0 = 0$.

If $\{\sigma_\ell\}_{\ell\in\mathbb{N}_0}$ is as in the statement, then the two sums $\sum_\ell \sigma_\ell \psi(\pi(\mathcal{R}))$ and $\sum_\ell \psi(\pi(\mathcal{R}))\sigma_\ell$ make sense as symbols in S^{m_0} modulo $S^{-\infty}$, see (3.6) and Theorem 3.5. Furthermore, they yield the same symbol modulo $S^{-\infty}$ by Corollary 4.7. The independence in $\mathcal{R}$ and ψ is a direct consequence from the following property:

Lemma 4.15. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two positive Rockland operators and if $\psi_1, \psi_2 \in C^\infty(\mathbb{R})$ are two real-valued functions such that $\psi_j \equiv 0$ on a neighbourhood of 0 and $\psi_j \equiv 1$ on $(\Lambda_j, +\infty)$ for some $\Lambda_j > 0$ and $j = 1, 2$, then $\{\psi_2(\pi(\mathcal{R}_2)) - \psi_1(\pi(\mathcal{R}_1)), \pi \in \widehat{G}\} \in S^{-\infty}$.*

Proof of Lemma 4.15. We write

$$\psi_2(\mathcal{R}_2) - \psi_1(\mathcal{R}_1) = (\psi_2 - \psi_1)(\mathcal{R}_2) \;+\; (1 - \psi_1)(\mathcal{R}_1) \;+\; (\psi_1 - 1)(\mathcal{R}_2).$$

The result follows from the application of Corollary 3.8 to $\psi_2 - \psi_1$ and to $(1 - \psi_1)$. □

It remains to prove the last claim in Proposition 4.14.

End of the proof of Proposition 4.14. Now let us assume that the symbol defined by $\sum_\ell \sigma_\ell \psi(\pi(\mathcal{R}))$ is in S^m with $m < m_0$. We may assume that $m_1 < m < m_0$. Then $\sigma_0 \psi(\pi(\mathcal{R})) \in S^m$. Denoting by ν the homogeneous degree of $\mathcal{R}$, we have $\psi(\pi(\mathcal{R}))\pi(\mathcal{R})^{-\frac{m}{\nu}} \in S^m$ and

$$S^0 \ni \sigma_0 \psi(\pi(\mathcal{R}))\psi(\pi(\mathcal{R}))\pi(\mathcal{R})^{-\frac{m}{\nu}} = \sigma_0 \pi(\mathcal{R})^{-\frac{m_0}{\nu}}\ \psi^2(\pi(\mathcal{R}))\pi(\mathcal{R})^{\frac{m'}{\nu}}$$

with $m' := -m + m_0 > 0$. For each $x \in G$, the 0-homogeneous field

$$\left\{\sigma_0(x, \pi)\pi(\mathcal{R})^{-\frac{m_0}{\nu}}, \pi \in \widehat{G}\right\}$$

satisfies the hypotheses of Lemma 4.9 (2) and thus must be zero. This conclude the proof of Proposition 4.14. □

Proposition 4.14 allows us to give a meaning to a homogeneous expansion as in (4.5):

Definition 4.16. Let $\{\sigma_\ell\}_{\ell\in\mathbb{N}_0}$ be a sequence of homogeneous symbols such that $\sigma_\ell \in \dot{S}^{m_\ell}$ with m_ℓ strictly decreasing to $-\infty$. Then $\sum_{\ell=0}^\infty \sigma_\ell$ denotes the symbol σ in S^{m_0} modulo $S^{-\infty}$ given by the asymptotic sum $\sum_\ell \psi(\pi(\mathcal{R}))\sigma_\ell$ or $\sum_\ell \sigma_\ell \psi(\pi(\mathcal{R}))$ in the sense of (3.6) where $\mathcal{R}$ is any positive Rockland operator and $\psi \in C^\infty(\mathbb{R})$ any real-valued function satisfying $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on (Λ, ∞) for some $\Lambda > 0$. We then write (4.5).

We denote by $S^{m_0}_{\text{asymp}}$ the set of symbols $\sigma \in S^{m_0}$ which admits such an homogeneous expansion.

The last part of Proposition 4.14 also shows that the first term of an expansion $\sigma \sim \sum_{\ell=0}^{\infty} \sigma_\ell$ is unique (hence, proceeding recursively and up to writing zero terms, the expansion itself is unique). This allows us to define the principal part of a symbol:

Definition 4.17. If $\sigma \sim \sum_{\ell=0}^{\infty} \sigma_\ell$ is in $S^{m_0}_{\rm asymp}$, then its first term σ_0 is called its principal part and we write:

$$\mathrm{princ}_{m_0}(\sigma) := \sigma_0.$$

Example 4.18. If $\sigma = \sum_\alpha c_\alpha(x)\pi(X)^\alpha$ where $(c_\alpha)_{\alpha\in\mathbb{N}^n}$ is a sequence of functions in $C^\infty(G)$ such that c_α and all the left derivatives $X^\beta c_\alpha$ are bounded while all but a finite number of c_α are zero, then $\sigma \in S^m_{\rm asymp}$ where $m = \max\{[\alpha], c_\alpha \neq 0\}$ and

$$\sigma = \sum_{\ell=0}^{m} \sigma_{m-\ell} \quad \text{with} \quad \sigma_{m-\ell} = \sum_{[\alpha]=m-\ell} c_\alpha(x)\pi(X)^\alpha \in S^\ell \cap \dot{S}^\ell.$$

Moreover the principal part coincides with the top part of the left-invariant differential operator:

$$\mathrm{princ}_m(\sigma) = \sum_{[\alpha]=m} c_\alpha(x)\pi(X)^\alpha.$$

The asymptotic expansion and the principal part satisfy the analogue properties to its Euclidean counterpart:

Proposition 4.19. *The set $S^m_{\rm asymp}$ is a linear subspace of S^m and the mapping $\mathrm{princ}_m : S^m_{\rm asymp} \to \dot{S}^m$ is linear. Moreover if $\sigma \in S^m_{\rm asymp}$ and $\sigma' \in S^{m'}_{\rm asymp}$ with asymptotic expansion $\sigma \sim \sum_\ell \sigma_\ell$ and $\sigma' \sim \sum_{\ell'} \sigma'_{\ell'}$ then $\sigma^* \in S^m_{\rm asymp}$ and $\sigma\sigma' \in S^{m+m'}_{\rm asymp}$ with asymptotic expansions*

$$\sigma^* \sim \sum_\ell \sigma^*_\ell \qquad \textit{and} \qquad \sigma\sigma' \sim \sum_{\ell,\ell'} \sigma_\ell \sigma'_{\ell'}.$$

In particular,

$$\mathrm{princ}_m\left(\sigma^*\right) = \mathrm{princ}_m(\sigma)^* \quad \textit{and} \quad \mathrm{princ}_{m+m'}\left(\sigma\sigma'\right) = \mathrm{princ}_m(\sigma)\mathrm{princ}_{m'}(\sigma').$$

Proof. The linearity of $S^m_{\rm asymp}$ and of princ_m is easy to check. The property regarding the adjoint follows from Theorem 3.4 (3). Let σ and σ' be as in the statement. We fix a positive Rockland operator $\mathcal{R}$ and a real-valued function $\psi \in C^\infty(\mathbb{R})$ satisfying $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on (Λ, ∞) for some $\Lambda > 0$. Then we can develop

$$\left(\sum_{\ell \le M} \sigma_\ell \psi(\pi(\mathcal{R}))\right)\left(\sum_{\ell' \le M'} \psi(\pi(\mathcal{R}))\sigma'_{\ell'}\right)$$

on one hand as (see Proposition 4.14 (1))

$$\left(\sigma \bmod S^{m_M}\right)\left(\sigma' \bmod S^{m'_M}\right) = \sigma\sigma' \bmod S^{\tilde{M}}$$

where $\tilde{M} := \max(m + m'_{M'}, m' + m_M, m_M, m'_{M'})$, and on the other hand by Corollary 4.7

$$\sum_{\substack{\ell \le M \\ \ell' \le M'}} \sigma_\ell \psi^2(\pi(\mathcal{R}))\sigma'_{\ell'} = \sum_{\substack{\ell \le M \\ \ell' \le M'}} \psi^2(\pi(\mathcal{R}))\sigma_\ell\sigma'_{\ell'} \bmod S^{-\infty}.$$

Hence $\sigma\sigma' = \sum_{\substack{\ell \le M, \ell' \le M' \\ m_\ell + m'_{\ell} \ge \tilde{M}}} \psi^2(\pi(\mathcal{R}))\sigma_\ell\sigma'_{\ell'} \bmod S^{\tilde{M}}$.

This implies $\sigma\sigma' \sim \sum_{\tilde{\ell}} \tilde{\sigma}_{\tilde{\ell}}$ where $\tilde{\sigma}_{\tilde{\ell}} := \sum_{m_\ell + m_{\ell'} = \tilde{m}_{\tilde{\ell}}} \sigma_\ell\sigma'_{\ell'} \in S^{\tilde{m}_{\tilde{\ell}}}$, and in particular $\tilde{\sigma}_0 = \sigma_0\sigma'_0$. □

4.5. The classical calculus $\cup_m \Psi^m_{\mathrm{cl}}(\Omega)$

We can now define the classical classes of symbols and of operators.

Definition 4.20. Let $\Omega \subset G$ be an open subset. We denote by $S^m_{\mathrm{cl}}(\Omega)$ the class of symbol $\sigma \in S^m_{\mathrm{asymp}}$ such that the integral kernel of $\mathrm{Op}(\sigma)$ is compactly supported in $\Omega \times \Omega$. The corresponding class of operators is denoted by

$$\Psi^m_{\mathrm{cl}}(\Omega) := \mathrm{Op}\big(S^m_{\mathrm{cl}}(\Omega)\big).$$

The operation of taking the principal part is denoted by princ_m:

$$\mathrm{princ}_m(\mathrm{Op}(\sigma)) = \mathrm{Op}(\mathrm{princ}_m(\sigma)), \quad \sigma \in S^m_{\mathrm{cl}}(\Omega).$$

If $\Omega = G$ we may allow ourselves the shortcuts $S^m_{\mathrm{cl}}(G) = S^m_{\mathrm{cl}}$ and $\Psi^m_{\mathrm{cl}}(G) = \Psi^m_{\mathrm{cl}}$. Naturally the differential operators in the calculus with support in Ω are classical:

Example 4.21. If $(c_\alpha)_{\alpha \in \mathbb{N}^n}$ is a sequence of functions in $\mathcal{D}(\Omega)$ and all but a finite number of c_α are zero, then $\sum_\alpha c_\alpha(x) X^\alpha$ is in $\Psi^m_{\mathrm{cl}}(\Omega)$ where $m = \max\{[\alpha], c_\alpha \neq 0\}$. Indeed the (right convolution) kernel is $\sum_\alpha c_\alpha(x)\delta_0^{(\alpha)}$ which is supported in $\{0\}$. Moreover

$$\mathrm{princ}_m\left(\sum_\alpha c_\alpha(x) X^\alpha\right) = \sum_{[\alpha]=m} c_\alpha(x) X^\alpha.$$

We will often use the following easy lemma without referring to it.

Lemma 4.22. *Let $\Omega \subset G$ be an open subset. If $A \in \Psi^0_{\mathrm{cl}}(\Omega)$ then the operator A extends uniquely into a continuous mapping $L^2(\Omega, \mathrm{loc}) \to L^2(\Omega)$.*

As is customary, $L^2(\Omega, \mathrm{loc})$ denotes the space of distributions $f \in \mathcal{D}'(\Omega)$ such that for all $\chi \in \mathcal{D}(\Omega)$, $f\chi \in L^2(\Omega)$. Later on, we will need the more general definition:

Definition 4.23. Let $\Omega \subset G$ be an open subset. We denote by $L^2_s(\Omega, \text{loc})$ the space of distributions $f \in \mathcal{S}'(\Omega)$ such that for all $\chi \in \mathcal{D}(\Omega)$, $f\chi \in L^2_s(G)$. It is equipped with its natural structure of Fréchet space.

Proof of Lemma 4.22. Let $A \in \Psi^0_{\text{cl}}(\Omega)$. Its integral kernel is supported in a compact $K \subset \Omega$. We can always find $\chi \in \mathcal{D}(\Omega)$ such that $\chi \equiv 1$ on K. Hence if $\phi \in \mathcal{D}(\Omega)$, then $A\phi = A(\chi\phi)$.

Let $f \in L^2(\Omega, \text{loc})$. Then $f\chi \in L^2(\Omega) \subset L^2(G)$ and we define $Af := A(f\chi)$, as $A \in \Psi^0$, it is bounded on $L^2(G)$ (see Theorem 3.4). It is easy to show that this does not depend on the choice of χ and that we have:

$$\forall f \in L^2(\Omega, \text{loc}) \qquad \|Af\|_{L^2(G)} \le \|A\|_{\mathscr{L}(L^2(G))} \|f\chi\|_{L^2(G)}.$$

Since $f \in L^2(\Omega, \text{loc}) \mapsto f\chi \in L^2(G)$ is continuous, the operator $A : L^2(\Omega, \text{loc}) \to L^2(\Omega)$ is continuous. □

We now state and prove a theorem which in the Euclidean setting is a consequence of Rellich's theorem (which states that if $t < s$ and $K \subset \mathbb{R}^n$ is a compact subset, then the inclusion map $H^s(K) \to H^t$ is compact).

Theorem 4.24. *Let $\Omega \subset G$ be an open subset. If $A \in \Psi^m_{\text{cl}}(\Omega)$ with $m < 0$ then the operator*

$$A : L^2(\Omega, \text{loc}) \to L^2(\Omega)$$

is compact, i.e. *if $u_k \underset{k\to\infty}{\rightharpoonup} u$ in $L^2(\Omega, \text{loc})$ then $Au_k \underset{k\to\infty}{\longrightarrow} Au$ in the L^2-norm.*

The notation $u_k \underset{k\to\infty}{\rightharpoonup} u$ in $L^2(\Omega, \text{loc})$ means that the sequence (u_k) of distributions in $L^2(\Omega, \text{loc})$ converges towards u for the Fréchet topology of $L^2(\Omega, \text{loc})$. Consequently, $u_k \underset{k\to\infty}{\rightharpoonup} u$ in $L^2(\Omega, \text{loc})$ if and only if for every $v \in L^2(\Omega)$ compactly supported, $(u_k, v)_{L^2} \underset{k\to\infty}{\longrightarrow} (u, v)_{L^2}$.

Proof of Theorem 4.24. As $A \in \Psi^m_{\text{cl}}(\Omega)$, its integral kernel is supported in a compact $K \subset \Omega \times \Omega$ that we can assume of the form $K = K_1 \times K_2$. We can always find $\chi \in \mathcal{D}(\Omega)$ such that $\chi \equiv 1$ on K_2. As the integral kernel of A is supported in K, we have $A\phi = A(\chi\phi)$, for any $\phi \in L^2(\Omega, \text{loc})$. Let $\mathcal{R}$ be a (fixed) positive Rockland operator of homogeneous degree ν. As $A \in \Psi^m$, $A(\text{I} + \mathcal{R})^{-m/\nu} \in \Psi^0$ is bounded on $L^2(G)$. Let (u_k) be a sequence in $L^2(\Omega, \text{loc})$ with $u_k \underset{k\to\infty}{\rightharpoonup} u$. We have

$$\begin{aligned}\|Au_k - Au\|_{L^2} = \|A(\chi u_k - \chi u)\|_{L^2} &= \left\| A(\text{I} + \mathcal{R})^{-\frac{m}{\nu}} (\text{I} + \mathcal{R})^{\frac{m}{\nu}} (\chi u_k - \chi u) \right\|_{L^2} \\ &\le \left\| A(\text{I} + \mathcal{R})^{-\frac{m}{\nu}} \right\|_{\mathscr{L}(L^2(G))} \left\| (\text{I} + \mathcal{R})^{\frac{m}{\nu}} (\chi u_k - \chi u) \right\|_{L^2}.\end{aligned}$$

By Proposition 2.15,

$$\left\| (\text{I} + \mathcal{R})^{\frac{m}{\nu}} (\chi u_k - \chi u) \right\|_{L^2} = \|\chi(u_k - u)\|_{L^2_m(G)} \le C_m \|\chi\ (u_k - u)\|_{H^{m/\upsilon_1}}.$$

As $u_k \underset{k\to\infty}{\rightharpoonup} u$ in $L^2(\Omega, \text{loc})$ and $m/\upsilon_1 < 0$, by Rellich's Theorem and the uniqueness of the limit on a compact, $\chi u_k \underset{k\to\infty}{\longrightarrow} \chi u$ in the Sobolev norm of H^{m/υ_1}. Therefore

$$\|Au_k - Au\|_{L^2} \leq \left\| A(\mathrm{I}+\mathcal{R})^{-\frac{m}{\nu}} \right\|_{\mathscr{L}(L^2(G))} C_m \|\chi\,(u_k-u)\|_{H^{m/\upsilon_1}} \underset{k\to\infty}{\longrightarrow} 0,$$

and $Au_k \underset{k\to\infty}{\longrightarrow} Au$ in the L^2-norm. □

5. C^*-algebras generated by 0-homogeneous regular symbols

In this section, we study the regular 0-homogeneous symbols, that is, the symbols in $\dot{S}^0$, and the C^*-algebra it generates. We give a particular attention to those that do not depend on x.

5.1. The Fréchet space $\tilde{S}^0$

In this section, we study the invariant regular 0-homogeneous symbols, or in other words the symbol in $\dot{S}^0$ independent of x. They form the space $\tilde{S}^0$:

Definition 5.1. We denote by $\tilde{S}^0$ the set of measurable fields $\sigma = \{\sigma(\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty,\ \pi \in \widehat{G}\}$ satisfying

(1) σ is 0-homogeneous, *i.e.* $\sigma(r \cdot \pi) = \sigma(\pi)$ for all $r > 0, \pi \in \widehat{G}$;
(2) If $\mathcal{R}$ is a positive Rockland operator of degree ν and $\alpha \in \mathbb{N}_0^n$ and $\gamma \in \mathbb{R}$, then

$$\sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{[\alpha]+\gamma}{\nu}} \Delta^\alpha \sigma(\pi) \pi(\mathcal{R})^{-\frac{\gamma}{\nu}} \right\| < \infty.$$

Naturally, the second condition is independent of $\mathcal{R}$ and it suffices to show it for a sequence $(\gamma_\ell)_{\ell\in\mathbb{Z}}$ with $\lim_{\ell\to\pm\infty} = \pm\infty$. This equips naturally the vector space $\tilde{S}^0$ with a Fréchet topology which is the same as the one obtained with viewing $\tilde{S}^0$ as a closed sub-vector space of $\dot{S}^0$. Note that $\tilde{S}^0$ is also an algebra, in fact a sub-algebra of $\dot{S}^0$.

By Corollary 4.12, a 0-homogeneous symbol $\sigma = \{\sigma(\pi)\}$ is in $\tilde{S}^0$ if and only if for each $\alpha \in \mathbb{N}_0^n$, the following suprema are finite

$$\sup_{\pi\in\widehat{G}} \left\| \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha \sigma(\pi) \right\|_{\mathscr{L}(\mathcal{H}_\pi)}, \quad \text{and} \quad \sup_{\pi\in\widehat{G}} \left\| \Delta^\alpha \sigma(\pi)\, \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)}. \tag{5.1}$$

Here $\mathcal{R}$ is a fixed positive Rockland operator of degree ν.

Proposition 4.11 implies that the kernel associated to a symbol σ in $\tilde{S}^0$, *i.e.* $\kappa \in \mathcal{S}'(G)$ such that $\widehat{\kappa} = \sigma$, is smooth on $G \setminus \{0\}$.

Lemma 5.2 below shows the converse to Proposition 4.11 in the following way:

Lemma 5.2. *Let $\sigma = \{\sigma(\pi) \in \mathscr{L}(\mathcal{H}_\pi), \pi \in \widehat{G}\}$ be a measurable field of operators such that:*

- *σ is 0-homogeneous,* i.e. *$\sigma(r\pi) = \sigma(\pi)$ for (almost) all $\pi \in \widehat{G}$ and all $r > 0$;*
- *σ is bounded,* i.e. $\sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty$*;*
- *The kernel associated with σ coincides with a smooth function on $G\backslash\{0\}$.*

Then $\sigma \in \tilde{S}^0$.

Note that the proof of Lemma 5.2 given below does not produce any bounds for the suprema in (5.1) in terms of κ or σ. The main ingredient is the analysis of operators of type ν, see Section 2.7.

Proof of Lemma 5.2. If κ_1 is any tempered distribution, then T_{κ_1} denotes the convolution operator with right-convolution kernel κ_1, *i.e.* $T_{\kappa_1}(\phi) = \phi * \kappa_1, \phi \in \mathcal{S}(G)$. Recall that $X_1, \dots, X_n$ is a basis of $\mathfrak{g}$.

Let σ, κ satisfy the hypotheses of the statement. We fix $\alpha \in \mathbb{N}_0^n$, $\alpha \neq 0$. By Lemma 2.13, we may replace $\mathcal{R}$ by one of its power and thus we assume that $\nu > [\alpha]$. The operator $\mathcal{R}$ is a linear combination of X^β with $[\beta] = \nu$. Let us write one X^β as $Y_r \dots Y_1$ with $Y_j \in \{X_1, \dots, X_n\}$ for $j = 1, \dots, r$. We also denote by $[Y_j]$ the homogeneous degree of Y_j, so that $\nu = [\beta] = [Y_1] + \dots + [Y_r] > [\alpha]$. Let $r' \in \mathbb{N}_0$, $0 \leq r' \leq r$, be such that $[\alpha] - ([Y_1] + \dots + [Y_{r'}]) > 0$ but $[\alpha] - ([Y_1] + \dots + [Y_{r'+1}]) \leq 0$, with the convention that $[Y_1] + \dots + [Y_{r'}] = 0$ if $r' = 0$ and in this case $Y_{r'} \dots Y_1 = \text{I}$. By Proposition 2.24 (4), the operator $Y_{r'} \dots Y_1 T_{x^\alpha\kappa} = T_{Y_1 \dots Y_r x^\alpha \kappa}$ is of type $[\alpha] - ([Y_1] + \dots + [Y_{r'}]) \in (0, Q)$. As the operator $\mathcal{R}^{\frac{[\alpha]-([Y_1]+\dots+[Y_{r'+1}])}{\nu}}$ is of type $[Y_1] + \dots + [Y_{r'+1}] - [\alpha] \in (0, Q)$, see Example 2.23, the operator $Y_{r'} \dots Y_1 T_{x^\alpha\kappa} \mathcal{R}^{\frac{[\alpha]-([Y_1]+\dots+[Y_{r'+1}])}{\nu}}$ is of type $[Y_{r'+1}]$. Thus the operator $Y_{r'+1} \dots Y_1 T_{x^\alpha\kappa} \mathcal{R}^{\frac{[\alpha]-([Y_1]+\dots+[Y_{r'+1}])}{\nu}}$ is of type 0. Then $Y_{r'+1} \dots Y_1 T_{x^\alpha\kappa} \mathcal{R}^{\frac{[\alpha]-([Y_1]+\dots+[Y_{r'+2}])}{\nu}}$ is of type $[Y_{r'+2}]$ and

$$Y_{r'+2} \dots Y_1 T_{x^\alpha\kappa} \mathcal{R}^{\frac{[\alpha]-([Y_1]+\dots+[Y_{r'+2}])}{\nu}}$$

is of type 0. Proceeding recursively, we obtain that

$$X^\beta T_{x^\alpha\kappa} \mathcal{R}^{-\frac{\nu-[\alpha]}{\nu}} = Y_r \dots Y_1 T_{x^\alpha\kappa} \mathcal{R}^{-\frac{[Y_1]+\dots+[Y_r]-[\alpha]}{\nu}}$$

is of type 0. Thus $\mathcal{R} T_{x^\alpha\kappa} \mathcal{R}^{-\frac{\nu-[\alpha]}{\nu}}$ is bounded on $L^2(G)$. We can apply the same reasoning to $T^*_{x^\alpha\kappa} = (-1)^{|\alpha|} T_{x^\alpha\kappa^*}$ where $\kappa^*(x) = \bar{\kappa}(x^{-1})$. This shows that $T_{x^\alpha\kappa}$ and its adjoint $T^*_{x^\alpha\kappa}$ map $\dot{L}^2_{\nu-[\alpha]} \to \dot{L}^2_\nu$ continuously. By duality and interpolation (see Theorem 2.14), we obtain that $T_{x^\alpha\kappa}$ maps $\dot{L}^2 \to \dot{L}^2_{[\alpha]}$ and $\dot{L}^2_{-[\alpha]} \to \dot{L}^2$ continuously. The Plancherel theorem, see Section 2.2, now implies that the suprema in (5.1) are finite. This concludes the proof of Lemma 5.2. □

We can now describe the symbols in $\tilde{S}^0$ via their kernels. We will use the following conventions:

Definition 5.3. We denote by $C^\infty_{(-Q)-\text{hom}}(G\backslash\{0\})$ the set of functions which are smooth and $(-Q)$-homogeneous on $G\backslash\{0\}$, and by $\mathcal{F}$ the subset of functions in $C^\infty_{(-Q)-\text{hom}}(G\backslash\{0\})$ with zero mean value.

The definition and properties of having zero mean value were given in Proposition 2.24 (2). The vector space $C^\infty_{(-Q)-\text{hom}}(G\backslash\{0\})$ is naturally a Fréchet space isomorphic to the Fréchet space of smooth functions on the unit sphere given by a smooth quasi-norm; this latter Fréchet space is well-known to be separable. Note that by a smooth quasi-norm, we mean a quasi-norm which is smooth away from 0; such a quasi-norm exists. We also observe that $\mathcal{F}$ is a closed subspace of $C^\infty_{(-Q)-\text{hom}}(G\backslash\{0\})$. Hence $\mathcal{F}$ and $C^\infty_{(-Q)-\text{hom}}(G\backslash\{0\})$ are separable.

Corollary 5.4. *If $\sigma \in \tilde{S}^0$, we denote by $\kappa_\sigma \in \mathcal{F}$ the smooth function obtained by restriction of the associated kernel to $G\backslash\{0\}$ and by Lemma* 2.25 *we have*

$$\sigma = \sum_{j=-\infty}^{+\infty} \mathcal{F}_G\left\{\kappa_\sigma 1_{2^j \le |x| \le 2^{j+1}}\right\} + c_\sigma \mathrm{I},$$

with $c_\sigma \in \mathbb{C}$. The map

$$\Theta : \begin{cases} \tilde{S}^0 \longrightarrow \mathcal{F} \times \mathbb{C} \\ \sigma \longmapsto (\kappa_\sigma, c_\sigma) \end{cases}$$

is an isomorphism of Fréchet vector spaces. Consequently, the Fréchet space $\tilde{S}^0$ is separable.

Proof of Corollary 5.4. The fact that the map Θ is well defined, linear, continuous, and injective follows easily from Proposition 4.11 and Lemma 2.25. Let us show that Θ maps $\tilde{S}^0$ onto $\mathcal{F} \times \mathbb{C}$. Given $(\kappa, c) \in \mathcal{F} \times \mathbb{C}$, we want to construct $\sigma \in \dot{S}^0$ such that $\Theta(\sigma) = (\kappa, c)$. Defining κ_j as in Lemma 2.25, we then set $\sigma(\pi) := \sum_{j\in\mathbb{Z}} \widehat{\kappa}_j + c\mathrm{I}_{\mathcal{H}_\pi}$. The proof of Lemma 2.25 shows that this defines a field of operators $\{\sigma(\pi), \pi \in \mathcal{H}_\pi\}$ which is bounded by:

$$\sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le |c| + C \sup_{|z|=1, |\alpha|\le 1} \left|X^\alpha \kappa(z)\right|.$$

One easily checks that σ is 0-homogeneous and that the kernel associated with σ coincides with κ on $G\backslash\{0\}$ and that $c_\sigma = c$. By Lemma 5.2, $\sigma \in \tilde{S}^0$. Thus Θ is surjective. As the map Θ is a linear and continuous bijection between Fréchet spaces, it is an isomorphism by the open mapping theorem. $\square$

5.2. An important example

This section is devoted to a more concrete example of a symbol in $\tilde{S}^0$, more precisely to the symbol σ_f defined and studied in Lemma 5.5. It will be useful later.

Lemma 5.5. *We fix a quasi-norm $|\cdot|$ on G. Let $|\cdot|$ be the associated mapping on $\widehat{G}$, see Section 2.3. For any $f \in \mathcal{S}(G)$, $\tilde{S}^0$ contains the symbol σ_f defined via*

$$\sigma_f(\pi) = \widehat{f}\big(|\pi|^{-1} \cdot \pi\big), \quad \pi \in \widehat{G}\backslash\{1\}.$$

Strategy of the proof of Lemma 5.5. Since $f \in \mathcal{S}(G)$, for each $\pi \in \widehat{G}$, the operator $\widehat{f}(\pi)$ maps $\mathcal{H}_\pi^\infty$ to itself and has operator norm $\leq \|f\|_{L^1}$. Moreover the field of operators $\widehat{f}$ is measurable. Since the map $\pi \mapsto |\pi|^{-1}\cdot\pi$ is continuous $\widehat{G}\backslash\{1\} \to \widehat{G}$, see Section 2.3, we have $\sigma_f \in L^\infty(\widehat{G})$ with $\sup_{\pi\in\widehat{G}} \|\sigma_f(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq \|f\|_{L^1}$. One easily checks that σ_f is 0-homogeneous.

We denote by $\kappa \in \mathcal{S}'(G)$ the kernel of $\sigma_f \in L^\infty(\widehat{G})$, *i.e.* $\sigma_f = \widehat{\kappa}$. By Lemma 5.2, it suffices to show that its kernel $\kappa \in \mathcal{S}'(G)$ is smooth away from 0. And for this, it suffices to show that for every $M \in \mathbb{N}_0$ there exist $N \in \mathbb{N}_0$ and $C = C(\sigma, M) > 0$ such that for any $\phi \in \mathcal{D}(G\backslash\{0\})$, we have:

$$\left|\big(\kappa, \mathcal{R}^M\phi\big)\right| \leq C\left(\|\phi\|_{L^1(G)} + \|\phi\|_{L^2(G)} + \left\||x|^{-N}\phi\right\|_{L^2(G)}\right). \tag{5.2}$$

Indeed (5.2) will imply that $X^\alpha\kappa$ is locally square integrable on $G\backslash\{0\}$ for any $\alpha \in \mathbb{N}_0^n$, and thus that it is smooth away from 0. □

Proof of (5.2) *in the case $M = 0$.* We fix $\chi \in \mathcal{D}(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[-1, 1]$ and $\chi(\lambda) \equiv 0$ for $|\lambda| \geq 2$. Let $\phi \in \mathcal{D}(G)$. Since $\chi(\mathcal{R})\delta_0 \in \mathcal{S}(G)$, see Corollary 3.8, $\chi(\mathcal{R})\phi = \phi * \chi(\mathcal{R})\delta_0$ is Schwartz and so is $(1-\chi)(\mathcal{R})\phi$. As $\kappa \in \mathcal{S}'(G)$, we can write:

$$\langle\kappa, \phi\rangle = \big\langle\kappa, \chi(\mathcal{R})\phi\big\rangle + \big\langle\kappa, (1-\chi)(\mathcal{R})\phi\big\rangle.$$

Note that if $\psi \in \mathcal{S}(G)$, then $\sigma_f\widehat{\psi}\pi(\mathrm{I}+\mathcal{R})^N \in L^\infty(\widehat{G})$ for any $N \in \mathbb{N}$, and $\sigma_f\widehat{\psi}$ satisfies the hypotheses of Proposition 2.16. So the Fourier inversion formula yields

$$\langle\kappa, \psi\rangle = \int_{\widehat{G}} \mathrm{tr}\left(\sigma_f(\pi)\mathcal{F}\big(\check{\psi}\big)(\pi)\right) d\mu(\pi),$$

where $\check{\psi}(x) = \psi(x^{-1})$, and

$$\begin{aligned} |\langle\kappa, \psi\rangle| &\leq \|\sigma_f(\pi)\|_{L^\infty(\widehat{G})} \int_{\widehat{G}} \mathrm{tr}\left|\mathcal{F}\big(\check{\psi}\big)(\pi)\right| d\mu(\pi) \\ &\leq \|f\|_{L^1(G)} \int_{\widehat{G}} \mathrm{tr}\,|\mathcal{F}(\psi)(\pi)| d\mu(\pi). \end{aligned}$$

Applying this to $\psi = \chi(\mathcal{R})\phi \in \mathcal{S}(G)$ implies that

$$|\langle \kappa, \chi(\mathcal{R})\phi\rangle| \leq \|f\|_{L^1(G)}\|\phi\|_{L^1(G)} \int_{\widehat{G}} \operatorname{tr}|\chi(\pi(\mathcal{R}))|d\mu(\pi),$$

and the last integral is finite, see Corollary 3.8.

We now turn our attention to $\langle \kappa, (1-\chi)(\mathcal{R})\phi\rangle$. For the same reason as above,

$$\int_{\pi\in\widehat{G}} \operatorname{tr}\left|\sigma_f(\pi)(1-\chi)\big(\pi(\bar{\mathcal{R}})\big)\mathcal{F}\check{\phi}(\pi)\right| d\mu(\pi) < \infty$$

and

$$\begin{aligned}
&\langle \kappa, (1-\chi)(\mathcal{R})\phi\rangle \\
&= \int_{\pi\in\widehat{G}} \operatorname{tr}\left\{\sigma_f(\pi)(1-\chi)(\pi(\bar{\mathcal{R}}))\mathcal{F}\check{\phi}(\pi)\right\} d\mu(\pi) \\
&= \int_{r=0}^{\infty}\int_{\pi\in\Sigma_{1,|\cdot|}} \operatorname{tr}\left\{\sigma_f(r\cdot\pi)(1-\chi)(r\cdot\pi(\bar{\mathcal{R}}))\mathcal{F}\check{\phi}(r\cdot\pi)\right\} d\varsigma_{|\cdot|}(\pi) r^{Q-1}dr,
\end{aligned}$$

having used the polar decomposition on $\widehat{G}$, see Lemma 2.6. We now write:

$$\int_{r=0}^{\infty} = \int_{r=0}^{1} + \int_{r=1}^{\infty} = I_1 + I_2.$$

We have $r\cdot\pi(\mathcal{R}) = r^{\nu}\pi(\mathcal{R})$ and $\sigma_f(r\cdot\pi) = \widehat{f}(\pi)$, $\pi \in \Sigma_{1,|\cdot|}$, so fixing $N_0 \in \mathbb{N}$

$$\sigma_f(r\cdot\pi)(1-\chi)\big(r\cdot\pi(\bar{\mathcal{R}})\big) = r^{\nu N_0}\widehat{\tilde{\bar{\mathcal{R}}}^{N_0} f}(\pi)\chi_{N_0}\big(r^{\nu}\pi(\bar{\mathcal{R}})\big)$$

where $\chi_{N_0}(\lambda) = (1-\chi(\lambda))\lambda^{-N_0}$, and

$$\begin{aligned}
|I_1| &\leq \int_{r=0}^{1}\int_{\pi\in\Sigma_{1,|\cdot|}} \operatorname{tr}\left|\widehat{\tilde{\bar{\mathcal{R}}}^{N_0} f}(\pi)\chi_{N_0}\big(r^{\nu}\pi(\bar{\mathcal{R}})\big)\mathcal{F}\check{\phi}(r\cdot\pi)\right| d\varsigma_{|\cdot|}(\pi) r^{N_0\nu+Q-1}dr \\
&\leq \left\|\widehat{\tilde{\bar{\mathcal{R}}}^{N_0} f}\right\|_{L^\infty(\widehat{G})} \int_{r=0}^{\infty}\int_{\pi\in\Sigma_{1,|\cdot|}} \left\|\chi_{N_0}(r^{\nu}\pi(\bar{\mathcal{R}}))\right\|_{HS(\mathcal{H}_\pi)} \\
&\qquad \times \left|\mathcal{F}\check{\phi}(r\cdot\pi)\right\|_{HS(\mathcal{H}_\pi)} d\varsigma_{|\cdot|}(\pi) r^{Q-1}dr \\
&\leq \left\|\tilde{\bar{\mathcal{R}}}^{N_0} f\right\|_{L^1} \|\chi_{N_0}(\pi(\mathcal{R}))\|_{L^2(\widehat{G})} \|\widehat{\phi}\|_{L^2(\widehat{G})},
\end{aligned}$$

by the Cauchy-Schwartz inequality. The Plancherel formula yields $\|\widehat{\phi}\|_{L^2(\widehat{G})} = \|\phi\|_{L^2(G)}$ and, together with the functional calculus of $\mathcal{R}$,

$$\begin{aligned}
\|\chi_{N_0}(\pi(\mathcal{R}))\|_{L^2(\widehat{G})} &= \|\chi_{N_0}(\mathcal{R})\delta_0\|_{L^2(G)} \\
&\leq \sup_{\lambda\geq 0}(1-\chi(\lambda))\left(\frac{1+\lambda}{\lambda}\right)^{N_0} \left\|(\mathrm{I}+\mathcal{R})^{-N_0}\delta_0\right\|_{L^2(G)}.
\end{aligned}$$

This last quantity is finite when $N_0\nu > Q/2$ since $(\mathrm{I}+\mathcal{R})^{\frac{s_1}{\nu}}\delta_0 \in L^2(G)$ for $s_1 < -Q/2$, see [25, Section 4.3.3].

For the second integral, we see

$$I_2 = \int_{r=1}^{\infty}\int_{\pi\in\Sigma_{1,|\cdot|}} \mathrm{tr}\left\{\widehat{f}(\pi)(1-\chi)\big(r^\nu\pi(\bar{\mathcal{R}})\big)\mathcal{F}\check{\phi}(r\cdot\pi)\right\} d\varsigma_{|\cdot|}(\pi) r^{Q-1}dr.$$

For each $r > 0$, we define $g_r := (1-\chi)(r^\nu\tilde{\mathcal{R}})f \in L^2(G)$ so that $\widehat{g_r}(\pi) = \widehat{f}(\pi)(1-\chi)(r^\nu\pi(\bar{\mathcal{R}}))$ also defines an operator for each $\pi \in \widehat{G}$ with $\widehat{g_r} \in L^\infty(\widehat{G}) \cap L^2(\widehat{G})$. We observe that if $\alpha_2 \in \mathbb{N}_0^n$, $\alpha_2 \neq 0$, then

$$\Delta^{\alpha_2}(1-\chi)(\pi(\bar{\mathcal{R}})) = -\Delta^{\alpha_2}\chi(\pi(\bar{\mathcal{R}})) \in \mathcal{F}_G\mathcal{S}(G).$$

This together with the Leibniz formula easily implies that for any $\alpha \in \mathbb{N}_0^n$, we have that $\Delta^\alpha\widehat{g_r}(\pi)$ is a well defined bounded operator on $\mathcal{H}_\pi$ for each $\pi \in \widehat{G}$ and that $\Delta^\alpha\widehat{g_r} \in L^\infty(\widehat{G})$.

Let $N \in \mathbb{N}_0$. For any $x \in G\backslash\{0\}$, we set $\phi_N(x) = |x|^{-N}\phi(x)$ and assume $\phi = 0$ in a neighbourhood of 0, so that $\phi_N \in \mathcal{D}(G)$. We may assume that the integer N and the quasi-norm $|\cdot|$ are such that $|\cdot|^N$ is a polynomial in x, which is then necessarily homogeneous of degree N. One checks readily that $\phi = |x|^N\phi_N$ and by (3.4),

$$\mathcal{F}\check{\phi}(r\cdot\pi) = r^{-N}\Delta_{|x|^N}\mathcal{F}\big(\check{\phi}_N\big)_{(r)}(\pi).$$

From (3.3), we have

$$\begin{aligned}
&\mathrm{tr}\left\{\widehat{f}(\pi)(1-\chi)\big(r^\nu\pi(\bar{\mathcal{R}})\big)\mathcal{F}\check{\phi}(r\cdot\pi)\right\} = r^{-N}\mathrm{tr}\left\{\widehat{g_r}(\pi)\Delta_{|x|^N}\mathcal{F}\big(\check{\phi}_N\big)_{(r)}(\pi)\right\}\\
&= r^{-N}\sum_{[\alpha_1]+[\alpha_2]=N} c_{\alpha_1,\alpha_2}\mathrm{tr}\left\{\Delta^{\alpha_1}\left\{\Delta^{\alpha_2}\widehat{g_r}(\pi)\ \mathcal{F}\big(\check{\phi}_N\big)_{(r)}(\pi)\right\}\right\}\\
&= \sum_{[\alpha_1]+[\alpha_2]=N} c_{\alpha_1,\alpha_2}\mathrm{tr}\left\{\Delta^{\alpha_1}\left\{\sigma_{\alpha_2}\ \mathcal{F}\big(\check{\phi}_N\big)\right\}(r\cdot\pi)\right\}
\end{aligned}$$

where $\sigma_{\alpha_2}(r\cdot\pi) := r^{-[\alpha_2]}1_{r\geq 1}\Delta^{\alpha_2}\widehat{g_r}(\pi)$ $(r > 0,\ \pi \in \Sigma_{1,|\cdot|})$ is in $L^\infty(\widehat{G})$. By Lemmata 2.6 and 3.2,

$$\begin{aligned}
I_2 &= \sum_{[\alpha_1]+[\alpha_2]=N} c_{\alpha_1,\alpha_2}\int_{r=0}^{\infty}\int_{\pi\in\Sigma_{1,|\cdot|}}\mathrm{tr}\left\{\Delta^{\alpha_1}\left\{\sigma_{\alpha_2}\ \mathcal{F}\big(\check{\phi}_N\big)\right\}(r\cdot\pi)\right\} d\varsigma_{|\cdot|}(\pi)r^{Q-1}dr\\
&= \sum_{[\alpha_1]+[\alpha_2]=N} c_{\alpha_1,\alpha_2}\int_{r=0}^{\infty}\int_{\pi\in\Sigma_{1,|\cdot|}}\mathrm{tr}\left\{\Delta^{\alpha_1}\left\{\sigma_{\alpha_2}\ \mathcal{F}\big(\check{\phi}_N\big)\right\}(\pi)\right\} d\mu(\pi)\\
&= \sum_{[\alpha_2]=N} c_{0,\alpha_2}\int_{\widehat{G}}\mathrm{tr}\left\{\sigma_{\alpha_2}\ \mathcal{F}\big(\check{\phi}_N\big)\right\}(\pi)d\mu(\pi).
\end{aligned}$$

So we have obtained:

$$|I_2| \lesssim \sum_{[\alpha_2]=N} \int_{\widehat{G}} \operatorname{tr} \left| \sigma_{\alpha_2}\, \mathcal{F}\big(\check{\phi}_N\big) \right| (\pi)\, d\mu(\pi) \lesssim \sum_{[\alpha_2]=N} \|\sigma_{\alpha_2}\|_{L^2(\widehat{G})} \left\| \mathcal{F}\big(\check{\phi}_N\big) \right\|_{L^2(\widehat{G})}.$$

By the Plancherel formula, $\|\mathcal{F}(\check{\phi}_N)\|_{L^2(\widehat{G})} = \|\phi_N\|_{L^2(G)}$. We have with $\alpha_2 \in \mathbb{N}_0^n$, $[\alpha_2] = N$:

$$\left\|\sigma_{\alpha_2}\right\|^2_{L^2(\widehat{G})} = \int_{r=1}^{\infty} \int_{\pi \in \Sigma_{1,|\cdot|}} \|\Delta^{\alpha_2} \widehat{g}_r(\pi)\|^2_{HS(\mathcal{H}_\pi)} r^{-2N+Q-1} dr\, d\varsigma_{|\cdot|}(\pi),$$

and by the Leibniz formula

$$\left\|\Delta^{\alpha_2} \widehat{g}_r(\pi)\right\|_{HS(\mathcal{H}_\pi)} \lesssim \sum_{[\alpha_0]+[\alpha_1]=N} \left\|\Delta^{\alpha_0} \widehat{f}(\pi)\ \Delta^{\alpha_1} (1-\chi)(r^\nu \pi(\mathcal{R}))\right\|_{HS(\mathcal{H}_\pi)}.$$

In the sum above, for $\alpha_1 \neq 0$, we have

$$\begin{aligned} &\left\|\Delta^{\alpha_0} \widehat{f}(\pi)\ \Delta^{\alpha_1} (1-\chi)(r^\nu \pi(\mathcal{R}))\right\|_{HS(\mathcal{H}_\pi)} \\ &\leq \left\| \widehat{x^{\alpha_0} f}(\pi) \right\|_{L^\infty(\widehat{G})} \left\|\Delta^{\alpha_1} \chi(r^\nu \pi(\mathcal{R}))\right\|_{HS(\mathcal{H}_\pi)} \end{aligned}$$

whereas for $\alpha_1 = 0$, we have

$$\begin{aligned} &\left\|\Delta^{\alpha_0} \widehat{f}(\pi)\ (1-\chi)\big(r^\nu \pi(\mathcal{R})\big)\right\|_{HS(\mathcal{H}_\pi)} \\ &\leq r^{N_0 \nu} \left\| \mathcal{F}\left\{ \bar{\tilde{\mathcal{R}}}^{N_0} x^{\alpha_0} f \right\} (\pi) \chi_{N_0}\big(r^\nu \pi(\mathcal{R})\big) \right\|_{HS(\mathcal{H}_\pi)}. \end{aligned}$$

When $N_0 \nu \leq 2N$, these estimates yield

$$\begin{aligned} \sum_{[\alpha_2]=N} \|\sigma_{\alpha_2}\|_{L^2(\widehat{G})} &\lesssim \sum_{[\alpha_0]\leq N} \left\| \widehat{x^{\alpha_0} f}(\pi) \right\|_{L^\infty(\widehat{G})} \sum_{0<[\alpha_1]\leq N} \|\Delta^{\alpha_1} \chi(\pi(\mathcal{R}))\|_{L^2(\widehat{G})} \\ &\quad + \sum_{[\alpha_0]=N} \left\| \mathcal{F}\left\{ \bar{\tilde{\mathcal{R}}}^{N_0} x^{\alpha_0} f \right\} \right\|_{L^\infty(\widehat{G})} \|\chi_{N_0}(\pi(\mathcal{R}))\|_{L^2(\widehat{G})} \\ &\lesssim \left\| (1+|x|)^N f \right\|_{L^1(G)} \left\| (1+|x|)^N \chi(\mathcal{R}) \delta_0 \right\|_{L^1(G)} \\ &\quad + \sum_{[\alpha_0]=N} \left\| \bar{\tilde{\mathcal{R}}}^{N_0} x^{\alpha_0} f \right\|_{L^1(G)} \|\chi_{N_0}(\mathcal{R})\delta_0\|_{L^2(G)}. \end{aligned}$$

Recall that $\chi(\mathcal{R})\delta_0 \in \mathcal{S}(G)$ and we have already seen that $\chi_{N_0}(\mathcal{R})\delta_0 \in L^2(G)$ when $\nu N_0 > Q/2$. This implies that I_2 is bounded up to a constant of f by $\|\phi_N\|_{L^2(G)}$. Hence (5.2) is proved in the case $M = 0$. □

Proof of (5.2) *for* $M \in \mathbb{N}$. If $M \in \mathbb{N}$, then we modify the proof above. We write

$$\left(\kappa, \mathcal{R}^M \phi\right) = \left(\kappa, \mathcal{R}^M \chi(\mathcal{R})\phi\right) + \left(\kappa, \mathcal{R}^M (1-\chi)(\mathcal{R})\phi\right).$$

For the first term, we have

$$\left(\kappa, \mathcal{R}^M \chi(\mathcal{R})\phi\right) = \int_{\pi\in\widehat{G}} \operatorname{tr}\left\{\sigma_f(\pi)\pi(\bar{\mathcal{R}})^M \chi\big(\pi(\bar{\mathcal{R}})\big)\mathcal{F}\check{\phi}(\pi)\right\} d\mu(\pi)$$

$$|(\kappa, \chi(\mathcal{R})\phi)| \leq \|f\|_{L^1}\|\phi\|_{L^1} \int_{\pi\in\widehat{G}} \operatorname{tr}\left|\big(\lambda^M \chi\big)\big(\pi(\bar{\mathcal{R}})\big)\right| d\mu(\pi),$$

and this last integral is finite since $(\lambda^M\chi)(\pi(\bar{\mathcal{R}}))$ is the Fourier transform of a Schwartz function by Hulanicki's theorem. For the second term, we have:

$$\begin{aligned}&(\kappa, \mathcal{R}^M (1-\chi)(\mathcal{R})\phi)\\ &= \int_{r=0}^{\infty}\int_{\pi\in\Sigma_{1,|\cdot|}} \operatorname{tr}\left\{\widehat{f}(\pi) r^{\nu M}\pi\big(\bar{\mathcal{R}}\big)^M (1-\chi)\Big(r^\nu \pi\big(\bar{\mathcal{R}}\big)\Big)\mathcal{F}\check{\phi}(r\cdot\pi)\right\} d\varsigma_{|\cdot|}(\pi) r^{Q-1} dr.\end{aligned}$$

We decompose again $\int_{r=0}^{\infty} = \int_{r=0}^{1} + \int_{r=1}^{\infty}$, and a modification of the argument yields:

$$\int_{r=0}^{1} \leq \left\|\tilde{\bar{\mathcal{R}}}^{N_0} f\right\|_{L^1} \left\|\chi_{N_0+M}(\pi(\mathcal{R}))\right\|_{L^2(\widehat{G})} \left\|\widehat{\phi}\right\|_{L^2(\widehat{G})},$$

and $\int_{r=1}^{\infty} \leq \|\phi_N\|_{L^2(G)} I_2'$ with

$$\begin{aligned}(I_2')^2 \lesssim \sum_{[\alpha_0]+[\alpha_1]=N} \int_{\Sigma_{1,|\cdot|}} \int_1^{\infty} &\left\|\Delta^{\alpha_0}\mathcal{F}_G\{\tilde{\mathcal{R}}^M f(\pi)\}\ \Delta^{\alpha_1}(1-\chi)(r^\nu\pi(\mathcal{R}))\right\|^2_{HS(\mathcal{H}_\pi)}\\ &\times r^{\nu M-2N+Q-1} dr\, d\varsigma_{|\cdot|}(\pi).\end{aligned}$$

This implies (5.2) for $M \in \mathbb{N}$ and concludes the proof of Lemma 5.5. □

5.3. The C^*-algebra $C^*(\tilde{S}^0)$ and its spectrum

In this section, we study the closure of $\tilde{S}^0$ for $\sup_{\pi\in\widehat{G}} \|\cdot\|_{\mathscr{L}(\mathcal{H}_\pi)}$. It is denoted by $C^*(\tilde{S}^0)$. More precisely, we prove:

Proposition 5.6. *The closure* $C^*(\tilde{S}^0)$ *of* $\tilde{S}^0$ *for* $\sup_{\pi\in\widehat{G}} \|\cdot\|_{\mathscr{L}(\mathcal{H}_\pi)}$ *is a separable* C^**-algebra and a sub-*C^**-algebra of the von Neumann algebra* $L^\infty(\widehat{G})$. *It is of type 1 and its unit is the identity field* I.

If $\pi_0 \in \widehat{G}\backslash\{1\}$, *then the mapping*

$$\begin{cases}\tilde{S}^0 \longrightarrow \mathscr{L}(\mathcal{H}_{\pi_0})\\ \sigma \longmapsto \sigma(\pi_0).\end{cases}$$

extends to a continuous mapping $\rho_{\pi_0} : C^*(\tilde{S}^0) \to \mathscr{L}(\mathcal{H}_{\pi_0})$ *which is an irreducible representation of* $C^*(\tilde{S}^0)$*. For any* $r > 0$*, we have* $\rho_{\pi_0} = \rho_{r\cdot\pi_0}$*. Denoting by* $\dot{\pi}_0 \in \Sigma_1 = (\widehat{G}\backslash\{1\})/\mathbb{R}^+$ *the class of representations* $\{r \cdot \pi_0, r > 0\}$*, the mapping*

$$R : \begin{cases} \Sigma_1 \longrightarrow \widehat{C^*(\tilde{S}^0)} \\ \dot{\pi}_0 \longmapsto \rho_{\pi_0} \end{cases}$$

is a homeomorphism.

Consequently, we may identify the spectrum of $C^*(\tilde{S}^0)$ with Σ_1. Recall that Σ_1 may be viewied as the sphere from the polar decomposition on $\widehat{G}$, see Section 2.3.

One easily checks that $C^*(\tilde{S}^0)$ is a C^*-algebra. Its separability follows from Corollary 5.4. The essential point in the proof of Proposition 5.6 is the following lemma:

Lemma 5.7. *Let* ρ *be a representation of the* C^* *algebra* $C^*(\tilde{S}^0)$*. For any* $f \in \mathcal{S}(G)$*, we set*

$$\pi_\rho(f) := \rho\big(\sigma_f{}^*\big),$$

where the symbol $\sigma_f \in \tilde{S}^0$ *is defined as in Lemma* 5.5 (*we assume that a quasi-norm on G has been fixed*)*. Then the mapping* $\pi_\rho : \mathcal{S}(G) \to \mathscr{L}(\mathcal{H}_\rho)$ *extends to a continuous representation* π_ρ *of* $C^*(G)$*.*

If ρ *is non-zero irreducible, then* π_ρ *is non-zero irreducible,* i.e. $\pi_\rho \in \widehat{G}$*. Furthermore,* $\pi_\rho \neq 1$ *and for any symbol* $\sigma \in \tilde{S}^0$*, we have* $\rho(\sigma) = \sigma(\pi_\rho)$*.*

Proof of Lemma 5.7. We keep the notation of the statement. We check readily that for any $f, f_1, f_2 \in \mathcal{S}(G)$, we have:

$$\|\sigma_f\|_{L^\infty(\widehat{G})} \le \big\|\widehat{f}\big\|_{L^\infty(\widehat{G})}, \quad \sigma_{f_1 * f_2} = \sigma_{f_2}\,\sigma_{f_1}, \quad \text{and} \quad \sigma_f{}^* = \sigma_{f^*}.$$

Let ρ be a representation of $C^*(\tilde{S}^0)$. We check easily that the mapping $\pi_\rho : \mathcal{S}(G) \to \mathscr{L}(\mathcal{H}_\rho)$ defined via $\pi_\rho(f) = \rho(\sigma_f{}^*)$ extends to a continuous representation of $C^*(G)$.

We assume that ρ is irreducible and non-zero. Let us show that π_ρ is irreducible and non-zero. We will need the following preliminary step. Let $\chi_1 \in \mathcal{D}(\mathbb{R})\backslash\{0\}$ be supported in $[1/2, 2]$ and valued in $[0, +\infty)$. We set $\chi_\epsilon(\lambda) = \chi_1(\epsilon\lambda)$ for each $\epsilon > 0, \lambda \in \mathbb{R}$. The properties of $\mathcal{R}$ implies that $\chi_\epsilon(\mathcal{R})\delta_0 = (\chi_1(\mathcal{R})\delta_0)_{(\epsilon)} \in \mathcal{S}(G)$. Since

$$c := \int_G \chi_\epsilon(\mathcal{R})\delta_0 = \int_G \chi_1(\mathcal{R})\delta_0 = \big\|\sqrt{\chi_1}(\mathcal{R})\delta_0\big\|^2_{L^2(G)} > 0,$$

we may replace χ_1 with $c^{-1}\chi_1$, and assume $c = 1$. Let us show that

$$\pi_\rho(\chi_\epsilon(\mathcal{R})\delta_0) = \rho\big(\sigma_{\chi_\epsilon(\mathcal{R})\delta_0}\big) \underset{\epsilon\to 0}{\longrightarrow} \mathrm{I}_{\mathcal{H}_\rho} \quad \text{in SOT on } \mathcal{H}_\rho. \tag{5.3}$$

By Lemma 2.17, if π_1 is a continuous unitary representation of G, then

$$\widehat{\chi}_\epsilon(\pi_1(\mathcal{R})) = \pi_1\{\chi_\epsilon(\mathcal{R})\delta_0\} \underset{\epsilon\to 0}{\longrightarrow} \mathrm{I}_{\mathcal{H}_{\pi_1}} \quad \text{in SOT on } \mathcal{H}_{\pi_1}. \tag{5.4}$$

For any $v \in \mathcal{H}_\rho$, the representation π_ρ restricted to the closure of the subspace $\pi_\rho(C^*(G))v$ can be identified with a continuous unitary representation of G when $\pi_\rho(C^*(G))v \neq \{0\}$, see [21, Section 13.9.3]; we can then apply (5.4) to this representation. By [21, Proposition 2.2.6], the space $\mathcal{H}_{\pi_\rho} = \mathcal{H}_\rho$ of the representation π_ρ decomposes into $\mathcal{H}_{\pi_\rho,0} \oplus^\perp \mathcal{H}_{\pi_\rho,0}^\perp$ where $\mathcal{H}_{\pi_\rho,0}$ denotes the closure of the subspace of $v \in \mathcal{H}_\rho$ satisfying $\pi_\rho(C^*(G))v = \{0\}$. Hence we have obtained that, as $\epsilon \to 0$, $\pi_\rho(\chi_\epsilon(\mathcal{R})\delta_0) \to \mathrm{I}_{\mathcal{H}_{\pi_\rho,0}^\perp}$ in SOT on $\mathcal{H}_\rho = \mathcal{H}_{\pi_\rho}$ and on $\mathcal{H}_{\pi_\rho,0}^\perp$. If $v \in \mathcal{H}_{\pi_\rho,0}^\perp$ and $\sigma \in \tilde{S}^0$, then $\pi_\rho(\chi_\epsilon(\mathcal{R})\delta_0)\rho(\sigma)v$ is in $\mathcal{H}_{\pi_\rho,0}^\perp$ and converges to $\rho(\sigma)v$ which is necessarily in $\mathcal{H}_{\pi_\rho,0}^\perp$. Thus the closed subspace $\mathcal{H}_{\pi_\rho,0}^\perp$ is invariant under ρ. As ρ is irreducible and non-zero, we must have $\mathcal{H}_{\pi_\rho,0}^\perp = \mathcal{H}_\rho$. Thus we have obtained (5.3).

Let us now show that (5.3) implies the irreducibility of π_ρ. Let τ be a symbol in $\tilde{S}^0$. For every $\epsilon > 0$ and $\pi \in \widehat{G}$, we set $\widehat{f_{\epsilon,\tau}}(\pi) := \chi_\epsilon(\pi(\mathcal{R}))\tau(\pi)$. By Corollary 4.13, $f_{\epsilon,\tau} := f_\epsilon$ is Schwartz. We check easily that $\sigma_{f_\epsilon} = \sigma_{\chi_\epsilon(\mathcal{R})\delta_0}\tau$ thus

$$\pi_\rho(f_\epsilon) = \rho(\sigma_{f_\epsilon}) = \rho\big(\sigma_{\chi_\epsilon(\mathcal{R})\delta_0}\big)\rho(\tau) \underset{\epsilon\to 0}{\longrightarrow} \rho(\tau) \quad \text{in SOT on } \mathcal{H}_\rho. \tag{5.5}$$

This convergence implies that any π_ρ-invariant subspace of $\mathcal{H}_\rho$ is also invariant under ρ. Thus the representation π_ρ of $C^*(G)$ is irreducible.

We keep the same notation for the corresponding representation (class) $\pi_\rho \in \widehat{G}$ of G. We observe that

$$\pi_\rho(f_\epsilon) = \widehat{f_\epsilon}(\pi_\rho) = \chi_\epsilon(\pi_\rho(\mathcal{R}))\tau(\pi_\rho)$$

and, for SOT on $\mathcal{H}_\rho$, the left-hand side converges to $\rho(\tau)$ by (5.5) whereas the right-hand side tends to $\tau(\pi_\rho)$ by (5.4). Hence $\rho(\tau) = \tau(\pi_\rho)$ for any $\tau \in \tilde{S}^0$.

If $\pi_\rho = 1$ then $\pi_\rho(\mathcal{R}) = 0$ so $\chi_\epsilon(\pi_\rho(\mathcal{R})) = 0$ and, with the notation above, $f_{\epsilon,\tau} = 0$ for every $\epsilon > 0$ and $\tau \in \tilde{S}^0$, so ρ is zero on $\tilde{S}^0$ because of the convergence in (5.5), and $\rho = 0$. If ρ is non-zero, then (5.5) shows that π_ρ is also non-zero. This concludes the proof of Lemma 5.7. □

We can now prove Proposition 5.6.

End of the proof of Proposition 5.6. We fix $\pi_0 \in \widehat{G}\backslash\{1\}$. By Lemma 2.25, if $\sigma \in \tilde{S}^0$, we can consider $\sigma(\pi_0) \in \mathscr{L}(\mathcal{H}_{\pi_0})$. One checks readily that $\rho_{\pi_0} : \sigma \mapsto \sigma(\pi_0)$ is a representation of the algebra $\tilde{S}^0$ which extends to a continuous representation ρ_{π_0} of $C^*(\tilde{S}^0)$. This defines an injective mapping $R : \dot{\pi}_0 \mapsto \rho_{\pi_0}$ which is continuous. By Lemma 5.7, R is surjective. As Σ_1 is compact, see Lemma 2.4, R is a homeomorphism.

If $\rho \in \widehat{C^*(\tilde{S}^0)}$, then $\rho(C^*(\tilde{S}^0))$ contains $\pi_\rho(C^*(G))$ having used Lemma 5.7 and its notation. As $C^*(G)$ is of type 1, by [21, Theorem Dixmier 9.1], $\pi_\rho(C^*(G))$ contains the space of compact operators in $\mathscr{L}(\mathcal{H}_{\pi_\rho})$ thus so does $\rho(C^*(\tilde{S}^0))$. Again by [21, Theorem Dixmier 9.1], this shows that $C^*(\tilde{S}^0)$ is type 1. This concludes the proof of Proposition 5.6. □

5.4. The C^*-algebra $C^*(\dot{S}^0_c(\Omega))$ and its spectrum

We can use the results in Section 5.3 on invariant symbols to analyse 0-homogeneous regular symbols which depends smoothly in x and are compactly supported in x. We introduce the following definitions.

Definition 5.8. Let Ω be an open set of G.

- A symbol $\sigma = \{\sigma(x,\pi) : \mathcal{H}^\infty_\pi \to \mathcal{H}^\infty_\pi, (x,\pi) \in \Omega\times\widehat{G}\}$ is compactly supported in Ω when there exists a compact $K \subset \Omega$ such that $\sigma(x,\pi) = 0$ for almost all $(x,\pi) \in (G\backslash K) \times \widehat{G}$;
- We denote by $\dot{S}^0_c(\Omega)$ the space of symbols in $\dot{S}^0$ which are compactly supported in Ω.

One easily checks that $\dot{S}^0_c(\Omega)$ is an algebra without unity. Note that the symbol I, and more generally any invariant symbol $\sigma \in \tilde{S}^0$, is not a symbol in $\dot{S}^0_c(\Omega)$ where Ω is an open subset of G.

The vector space $\dot{S}^0_c(\Omega)$ is equipped with the following topology. If Ω' is an open subset of Ω such that $\bar{\Omega}' \subset \Omega$, the subspace of $\sigma \in \dot{S}^0(\Omega)$ which are x-supported in $\bar{\Omega}'$ is a Fréchet space when equipped with the $\dot{S}^0$-semi-norms. The union of these subspaces when Ω' runs over the open bounded subsets of Ω such that $\bar{\Omega}' \subset \Omega$ is $\dot{S}^0_c(\Omega)$. In fact, it suffices to consider a sequence of growing open subsets Ω'_j which exhausts Ω. Hence, we can now equip $\dot{S}^0_c(\Omega)$ with the inductive topology, so that it becomes a complete locally convex topological vector space.

Example 5.9. If $f \in \mathcal{D}(\Omega)$ then the symbol $f(x)\mathrm{I}$ is in $\dot{S}^0_c(\Omega)$. This yields a continuous inclusion of $\mathcal{D}(\Omega) \hookrightarrow \dot{S}^0_c(\Omega)$ for any open subset $\Omega \subset G$.

As in the invariant case (see Corollary 5.4), we can describe the elements of $\dot{S}^0(\Omega)$ in terms of their kernels and this implies that $\dot{S}^0(\Omega)$ is separable. We will use the space $\mathcal{F}$ defined in Definition 5.3 and the following convention:

Definition 5.10. If Ω is an open subset of G and $\mathcal{F}_1$ a Fréchet space, then $\mathcal{D}(\Omega; \mathcal{F}_1)$ denotes the set of functions from Ω into $\mathcal{F}_1$ which are smooth and compactly supported in Ω. Equipped with the inductive topology, it is a complete locally convex topological vector space.

For instance $\mathcal{D}(\Omega, \mathbb{C}) = \mathcal{D}(\Omega)$. Note that if $\mathcal{F}_1$ is separable then so is $\mathcal{D}(\Omega; \mathcal{F}_1)$. This together with Corollary 5.4 easily imply:

Proposition 5.11. *Let Ω be a open subset of G. If $\sigma \in \dot{S}^0_c(\Omega)$ then for each $x \in \Omega$ the symbol $\sigma(x,\cdot)$ is in $\tilde{S}^0$ and, as in Corollary* 5.4*, we denote by $\kappa_{\sigma,x} \in \mathcal{F}$ the* *smooth function obtained by restriction of the associated kernel to* $G\backslash\{0\}$ *and by Lemma* 2.25 *we have*

$$\sigma(x,\cdot) = \sum_{j=-\infty}^{+\infty} \mathcal{F}_G\left\{\kappa_{\sigma,x} 1_{2^j\leq|x|\leq 2^{j+1}}\right\} + c_{\sigma,x}\mathrm{I},$$

with $c_{\sigma,x} \in \mathbb{C}$*; then the function* $\kappa_\sigma : x \mapsto \kappa_{\sigma,x}$ *is in* $\mathcal{D}(\Omega;\mathcal{F})$ *and the function* $c_\sigma : x \mapsto c_{\sigma,x}$ *is in* $\mathcal{D}(\Omega)$. *The map*

$$\Theta_0 : \begin{cases} \dot{S}^0_c(\Omega) \longrightarrow \mathcal{D}(\Omega;\mathcal{F}) \times \mathcal{D}(\Omega) \\ \sigma \longmapsto (\kappa_\sigma, c_\sigma) \end{cases}$$

is an isomorphism of topological vector spaces. Consequently, $\dot{S}^0_c(\Omega)$ *is separable.*

Definition 5.12. If Ω is an open subset of G, we denote by $C^*(\dot{S}^0_c(\Omega))$ the closure of $\dot{S}^0(\Omega)$ for

$$\tau \longmapsto \sup_{(x,\pi)\in\Omega\times\widehat{G}} \|\tau(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}.$$

We view $C^*(\dot{S}^0_c(\Omega))$ as a space of fields on $\Omega\times\widehat{G}$. Let us summarize its properties:

Proposition 5.13. *The space* $C^*(\dot{S}^0_c(\Omega))$ *is a separable* C^**-algebra of type* 1. *It is non-unital but admits an approximate unity.*

If $\pi_0 \in \widehat{G}\backslash\{1\}$ *and* $x_0 \in \Omega$*, then the mapping*

$$\begin{cases} \dot{S}^0_c(\Omega) \longrightarrow \mathscr{L}(\mathcal{H}_{\pi_0}) \\ \sigma \longmapsto \sigma(x_0,\pi_0), \end{cases}$$

extends to a continuous mapping $\rho_{x_0,\pi_0} : C^*(\dot{S}^0(\Omega)) \to \mathscr{L}(\mathcal{H}_{\pi_0})$ *which is an irreducible non-zero representation of* $C^*(\dot{S}^0(\Omega))$. *For any* $r>0$*, we have* $\rho_{x_0,\pi_0} = \rho_{x_0,r\cdot\pi_0}$. *Denoting by* $\dot{\pi}_0$ *the class of representations* $\{r\cdot\pi_0, r>0\}$*, the mapping*

$$R : \begin{cases} \Omega\times\Sigma_1 \longrightarrow \widehat{C^*(\dot{S}^0(\Omega))} \\ (x_0,\dot{\pi}_0) \longmapsto \rho_{x_0,\pi_0} \end{cases}$$

is a homeomorphism.

Proposition 5.13 is the non-invariant version of Proposition 5.6 whose proofs we adapt. We will need the following notation: if $\mathcal{A}$ is a C^*-algebra, we denote by $C_0(\Omega;\mathcal{A})$ the C^*-algebra of continuous functions $f:\Omega\to C^*(\tilde{S}^0)$ such that for every $\epsilon>0$ there exists a compact of Ω outside of which $\|f(x)\|_{C^*(\tilde{S}^0)} < \epsilon$. For instance, $C_0(\Omega,\mathbb{C}) = C_0(\Omega)$.

Proof. Fixing a sequence $(f_j)_{j\in\mathbb{N}}$ of $C_c^\infty(\Omega)$ valued in $[0,1]$ and satisfying $\operatorname{supp}(f_j) \subset \{f_{j+1} = 1\}$ together with $\cup_{j\in\mathbb{N}}\{f_j = 1\} = \Omega$, we check easily that $f_j\mathrm{I} \in \dot{S}_c^0(\Omega)$ with

$$\|f_j\mathrm{I}\|_{C^*(\dot{S}_c^0(\Omega))} = \sup_\Omega |f_j| \le 1$$

and

$$\forall\sigma \in C^*(\dot{S}_c^0(\Omega)) \qquad \left\|(f_j\mathrm{I})\,\sigma - \sigma\right\|_{C^*(\dot{S}_c^0(\Omega))} = \left\|\sigma\,(f_j\mathrm{I}) - \sigma\right\|_{C^*(\dot{S}_c^0(\Omega))} \underset{j\to\infty}{\longrightarrow} 0.$$

Hence, $(f_j\mathrm{I})_{j\in\mathbb{N}}$ is an approximate unity of the C^*-algebra $C^*(\dot{S}_c^0(\Omega))$.

In order to show the rest of the statement, let us first show the analogue of Lemma 5.7. Given a representation ρ of the C^*-algebra $C^*(\dot{S}^0(\Omega))$, we consider

$$\pi_\rho(\phi f) = \rho\Big(\phi\sigma_f^*\Big) = \rho(\phi)\rho\Big(\sigma_f^*\Big) \quad \phi \in \mathcal{D}(\Omega),\ f \in \mathcal{S}(G).$$

Proceeding as in Lemma 5.7, we see that π_ρ extends to a representation of the C^*-algebra $C_0(\Omega; C^*(\tilde{S}^0))$. Note that the linear combinations of symbols of the form $\phi(x)\tau(\pi)$ with $\phi \in C_0(\Omega)$ and $\tau \in C^*(\tilde{S}^0)$ form a dense subspace of $C_0(\Omega; C^*(\tilde{S}^0))$ and that the spectrum of the C^*-algebra $C(\Omega; C^*(\tilde{S}^0))$ is $\Omega \times \Sigma_1$.

We now assume that ρ is irreducible and non-zero. As for any $\phi \in C_0(\Omega)$, the operator $\rho(\phi\mathrm{I}) \in \mathscr{L}(\mathcal{H}_\rho)$ commutes with any $\rho(\tau)$, $\tau \in C_0(\Omega; C^*(\tilde{S}^0))$, it must be scalar, *i.e.* $\rho(\phi\mathrm{I}) \in \mathbb{C}\mathrm{I}_{\mathcal{H}_\rho}$. This yields the one-dimensional non-zero representation $C_0(\Omega) \ni \phi \mapsto \rho(\phi\mathrm{I}) \in \mathbb{C}\mathrm{I}_{\mathcal{H}_\rho}$. Hence it is given by $x_0 \in \Omega$, *i.e.*

$$\forall\phi \in C_0(\Omega) \qquad \rho(\phi\mathrm{I}) = \phi(x_0)\mathrm{I}_{\mathcal{H}_\rho}.$$

We fix a function $\phi \in C_0(\Omega)$ such that $\phi(x_0) = 1$. The restriction of ρ to $\phi C^*(\tilde{S}^0)$ yields an irreducible representation of $C^*(\tilde{S}^0)$, which may be identified with an irreducible non-trivial representation (class) $\pi_0 \in \widehat{G}\backslash\{1\}$ of $C^*(G)$ by Lemma 5.7. This easily implies

$$\rho(\tau) = \tau(x_0, \pi_0).$$

We have obtained that any irreducible non-zero representation of $C_0(\Omega; C^*(\tilde{S}^0))$ is of the form $\delta_{x_0} \otimes \pi_0$ with $x_0 \in \Omega$ and $\pi_0 \in \widehat{G}\backslash\{1\}$. Conversely, if $\rho = \delta_{x_0} \otimes \pi_0$, then it is an irreducible non-zero representation of $C_0(\Omega; C^*(G))$.

The rest of the proof is obtained easily by adapting the arguments given in the proof of Proposition 5.6. □

5.5. The states of $C^*(\tilde{S}^0)$ and $C^*(\dot{S}_c^0(\Omega))$

In Propositions 5.6 and 5.13, we described the spectra of the C^*-algebras In this section, we show that this allows us to describe the states (*i.e.* the continuous positive forms) of these C^*-algebra in terms of objects depending on $\widehat{G}$.

Definition 5.14. Let Z be a complete separable metric space, and let $\xi \mapsto \mathcal{H}_\xi$ a measurable field of complex Hilbert spaces of Z.

- The set $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi\in Z})$ is the set of pairs (γ, Γ) where γ is a positive Radon measure on Z and $\Gamma = \{\Gamma(\xi) \in \mathcal{L}(\mathcal{H}_\xi) : \xi \in Z\}$ is a measurable field of trace-class operators such that for all compact set $K \subset Z$,

$$\int_K \mathrm{Tr}\, |\Gamma(\xi)|\, d\gamma(\xi) < +\infty;$$

- Two pairs (γ, Γ) and (γ', Γ') in $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi\in Z})$ are equivalent when there exists a measurable function $f : Z \to \mathbb{C} \setminus \{0\}$ such that

$$d\gamma'(\xi) = f(\xi) d\gamma(\xi) \quad \text{and} \quad \Gamma'(\xi) = \frac{1}{f(\xi)}\Gamma(\xi)$$

  for γ-almost every $\xi \in Z$. The equivalence class of (γ, Γ) is denoted by $\Gamma d\gamma$;
- A pair (γ, Γ) in $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi\in Z})$ is positive when $\Gamma(\xi) \geq 0$ for γ-almost all $\xi \in Z$. In this case, we may write $\Gamma d\gamma \geq 0$ or $(\gamma, \Gamma) \in \mathcal{M}_1^+(Z, (\mathcal{H}_\xi)_{\xi\in Z})$.

We start with $C^*(\tilde{S}^0)$. We will use the short-hands

$$\mathcal{M}_1^+(\Sigma_1) = \mathcal{M}_1^+(Z, (\mathcal{H}_\xi)_{\xi\in Z}) \quad \text{when} \quad Z = \{\dot{\pi} \in \Sigma_1\}, \quad \text{and} \quad \mathcal{H}_{\dot{\pi}} = \mathcal{H}_\pi,$$

where Σ_1 is the sphere coming from the polar decomposition together with its measure ς, see Section 2.3.

Proposition 5.15.

(1) *If ℓ is a state of $C^*(\tilde{S}^0)$, then there exists $(\gamma, \Gamma) \in \mathcal{M}_1^+(\Sigma_1)$ satisfying*

$$\int_{\Sigma_1} \mathrm{tr}\, (\Gamma(\dot{\pi}))\, d\gamma(\dot{\pi}) = 1, \tag{5.6}$$

and

$$\forall \sigma \in C^*(\tilde{S}^0) \qquad \ell(\sigma) = \int_{\Sigma_1} \mathrm{tr}\, (\sigma(\dot{\pi})\Gamma(\dot{\pi}))\, d\gamma(\dot{\pi}); \tag{5.7}$$

(2) *Conversely, given $(\gamma, \Gamma) \in \mathcal{M}_1^+(\Sigma_1)$ satisfying* (5.6)*, the linear form ℓ defined via* (5.7) *is a state of $C^*(\tilde{S}^0)$. Furthermore, if $(\gamma', \Gamma') \in \mathcal{M}_1^+(\Sigma_1)$ also satisfies* (5.6) *and* (5.7) *for the same state ℓ, then (γ', Γ') is equivalent to (γ, Γ).*

Proof of Part 1 *of Proposition* 5.15. Let ℓ be a state of the C^*-algebras $C^*(\tilde{S}^0)$. The GNS construction [21, Proposition 2.4.4] yields a representation ρ of $C^*(\tilde{S}^0)$ on the Hilbert space $\mathcal{H}_\ell := C^*(\tilde{S}^0)/\{\sigma : \ell(\sigma\sigma^*) = 0\}$ and

$$\ell(\sigma) = (\rho(\sigma)\xi, \xi)_{\mathcal{H}_\ell}, \quad \sigma \in C^*\big(\tilde{S}^0\big),$$

where the unit vector ξ is the image of $\mathrm{I} \in \tilde{S}^0$ via the canonical projection $C^*(\tilde{S}^0) \mapsto \mathcal{H}_\ell$. We then decompose [21, Theorem 8.6.6] the representation ρ (taking into account the possible multiplicities) as

$$(\rho, \mathcal{H}_\ell) \sim (\rho_1, \mathcal{H}_1) \oplus 2(\rho_2, \mathcal{H}_2) \oplus \ldots \oplus \aleph_0(\rho_\infty, \mathcal{H}_\infty),$$

and each ρ_r, $r \in \mathbb{N} \cup \{\infty\}$, may be disintegrated as

$$\rho_r \sim \int_{\widehat{C^*(\tilde{S}^0)}} \zeta \, d\gamma_r(\zeta);$$

furthermore, the positive measures $\gamma_1, \gamma_2, \ldots, \gamma_\infty$ are mutually singular in $\widehat{C^*(\tilde{S}^0)}$. Consequently we can write $\xi \in \mathcal{H}_\ell$ as

$$\xi \sim (\xi_1, \xi_2, \ldots, \xi_\infty), \quad \text{with } \xi_r = (\xi_{r,s})_{1 \le s \le r} \text{ for each } r \in \mathbb{N} \cup \{\infty\}, \text{ and } \xi_{r,s} \in \mathcal{H}_r.$$

Note that

$$1 = |\xi|^2_{\mathcal{H}_\ell} = \sum_{r \in \mathbb{N} \cup \{\infty\}} \sum_{s=1}^{r} |\xi_{r,s}|^2_{\mathcal{H}_r} \quad \text{with} \quad |\xi_{r,s}|^2_{\mathcal{H}_r} = \int_{\widehat{C^*(\tilde{S}^0)}} |\xi_{r,s}(\zeta)|^2_{\mathcal{H}_\zeta} d\gamma_r(\zeta).$$

Since we have identifed $\widehat{C^*(\tilde{S}^0)}$ with Σ_1:

$$\rho_r \sim \int_{\Sigma_1} \dot{\pi} \, d\gamma_r(\dot{\pi}), \quad \mathcal{H}_r \sim \int_{\Sigma_1} \mathcal{H}_\pi d\gamma_r(\dot{\pi}), \quad \sum_{r=1}^{\infty} \sum_{s=1}^{r} \int_{\Sigma_1} |\xi_{r,s}(\dot{\pi})|^2_{\mathcal{H}_\pi} d\gamma_r(\dot{\pi}) = 1.$$

Hence $\Gamma_r := \sum_{s=1}^r \xi_{r,s} \otimes \xi_{r,s}{}^*$ is a γ_r-measurable field on Σ_1 of positive trace-class operators of rank r. We have obtained:

$$\begin{aligned} \ell(\sigma) = (\rho(\sigma)\xi, \xi) &= \sum_{r \in \mathbb{N} \cup \{\infty\}} \sum_{s=1}^{r} \int_{\Sigma_1} (\sigma(\dot{\pi})\xi_{r,s}(\dot{\pi}), \xi_{r,s}(\dot{\pi}))_{\mathcal{H}_r} d\gamma_r(\dot{\pi}) \\ &= \sum_{r \in \mathbb{N} \cup \{\infty\}} \int_{\Sigma_1} \mathrm{tr}\, (\sigma(\dot{\pi})\Gamma_r(\dot{\pi}))\, d\gamma_r(\dot{\pi}). \end{aligned}$$

We now define the positive measure $\gamma := \sum_r \gamma_r$. As the measures γ_r are mutually singular, the field $\Gamma := \sum_r \Gamma_r$ is measurable and satisfies

$$\Gamma(\dot{\pi}) \geq 0, \quad \mathrm{tr}\, \Gamma(\dot{\pi}) < \infty, \qquad \int_{\Sigma_1} \mathrm{tr}\, \Gamma(\dot{\pi}) d\gamma(\dot{\pi}) = 1\,.$$

This shows Part 1. □

Proof of Part 2 *of Proposition* 5.15. Given $(\gamma, \Gamma) \in \mathcal{M}_1^+(\Sigma_1)$ satisfying (5.6), one easily checks that the linear form ℓ defined via (5.7) is a state of $C^*(\tilde{S}^0)$. To prove the last part of the statement, we consider $(\gamma', \Gamma') \in \mathcal{M}_1^+(\Sigma_1)$ which also satisfies (5.6) and (5.7) for the same state ℓ. It suffices to consider the case of γ and Γ obtained as in Part 1; in particular γ and Γ have the same support in Σ_1. We may also assume that γ' and Γ' have the same support in Σ_1. For each $r \in \mathbb{N} \cup \{\infty\}$, let B_r be the measurable subset of $\widehat{G}/\mathbb{R}^+$ where $\Gamma'(\dot{\pi})$ is of rank r a.e. We may assume these subsets disjoint. We define the measure $\gamma_r' = 1_{B_r}\gamma'$ and the field $\Gamma_r' := 1_{B_r}\Gamma'$ as the restrictions of γ' and Γ' to B_r. As Γ_r' is a measurable field of positive operators of rank r, there exists a measurable field of orthogonal vectors $(\xi_{r,s})_{s=1}^r$ such that $\Gamma_r' = \sum_{s=1}^r \xi_{r,s}' \otimes {\xi_{r,s}'}^*$. We have $\operatorname{tr}\Gamma_r' = \sum_{s=1}^r |\xi_{r,s}'|^2$.

We define the representation ρ' of $C^*(\tilde{S}^0)$ and the vector ξ' of ρ' via

$$\rho' := \oplus_{r\in\mathbb{N}\cup\{\infty\}} r \int_{\Sigma_1} \pi \, d\gamma_r'(\dot{\pi}), \quad \text{and} \quad \xi' := \oplus_{r\in\mathbb{N}\cup\{\infty\}} \oplus_{s=1}^r \int_{\Sigma_1} \xi_{r,s}'(\dot{\pi}) \, d\gamma_r'(\dot{\pi}).$$

We observe that ξ' is a unit vector:

$$|\xi'|^2 = \sum_{r\in\mathbb{N}\cup\{\infty\}} \sum_{s=1}^r |\tilde{\xi}_{r,s}'|^2 = \sum_{r\in\mathbb{N}\cup\{\infty\}} \int_{\Sigma_1} \operatorname{tr}\Gamma_r' \, d\gamma_r' = \int_{\Sigma_1} \operatorname{tr}\Gamma' \, d\gamma' = 1.$$

Moreover for any $\sigma \in C^*(\tilde{S}^0)$:

$$\begin{aligned}(\rho'(\sigma)\xi', \xi') &= \sum_{r\in\mathbb{N}\cup\{\infty\}} \sum_{s=1}^r \int_{\Sigma_1} \left(\sigma\xi_{r,s}', \xi_{r,s}'\right) d\gamma_r' = \sum_{r\in\mathbb{N}\cup\{\infty\}} \int_{\Sigma_1} \operatorname{tr}\left(\sigma\Gamma_r'\right) d\gamma_r' \\ &= \int_{\Sigma_1} \operatorname{tr}\left(\sigma\Gamma'\right) d\gamma' = \ell(\sigma).\end{aligned}$$

In other words, the state associated with ρ' and ξ' coincides with ℓ. This implies that ρ' and ρ are equivalent [21, Proposition 2.4.1], therefore the measures γ_r' and γ_r are equivalent for every $r \in \mathbb{N} \cup \{\infty\}$ [21, Theorem 8.6.6]. In other words, there exists a measurable positive function f_r supported in B_r such that $d\gamma_r'(\dot{\pi}) = f_r(\dot{\pi})d\gamma_r(\dot{\pi})$. As ξ' corresponds to ξ via the (ρ', ρ)-equivalence, we must have $\Gamma_r(\dot{\pi}) = f_r(\dot{\pi})\Gamma_r'(\dot{\pi})$. This concludes the proof of Part 2. □

From the proof of Proposition 5.15, we can determine easily the pure states, that is, the states corresponding to the irreducible representations:

Corollary 5.16. *The pure states of the C^*-algebra $C^*(\tilde{S}^0)$ are the functionals $\ell = \ell_{\pi_0,v_0}$ of the form:*

$$\ell(\sigma) = (\sigma(\dot{\pi}_0)v_0, v_0)_{\mathcal{H}_{\pi_0}}, \quad \sigma \in C^*\big(\tilde{S}^0\big),$$

where $\pi_0 \in \widehat{G}$ and $v_0 \in \mathcal{H}_{\pi_0}$ is a unit vector. The states $\ell = \ell_{\pi_0,v_0}$ where $\pi_0 \in \widehat{G}$ and $v_0 \in \mathcal{H}_{\pi_0}^\infty$ is a smooth unit vector, form a dense subset of the set of states of $C^*(\tilde{S}^0)$.

We observe that ℓ_{π_0,v_0} corresponds to $\gamma(\dot\pi) = \delta_{\dot\pi_0}(\dot\pi)$ and $\Gamma(\dot\pi_0) = v_0 \otimes v_0^*$.

Using the short-hand

$$\begin{aligned} &\mathcal{M}_1^+(\Omega \times \Sigma_1) = \mathcal{M}_1^+(Z, (\mathcal{H}_\xi)_{\xi\in Z}) \\ \text{when} \quad &Z = \{(x,\dot\pi) \in \Omega \times \Sigma_1\}, \quad \text{and} \quad \mathcal{H}_{x,\dot\pi} = \mathcal{H}_\pi, \end{aligned} \tag{5.8}$$

we also have a similar description of the states of $C^*(\dot S_c^0(\Omega))$:

Proposition 5.17. *Let Ω be an open set of G.*

(1) *If ℓ is a state of $C^*(S_c^0(\Omega))$, then there exists $(\gamma,\Gamma) \in \mathcal{M}_1^+(\Omega\times\Sigma_1)$ satisfying*

$$\int_{\Omega\times\Sigma_1} \text{tr}\ (\Gamma(x,\dot\pi))\, d\gamma(x,\dot\pi) = 1, \tag{5.9}$$

and

$$\forall \sigma \in C^*\big(\dot S_c^0(\Omega)\big) \qquad \ell(\sigma) = \int_{\Omega\times\Sigma_1} \text{tr}\ (\sigma(x,\dot\pi)\Gamma(x,\dot\pi))\, d\gamma(x,\dot\pi); \tag{5.10}$$

(2) *Conversely, given $(\gamma,\Gamma) \in \mathcal{M}_1^+(\Omega\times\Sigma_1)$ satisfying* (5.9)*, the linear form ℓ defined via* (5.10) *is a state of $C^*(\dot S_c^0(\Omega))$. Furthermore, if $(\gamma',\Gamma') \in \mathcal{M}_1^+(\Omega \times \Sigma_1)$ also satisfies* (5.9) *and* (5.10) *for the same state ℓ, then (γ',Γ') is equivalent to (γ,Γ);*

(3) *The pure states of the C^*-algebra $C^*(\dot S_c^0(\Omega))$ are the functionals $\ell = \ell_{x_0,\pi_0,v_0}$ of the form:*

$$\ell(\sigma) = (\sigma(x_0,\dot\pi_0)v_0, v_0)_{\mathcal{H}_{\pi_0}}, \quad \sigma \in C^*(\dot S_c^0(\Omega)),$$

where $x_0 \in \Omega$, $\pi_0 \in \widehat G$ and $v_0 \in \mathcal{H}_{\pi_0}$ is a unit vector. The states $\ell = \ell_{x_0,\pi_0,v_0}$ where $x_0 \in \Omega$, $\pi_0 \in \widehat G$ and $v_0 \in \mathcal{H}_{\pi_0}^\infty$ is a smooth unit vector, form a dense subset of the set of states of $C^*(\dot S_c^0(\Omega))$.

Proof. The proof of Proposition 5.17 is a simple modification of the proof of Proposition 5.15; indeed, it suffices to replace the characterisation of the spectrum of $C^*(\tilde S^0)$ with the one of $C^*(\dot S_c^0(\Omega))$ given in Proposition 5.13, using an approximate unity of $C^*(\dot S_c^0(\Omega))$ instead of the unity of $C^*(\tilde S^0)$. It is left to the reader. □

We observe that ℓ_{x_0,π_0,v_0} corresponds to $\gamma(x,\dot\pi) = \delta_{x_0}(x) \otimes \delta_{\dot\pi_0}(\dot\pi)$ and $\Gamma(x_0,\dot\pi_0) = v_0 \otimes v_0^*$.

6. Defect measures

In this section, we state and prove our main results, that is, the existence of defect measures. We also give examples of such measures and prove the consistency of our description.

6.1. Main result

The microlocal defect measures that we are going to define are elements in $\mathcal{M}_1^+(\Omega\times\Sigma_1)$, see Definitions 2.12 and 5.14 and the shorthand (5.8).

Theorem 6.1. *Let Ω be a non-empty open set of G. Let (u_k) be a sequence in $L^2(\Omega,\text{loc})$ and $u\in L^2(\Omega,\text{loc})$. We assume that $u_k\rightharpoonup_{k\to\infty} u$ a.e. in $L^2(\Omega,\text{loc})$. Then there exist a subsequence $(u_{k(j)})_{j\in\mathbb{N}}$ of (u_k) and a positive measure $(\gamma,\Gamma)\in\mathcal{M}_1^+(\Omega\times\Sigma_1)$ such that for any $A\in\Psi^0_{\text{cl}}(\Omega)$, we have the convergence*

$$\lim_{j\to\infty}\big(A(u_{k(j)}-u),(u_{k(j)}-u)\big)_{L^2(\Omega)}$$
$$=\int_{\Omega\times\Sigma_1}\text{tr}\,\left(\text{princ}_0(A)(x,\dot\pi)\,\Gamma(x,\dot\pi)\right)d\gamma(x,\dot\pi)\,.$$

Moreover, once the subsequence $(k(j))$ is fixed, the positive trace-class-valued measure (Γ,γ) is unique up to equivalence.

Definition 6.2. With the notation of Theorem 6.1, a sequence (u_k) is *pure* when the subsequence is trivial, *i.e.* $k(j)=j$. In this case, the equivalence class $\Gamma d\gamma$ is called the microlocal defect measure, or *MDM*, of (u_k).

The definition of pure sequence follows the vocabulary set in [32]; it bears no relation with pure states.

The main step in the proof of Theorem 6.1 is the following property:

Lemma 6.3. *Let Ω be an open set of G. Let (u_k) be a sequence in $L^2(\Omega,\text{loc})$ satisfying $u_k\rightharpoonup_{k\to\infty}0$ in $L^2(\Omega,\text{loc})$. Let $\chi\in\mathcal{D}(\Omega)$. We fix a positive Rockland operator $\mathcal{R}$ and a function $\psi\in C^\infty(\mathbb{R})$ such that $\psi\equiv 0$ on a neighbourhood of 0 and $\psi\equiv 1$ on a neighbourhood of $+\infty$. For any $\sigma\in\dot S^0_c(\Omega)$, we set:*

$$v_k^{(\sigma,\chi)}:=\left(\text{Op}(\sigma\psi(\pi(\mathcal{R})))(\chi u_k),(\chi u_k)\right)_{L^2(\Omega)}\,.$$

1. *The sequence $(v_k^{(\sigma,\chi)})_{k\in\mathbb{N}}$ is bounded;*
2. *We can extract a subsequence $(k_j)_{j\in\mathbb{N}}$ such that the sequence $(v_{k_j}^{(\sigma,\chi)})_{j\in\mathbb{N}}$ has a finite limit in $\mathbb{C}$;*
3. *If (k_j) is as in Part (2), then the limit is independent of ψ and $\mathcal{R}$ and it is the same with $\text{Op}(\psi(\pi(\mathcal{R})\sigma)$ or with $\text{Op}(\psi(\pi(\mathcal{R})\sigma\psi(\pi(\mathcal{R}))$ instead of $\text{Op}(\sigma\psi(\pi(\mathcal{R}))$;*
4. *If (k_j) is as in Part (2), then the sequence $(v_{k_j}^{(\sigma^*,\bar\chi)})_{j\in\mathbb{N}}$ has also a finite limit and*
$$\lim_{j\to\infty}v_{k_j}^{(\sigma^*,\bar\chi)}=\overline{\lim_{j\to\infty}v_{k_j}^{(\sigma,\chi)}};$$
5. *If σ is of the form $\sigma=\tau^*\tau$ with $\tau\in\dot S^0_c(\Omega)$ then any limit obtained as in Part (2) is non-negative.*

Proof of Lemma 6.3. As $\mathrm{Op}(\sigma\psi(\pi(\mathcal{R})))$ is bounded on $L^2(\Omega)$ and

$$\sup_{k'\in\mathbb{N}} \|\chi u_{k'}\|_{L^2(\Omega)} < \infty,$$

the sequence $(v_k^{(\sigma,\chi)})$ is bounded by

$$\left|v_k^{(\sigma,\chi)}\right| \leq \|\mathrm{Op}(\sigma\psi(\pi(\mathcal{R}))\|_{\mathscr{L}(L^2(\Omega))} \sup_{k'\in\mathbb{N}} \|\chi u_{k'}\|^2_{L^2(\Omega)}. \tag{6.1}$$

This proves Part (1) and thus Part (2).

By Proposition 4.14 and its proof together with the properties of the pseudo-differential calculus,

$$\begin{aligned}\mathrm{Op}(\sigma^*\psi(\pi(\mathcal{R}))) &= \mathrm{Op}(\psi(\pi(\mathcal{R}))\sigma^*) \bmod \Psi^{-\infty}\\ &= \mathrm{Op}(\tilde{\sigma}\psi(\pi(\mathcal{R})))^* + E,\end{aligned}$$

where E is an error term in Ψ^{-1}. Using Rellich's theorem as in Theorem 4.24 and its proof shows Part (4). Similar arguments show Part (3).

If $\sigma = \tau^*\tau$ with $\tau \in \dot{S}_c^0(\Omega)$, by Proposition 4.14 and its proof together with the properties of the pseudo-differential calculus,

$$\begin{aligned}\mathrm{Op}(\sigma\psi(\pi(\mathcal{R}))) &= \mathrm{Op}(\psi(\pi(\mathcal{R}))\sigma\psi(\pi(\mathcal{R}))) \bmod \Psi^{-\infty}\\ &= \mathrm{Op}(\psi(\pi(\mathcal{R}))\tau^*)\mathrm{Op}(\tau\psi(\pi(\mathcal{R}))) \bmod \Psi^{-1}\\ &= \mathrm{Op}(\tau\psi(\pi(\mathcal{R})))^*\mathrm{Op}(\tau\psi(\pi(\mathcal{R}))) + E,\end{aligned}$$

where E is an error term in Ψ^{-1}. Thus we have

$$v_k^{(\sigma,\chi)} = \|\mathrm{Op}(\psi(\pi(\mathcal{R}))\tau)(\chi u_k)\|^2_{L^2(\Omega)} + (E(\chi u_k), \chi u_k)_{L^2(\Omega)}.$$

The first term of the right-hand side is non-negative for all $k \in \mathbb{N}$ whereas the second term tends to 0 as $k \to \infty$ by Theorem 4.24 and its proof. This shows Part (5). □

We can now prove our main result.

Proof of Theorem 6.1. Let Ω be a non-empty open set of G. Let (u_k) be a converging sequence in $L^2(\Omega, \mathrm{loc})$. We may assume that the weak limit is $u = 0$. We fix a positive Rockland operator $\mathcal{R}$ and a function $\psi \in C^\infty(\mathbb{R})$ such that $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on a neighbourhood of $+\infty$.

By Proposition 5.11, there exists a dense and countable $(\mathbb{Q} + i\mathbb{Q})$-subspace V_0 of $\dot{S}_c^0(\Omega)$. By diagonal extraction, we may assume that the subsequence $(k_j)_{j\in\mathbb{N}}$ obtained in Lemma 6.3 is the same for any element of V_0. Let us consider a real-valued function $\chi \in \mathcal{D}(\Omega)$. As $\chi u_k \rightharpoonup_{k\to\infty} 0$ in $L^2(\Omega)$, the sequence $(\|\chi u_k\|_{L^2(\Omega)})_{k\in\mathbb{N}}$

is bounded and we may extract further a converging subsequence still denoted by $(k_j)_{j\in\mathbb{N}}$. We set

$$s_\chi := \lim_{j\to+\infty} \|\chi u_{k_j}\|^2_{L^2(\Omega)} \quad \text{and when } s_\chi \neq 0, \quad \ell_\chi(\sigma) := \frac{1}{s_\chi} \lim_{j\to\infty} v_{k_j}^{(\sigma,\chi)}, \quad \sigma \in V_0.$$

Using the density of V_0 and the proof of Part (1) of Lemma 6.3, we extend ℓ_χ to a continuous linear functional on $\dot{S}^0_c(\Omega)$ satisfying

$$\forall \sigma \in \dot{S}^0_c(\Omega) \qquad |\ell_\chi(\sigma)| \lesssim \|\sigma\|_{\dot{S}^0(\Omega),a,b,c},$$

having kept the same notation for ℓ and its extension.

Note that we can construct the subspace V_0 of $\dot{S}^0_c(\Omega)$ as follows. We consider V a dense and countable $(\mathbb{Q}+i\mathbb{Q})$-subspace of $\tilde{S}^0$ and V_1 a dense and countable $(\mathbb{Q}+i\mathbb{Q})$-subspace of $\mathcal{D}(\Omega)$. The tensor product of V and V_1 yields V_0, the set of symbols which are finite linear combinations over $(\mathbb{Q}+i\mathbb{Q})$ of $\phi(x)\sigma(\dot\pi)$ with $\phi \in V_1$, $\sigma \in V_0$. Then V_0 is a dense countable subset of $\dot{S}^0_c(\Omega)$. The proof of Proposition 5.11 shows that V_0 is also dense in the Banach space $C^*(\dot{S}^0_c(\Omega))$ whose norm satisfies

$$\sup_{(x,\dot\pi)\in\Omega\times\Sigma_1} \|\sigma(x,\dot\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} = \inf\left\{\sum_i \sup_{\dot\pi\in\Sigma_1} \|\tau_i\|_{\mathscr{L}(\mathcal{H}_\pi)} \sup_{x\in\Omega} |f_i(x)| : \sigma = \sum_i f_i \tau_i\right\}.$$

If the symbol σ is of the form $f(x)\tau(\dot\pi)$, with $f \in V_1$, $\tau \in V$, then $\sigma \in \dot{S}^0_c(\Omega)$ and

$$\begin{aligned} v_k^{(\sigma,\chi)} &= \big(\mathrm{Op}(\tau\psi(\pi(\mathcal{R})))\chi u_k,\ \bar f \chi u_k\big)_{L^2(\Omega)} \\ \left|v_k^{(\sigma,\chi)}\right| &\le \|\mathrm{Op}(\tau\psi(\pi(\mathcal{R})))\|_{\mathscr{L}(L^2(G))} \|\chi u_k\|_{L^2(\Omega)} \|\bar f \chi u_k\|_{L^2(\Omega)} \\ &\le \|\tau\|_{L^\infty(\widehat{G})} \|\psi\|_{L^\infty(\mathbb{R})} \|\chi u_k\|^2_{L^2(\Omega)} \|f\|_{L^\infty(\Omega)}, \end{aligned}$$

thus,

$$\left|\ell_\chi(\sigma)\right| \le \|\tau\|_{L^\infty(\widehat{G})} \|f\|_{L^\infty(\Omega)} = \sup_{(x,\dot\pi)\in\Omega\times\Sigma_1} \|\sigma(x,\dot\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}.$$

Hence ℓ_χ admits a unique continuous extension to a linear functional of $C^*(\dot{S}^0_c(\Omega))$. It is not zero since $\ell_\chi(f\mathrm{I}) = 1$ for a function $f \in \mathcal{D}(\Omega)$ such that $f = 1$ on $\mathrm{supp}\,(\chi)$. Hence, ℓ is a non-zero continuous linear functional on $C^*(\dot{S}^0_c(\Omega))$ which is positive since Lemma 6.3 implies that $\ell(\tau^*\tau) \ge 0$ holds for all $\tau \in C^*(\dot{S}^0_c(\Omega))$. In other words, ℓ_χ is a state of the C^*-algebra $C^*(\dot{S}^0(\Omega))$ and therefore corresponds to a measure $(\gamma_\chi, \Gamma_\chi) \in \mathcal{M}_1^+(\Omega\times\Sigma_1)$ as in Proposition 5.17. This measure is unique up to equivalence. Furthermore, one easily checks that it is supported in x in $\mathrm{supp}\,(\chi)$.

We now consider a sequence of functions $(\chi_{j'})_{j\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ which is real-valued and such that $\operatorname{supp}(\chi_{j'}) \subset \{\chi_{j'+1} = 1\}$ and $\cup_{j'\in\mathbb{N}}\{\chi_{j'} = 1\} = \Omega$. By another diagonal extraction, we may assume that the subsequence $(k_j)_{j\in\mathbb{N}}$ taken above is the same for each $\chi_{j'}$. Furthermore, we may assume $s_{\chi_{j'}} \neq 0$ for every $j' \in \mathbb{N}$. Indeed, if this is not possible, then the sequence (u_k) admits a subsequence which is locally converging in L^2-norm and therefore admits the trivial MDM 0.

We denote by $\Omega_{j'}$ the interior of $\{\chi_{j'+1} = 1\}$. Let us show that if $j'_2 < j'_1 < j'$ then the restrictions of the states $\ell_{\chi_{j'}}$ and $\ell_{\chi_{j'_1}}$ of the C^*-algebras $C^*(\dot{S}^0(\Omega_{j'}))$ and $C^*(\dot{S}^0(\Omega_{j'_1}))$ to $C^*(\dot{S}^0(\Omega_{j'_2}))$ coincide up to the normalising constants $s_{\chi_{j'}}, s_{\chi_{j'_1}}$:

$$i.e. \qquad s_{\chi_{j'}}\ell_{\chi_{j'}}\Big|_{C^*(\dot{S}^0(\Omega_{j'_2}))} = s_{\chi_{j'_1}}\ell_{\chi_{j'_1}}\Big|_{C^*(\dot{S}^0(\Omega_{j'_2}))}. \tag{6.2}$$

We have for $j'_2 < j'_1 < j'$ and a symbol $\sigma \in \dot{S}^0_c(\Omega)$ which is x-supported in $\{\chi_{j'_2} = 1\}$

$$v_k^{\sigma,\chi_{j'}} = v_k^{\sigma,\chi_{j'_1}} + \big(\mathrm{Op}(\sigma\psi(\pi(\mathcal{R})))(\chi_{j'} - \chi_{j'_1})u_k\big), \big(\chi_{j'_1}u_k\big)\big)_{L^2(\Omega)}.$$

The integral kernel $K(x,y)$ of $\mathrm{Op}(\sigma\psi(\pi(\mathcal{R}))) \in \Psi^0$ is smooth (even Schwartz) away from the diagonal and x-supported in $\{\chi_{j'_2} = 1\}$. Since $\chi_{j'} - \chi_{j'_1}$ vanishes on the compact set $\{\chi_{j'_1} = 1\}$ which is a neighbourhood of $\{\chi_{j'_2} = 1\}$, the integral kernel of $\mathrm{Op}(\sigma\psi(\pi(\mathcal{R})))(\chi_{j'} - \chi_{j'_1})$ is smooth and compactly supported in $\{\chi_{j'_2} = 1\} \times \{\chi_{j'} - \chi_{j'_1} = 0\}$ in (x,y). Theorem 4.24 and its proof imply

$$\left\|\mathrm{Op}(\sigma\psi(\pi(\mathcal{R})))(\chi_{j'} - \chi_{j'_1})u_k\right\|_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0.$$

This yields (6.2). Therefore, the restriction of $(s_{\chi_{j'}}\gamma_{\chi_{j'}}, \Gamma_{\chi_{j'}})$ to $\Omega_{j'_2} \times \Sigma_1$ coincides with the restriction of $(s_{\chi_{j'_1}}\gamma_{\chi_{j'_1}}, \Gamma_{\chi_{j'_1}})$ to $\Omega_{j'_2} \times \Sigma_1$. This defines a unique measure $(\gamma, \Gamma) \in \mathcal{M}_1^+(\Omega\times\Sigma_1)$ such that for all $j' > j'_2+1$ and $\sigma \in C^*(\dot{S}^0_c(\Omega))$ x-supported in $\{\chi_{j'_2} = 1\}$ we have

$$s_{\chi_{j'}}\ell_{\chi_{j'}}(\sigma) = \int_{\Omega\times\Sigma_1} \mathrm{tr}\,(\sigma(x,\dot{\pi})\Gamma(x,\dot{\pi}))\,d\gamma(x,\dot{\pi}).$$

Let $A \in \Psi^0_{\mathrm{cl}}(\Omega)$. Let $j'_2 \in \mathbb{N}$ be such that the integral kernel of A is supported in $\{\chi_{j'_2} = 1\} \times \{\chi_{j'_2} = 1\}$. We set $j' = j'_2 + 2$ and $\sigma_0 := \mathrm{princ}_0(A) \in S^0_c(\Omega)$. Then σ_0 is x-supported in $\{\chi_{j'_2} = 1\}$ and

$$(Au_k, u_k)_{L^2(\Omega)} = v_k^{\sigma,\chi_{j'}} + (Bu_k, u_k)_{L^2(\Omega)},$$

with $B \in \Psi_{cl}^{-1}(\Omega)$. By Theorem 4.24, $(Bu_k, u_k)_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0$. Therefore,

$$\lim_{j\to\infty} (Au_{k_j}, u_{k_j})_{L^2(\Omega)} = s_{\chi_{j'}} \ell_{\chi_{j'}}(\sigma) = \int_{\Omega\times\Sigma_1} \operatorname{tr}\, (\sigma(x,\dot\pi)\Gamma(x,\dot\pi))\, d\gamma(x,\dot\pi).$$

This concludes the proof of Theorem 6.1. □

6.2. Example: spatial concentration

In this section, we study the example of a sequence of functions whose mass is concentrating at 0:

Proposition 6.4. *Let $u_1 \in \mathcal{S}(G)$. We define*

$$u_k(x) = k^{\frac{Q}{2}} u_1(kx), \quad x \in G,\ k \in \mathbb{N}_0\,.$$

Then $u_k \underset{k\to\infty}{\rightharpoonup} 0$ in $L^2(G, \text{loc})$ and this sequence is pure. Its MDM *is given by*

$$\gamma(x,\dot\pi) = \delta_{x=0} \otimes \varsigma(\pi), \quad \Gamma(\dot\pi) = \int_{r=0}^{\infty} \hat u_1(r\cdot\pi)\hat u_1(r\cdot\pi)^* r^{Q-1} dr.$$

Note that $\Gamma(\dot\pi) \geq 0$ and that the polar decomposition in Section 2.3 yields

$$\begin{aligned}\int_{\Sigma_1} \operatorname{tr}\Gamma(\dot\pi)\, d\varsigma(\pi) &= \int_{\Sigma_1}\int_{r=0}^{\infty} \|\hat u_1(r\cdot\pi)\|^2_{HS(\mathcal{H}_\pi)}\, r^{Q-1} dr\, d\varsigma(\pi)\\ &= \int_{\widehat G} \|\hat u_1(\pi)\|^2_{HS(\mathcal{H}_\pi)} d\mu(\pi) = \|u_1\|_{L^2(G)} < \infty.\end{aligned}$$

One easily checks that Γ on $\widehat G$ is a $(-Q)$-homogeneous field of operators.

Proof of Proposition 6.4. By Rellich's theorem (*cf.* Theorem 4.24 and its proof), we may assume that $A = \operatorname{Op}(\tilde\sigma_0)$ where $\tilde\sigma_0 = \sigma_0\psi(\pi(\mathcal{R}))$. Using (2.1), the group Fourier transform of u_k is

$$\widehat{u}_k(\pi) = k^{-\frac{Q}{2}} \widehat{u}_1\big(k^{-1}\cdot\pi\big).$$

Hence we have:

$$\begin{aligned}(Au_k, u_k) &= \int_G\int_{\widehat G} \operatorname{tr}\, (\pi(x)\tilde\sigma_0(x,\pi)\widehat{u}_k(\pi))\, d\mu(\pi)\bar u_k(x)dx\\ &= \int_G\int_{\widehat G} \operatorname{tr}\, \Big(\pi(x)\tilde\sigma_0(x,\pi)\widehat{u}_1(k^{-1}\cdot\pi)\Big)\, d\mu(\pi)\bar u_1(kx)dx\\ &= \int_G\int_{\widehat G} \operatorname{tr}\, \Big(\pi'(x')\tilde\sigma_0(k^{-1}x', k\cdot\pi')\widehat{u}_1(\pi')\Big)\, d\mu(\pi')\bar u_1(x')dx',\end{aligned}$$

after the change of variables $x' = kx$ and $\pi' = k^{-1} \cdot \pi$, using (2.1) and (2.9). We are going to prove that the following expression tends to 0 as $k \to \infty$:

$$\begin{aligned}
&(Au_k, u_k) - \int_{\widehat{G}} \operatorname{tr}\left(\widehat{u}_1(\pi')^* \sigma_0(0, \pi') \widehat{u}_1(\pi')\right) d\mu(\pi') \\
&= \int_G \int_{\widehat{G}} \operatorname{tr}\left(\pi(x)\left(\tilde{\sigma}_0(k^{-1}x, k\cdot\pi) - \sigma_0(0, k\cdot\pi)\right)\widehat{u}_1(\pi)\right) d\mu(\pi) \bar{u}_1(x) dx \\
&= T_1 + T_2,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \int_G \int_{\widehat{G}} \operatorname{tr}\left(\pi(x)\left(\sigma_0(k^{-1}x, k\cdot\pi) - \sigma_0(0, k\cdot\pi)\right)\psi(k\cdot\pi(\mathcal{R}))\,\widehat{u}_1(\pi)\right) \\
&\quad \times d\mu(\pi)\bar{u}_1(x)dx, \\
T_2 &= \int_G \int_{\widehat{G}} \operatorname{tr}\left(\pi(x)\sigma_0(0,\pi)(1-\psi)(k\cdot\pi(\mathcal{R}))\,\widehat{u}_1(\pi)\right) d\mu(\pi)\bar{u}_1(x)dx \\
&= \int_{\widehat{G}} \operatorname{tr}\left(\sigma_0(0,\pi)(1-\psi)(k\cdot\pi(\mathcal{R}))\,\widehat{u}_1(\pi)\widehat{u}_1(\pi)^*\right) d\mu(\pi),
\end{aligned}$$

and the function ψ is chosen as usual, $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on (Λ, ∞) for some $\Lambda > 0$.

For T_2, we use that by (2.17), for each $\pi \in \widehat{G}$, there exists $k_\pi \in \mathbb{N}$ such that $(1-\psi)(k\cdot\pi(\mathcal{R})) = 0$ for all $k \geq k_\pi$. Hence for $k \geq k_\pi$, we also have

$$\operatorname{tr}\left(\sigma_0(0,\pi)(1-\psi)(k\cdot\pi(\mathcal{R}))\,\widehat{u}_1(\pi)\widehat{u}_1(\pi)^*\right) = 0.$$

Since we have

$$\left|\operatorname{tr}\left((1-\psi)(k\cdot\pi(\mathcal{R}))\sigma_0(0,\pi)\,\widehat{u}_1(\pi)\widehat{u}_1(\pi)^*\right)\right| \leq C_{\psi,\sigma_0}\|\widehat{u}_1(\pi)\|^2_{HS(\mathcal{H}_\pi)}$$

with $C_{\psi,\sigma_0} := \sup_{\lambda>0}|1-\psi(\lambda)|\ \sup_{\pi'\in\widehat{G}}\|\sigma_0(0,\pi')\|_{\mathscr{L}(\mathcal{H}_{\pi'})} \in (0,\infty)$ and

$$\int_{\widehat{G}} \|\widehat{u}_1(\pi)\|^2_{HS(\mathcal{H}_\pi)} d\mu(\pi) = \|u_1\|_2^2 < \infty,$$

the Lebesgue dominated convergence theorem yields that T_2 tends to 0 as $k \to \infty$.

Let us now study T_1. The mean value theorem stated in Lemma 2.2 extends to Banach value functions. Hence fixing a homogeneous quasi-norm $|\cdot|$, there exists a constant $C > 0$ such that for any $\sigma \in \dot{S}^0$, $x \in G$ and $r > 0$, we have

$$\sup_{\pi\in\widehat{G}}\|\sigma_0(rx,\pi) - \sigma_0(0,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C\sum_{j=0}^{n}|rx|^{\nu_j}\sup_{y\in G,\pi\in\widehat{G}}\|X_j\sigma_0(y,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}.$$

We obtain

$$\left|\operatorname{tr}\left(\pi(x)\,\psi(k\cdot\pi(\mathcal{R}))\left(\sigma_0(k^{-1}x,k\cdot\pi)-\sigma_0(0,k\cdot\pi)\right)\widehat{u}_1(\pi)\right)\bar{u}_1(x)\right|$$
$$\leq CC'_{\psi,\sigma_0}|u_1(x)|\sum_{j=0}^{n}\left|k^{-1}x\right|^{\nu_j}\operatorname{tr}|\widehat{u}_1(\pi)|$$

where $C'_{\psi,\sigma_0}:=\sup_{\lambda>0}|\psi(\lambda)|\sup_{y\in G,\pi\in\widehat{G},j=1,\dots,n}\|X_j\sigma_0(y,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}\in(0,\infty)$. Since $u_1\in\mathcal{S}(G)$, we have

$$\int_G|u_1(x)|\sum_{j=0}^{n}|x|^{\nu_j}dx<\infty\quad\text{and}\quad\int_{\widehat{G}}\operatorname{tr}|\widehat{u}_1(\pi)|\,d\mu(\pi)<\infty,$$

and the Lebesgue dominated convergence theorem yields again that T_1 tends to 0 as $k\to\infty$.

We have shown that T_1 and T_2 tend to 0 as $k\to\infty$ and this implies

$$(Au_k,u_k)_{L^2(G)}\underset{k\to\infty}{\longrightarrow}\int_{\widehat{G}}\operatorname{tr}\left(\widehat{u}_1(\pi')^*\sigma_0(0,\pi')\widehat{u}_1(\pi')\right)d\mu(\pi'),$$

which gives the result by use of the polar decomposition for the Plancherel measure, see Section 2.3. □

6.3. Example: oscillations from square integrable representations

We now study another example, which may be viewed as a spectral or dual concentration. We consider a graded group G which admits an infinite dimensional (unitary irreducible) representation π_0 which is square integrable modulo its centre. We also fix a smooth unit vector $v_0\in\mathcal{H}^\infty_{\pi_0}$ and consider the associated matrix coefficient:

$$e_0(x):=(\pi_0(x)v_0,v_0)_{\mathcal{H}_{\pi_0}},\quad x\in G.$$

We may assume that the basis $\{X_1,\dots,X_n\}$ of the Lie algebra $\mathfrak{g}$ has been chosen so that a subset $\{X_{j_1},\dots,X_{j_{n_\mathfrak{z}}}\}$, form a basis for the centre $\mathfrak{z}$ of $\mathfrak{g}$. Therefore we can write any element x as

$$x=\exp_G(x_1X_1+\dots+x_nX_n)=x'x_\mathfrak{z}=x_\mathfrak{z}x',$$

where $x_\mathfrak{z}=\exp_G(x_{j_1}X_{j_1}+\dots+x_{n_\mathfrak{z}}X_{n_\mathfrak{z}})$ and $x'=\exp_G\left(\sum_{j\notin\{j_1,\dots,j_{n_\mathfrak{z}}\}}x_jX_j\right)$. Naturally, we identify the centre of the Lie algebra $\mathfrak{z}$ and the centre of the group $Z:=\exp_G\mathfrak{z}$ with $\mathbb{R}^{n_\mathfrak{z}}$. Note that we still consider anisotropic dilations in those directions. The quotient group $G':=G/Z$ is also graded and we denote by Q' its homogeneous dimensions, also given by

$$Q':=\sum_{j\notin\{j_1,\dots,j_{n_\mathfrak{z}}\}}\upsilon_j.$$

Finally, we denote by d_{π_0} the formal degree of π_0 for which we have for any $v_1, w_1, v_2, w_2 \in \mathcal{H}_{\pi_0}$:

$$d_{\pi_0} \int_{G/Z} (\pi_0(x')v_1, w_1)_{\mathcal{H}_{\pi_0}} \overline{(\pi_0(x')v_2, w_2)_{\mathcal{H}_{\pi_0}}} dx' = (v_1, v_2)_{\mathcal{H}_{\pi_0}} \overline{(w_1, w_2)_{\mathcal{H}_{\pi_0}}}, \quad (6.3)$$

see [18, page 169 and Theorem 4.5.11].

Proposition 6.5. *Let $u_0 \in \mathcal{S}(\mathbb{R}^{n_3})$. For each $k \in \mathbb{N}$, let $u_k : G \to \mathbb{C}$ be the square integrable function given by*

$$u_k(x) = k^{\frac{Q'}{2}} e_0(kx) u_0(x_3), \quad x \in G.$$

Then $\|u_k\|_{L^2(G)} = \|e_0\|_{L^2(G')} \|u_0\|_{L^2(Z)} < \infty$ and $u_k \underset{k\to\infty}{\rightharpoonup} 0$ in $L^2(G, \mathrm{loc})$. This sequence is pure and its MDM *is given by*

$$\gamma(x, \dot\pi) = \left(\frac{|u_0(x_3)|^2}{d_{\pi_0}} dx_3 \otimes \delta_{x'=0} \right) \otimes \delta_{\dot\pi = \dot\pi_0},$$

and $\Gamma(\dot\pi_0) = v_0 \otimes v_0^*$ being the orthogonal projection on $\mathbb{C}v_0$.

The Schwartz function on the centre is needed to contain the oscillations, as in the abelian case. Indeed, on the one hand, on the centre Z of the group, π_0 coincides with the character $e^{i\lambda_0 \cdot}$, *i.e.* $\pi_0(x_3) = e^{i\lambda_0 x_3}$ where we identify x_3 with an element of $\mathbb{R}^{n_3}$ and where $\lambda_0 x_3$ denotes the standard scalar product of the two elements λ_0 and x_3 of $\mathbb{R}^{n_3}$. Thus for any $x = x'x_3$ in G we have

$$e_0(x) = \big(\pi_0(x'x_3)v_0, v_0\big)_{\mathcal{H}_{\pi_0}} = e^{i\lambda_0 x_3} \big(\pi_0(x')v_0, v_0\big)_{\mathcal{H}_{\pi_0}} = e^{i\lambda_0 x_3} e_0(x').$$

On the other hand, $e_0\big|_{G'} \in \mathcal{S}(G')$. See again [18, page 169 and Theorem 4.5.11].

Before starting the proof, let us describe the more concrete case of the Heisenberg group and the matrix coefficient given by the bounded spherical functions, see, *e.g.*, [5]. More precisely, we realise the Heisenberg group as $\mathbb{H}_1 = \{(x, y, t) \in \mathbb{R}^3\}$ with law

$$(x, y, t)(x', y', t') = \big(x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)\big).$$

Let π_0 be the representation of $\mathbb{H}_1$ determined (up to equivalence) by the fact that it coincides with the character $t \mapsto e^{it}$ on the centre of $\mathbb{H}_1$. For the smooth vector, we choose the ℓ-th Hermite function (with $L^2(\mathbb{R})$-normalisation) if we realise this representation in the Schrödinger model, or equivalently, the (suitably normalised) monomial of degree ℓ in the Bergman-Fock model. In this case, the matrix coefficient is given by

$$e_0(x, y, t) = e^{it} \mathcal{L}_\ell \left(\frac{x^2 + y^2}{2} \right),$$

where $\mathcal{L}_\ell$ is the ℓ-th Laguerre function, that is, $\mathcal{L}_\ell(r) = e^{\frac{-r}{2}} L_\ell(r)$ and L_ℓ is the ℓ-th Laguere polynomial. Note that the e_0 in this particular case is of the form described above.

Proof of Proposition 6.5. First let us show that each function u_k is square integrable:

$$\begin{aligned}\|u_k\|^2_{L^2(G)} &= \int_{G'}\int_{\mathbb{R}^{n_3}} \left| e_0\big(kx'kx_3\big)u_0(x_3)k^{\frac{Q'}{2}}\right|^2 dx_3dx' \\ &= \int_{G'} \left|e_0\big(kx'\big)\right|^2 k^{Q'}dx' \int_{\mathbb{R}^{n_3}} \left|u_0(x_3)\right|^2 dx_3 \\ &= \int_{G'} \left|e_0\big(x''\big)\right|^2 dx'' \int_{\mathbb{R}^{n_3}} \left|u_0\big(x_3\big)\right|^2 dx_3,\end{aligned}$$

having used the change of variable $x'' = kx'$. As the functions e_0 and u_0 are Schwartz on G' and $\mathbb{R}^{n_3}$ respectively, the quantity above is finite, and $u_k \in L^2(G)$ with $\|u_k\|_{L^2(G)} = \|e_0\|_{L^2(G')}\|u_0\|_{L^2(Z)}$.

For any $\phi_1 \in \mathcal{D}(\mathbb{R}^{n_3})$, $\phi_2 \in \mathcal{D}(G')$, we have

$$\begin{aligned}&\int_{G'}\int_{\mathbb{R}^{n_3}} u_k\big(x_3x'\big)\phi_1\big(x_3\big)\phi_2(x')dx_3dx' \\ &= \int_{\mathbb{R}^{n_3}} e^{i\lambda_0(kx_3)}u_0\big(x_3\big)\phi_1(x_3)dx_3 \int_{G'} e_0\big(kx'\big)\phi_2(x')k^{\frac{Q'}{2}}dx'.\end{aligned}$$

By the Riemann-Lebesgue theorem, the integral over $\mathbb{R}^{n_3}$ tends to zero as $k \to \infty$. After the change of variable $x'' = kx'$, the integral over G' becomes

$$\begin{aligned}&\int_{G'} e_0\big(kx'\big)\phi_2\big(x'\big)k^{\frac{Q'}{2}}dx' \\ &= k^{\frac{-Q'}{2}} \int_{G'} e_0\big(x''\big)\phi_2\big(k^{-1}x'\big)dx'' \sim_{k\to\infty} k^{\frac{-Q'}{2}}\phi_2(0)\int_{G'} e_0\big(x''\big)dx'',\end{aligned}$$

thus this integral tends to zero as $k \to \infty$. Hence $u_k \underset{k\to\infty}{\rightharpoonup} 0$ in $L^2(G, \mathrm{loc})$.

Let us now compute the MDM of (u^k). Let $A = \mathrm{Op}(\sigma) \in \Psi^0_{\mathrm{cl}}(G)$. Let $\chi \in \mathcal{D}(G)$ be real valued and such that the support of the integral kernel of A is in $\{\chi = 1\} \times \{\chi = 1\}$.

$$\begin{aligned}(Au_k, u_k)_{L^2(G)} &= (A(\chi u_k), \chi u_k)_{L^2(G)} \\ &= \int_{\widehat{G}}\int_G \mathrm{tr}\,(\pi(x)\sigma_0(x,\pi)\psi(\pi_0(\mathcal{R}))\pi(\chi u_k))\,(\chi\bar{u}_k)(x)dxd\mu(\pi);\end{aligned}$$

here we understand the double integral over $\widehat{G}$ as in Proposition 2.21, that is, as the limit of the absolutely convergent double integral:

$$\lim_{N\to+\infty}\int_{N\cdot\mathcal{C}}\int_G \mathrm{tr}_N\,(\pi(x)\sigma_0(x,\pi)\psi(\pi_0(\mathcal{R}))\pi(\chi u_k))\,(\chi\bar{u}_k)(x)dxd\mu(\pi),$$

where $\mathcal{C}$ is a compact neighbourhood of $1 \in \widehat{G}$ such that $\cup_{N \in \mathbb{N}} N \cdot \mathcal{C} = \widehat{G}$, and tr_N denotes the trace of the operators projected on the subspace spanned by the first N vectors, having fixed a fundamental sequence of vector fields. Hence we are led to study:

$$\begin{aligned}
&\int_{\widehat{G}} \int_G \mathrm{tr}\ (\pi(x)\sigma(x,\pi)\pi(\chi u_k))\ (\chi \bar{u}_k)(x) dx d\mu(\pi) \\
&= \int_{\widehat{G}} \int_G \int_G \mathrm{tr}\ \left(\pi(x)\sigma(x,\pi)\pi(y)^*\right) (\chi \bar{u}_k)(x)(\chi u_k)(y) dx dy d\mu(\pi),
\end{aligned}$$

having expanded $\pi(\chi u_k)$. This multiple integral is again convergent. Applying the change of variables first $y \mapsto w = y^{-1}x$ and using the properties of the trace, the integral above becomes

$$\begin{aligned}
&\int_{\widehat{G}} \int_G \int_G \mathrm{tr}\ \left(\pi(x)\sigma(x,\pi)\pi\left(wx^{-1}\right)\right) (\chi \bar{u}_k)(x)\ (\chi u_k)\left(xw^{-1}\right) dx dw d\mu(\pi) \\
&= \int_{\widehat{G}} \int_G \int_G \mathrm{tr}\ (\pi(w)\sigma(x,\pi))\ (\chi \bar{u}_k)(x)\ (\chi u_k)\left(xw^{-1}\right) dx dw d\mu(\pi) \\
&= \int_{\widehat{G}} \int_G \int_G \mathrm{tr}\ \left(\pi'(w')\sigma(x, k \cdot \pi')\right) (\chi \bar{u}_k)(x)\ (\chi u_k)\left(x\, k^{-1} w'^{-1}\right) dx dw' d\mu\left(\pi'\right),
\end{aligned}$$

after the change of variable $(\pi, w) \mapsto (\pi', w') = (k^{-1} \cdot \pi, kw)$, whose Jacobian is 1 by (2.1) and (2.9). Let us write

$$\begin{aligned}
&(\chi \bar{u}_k)(x)\ (\chi u_k)\left(x\, k^{-1} w'^{-1}\right) \\
&= k^{Q'} \chi(x) \chi\left(x\, k^{-1} w'^{-1}\right) \bar{e}_0(kx) e_0\left(kx\ w'^{-1}\right) \bar{u}_0(x_{\mathfrak{z}}) u_0\left(\left(x\, k^{-1} w'^{-1}\right)_{\mathfrak{z}}\right) \\
&= k^{Q'} |\chi|^2(x) |u_0|^2\left(x_{\mathfrak{z}}\right) \bar{e}_0(kx) e_0\left(kx\ w'^{-1}\right) + \varepsilon_k\left(x, w'\right).
\end{aligned}$$

We claim that for any $\tau \in L^\infty(\widehat{G})$ such that $\mathcal{F}_G^{-1}\tau$ is a compactly supported distribution on G, we have for any $x \in G$

$$\int_{\widehat{G}} \int_G \mathrm{tr}\ \left(\pi(y)\tau(\pi) e_0\left(xy^{-1}\right)\right) dy d\mu(\pi) = (\pi_0(x)\tau(\pi_0) v_0, v_0)_{\mathcal{H}_{\pi_0}}. \tag{6.4}$$

Indeed by the Fourier inversion formula, the limit is equal to

$$\int_G \mathcal{F}_G^{-1}\tau(y)\ e_0\left(xy^{-1}\right) dy$$

interpreted in the sense of a compactly supported distribution at a smooth bounded function, and this is equal to the right hand side of (6.4). We can apply this to $\tau = \{\sigma(x,\pi), \pi \in \widehat{G}\}$ since $\mathcal{F}_G^{-1}\sigma(x,\cdot)$ is the convolution kernel of A which is compactly supported (as the integral kernel of A is compctly supported). Hence the claim in (6.4) is proved and we may apply it to obtain:

$$(Au_k, u_k)_{L^2(G)} = T(k) + \tilde{\varepsilon}(k)$$

where

$$T(k) := \int_G k^{Q'} |\chi|^2(x) |u_0|^2\big(x_3\big) \bar{e}_0(kx)(\pi_0(kx)\sigma(x, k\cdot\pi_0)v_0, v_0)_{\mathcal{H}_{\pi_0}} dx,$$
$$\tilde{\varepsilon}(k) := \lim_{R\to+\infty} \int_{(k^{-1}R)\cdot\mathcal{C}} \int_G \int_G \operatorname{tr}\big(\pi'(w')\sigma\big(x, k\cdot\pi'\big)\varepsilon_k\big(x, w'\big)\big)\, dx dw' d\mu\big(\pi'\big).$$

Let us show that $\tilde{\varepsilon}(k)$ tends to zero as $k\to\infty$. It is easy to see that in the Sobolev space $L^2_s(G)$ for any $s > Q/2$, we have the uniform convergence:

$$\sup_{x\in G} \|\varepsilon_k(x,\cdot)\|_{L^2_s(G)} \longrightarrow_{k\to\infty} 0.$$

From Section 2.6, we have

$$\forall \phi\in L^2_s(G),\ \tau\in L^\infty(\widehat{G}) \quad \left|\int_{\widehat{G}} \operatorname{tr}\big(\tau(\pi)\widehat{\phi}(\pi)\big)\, d\mu(\pi)\right| \le C_s \|\tau\|_{L^\infty(\widehat{G})} \|\phi\|_{L^2_s(G)}.$$

From the two properties above, we obtain easily

$$|\tilde{\varepsilon}(k)| \le \int_G \|\sigma(x,\cdot)\|_{L^\infty(\widehat{G})} \|\varepsilon_k(x,\cdot)\|_{L^2_s(G)} dx \longrightarrow_{k\to\infty} 0,$$

as the integrand has compact support in $x\in G$.

For $T(k)$, as we have

$$\begin{aligned}\bar{e}_0(kx)\pi_0(kx) &= e^{-i\lambda_0(kx_3)}\bar{e}_0\big(kx'\big)\, e^{i\lambda_0(kx_3)}\pi_0\big(kx'\big)\\ &= \bar{e}_0\big(kx'\big)\pi_0\big(kx'\big),\end{aligned}$$

the change of variable $x'' = kx'$ whose Jacobian is $k^{-Q'}$ yields:

$$\begin{aligned}T(k) &= \int_{G'} \int_{\mathbb{R}^{n_3}} k^{Q'} |\chi|^2(x) |u_0|^2\big(x_3\big)\bar{e}_0\big(kx'\big)\\ &\quad \times \Big(\pi_0\big(kx'\big)\sigma\big(x_3 x', k\cdot\pi_0\big)v_0, v_0\Big)_{\mathcal{H}_{\pi_0}} dx_3 dx'\\ &= \int_{G'} \int_{\mathbb{R}^{n_3}} |\chi|^2\big(x_3 k^{-1} x''\big) |u_0|^2\big(x_3\big)\bar{e}_0\big(x''\big)\\ &\quad \times \Big(\pi_0\big(x''\big)\sigma\big(x_3 k^{-1} x'', k\cdot\pi_0\big)v_0, v_0\Big)_{\mathcal{H}_{\pi_0}} dx_3 dx''.\end{aligned}$$

We claim that, by the Lebesgue theorem, this converges towards

$$\begin{aligned} T(k) \underset{k\to\infty}{\longrightarrow} \int_{G'} \int_{\mathbb{R}^{n_3}} |\chi|^2(x_3)|u_0|^2(x_3)\bar{e}_0(x'') \\ \times \Big(\pi_0(x'')\sigma_0(x_3,\pi_0)v_0, v_0\Big)_{\mathcal{H}_{\pi_0}} dx_3 dx'', \end{aligned} \tag{6.5}$$

where $\sigma_0 = \text{princ}(A)$. Indeed we fix a positive Rockland operator $\mathcal{R}$ (of homogeneous degree ν) and $\psi \in C^\infty(\mathbb{R})$ a smooth function such that $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on $[\Lambda, \infty)$. We know that the symbol

$$\rho := \sigma - \sigma_0 \psi(\pi(\mathcal{R})),$$

is in S^{m_1} with $m_1 < 0$. We may write

$$(\rho(x, k\cdot\pi_0)v_0, v_0)_{\mathcal{H}_{\pi_0}} = \big(\tilde{\rho}_{k,x} v_k, v_0\big)_{\mathcal{H}_{\pi_0}}$$

where $\tilde{\rho}_{k,x} = \rho(x, k\cdot\pi_0)k\cdot\pi_0(\mathrm{I}+\mathcal{R})^{-\frac{m_1}{\nu}}$ and $v_k := k\cdot\pi_0(\mathrm{I}+\mathcal{R})^{\frac{m_1}{\nu}}v_0$. The operator $\tilde{\rho}_{k,x}$ is uniformly bounded:

$$\|\tilde{\rho}_{k,x}\|_{\mathscr{L}(\mathcal{H}_{\pi_0})} \le \sup_{x\in G, \pi_1\in\widehat{G}} \left\|\rho(x,\pi_1)\pi_1(\mathrm{I}+\mathcal{R})^{-\frac{m_1}{\nu}}\right\|_{\mathscr{L}(\mathcal{H}_{\pi_0})} \le \|\rho\|_{S^{m_1},0,0,m_1},$$

and so is the vector v_k:

$$\|v_k\|_{\mathcal{H}_0} \le \left\|k\cdot\pi_0(\mathrm{I}+\mathcal{R})^{\frac{m_1}{\nu}}\right\|_{\mathscr{L}(\mathcal{H}_{\pi_0})} \|v_0\|_{\mathcal{H}_0} \le \sup_{\lambda>\lambda_{\min}(\pi_0)} (1+k^\nu\lambda)^{\frac{m_1}{\nu}} \le 1.$$

Here $\lambda_{\min}(\pi_0)$ is the smallest eigenvalue of $\pi_0(\mathcal{R})$, see Lemma 2.10 (2), so $\lambda_{\min}(\pi_0) \in (0,\infty)$ and $\|v_k\|_{\mathcal{H}_0}$ tends to 0 as $k\to\infty$. It is now a routine exercise left to the reader to apply the Lebesgue theorem and obtain the convergence in (6.5).

As A is compactly supported in $\{\chi = 1\}\times\{\chi = 1\}$, we may assume that σ_0 is compactly supported in x, and that this support is included in $\{\chi = 1\}$. Hence we have obtain that the $(Au_k, u_k)_{L^2(G)}$ has the same limit as $T(k)$ and, in view of (6.3), this limit can be rewritten as

$$\frac{1}{d_{\pi_0}} \left(\left(\int_G \text{princ}_0(A)(x_3,\pi_0)\ |u_0(x_3)|^2 dx_3 \right) v_0\,,\ v_0 \right)_{\mathcal{H}_{\pi_0}}. \qquad \square$$

6.4. Example: general oscillations

In Section 6.3, we constructed a pure sequence associated with a square integrable representation. In this section, we generalise the idea to any irreducible representation of G. If the representation is finite dimensional, then it is of dimension 1 (see [18]) and we may proceed as in the Euclidean case. Let us consider π_0 an irreducible representation of infinite dimension. We will replace the properties of square integrability with the general results on representations of nilpotent Lie groups due to Pedersen [46].

Unfortunately, the notation adapted to the presence of dilations is in conflict with the conventional notation for Jordan-Hölder bases. Indeed, our canonical basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, that is, a basis adapted to the gradation (see Section 2.1), is adapted to the Jordan-Hölder sequence:

$$\begin{aligned} &\mathfrak{g} =: \mathfrak{g}_n \subset \mathfrak{g}_{n-1} \subset \ldots \subset \mathfrak{g}_1 \subset \mathfrak{g}_0 := \{0\}, \\ \text{where} \quad &\mathfrak{g}_k := \mathbb{R}X_{n-k+1} \oplus \ldots \oplus \mathbb{R}X_n, \ k = 1, \ldots, n, \end{aligned}$$

except for the order of the labels in the basis; for instance $X_k \in \mathfrak{g}_{n-k+1} \backslash \mathfrak{g}_{n-k}$. We denote by J the set of jump indices of π_0:

$$J := \{1 \leq k \leq n \ : \ \pi_0(X_k) \notin \pi_0(\mathfrak{U}(\mathfrak{g}_{n-k}))\}.$$

We recall that we have denoted in the same manner the representations of G and $\mathfrak{g}$. We observe that the set of jump indices is the same for $r \cdot \pi_0$, for any $r > 0$. We set

$$\mathfrak{g}_J := \oplus_{k\in J}\mathbb{R}X_k \qquad \text{and} \qquad \mathfrak{g}_{J^c} := \oplus_{k\notin J}\mathbb{R}X_k.$$

The natural Haar measures on $\mathfrak{g}_J$ and $\mathfrak{g}_{J^c}$ are $\Pi_{k\in J} dx_j$ and $\Pi_{k\notin J} dx_j$ respectively; we will denote them by dx, or any other letter representing the variable of integration.

For any Schwartz function $\phi \in \mathcal{S}(\mathfrak{g}_J)$ on the vector space $\mathfrak{g}_J$, we define:

$$\tilde{\phi} := \int_{x=(x_k)_{k\in J}\in\mathfrak{g}_J} \phi(x)\, \pi_0\left(\exp\left(\sum_{k\in J} x_j X_j\right)\right) dx.$$

This is a smooth operator on $\mathcal{H}_{\pi_0}$, *i.e.* $\tilde{\phi} \in \mathscr{L}(\mathcal{H}_{\pi_0})_\infty$. Moreover (*cf.* [46]), $\phi \mapsto \tilde{\phi}$ is an isomorphism between the Fréchet spaces $\mathcal{S}(\mathfrak{g}_J)$ and $\mathscr{L}(\mathcal{H}_{\pi_0})_\infty$; its inverse is given by

$$\mathscr{L}(\mathcal{H}_{\pi_0})_\infty \ni A \mapsto f_A \circ \exp\big|_{\mathfrak{g}_J}, \quad \text{where} \quad f_A(x) := \text{tr}\,(\pi_0(x)A)\,, \quad x \in G. \tag{6.6}$$

Any element in $\mathscr{L}(\mathcal{H}_{\pi_0})_\infty$ is trace-class. An example of an element of $\mathscr{L}(\mathcal{H}_{\pi_0})_\infty$ is $v \otimes w^*$ where v and w are two smooth vectors of $\mathcal{H}_{\pi_0}$. For any $\phi \in \mathcal{S}(\mathfrak{g}_J)$ the operator $\tilde{\phi}$ is trace-class with [46]

$$\text{tr}\,\tilde{\phi} = \frac{1}{d_{\pi_0}}\phi(0). \tag{6.7}$$

Here $d_{\pi_0} > 0$ is the computable constant in (6.3). It depends on π_0 and on the choice of Jordan-Hölder basis (but not on ϕ), and generalises the notion of degree for square integrable representations.

We can now state and prove the following generalisation of Proposition 6.5:

Proposition 6.6. *Let π_0 be an irreducible representation of G of infinite dimension. We define its jump set J and the subspaces $\mathfrak{g}_J$, $\mathfrak{g}_{J^c}$ of $\mathfrak{g}$ as above. We set*

$$Q_J := \sum_{k\in J} \upsilon_k.$$

Let $u_0 \in \mathcal{S}(\mathfrak{g}_{J^c})$. Let $A \in \mathscr{L}(\mathcal{H}_{\pi_0})_\infty$ and define f_A as in (6.6). *For each $k \in \mathbb{N}$, let $u_k : G \to \mathbb{C}$ be the function given by*

$$u_k(x) = k^{\frac{Q_J}{2}} f_A(kx)\, u_0\big(x_{J^c}\big).$$

Then $\|u_k\|_{L^2} = d_{\pi_0}^{-1/2}\|A\|_{HS(\mathcal{H}_{\pi_0})}\|u_0\|_{L^2(\mathfrak{g}_{J^c})}$ and $u_k \underset{k\to\infty}{\rightharpoonup} 0$ in $L^2(G, \mathrm{loc})$. This sequence is pure and its MDM *is given by*

$$\gamma(x,\dot\pi) = \left(\frac{|u_0(x_{J^c})|^2}{d_{\pi_0}} dx_{J^c} \otimes \delta_0(x_J)\right) \otimes \delta_{\dot\pi_0}(\dot\pi), \qquad \Gamma(\dot\pi_0) = AA^*.$$

In the statement, we have used the following notation:

$$\text{if } x = \exp\left(\sum_{k=1}^n x_j X_j\right) \in G, \text{ then } x_{J^c} := (x_j)_{j\notin J^c} \in \mathfrak{g}_{J^c},\ x_J := (x_j)_{j\in J} \in \mathfrak{g}_J.$$

In the proof, we will use the following properties:

Lemma 6.7. *Let π_0 be an irreducible representation of infinite dimension of G. We define as above its jump set J, the subspaces $\mathfrak{g}_J$, $\mathfrak{g}_{J^c}$.*

(1) *There exists a linear function $F : \mathfrak{g}_J \times \mathfrak{g}_{J^c} \to \mathfrak{g}_J$ such that*

$$\pi_0(x) = \pi_0\Big(\exp\big(X_J + F(X_J, X_{J^c})\big)\Big),$$

where $x = \exp(\sum_{k=1}^n x_j X_j) \in G$, $X_J = \sum_{k\in J} x_j X_j \in \mathfrak{g}_J$, $X_{J^c} = \sum_{k\notin J} x_j X_j \in \mathfrak{g}_{J^c}$.
Furthermore, for any X_{J^c}, the change of variable $X_J \mapsto X_J' = X_J + F(X_J, X_{J^c})$ is a diffeomorphism of $\mathfrak{g}_J$ with determinant 1;

(2) *Let $A \in \mathscr{L}(\mathcal{H}_{\pi_0})_\infty$ and define f_A as in* (6.6). *We have*

$$\int_{(x_k)_{k\in J}\in\mathfrak{g}_J} f_A(x)\, \pi_0\left(\exp\left(\sum_{k\in J} x_j X_j\right)\right)^* dx = \frac{1}{d_{\pi_0}} A;$$

(3) *Let $\sigma \in L^\infty(\widehat{G})$ be such that $\mathcal{F}_G^{-1}\sigma$ is a compactly supported distribution on G. Then*

$$\int_G \int_{\widehat{G}} \operatorname{tr}\,(\sigma(\pi)\,\pi(w))\, f_A(w^{-1}x) d\mu(\pi) dw = \operatorname{tr}\,(\kappa(\pi_0)\pi_0(x)A)\,,$$

interpreting the left-hand side as in Proposition 2.21, *that is, as the limits* (*in this order*) *of the absolutely convergent double integral:*

$$\lim_{R\to\infty}\lim_{N\to+\infty}\int_{N\cdot\mathcal{C}}\int_G \mathrm{tr}_N\left(\sigma(\pi)\pi(w)\right) f_A\big(w^{-1}x\big)\chi_R(w)\, dw d\mu(\pi),$$

where $\chi\in\mathcal{D}(G)$ *with* $\chi\equiv 1$ *on a neighbourhood of 0 and* $\chi_R(x):=\chi(R^{-1}x)$, $\mathcal{C}$ *a compact neighbourhood of* $1\in\widehat{G}$ *such that* $\cup_{N\in\mathbb{N}}N\cdot\mathcal{C}=\widehat{G}$, *and* tr_N *denotes the trace of the operators projected on the subspace spanned by the first* N *vectors, having fixed a fundamental sequence of vector fields.*

Proof. Part (1) is a simple consequence of the definition of a jump set, it is left to the reader. Part (2) is in [46]. For Part (3), we apply Proposition 2.21 to obtain:

$$\int_G\int_{\widehat{G}}\mathrm{tr}\left(\sigma(\pi)\,\pi(w)\right) f_A\big(w^{-1}x\big)d\mu(\pi)dw=\int_G \mathcal{F}_G^{-1}\sigma(w)\, f_A\big(w^{-1}x\big)dw$$
$$=\int_G \mathrm{tr}\left(\mathcal{F}_G^{-1}\sigma(w)\pi_0(w)^*\pi_0(x)\ A\right)dw=\mathrm{tr}\left(\sigma(\pi_0)\pi_0(x)A\right),$$

since A is trace-class and $\sigma(\pi_0)\in\mathscr{L}(\mathcal{H}_{\pi_0})$. ☐

The arguments to show Proposition 6.6 follow the ones for Proposition 6.5. The main modifications come from replacing the properties of the centre with Lemma 6.7 Part (1). We will only outline the ideas, the technical details being very similar to the ones in the proof of Proposition 6.5.

Sketch of the proof of Proposition 6.6. Let $\chi\in\mathcal{D}(G)$ and $\sigma\in S^0_{\mathrm{cl}}(G)$.

$$(\mathrm{Op}(\sigma)(\chi u_k),\chi u_k)_{L^2(G)}$$
$$=k^{Q_J}\int_{\widehat{G}}\int_G\int_G \mathrm{tr}\left(\sigma(x,\pi)\pi(w)\right)(\chi u_k)\big(xw^{-1}\big)\,\overline{\chi u_k}(x)dwdxd\mu(\pi)$$
$$=k^{Q_J}\int_{\widehat{G}}\int_G\int_G \mathrm{tr}\left(\sigma(x,k\cdot\pi)\pi(w)\right)(\chi u_k)\big(x\ k^{-1}w^{-1}\big)\,\overline{\chi u_k}(x)\ dwdxd\mu(\pi),$$

after the change of variable $(\pi,w)\mapsto(k\cdot\pi,k^{-1}w)$. For k large,

$$(\chi u_k)\big(x\ k^{-1}w^{-1}\big)\sim(\chi u_0)(x)\ f_A\big((kx)w^{-1}\big).$$

Lemma 6.7 Part (3) implies

$$\int_{\widehat{G}}\int_G \mathrm{tr}\left(\sigma(x,k\cdot\pi)\pi(w)\right)f_A\big((kx)w^{-1}\big)\,dwd\mu(\pi)=\mathrm{tr}\left(\pi_0(kx)\sigma(x,k\pi_0)A\right).$$

Let us define $u_0(x) = u_0(x_{J^c})$ when $x = \exp(\sum_{k=1}^n x_j X_j) \in G$. Therefore

$$(\mathrm{Op}(\sigma)(\chi u_k), \chi u_k)_{L^2(G)} \sim k^{Q_J} \int_G \mathrm{tr}\,(\pi_0(kx)\sigma(x, k\pi_0)A)\,\overline{f_A(kx)}|\chi u_0|^2(x)dx$$
$$= \int_{\mathfrak{g}_{J^c}} \int_{\mathfrak{g}_J} \mathrm{tr}\,\Big(\pi_0\Big(e^{X_J + kX_{J^c}}\Big)\sigma\Big(e^{k^{-1}X_J + X_{J^c}}, k\pi_0\Big)A\Big)$$
$$\times\, \overline{f_A(e^{X_J + kX_{J^c}})}|\chi u_0|^2(e^{k^{-1}X_J + X_{J^c}})dX_J dX_{J^c},$$

having written $x = \exp(X_J + X_{J^c})$ and then performed the change of variable $X_J \mapsto k^{-1}X_J$. We have $k^{-1}X_J \to 0$, so

$$(\mathrm{Op}(\sigma)(\chi u_k), \chi u_k)_{L^2(G)}$$
$$\sim \int_{\mathfrak{g}_{J^c}} \int_{\mathfrak{g}_J} \mathrm{tr}\,\Big(\pi_0\Big(e^{X_J + kX_{J^c}}\Big)\sigma\Big(e^{X_{J^c}}, k\pi_0\Big)A\Big)\overline{f_A(e^{X_J + kX_{J^c}})}|\chi u_0|^2\Big(e^{X_{J^c}}\Big)dX_J dX_{J^c}$$
$$= \int_{\mathfrak{g}_{J^c}} \int_{\mathfrak{g}_J} \mathrm{tr}\,\Big(\pi_0\Big(e^{X'_J}\Big)\sigma\Big(e^{X_{J^c}}, k\pi_0\Big)A\Big)\,\overline{f_A(e^{X'_J})}|\chi u_0|^2\Big(e^{X_{J^c}}\Big)dX'_J dX_{J^c},$$

after having used the change of variable $X_J \mapsto X'_J = X_J + F(X_J, kX_{J^c})$, see Lemma 6.7 Part (1). Applying Lemma 6.7 Part (2) on the integral over $\mathfrak{g}_J$ concludes the (sketch of the) proof. □

In the next section, we will need the following limits which follow from similar computations to the ones above:

Corollary 6.8.

(1) *Let π_0, A, u_0 and u_k as in Proposition* 6.6. *If $x_0 \neq x_1$ and u_0 has a compact support small enough then*

$$\lim_{k\to\infty} (\mathrm{Op}(\sigma)(\chi u_k(x_1\cdot)), \chi u_k(x_0\cdot))_{L^2(G)} = 0.$$

(2) *Let π_0 and u_0 as in Proposition* 6.6 *and A, B be in $\mathscr{L}(\mathcal{H}_{\pi_0})_\infty$. We construct (u_k) and (v_k) as in Proposition* 6.6 *for A and B respectively. If $AB^* = 0$ then*

$$\lim_{k\to\infty} (\mathrm{Op}(\sigma)(\chi u_k), \chi v_k)_{L^2(G)} = 0.$$

(3) *Let π_0, A and u_0 as in Proposition* 6.6 *and consider $\dot{\pi}_1 \in \widehat{G}$ with $\dot{\pi}_1 \neq \dot{\pi}_0$, and v_0 and B in $\mathscr{L}(\mathcal{H}_{\pi_1})_\infty$. We construct (u_k) and (v_k) as in Proposition* 6.6 *for u_0, π_0, A and v_0, π_1, B respectively. Then*

$$\lim_{k\to\infty} (\mathrm{Op}(\sigma)(\chi(u_k + v_k)), \chi(u_k + v_k))_{L^2} = \lim_{k\to\infty} (\mathrm{Op}(\sigma)(\chi u_k), \chi u_k)_{L^2} + \lim_{k\to\infty} (\mathrm{Op}(\sigma)(\chi v_k), \chi v_k)_{L^2}.$$

Proof of Part (1). An argument of translation shows that it suffices to prove the case $x_1 = 0$. Proceeding as in the proof of Proposition 6.6, we obtain:

$$(\mathrm{Op}(\sigma)(\chi u_k), \chi u_k(x_0\cdot))_{L^2(G)}$$
$$\sim k^{Q_J} \int_G \mathrm{tr}\ (\pi_0(kx)\sigma(x, k\pi_0)A)\ \overline{f_A(k(x_0x))}(\chi u_0)(x)\ \overline{\chi u_0}(x_0x)dx.$$

Now $u_0(x)\bar{u}_0(x_0x) = 0$ for any $x \in G$ when u_0 has a support small enough and $x_0 \neq 0$. □

Proof of Part (2). Proceeding as in the proof of Proposition 6.6, we obtain:

$$(\mathrm{Op}(\sigma)(\chi u_k), \chi v_k)_{L^2(G)} \sim k^{Q_J} \int_G \mathrm{tr}\ (\pi_0(kx)\sigma(x, k\pi_0)A)\ \overline{f_B(kx)}|\chi u_0|^2(x)dx$$
$$\sim \int_{\mathfrak{g}_{J^c}} \int_{\mathfrak{g}_J} \mathrm{tr}\ \left(\pi_0(e^{X'_J})\sigma(e^{X_{J^c}}, k\pi_0)A\right)\ \overline{f_B(e^{X'_J})}|\chi u_0|^2(e^{X_{J^c}})dX'_J dX_{J^c}$$
$$= \int_{\mathfrak{g}_{J^c}} \mathrm{tr}\ \left(\frac{1}{d_{\pi_0}} B^*\sigma(e^{X_{J^c}}, k\pi_0)A\right) |\chi u_0|^2(e^{X_{J^c}})dX'_J dX_{J^c}.$$

Hence this is zero when $AB^* = 0$. □

Proof of Part (3). Proceeding as in the proof of Proposition 6.6, we obtain:

$$(\mathrm{Op}(\sigma)(\chi u_k), \chi v_k)_{L^2(G)}$$
$$\sim k^{Q_J} \int_G \mathrm{tr}\ (\pi_0(kx)\sigma(x, k\pi_0)A)\ \overline{f_B(kx)}|\chi u_0|^2(x)dx$$
$$\sim \int_{\mathfrak{g}_{J^c}} \int_{\mathfrak{g}_J} \mathrm{tr}\ \left(\pi_0\left(e^{X_J + kX_{J^c}}\right)\sigma\left(e^{X_{J^c}}, k\pi_0\right)A\right)\overline{f_B(e^{X_J+kX_{J^c}})}|\chi u_0|^2\left(e^{X_{J^c}}\right)dX_J dX_{J^c},$$

having used the jump set for π_0. And this is equivalent to the same quantity with $\mathrm{princ}_0(\sigma)$ replacing σ. So when $\mathrm{princ}_0(\sigma)$ is zero at $(x, \dot{\pi}_0)$ for all $x \in G$, we have

$$\lim_{k\to\infty} (\mathrm{Op}(\sigma)(\chi u_k), \chi v_k)_{L^2(G)} = 0.$$

Let us fix a continuous real-valued function on $\widehat{G}/\mathbb{R}^+$ such that $\eta(\dot{\pi}_0) = 0$ and $\eta(\dot{\pi}_1) = 1$. Considering a general symbol σ, we write $\sigma = \sigma\eta + (1-\eta)\sigma$ so

$$\Re(\mathrm{Op}(\sigma)(\chi u_k), \chi v_k)_{L^2} = \Re(\mathrm{Op}(\sigma\eta)(\chi u_k), \chi v_k)_{L^2}$$
$$+ \Re((\chi u_k), (\mathrm{Op}(\sigma(1-\eta))^*\chi v_k)_{L^2}.$$

As $\sigma\eta$ vanishes at $\dot{\pi}_0$, the limit of the first term on the right hand side is zero. For the second term, we have as in the proof of Lemma 6.3

$$\lim_{k\to\infty} ((\chi u_k), \left(\mathrm{Op}(\sigma(1-\eta))^*\chi v_k\right)_{L^2} = \lim_{k\to\infty} \left((\chi u_k), \mathrm{Op}\left(\sigma^*(1-\eta)\right)\chi v_k\right)_{L^2}$$

and it must be zero since $\sigma^*(1-\eta)$ vanishes at $\dot{\pi}_1$. □

6.5. Consistency of the description

Our main result describes MDMs as trace-class-valued positive measures, see Section 6.1. In this section, we will show the converse, that is, that any element of $\mathcal{M}_1^+(\Omega \times \Sigma_1)$ is a MDM:

Proposition 6.9. *Let Ω be a non-empty open set of G and $(\Gamma, \gamma) \in \mathcal{M}_1^+(\Omega \times \Sigma_1)$. Then, there exists a pure sequence (u_k) with $\Gamma d\gamma$ as* MDM.

Our proof of Proposition 6.9 will use the following description of the topological dual of $C^*(\dot{S}_c^0(\Omega))$.

Lemma 6.10. *Let Ω be a non-empty open subset of G.*

(1) *For any complex element (γ, Γ) of $\mathcal{M}_1(\Omega \times \Sigma_1)$, the linear form*

$$\dot{S}_c^0(\Omega) \ni \sigma \longmapsto \int \mathrm{tr}\,(\sigma\ \Gamma)\ d\gamma = \int_{\Omega\times\Sigma_1} \mathrm{tr}\,(\sigma(x, \dot{\pi})\ \Gamma(x, \dot{\pi}))\ d\gamma(x, \dot{\pi}),$$

extends uniquely to a continuous linear form $\ell_{(\gamma,\Gamma)}$ on $C^*(\dot{S}_c^0(\Omega))$;

(2) *Conversely, given any continuous form ℓ on $C^*(\dot{S}_c^0(\Omega))$, there exists a complex element (γ, Γ) of $\mathcal{M}_1(\Omega \times \Sigma_1)$, such that $\ell = \ell_{(\gamma,\Gamma)}$;*

(3) *If two complex elements (γ, Γ) and (γ', Γ') of $\mathcal{M}_1(\Omega \times \Sigma_1)$, yield the same linear form,* i.e. *$\ell_{(\gamma,\Gamma)} = \ell_{(\gamma',\Gamma')}$, then they are equivalent;*

(4) *The map $\Phi : \ell = \ell_{(\gamma,\Gamma)} \mapsto \Gamma d\gamma$ is an isomorphism from the topological dual of the Banach space $C^*(\dot{S}_c^0(\Omega))$ onto the Banach space of the equivalence classes $\Gamma d\gamma$ in $\mathcal{M}_1(\Omega \times \Sigma_1)$ equipped with the norm*

$$\|\Gamma d\gamma\| := \sup_{\sigma\in C^*(\dot{S}_c^0(\Omega))\setminus\{0\}} \frac{|\ell_{\gamma,\Gamma}(\sigma)|}{\|\sigma\|_{C^*(\dot{S}_c^0(\Omega))}};$$

(5) *The states of the C^*-algebra $C^*(\dot{S}_c^0(\Omega))$ are mapped by Φ onto the measures $\Gamma d\gamma \geq 0$ with $\int \mathrm{tr}\,(\Gamma) d\gamma = 1$. The pure states corresponds to $(\delta_{x_0}(x) \otimes \delta_{\dot{\pi}_0}(\pi), v_0 \otimes v_0^*)$ where $x_0 \in \Omega$, $\dot{\pi}_0 \in \Sigma_1$ and v_0 a unit vector in $\mathcal{H}_{\pi_0}$;*

(6) *The positive forms of the C^*-algebra $C^*(\dot{S}_c^0(\Omega))$ are mapped by Φ onto the measures $\Gamma d\gamma \geq 0$ with $(\gamma, \Gamma) \in \mathcal{M}_1^+(\Omega \times \Sigma_1)$.*

Proof. The states were characterised in Proposition 5.17. The properties are easily proved from well-known facts or routine exercises in functional analysis. ☐

Lemma 6.10 allows us to identify the topological dual of $C^*(\dot{S}_c^0(\Omega))$ with $\mathcal{M}_1^+(\Omega \times \Sigma_1)$ modulo equivalence. We can now prove Proposition 6.9.

Proof of Proposition 6.9. Let Ω be a non-empty bounded open set of G. We fix a positive Rockland operator $\mathcal{R}$, and a function $\psi \in C^\infty(\mathbb{R})$ such that $\psi \equiv 0$ on a neighbourhood of 0 and $\psi \equiv 1$ on a neighbourhood of $+\infty$. We denote by $\widetilde{\mathcal{M}}$ the subset of $\mathcal{M} := \mathcal{M}_1(\Omega \times \Sigma_1)$, of $\Gamma d\gamma \geq 0$ for which there exists a sequence (u_k) in $L^2(\Omega)$ satisfying $u_k \rightharpoonup_{k\to\infty} 0$ and

$$\begin{aligned} &\forall \sigma \in \dot{S}^0_c(\Omega), \\ &\lim_{k\to\infty} (\mathrm{Op}(\sigma\psi(\pi(\mathcal{R})))u_k, u_k)_{L^2(\Omega)} = \int_{\Omega\times\Sigma_1} \mathrm{tr}\ (\sigma(x,\dot\pi)\ \Gamma(x,\dot\pi))\, d\gamma(x,\dot\pi)\,. \end{aligned} \tag{6.8}$$

We want to show that $\widetilde{\mathcal{M}}$ is equal to $\mathcal{M}^+ := \mathcal{M}^+_1(\Omega\times\Sigma_1)$. Clearly, $\widetilde{\mathcal{M}}$ is included in $\mathcal{M}^+$ and we already know that $\widetilde{\mathcal{M}}$ contains 0, and the examples in Propositions 6.4 and 6.6.

Claim 1: Let us show that $\widetilde{\mathcal{M}}$ is convex. Indeed, one easily checks that if the sequence (u_k) in $L^2(\Omega)$ satisfies $u_k \rightharpoonup_{k\to\infty} 0$ and (6.8) with $\Gamma d\gamma \geq 0$, then for any $r > 0$ the sequence (ru_k) satisfies the same property with $r^2\Gamma d\gamma \geq 0$.

Claim 2: One easily checks that if $x_0 \in G$ and if the sequence (u_k) in $L^2(\Omega)$ satisfies $u_k \rightharpoonup_{k\to\infty} 0$ and (6.8) with $\Gamma d\gamma \geq 0$, then the sequence $(u_k(x_0\,\cdot))$ satisfies the analogous properties with $\Gamma(x_0x,\dot\pi)d\gamma(x_0x,\dot\pi)$, which thus is in $\widetilde{\mathcal{M}}$. In this sense, $\widetilde{\mathcal{M}}$ is invariant under spatial translations.

Lemma 6.10 allows us to identify $\mathcal{M}$ with the topological dual of $C^*(\dot{S}^0_c(\Omega))$; we now equip it with the weak-* topology. By the Krein-Milman Theorem, $\mathcal{M}^+$ is the positive span of the pure states and 0 (*i.e.* the closure of the set of all non-negative linear combinations of pure states).

Claim 3: Let us show that $\widetilde{\mathcal{M}}$ is closed in $\mathcal{M}^+$. Indeed, let $(\Gamma^{(j)}d\gamma^{(j)})_{j\in\mathbb{N}}$ be a sequence in $\widetilde{\mathcal{M}}$ converging to $\Gamma d\gamma$ in $\mathcal{M}$. Considering corresponding sequences $(u^{(j)}_k)_{k\in\mathbb{N}}$ in $L^2(\Omega)$ satisfying $u^{(j)}_k \rightharpoonup_{k\to\infty} 0$ and (6.8) with $\Gamma^{(j)}d\gamma^{(j)} \geq 0$, then we extract a diagonal subsequence $(u^{(k(j))}_{k(j)})_{j\in\mathbb{N}}$ satisfying $u^{(k(j))}_{k(j)} \rightharpoonup_{j\to\infty} 0$. By Theorem 6.1, we may assume that this subsequence satisfies (6.8) for a positive trace-class-valued measure which is unique up to equivalence, so this positive trace-class-valued measure coincides with $\Gamma d\gamma$ by Lemma 6.10 (3). Hence the limit $\Gamma d\gamma$ is in $\widetilde{\mathcal{M}}$ which is thus closed.

Conclusion: Considering a sequence as in Proposition 6.6 with $\pi_0 \in \widehat{G}$ of infinite dimension, $A \in \mathscr{L}(\mathcal{H}_{\pi_0})_\infty$, $u^{(\epsilon)}_0 = \epsilon^{-Q_{J^c}/2}u_0(\epsilon^{-1}x)$ where $\epsilon > 0$, $u_0 \in \mathcal{D}(\mathfrak{g}_{J^c})$ with a support small enough and $Q_{J^c} = \sum_{k\in J^c} \upsilon_k$, the proof of Claim 3 shows that $(\delta_{x=0}\otimes\delta_{\dot\pi=\pi_0}, AA^*) \in \widetilde{\mathcal{M}}$ when $0 \in \Omega$. Using the invariance under spatial translation (*cf.* Claim 2), $(\delta_{x=x_0}\otimes\delta_{\dot\pi=\dot\pi_0}, AA^*) \in \widetilde{\mathcal{M}}$ for any $x_0 \in \Omega$ and $\pi_0 \in \widehat{G}$ of infinite dimension. Note that this membership also holds when π_0 is of finite dimension, therefore of dimension one; it suffices then to adapt the Euclidean case, and we view it as a degenerate case of Proposition 6.6. We choose $A = v_0 \otimes v_0^*$ with $v_0 \in \mathcal{H}_{\pi_0}$ smooth and unitary. We can remove the hypothesis "smooth" by considering a sequence of such vectors and Claim 3. This shows that $\widetilde{\mathcal{M}}$ contains all the pure states, see Lemma 6.10 (5). Moreover, they can all be obtained as MDM of sequences obtained by diagonal extractions of suitable sequences constructed in

Proposition 6.6. This together with Corollary 6.8 and Claim 1 show that $\widetilde{\mathcal{M}}$ also contains the positive span of the pure states and 0. Therefore $\widetilde{\mathcal{M}} = \mathcal{M}^+$. □

7. Applications

In this section, we investigate the properties of the MDM of a sequence of functions that satisfy a differential equation. In particular, we are concerned with *Div-Curl* type results and, as a consequence, we shall focus on vector-valued sequences.

Let Ω be an open subset of G and let us consider a vector-valued sequence of functions of $L^2(\Omega)$, $(U_k)_{k\in\mathbb{N}} = (u_1^k, \cdots, u_N^k)_{k\in\mathbb{N}}$, $N \in \mathbb{N}$. We assume that (U_k) converges weakly to some vector valued function $U = (u_1, \cdots, u_N)$ of $L^2(\Omega)^N$, in the sense that for all $j \in \{1, \cdots, N\}$, u_j^k tends weakly to u_j in $L^2(G)$. In order to study the defects of compactness of a family of the form $(U_k)_{k\in\mathbb{N}}$, we shall use matrices of operators in $\Psi^0_{\mathrm{cl}}(\Omega)$. We shall denote by $A^{P,Q}$ the set of matrices with P lines and Q rows and with entries in a given algebra A, for instance $A = \mathbb{C}$ or $\dot{S}^0$ or $C^*(\dot{S}^0)$. We shall need basic notions about the C^*-algebra $A^{N,N}$ for a general C^*-algebra A and $N \in \mathbb{N}$, and this is done in the first subsection. Then, we shall define MDM for vector-valued sequences and discuss localisation property of MDM whenever $(U_k)_{k\in\mathbb{N}}$ satisfies a system of differential equations. Finally, the last subsection is devoted to compensated compactness results and application to *Div-Curl* Lemma.

7.1. Matrices of a C^*-algebra

Let A be an algebra. When A has a unit, we will denote the unit by 1_A. Let $N \in \mathbb{N}$. We denote by $\mathrm{I}_N \in \mathbb{C}^{N,N}$ the identity matrix and by E_{ij} the $N \times N$ complex matrix with 0 in every entry except for the ith row and jth column where the entry is 1. Therefore, we will denote by $A\mathrm{I}_N$ the set of diagonal matrices in $A^{N,N}$ with the same repeated entry on the diagonal, and by $aE_{ij} \in A^{N,N}$ the matrix with 0 in every entry except for the i-th row and k-th column where the entry is $a \in A$. If the algebra A is also a normed vector space, we set the following norm on $A^{N,1}$:

$$\|V\|^2_{A^{N,1}} = \left\| \sum_{j=1}^N v_j v_j^* \right\|_A \quad \text{when} \quad V = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}.$$

The next lemma gives the main properties of $A^{N,N}$ when A is a C^*-algebra.

Lemma 7.1. *Let A be a C^* algebra and let $N \in \mathbb{N}$.*

(i) *Equipped with the norm given by*

$$\|M\|_{A^{N,N}} = \sup\{\|V_1^* M V_2\|_A : V_1, V_2 \in A^{N,1}, \|V_1\|_{A^{N,1}} \le 1, \|V_2\|_{A^{N,1}} \le 1\},$$

$A^{N,N}$ is a C^*-algebra. The sub-C^*-algebra $A\mathrm{I}_N$ of $A^{N,N}$ is isomorphic to the C^*-algebra A.
If A has a unit, then $1_A \mathrm{I}_N$ is the unit of $A^{N,N}$.
If A is a C^*-algebra with approximate unit $(a_k)_{k\in\mathbb{N}}$ then $A^{N,N}$ is a C^*-algebra with approximate unit $(a_k \mathrm{I}_N)_{k\in\mathbb{N}}$;

(ii) *Assume here that A has a unit. Let π be a representation of the C^*-algebra $A^{N,N}$. Let ξ_1 be a non-zero vector of this representation with $\xi_1 \in \pi(1_A E_{11})\mathcal{H}_\pi$. We denote by W the closed subspace of $\mathcal{H}_\pi$ generated by ξ_1. As Hilbert spaces, W is isomorphic to the orthogonal sum of N copies of $\pi(A\mathrm{I}_N)\xi_1$. Furthermore the representation π of the C^*-algebra $A^{N,N}$ on W is completely determined by its restriction $\pi\big|_{A\mathrm{I}_N}$ to $A\mathrm{I}_N$;*

(iii) *The spectrum of the C^*-algebra $A^{N,N}$ may be identified with the spectrum of the C^*-algebra A via the homeomorphism which maps an irreducible representation π of $A^{N,N}$ to the irreducible representation of A defined by the restriction $\pi\big|_{A\mathrm{I}_N}$;*

(iv) *If ℓ is a state of A and $V \in A^{N,1}$ with $\ell(V^*V) = 1$, then the functional $L = L_{\ell,V}$ defined on $A^{N,N}$ via*

$$L(M) = \ell(V^*MV), \quad M \in A^{N,N},$$

is a state of $A^{N,N}$. The pure states of $A^{N,N}$ are of the form $L_{\ell,V}$ with ℓ a pure state of A and $V \in A^{N,1}(\mathbb{C})1_A$ a complex vector satisfying $\ell(V^*V) = 1$.

Proof. Part (i) is left to the reader. Let us prove Part (ii). Let π be a representation of the C^*-algebra $A^{N,N}$. For each $j = 1, \dots, N$, we set $\mathcal{H}_\pi^{(j)} := \pi(1_A E_{jj})\mathcal{H}_\pi$. The subspaces $\mathcal{H}_\pi^{(j)}$ are closed, orthogonal and their sum is $\mathcal{H}_\pi = \oplus^{\perp}_{1\le j\le N}\mathcal{H}_\pi^{(j)}$ since $E_{ii}E_{jj} = \delta_{i=j}E_{ii}$ and $\mathrm{I}_N = \sum_{j=1}^N E_{jj}$ in $\mathbb{C}^{N,N}$. Furthermore, since $\pi(m\mathrm{I}_N)$ and $\pi(1_A E_{jj})$ commutes for j fixed and any $m \in A$, the algebra A acts on $\mathcal{H}_\pi^{(j)}$ via $m \mapsto \pi(m\mathrm{I}_N)$. We also observe that for any $1 \le i, j \le N$, $\pi(1_A E_{ji})$ maps $\mathcal{H}_\pi^{(i)}$ to $\mathcal{H}_\pi^{(j)}$ since $E_{ji}E_{ii} = E_{jj}E_{ji}E_{ii}$. In fact, $\pi(1_A E_{ji})$ maps unitarily $\mathcal{H}_\pi^{(i)}$ onto $\mathcal{H}_\pi^{(j)}$ with inverse $\pi(1_A E_{ij})$ since $E_{ij}E_{ji} = E_{ii}$ and $E_{ji}E_{ij} = E_{jj}$.

Let us fix a non-zero vector $\xi^{(1)} \in \mathcal{H}_\pi^{(1)}$. Let $\overline{\pi(A^{N,N})\xi^{(1)}}$ be the closed subspace of $\mathcal{H}_\pi$ generated by $\xi^{(1)}$ under π. Its orthogonal projection on $\mathcal{H}_\pi^{(j)}$ is

$$\begin{aligned}
\pi\big(1_A E_{jj}\big)\overline{\pi(A^{N,N})\xi^{(1)}} &= \overline{\pi(1_A E_{jj}) \oplus_{l,k} \pi(A\mathrm{I}_N)\pi(1_A E_{lk})\xi^{(1)}} \\
&= \overline{\pi(A\mathrm{I}_N)\pi(1_A E_{j1})\xi^{(1)}},
\end{aligned}$$

that is, the closed subspace in $\mathcal{H}_\pi^{(j)}$ generated by $\xi^{(j)} := \pi(1_A E_{j1})\xi^{(1)}$ under the action of A given by the restriction of π to $A\mathrm{I}_N$. All these orthogonal projections are unitarily isomorphic:

$$\pi\big(1_A E_{jj}\big)\overline{\pi(A^{N,N})\xi^{(1)}} = \overline{\pi(A\mathrm{I}_N)\xi^{(j)}} = \pi(1_A E_{j1})\overline{\pi(A\mathrm{I}_N)\xi^{(1)}}.$$

So, as vector spaces and in terms of actions of A via $\pi|_{A\mathrm{I}_N}$, $\overline{\pi(A^{N,N})\xi^{(1)}}$ is isomorphic to N copies of $\overline{\pi(A\mathrm{I}_N)\xi^{(1)}}$. Furthermore, writing any matrix $M \in A^{N,N}$ as $M = \sum_{1\leq l,k\leq N} m_{lk} 1_A E_{lk}$ with $m_{l,k} \in A$, we have

$$\begin{aligned}\pi(M)\xi^{(j)} &= \sum_{1\leq l,k\leq N} \pi(m_{lk}1_A E_{lk})\xi^{(j)} = \sum_{1\leq l,k\leq N} \pi(m_{lk}\mathrm{I}_N)\pi\big(1_A E_{lk}E_{j1}\big)\xi^{(1)} \\ &= \sum_{1\leq l\leq N} \pi\big(m_{lj}\mathrm{I}_N\big)\pi(1_A E_{l1})\xi^{(1)} = \sum_{1\leq l\leq N} \pi\big(m_{lj}\mathrm{I}_N\big)\xi^{(l)},\end{aligned}$$

is in $\overline{\pi(A^{N,N})\xi^{(1)}}$. So π acts on $\overline{\pi(A^{N,N})\xi^{(1)}}$ where it is completely determined by its restriction to $\pi(AI_N)$. This shows Part (ii).

Before continuing the proof, let us observe that these last computations imply

$$\Big(\pi(M)\xi^{(1)}, \xi^{(1)}\Big)_{\mathcal{H}_\pi} = \Big(\pi(m_{11}\mathrm{I}_N)\xi^{(1)}, \xi^{(1)}\Big)_{\mathcal{H}_\pi}.$$

A form of converse of this property consists in noticing that if ℓ is a state of the C^*-algebra A, then the functional L defined on $A^{N,N}$ via $L(M) = \ell(m_{11})$ is a state of the C^*-algebra $A^{N,N}$. More generally, if $V \in A^{N,1}$ is a fixed vector valued in A and ℓ is a state of the C^*-algebra A, then the functional L defined on $A^{N,N}$ via $L(M) = \ell(V^*MV)$ is a state of the C^*-algebra $A^{N,N}$ provided that $\ell(V^*V) = 1$.

For Part (iii) and (iv), if A has no unit, we consider instead its extension $\tilde{A} = A\oplus\mathbb{C}$, see, *e.g.*, [28, Section 1.3]. Hence, we may assume that A has a unit. Part (iii) then follows from Part (ii) and its proof.

Let us now prove Part (iv) under the hypotheses that A has a unit. Let L be a state of $A^{N,N}$. We set $\Gamma_{ji} := L(1_A E_{ij})$ for $1 \leq i, j \leq N$, and consider the matrix $\Gamma = (\Gamma_{ij}) \in \mathbb{C}^{N,N}$. Since $\overline{L(M)} = L(M^*)$, the matrix $\Gamma = \Gamma^*$ is Hermitian so there exists a $N \times N$ unitary matrix P such that $P^*\Gamma P$ is diagonal. We may replace L by $M \mapsto L(PMP^*)$ and assume that Γ is diagonal. Since L is a positive linear functional, so is its restriction $M \mapsto \mathrm{tr}\,(M\Gamma)$ to $\mathbb{C}^{N,N}$ and this implies $\Gamma \geq 0$. Furthermore as L is a state, $\mathrm{tr}\,\Gamma = 1$. So we may assume that $\Gamma = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$ and $\lambda_1 + \ldots + \lambda_N = 1$.

We now assume that L is pure. Let π be the irreducible representation of the C^*-algebra $A^{N,N}$ and ξ the unit vector associated with L. We can decompose

$$\xi = \xi_1 + \ldots + \xi_N, \quad \text{where} \quad \xi_j := \pi\big(1_A E_{jj}\big)\xi \in \mathcal{H}_\pi^{(j)} := \pi\big(1_A E_{jj}\big)\mathcal{H}_\pi.$$

We have $1 = \|\xi\|^2_{\mathcal{H}_\pi} = \|\xi_1\|^2_{\mathcal{H}_\pi} + \ldots + \|\xi_N\|^2_{\mathcal{H}_\pi} = \lambda_1 + \ldots + \lambda_N$ and more generally

$$\big(\xi_i, \xi_j\big)_{\mathcal{H}_\pi} = \big(\pi(1_A E_{ij})\xi, \xi\big)_{\mathcal{H}_\pi} = L\big(1_A E_{ij}\big) = \lambda_j \delta_{i=j}.$$

Necessarily $\xi_1 \neq 0$. Naturally, $\xi_i \in \mathcal{H}_\pi^{(i)}$ is orthogonal to $\xi^{(j)} := \pi(1_A E_{j1})\xi_1 = \pi(1_A E_{j1})\xi \in \mathcal{H}_\pi^{(j)}$ if $i \neq j$. And for $j = 2, \ldots, N$, ξ_j is orthogonal to $\xi^{(j)}$ since

$$\begin{aligned}\Big(\xi_j, \xi^{(j)}\Big)_{\mathcal{H}_\pi} &= \big(\pi\big(1_A E_{jj}\big)\xi, \pi\big(1_A E_{j1}\xi\big)_{\mathcal{H}_\pi} = \big(\pi(1_A E_{1j})\xi, \xi\big)_{\mathcal{H}_\pi} \\ &= L\big(1_A E_{1j}\big) = \delta_{1=j}.\end{aligned}$$

This shows that if $\xi_2 \neq 0$ then the space $\overline{\pi(A^{N,N})\xi_2}$ generated by ξ_2 under the representation π of $A^{N,N}$ will be non-zero and distinct from $\overline{\pi(A^{N,N})\xi_1}$, contradicting the irreducibility of π. Therefore $0 = \xi_2 = \ldots = \xi_N$ and $\xi = \xi_1$ is unitary. Furthermore we have:

$$\begin{aligned} L(M) &= \sum_{1\leq i,j\leq N} \big(\pi\big(m_{ij}E_{ij}\big)\xi, \xi\big) = \sum_{1\leq i,j\leq N} \big(\pi\big(m_{ij}E_{ij}\big)\xi_j, \xi_i\big) \\ &= \big(\pi(m_{11}E_{11})\xi_1, \xi_1\big) = \ell_1(m_{11}), \end{aligned}$$

where ℓ_1 is the state of A associated with the restriction of π to A on $\overline{\pi(A\mathrm{I}_N)\xi_1}$ and the unit vector ξ_1. With the notation of the statement, this shows $L = L_{\ell_1,V}$ with $V = e_1 1_A$ where e_1 is the first vector of the canonical basis of $\mathbb{C}^N$; one easily checks $\ell_1(V^*V) = \ell_1(1_A) = 1$. This concludes the proof of Part (iv) and of Lemma 7.1. □

7.2. Microlocal defect measures of vector-valued sequences and localisation properties

Let us now go back to the family $(U_k)_{k\in\mathbb{N}}$ in $L^2(\Omega, \mathrm{loc})^N$ where Ω is an open subset of G. We are concerned with the limit of quantities of the form $(AU_k, U_k)_{L^2(\Omega)^N}$ for $A \in \big(\Psi^0_{\mathrm{cl}}(\Omega)\big)^{N,N}$. We shall denote by Tr_N the trace of operators acting on the Hilbert space of $\mathcal{H}^N_\pi$: if $\sigma = (\sigma_{i,j})_{1\leq i,j\leq N}$, $\Gamma = (\Gamma_{i,j})_{1\leq i,j\leq N}$,

$$\mathrm{Tr}_N\, (\sigma(x,\dot\pi)\ \Gamma(x,\dot\pi)) = \sum_{1\leq i,j\leq N} \mathrm{tr}\, \big(\sigma_{i,j}(x,\dot\pi)\ \Gamma_{j,i}(x,\dot\pi)\big).$$

A simple adaptation of the proof of our main result in Theorem 6.1 for scalar-valued sequences and pseudo-differential operators yields:

Theorem 7.2. *Let Ω be a non-empty open set of G. Let (U_k) be a sequence in $L^2(\Omega, \mathrm{loc})^N$ and $U \in L^2(\Omega, \mathrm{loc})^N$. We assume that $U_k \rightharpoonup_{k\to\infty} U$ a.e. in $L^2(\Omega, \mathrm{loc})^N$. Then there exist a subsequence $(U_{k(j)})_{j\in\mathbb{N}}$ of (U_k) and a positive measure $(\gamma, \Gamma) \in \mathcal{M}_1^+(\Omega\times\Sigma_1, (\mathcal{H}^N_\pi)_{\dot\pi\in\Sigma_1})$ such that for any $A \in (\Psi^0_{\mathrm{cl}}(\Omega))^{N,N}$, we have the convergence*

$$\begin{aligned} &\lim_{j\to\infty} \big(A(U_{k(j)} - U), (U_{k(j)} - U)\big)_{L^2(\Omega)^N} \\ &= \int_{\Omega\times\Sigma_1} \mathrm{Tr}_N\, (\mathrm{princ}_0(A)(x,\dot\pi)\ \Gamma(x,\dot\pi))\, d\gamma(x,\dot\pi). \end{aligned}$$

Moreover, once the subsequence $(k(j))$ is fixed, the positive trace-class-valued measure (Γ, γ) is unique up to equivalence.

Let us now consider a matrix-valued operator P consisting of K lines and N rows of differential operators of order m such that $(PU_k)_{k\in\mathbb{N}}$ converges to 0 in $L^2_{-m}(\Omega, \mathrm{loc})^K$ as $k \to +\infty$ (recall that $L^2_s(\Omega, \mathrm{loc})$ was defined in Definition 4.23). The MDMs $\Gamma d\gamma$ of $(U_k)_{k\in\mathbb{N}}$ satisfy the following localization property.

Proposition 7.3. *Let $p(x,\pi)$ be the principal symbol of the matrix-valued differential operator $P \in (S^m(\Omega))^{K,N}$ where Ω is an open subset of G. We assume that the family $(U_k)_{k\in\mathbb{N}}$ in $L^2(\Omega)^N$ is such that PU_k tends to PU in $L^2_{-m}(\Omega,\mathrm{loc})^N$. Let $\Gamma d\gamma$ be a MDM of $(U_k)_{k\in\mathbb{N}}$, then*

$$p_0(x,\dot\pi)\Gamma(x,\dot\pi)p_0(x,\dot\pi)^* = 0,\ \ d\gamma(x,\dot\pi)\ a.e.,$$

where $p_0(x,\dot\pi) := \pi(\mathcal{R})^{-\frac{m}{\nu}}p(x,\dot\pi)$ for any positive Rockland operator $\mathcal{R}$ (with homogeneous degree ν).

Proof. We may assume that the sequence $(U_k)_{k\in\mathbb{N}}$ is pure. The equation satisfied by $(PU_k)_{k\in\mathbb{N}}$ implies that for any $\sigma \in (S^0_{\mathrm{cl}}(\Omega))^{K,K}$ we have

$$\lim_{k\to\infty}\Big(\mathrm{Op}(\sigma)(\mathrm{I}+\mathcal{R})^{-\frac{m}{\nu}}P(U_k-U),(\mathrm{I}+\mathcal{R})^{-\frac{m}{\nu}}P(U_k-U)\Big)_{L^2(\Omega)^K} = 0.$$

By the definition of $\Gamma d\gamma$, we deduce

$$\int_{\Omega\times\Sigma_1}\mathrm{Tr}_N\Big(p(x,\dot\pi)^*\pi(\mathcal{R})^{-\frac{m}{\nu}}\sigma_0(x,\dot\pi)\pi(\mathcal{R})^{-\frac{m}{\nu}}p(x,\dot\pi)\Gamma(x,\dot\pi)\Big)d\gamma(x,\dot\pi) = 0,$$

and this relation holds for any $\sigma_0 \in (\dot S^0_c(\Omega))^{K,K}$, and this implies the result. □

7.3. Compensated compactness

The issue of compensated compactness result is to pass to the limit in quantities of the form

$$\int_\Omega \phi(x)(q(x)U_k(x),U_k(x))_{\mathbb{C}^N}dx,$$

for some compactly supported scalar-valued smooth function ϕ and for smooth bounded matrix-valued function $q \in (\mathcal{D}(\Omega))^{N,N}$. The aim is to find conditions on the matrix q which allow to pass to the limit in terms of the weak limits U of $(U_k)_{k\in\mathbb{N}}$. The next proposition is a compensated compactness result. Recall that the spaces $L^2_s(\Omega,\mathrm{loc})$ were defined in Definition 4.23.

Proposition 7.4. *Let $p(x,\pi)$ be the principal symbol of the matrix-valued differential operator $P \in (S^m(\Omega))^{K,N}$ where Ω is an open subset of G. Let $(U_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega,\mathrm{loc})^N$ which converges to U weakly in $L^2(\Omega,\mathrm{loc})^N$ and such that $(PU_k)_{k\in\mathbb{N}}$ converges to PU in $L^2_{-m}(\Omega,\mathrm{loc})^K$.*

(i) *Let $q \in (C^\infty(\Omega))^{N,N}$ be such that for all $x\in\Omega$, $\pi\in\widehat G$ and $h\in(\mathcal{H}^\infty_\pi)^N$, we have*

$$p(x,\pi)h = 0 \Longrightarrow (q(x)h,h)_{\mathcal{H}^N_\pi} = 0.$$

Then the sequence of smooth functions given by $x\mapsto (q(x)U_k(x),U_k(x))_{\mathbb{C}^N}$ converges to $x\mapsto(q(x)U(x),U(x))_{\mathbb{C}^N}$ in $\mathcal{D}'(\Omega)$;

(ii) *Let $q \in (C^\infty(\Omega))^{N,N}$ be such that $q^* = q$ and satisfying for all $x \in \Omega, \pi \in \widehat{G}$ and $h \in (\mathcal{H}_\pi^\infty)^N$,*

$$p(x,\pi)h = 0 \Longrightarrow (q(x)h, h)_{\mathcal{H}_\pi^N} \geq 0.$$

Then, for any non-negative $\phi \in \mathcal{D}(\Omega)$,

$$\liminf_{k\to\infty} \int_\Omega \phi(x)(q(x)U_k(x), U_k(x))_{\mathbb{C}^N} dx \geq \int_\Omega \phi(x)(q(x)U(x), U(x))_{\mathbb{C}^N} dx.$$

Proof of Proposition 7.4. The proof follows the lines of [32, Theorem 2]. Part (i) follows from Part (ii), using in particular the decomposition of $q = q_1 + iq_2$ with q_1, q_2 smooth functions valued in the space of Hermitian $N \times N$-matrices. So we just have to prove Part (ii). We may assume that the sequence (U_k) is pure and that $U = 0$. As a consequence, we know that

$$\lim_{k\to\infty} \int_\Omega \phi(x)(q(x)U_k(x), U_k(x))_{\mathbb{C}^N} dx = \int_{\Omega\times\Sigma_1} \phi(x)\mathrm{Tr}_N(q(x)\Gamma(x,\dot\pi))d\gamma(x,\dot\pi),$$

and our aim is to show that the right-hand side of the preceding equality is nonnegative. The proof comes from the following observation:

$$\begin{aligned} &\forall\varepsilon > 0, \quad \exists C_\varepsilon > 0, \quad \forall(x,\dot\pi) \in \Omega\times\Sigma_1, \quad \forall h \in \mathcal{H}_\pi, \\ &(q(x)h, h)_{\mathcal{H}_\pi^N} + C_\varepsilon \|p_0(x,\dot\pi)h\|^2_{\mathcal{H}_\pi^K} \geq -\varepsilon\|h\|^2_{\mathcal{H}_\pi^N}, \end{aligned} \tag{7.1}$$

where $p_0(x,\dot\pi) = \pi(\mathcal{R})^{-\frac{m}{\nu}} p(x,\dot\pi)$. Indeed, this equation yields the positivity of the operator

$$R_\varepsilon(x,\dot\pi) = q(x) + C_\varepsilon p_0^*(x,\dot\pi)p_0(x,\dot\pi) + \varepsilon\mathrm{Id}$$

and we deduce for all non-negative $\phi \in \mathcal{D}(\Omega)$ and for all $\varepsilon > 0$

$$\int_{\Omega\times\Sigma_1} \phi(x)\mathrm{Tr}_N(R_\varepsilon(x,\dot\pi)\Gamma(x,\dot\pi))d\gamma(x,\dot\pi) \geq 0.$$

On the other hand,

$$\begin{aligned} \mathrm{Tr}_N(R_\varepsilon(x,\dot\pi)\Gamma(x,\dot\pi)) &= \mathrm{Tr}_N(q(x)\Gamma(x,\dot\pi)) \\ &\quad + C_\varepsilon \mathrm{Tr}_K(p_0(x,\dot\pi)\Gamma(x,\dot\pi)p_0^*(x,\dot\pi)) + \varepsilon\,\mathrm{Tr}_N(\Gamma(x,\dot\pi)). \end{aligned}$$

By Proposition 7.3, we obtain that

$$\int_{\Omega\times\Sigma_1} \phi(x)\mathrm{Tr}_N(q(x)\Gamma(x,\dot\pi))d\gamma(x,\dot\pi) \geq -\varepsilon\int_{\Omega\times\Sigma_1} \phi(x)\mathrm{Tr}_N\Gamma(x,\dot\pi)\, d\gamma(x,\dot\pi),$$

which allows us to conclude.

It remains to prove (7.1). We first note that the injectivity of $\pi(\mathcal{R})$ implies that the property on q with respect to p is also satisfied by p_0, *i.e.* for all $x \in \Omega, \pi \in \widehat{G}$ and $h \in \mathcal{H}_\pi^N$,

$$p_0(x,\pi)h = 0 \Longrightarrow (q(x)h, h)_{\mathcal{H}_\pi^N} \geq 0.$$

We argue by contradiction: if (7.1) is false, then there exists $\varepsilon_0 > 0$ and sequences $(x_n)_{n\in\mathbb{N}}$ of Ω and $(\dot{\pi}_n)_{n\in\mathbb{N}}$ of Σ_1, together with unit vectors $h_n \in \mathcal{H}_{\pi_n}^N$ for each $n \in \mathbb{N}$ such that

$$(q(x_n)h_n, h_n)_{\mathcal{H}_{\pi_n}^N} + n\|p_0(x_n, \dot{\pi}_n)h_n\|^2_{\mathcal{H}_{\pi_n}^N} < -\varepsilon_0.$$

We interpret these sequences as the data of a sequence of states $(L_n)_{n\in\mathbb{N}}$ of the C^*-algebra $C^*((\dot{S}_c^0(\Omega))^{N,N}) = C^*(\dot{S}_c^0(\Omega))^{N,N}$, defined by

$$\forall \sigma \in C^*\left(\left(\dot{S}_c^0(\Omega)\right)^{N,N}\right) \qquad L_n(\sigma) = (\sigma(x_n, \dot{\pi}_n)h_n, h_n)_{\mathcal{H}_\pi^N}.$$

We have

$$\forall n \in \mathbb{N}, \quad L_n(q) + nL_n(p_0^* p_0) < -\varepsilon_0,$$

in particular $L_n(q) < -\varepsilon_0$ for all $n \in \mathbb{N}$. We extract a weak-$*$ converging subsequence from of $(L_n)_{n\in\mathbb{N}}$ and we denote its weak limit by L. Note that L is a state and it satisfies $L(q) \leq -\varepsilon_0$ and $L(p_0^* p_0) = 0$. Desintegrating L into pure states [21, §8.8] and combining Lemma 7.1 with Proposition 5.17, we obtain L as an integral of states of the form $\sigma \mapsto (V^*(x_0, \dot{\pi}_0)\sigma(x_0, \dot{\pi}_0)V(x_0, \dot{\pi}_0)v_0, v_0)_{\mathcal{H}_{\pi_0}}$ against a positive measure ν. Since $L(p_0^* p_0) = 0$, we have:

$$\begin{aligned}&(V^*(x_0, \dot{\pi}_0)p_0^* p_0(x_0, \dot{\pi}_0)V(x_0, \dot{\pi}_0)v_0, v_0)_{\mathcal{H}_{\pi_0}}\\ &= \|p_0(x_0, \dot{\pi}_0)V(x_0, \dot{\pi}_0)v_0\|^2_{\mathcal{H}_{\pi_0}^K} = 0 \quad \nu\text{-a.e.}\end{aligned}$$

But our hypothesis implies

$$(q(x_0)V(x_0, \dot{\pi}_0)v_0, V(x_0, \dot{\pi}_0)v_0)_{\mathcal{H}_\pi^N} \geq 0 \quad \nu\text{-a.e.},$$

and therefore $L(q) \geq 0$. This contradicts $L(q) \leq -\varepsilon_0 < 0$. Hence (7.1) is proved and this concludes the proof of Proposition 7.4. □

7.4. Link with *div-curl* results

Our result below gives a new approach to *div-curl lemma*, which had already been considered from a more geometric (sub-Riemannian) perspective in [6,7]. We assume that G is a stratified Lie group. We fix a canonical basis $X_1, \ldots, X_{n_1}$ on the first stratum. Then the divergence operator is defined by

$$\forall f = (f_1, \cdots, f_{n_1}) \in \mathcal{S}(G)^{n_1}, \quad \operatorname{div}(f) = X_1 f_1 + \cdots + X_{n_1} f_{n_1}.$$

We denote by $\pi(\operatorname{div})$ the symbol of the operator div. This symbol is a vector of n_1 symbols of order 1, *i.e.* $\operatorname{div} \in (S^1_{\rm cl}(G))^{1,n_1}$. We define the curl-property as follows:

Definition 7.5. Let Ω be an open subset of G. A $n_1 \times n_1$ matrix $\rho(x,\pi) \in (S^m_{\rm cl}(\Omega))^{n_1,n_1}$ of symbols of order m satisfies the *curl*-property when, for all $x \in \Omega$, $\pi \in \widehat{G}$, $h_1, h_2 \in (\mathcal{H}^\infty_\pi)^{n_1}$:

$$\pi(\mathrm{div}) \cdot h_1 = 0 \text{ and } \rho(x,\pi)h_2 = 0 \Longrightarrow (h_1,h_2)_{\mathcal{H}_\pi^{n_1}} = 0.$$

Recall that the spaces $L^2_s(\Omega, \mathrm{loc})$ were defined in Definition 4.23. We have the following *div-curl* type result.

Proposition 7.6. *Let Ω be an open subset of G. Let $(V_k)_{k\in\mathbb{N}}$ and $(W_k)_{k\in\mathbb{N}}$ be two pure families of $L^2(\Omega,\mathrm{loc})^{n_1}$ with weak limits V and W respectively. We assume that the sequence of scalar functions* $(\mathrm{div}(V_k))_{k\in\mathbb{N}}$ *converges to* $\mathrm{div}(V)$ *in $L^2_{-1}(\Omega,\mathrm{loc})$ as $k \to +\infty$ and that there exists $\rho(\pi) \in (S^m_{\rm cl}(\Omega))^{n_1,n_1}$ satisfying the curl-property* (cf. *Definition* 7.5) *such that* $(\mathrm{Op}(\rho)W_k)_{k\in\mathbb{N}}$ *converges to* $\mathrm{Op}(\rho)W$ *in $L^2_{-m}(\Omega,\mathrm{loc})^{n_1}$. Then the sequence of functions given by $x \mapsto \phi(x)(V_k(x),W_k(x))_{\mathbb{C}^{n_1}}$ converges to $x \mapsto \phi(x)(V(x),W(x))_{\mathbb{C}^{n_1}}$ in the sense of distribution on Ω as $k \to \infty$.*

Proof. We set:

$$p(x,\pi) := \begin{pmatrix} \mathcal{R}(\pi)^{m-1}\pi(\mathrm{div}) & 0 \\ 0 & \rho(x,\pi) \end{pmatrix} \in \mathcal{M}_{n_1+1,2n_1}\big(S^m_{\rm cl}(\Omega)\big)$$
$$\text{and} \quad q(x) := \begin{pmatrix} 0 & 0 \\ \mathrm{I}_{n_1} & 0 \end{pmatrix} \in (\mathbb{C})^{2n_1,2n_1}.$$

Then, for any $h = (h_1,h_2) \in (\mathcal{H}^\infty_\pi)^{n_1} \times (\mathcal{H}^\infty_\pi)^{n_1} = (\mathcal{H}^\infty_\pi)^{2n_1}$, we have

$$(q(x)h,h)_{\mathcal{H}_\pi^{2n_1}} = (h_1,h_2)_{\mathcal{H}_\pi^{n_1}},$$

and

$$p(x,\pi)h = 0 \quad \Longleftrightarrow \quad \pi(\mathrm{div}) \cdot h_1 = 0 \text{ and } \rho(x,\pi)h_2 = 0.$$

The curl-property allows us to apply Proposition 7.4 Part (i) to the sequence $(U_k)_{k\in\mathbb{N}}$ given by $U_k = (V_k, W_k) \in \mathbb{C}^{2n_1}$, the operator $P = \mathrm{Op}(p)$ and the matrix-valued function $q(x)$. The statement follows. □

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LAMA
Université Paris Est Créteil
Université Gustave Eiffel, CNRS
94010 Créteil, France
clotilde.fermanian@u-pec.fr

University of Bath
Department of Mathematical Sciences
Bath, BA2 7AY, UK
v.c.m.fischer@bath.ac.uk