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A uniqueness result for functions with zero fine gradient on quasiconnected and finely connected sets

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Abstract. We show that every Sobolev function in $W^{1,p}_{\mathrm{loc}}(U)$ on a p-quasiopen set $U \subset \mathbb{R}^n$ with a.e.-vanishing p-fine gradient is a.e.-constant if and only if U is p-quasiconnected. To prove this we use the theory of Newtonian Sobolev spaces on metric measure spaces, and obtain the corresponding equivalence also for complete metric spaces equipped with a doubling measure supporting a p-Poincaré inequality. On unweighted $\mathbb{R}^n$, we also obtain the corresponding result for p-finely open sets in terms of p-fine connectedness, using a deep result by Latvala.

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1. Introduction

One of the basic properties of derivatives and gradients is that they control the oscillation, so that every sufficiently nice function with vanishing gradient in an open connected set must be constant therein. This is used in many proofs and holds in rather general situations, such as for distributions (Hörmander [11, Theorem 3.1.4]) and Sobolev functions, including those on weighted $\mathbb{R}^n$ with a p-admissible measure (Heinonen-Kilpeläinen-Martio [10, Lemma 1.16]).

In this note we address a similar question on quasiopen and finely open sets in the context of the corresponding Sobolev spaces. Such sets and spaces are fundamental in the fine potential theory as well as for fine properties of solutions of various partial differential equations, see, *e.g.*, Malý-Ziemer [14] and the references therein. In fact, our main uniqueness result (Theorem 1.1) is used in the recent paper Fusco-Mukherjee-Zhang [9] in connection with eigenvalue problems on quasiopen sets; and it was a question by Fusco [8] on the validity of the implication (c) $\Rightarrow$ (a) in Theorem 1.1 that triggered this note.

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We will be able to prove our results in the setting of metric spaces, under the usual assumptions, but this is not the primary goal of this note. This general approach demonstrates the strength of the metric space theory, since the proofs below turn out to be very natural. In particular, arguments using upper gradients and curve families play a crucial role. We will also rely on several recent results from the fine potential theory on metric spaces.

Let $1 < p < \infty$. A set $U \subset \mathbb{R}^n$ is *p-quasiopen* if for every $\varepsilon > 0$ there is an open set G such that $G \cup U$ is open and $C_p(G) < \varepsilon$, where C_p is the Sobolev p-capacity associated with the Sobolev space $W^{1,p}(\mathbb{R}^n)$. (This can equivalently be defined using various other related capacities.) The study of Sobolev spaces and nonlinear partial differential equations on p-quasiopen sets was initiated by Kilpeläinen and Malý in [12]; see the introductions in [4] and [5] for more on the history.

The p-quasiopen sets are preserved under taking finite intersections and countable unions, but not under arbitrary unions. (For example, since points in $\mathbb{R}^n$ have zero p-capacity, $1 < p \le n$, and are thus p-quasiopen, every set is a union of p-quasiopen sets, but not all sets are p-quasiopen.)

Since the p-quasiopen sets *do not* form a topology, it is not completely obvious how to define p-quasiconnectedness, and there seem to be at least two natural definitions. Following Fuglede [7, page 164] we say that a p-quasiopen set U is *p-quasiconnected* if the only subsets of U, that are both p-quasiopen and relatively p-quasiclosed (*i.e.* their complement within U is also p-quasiopen), are the sets with zero p-capacity and their complements (within U). This definition was also used by Adams-Lewis [1] in the nonlinear potential theory. Equivalently, U is p-quasiconnected if it cannot be written as a union of two disjoint p-quasiopen sets with positive p-capacity.

The p-quasiopen sets are closely related to p-finely open sets, which are defined using the Wiener type integral (2.3) and form the coarsest topology making all p-superharmonic functions continuous; called the *p-fine topology*. More precisely, U is p-quasiopen if and only if it can be written as a union $U = V \cup E$, where V is p-finely open and $C_p(E) = 0$. Another recent characterization of p-quasiopen sets is that they are precisely the p-path open sets, see Björn-Björn-Malý [6, Theorem 1.1] and Shanmugalingam [15, Remark 3.5].

With this connection to p-finely open sets in mind it seems natural to say that a p-quasiopen set U is *weakly p-quasiconnected* if it can be written as a union $U = V \cup E$, where $C_p(E) = 0$ and V is a p-finely connected p-finely open set (*i.e.* connected in the p-fine topology). A consequence is that a p-finely open set is p-finely connected if and only if it is weakly p-quasiconnected; the nontrivial "if" part follows from Lemma 3.4.

The following is our main result.

Theorem 1.1. *Assume that $U \subset \mathbb{R}^n$ is a p-quasiopen set in unweighted $\mathbb{R}^n$. Then the following are equivalent*:

(a) *If $u \in W^{1,p}_{\mathrm{loc}}(U)$ and $\nabla u = 0$ a.e., then there is a constant c such that $u = c$ a.e. in U;*

(b) *U is p-quasiconnected;*
(c) *U is weakly p-quasiconnected;*
(d) *If $U = V \cup E$, where V is p-finely open and $C_p(E) = 0$, then V is p-finely connected.*

Here, $W^{1,p}_{\rm loc}(U)$ is the local Sobolev space defined by Kilpeläinen-Malý [12] for p-quasiopen sets U, and ∇u is the p-fine gradient also introduced in [12]. On open sets, these notions coincide with the usual Sobolev space and weak (or distributional) gradient, respectively.

If U is open and $u \in W^{1,p}_{\rm loc}(U)$ with $\nabla u = 0$ a.e., then it follows from the p-Poincaré inequality (2.1) that u is locally a.e.-constant, and hence necessarily a.e.-constant if and only if U is connected, *i.e.* (a) is equivalent to U being connected in this case. (One direction of this is [10, Lemma 1.16].) Hence an *open* set is connected if and only if it is p-quasiconnected. We thus recover Corollary 1 in Adams-Lewis [1] for first-order Sobolev spaces; which is the "only if" part of this equivalence, and which they obtain also for higher-order Sobolev spaces in unweighted $\mathbb{R}^n$. Similar facts are true also in metric spaces.

To prove Theorem 1.1 we will use the theory of Newtonian Sobolev spaces on metric measure spaces, including several recent results in the fine potential theory on metric spaces. In fact, in general metric spaces (assuming the rather standard assumptions of completeness, doubling and a p-Poincaré inequality), we show that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d) $\Rightarrow$ (c). (The statement (a) needs to be slightly reformulated, see Theorem 3.1.) On the other hand, the implication (c) $\Rightarrow$ (d) is equivalent to a statement about p-fine connectedness, whose truth on unweighted $\mathbb{R}^n$ follows from a deep result by Latvala [13]. We do not know whether Latvala's result can be generalized to metric spaces, or even to weighted $\mathbb{R}^n$.

For p-finely open sets, (part of) Theorem 1.1 takes the following form.

Theorem 1.2. *Assume that $V \subset \mathbb{R}^n$ is a p-finely open set in unweighted $\mathbb{R}^n$. Then the following are equivalent*:

(a) *If $u \in W^{1,p}_{\rm loc}(V)$ and $\nabla u = 0$ a.e., then there is a constant c such that $u = c$ a.e. in V;*
(b) *V is p-quasiconnected;*
(c) *V is p-finely connected.*

On metric spaces we know that (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c), but whether (c) $\Rightarrow$ (b) remains an open question.

2. Preliminaries

To keep this note short, we follow the notation from Björn-Björn-Latvala [4], without repeating all the discussion here; see [4] for more references. As usual, we assume that $1 < p < \infty$ and that X is a complete metric space equipped with a doubling measure μ which supports a p-Poincaré inequality, i.e there are constants $C, \lambda > 0$ such that for all open balls $B = B(x,r)$, we have

$$\mu(2B) \le C\mu(B)$$

and, setting $u_B = \int_B u\,d\mu/\mu(B)$,

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \le Cr\biggl(\frac{1}{\mu(B)}\int_{\lambda B} g^p\,d\mu\biggr)^{1/p} \tag{2.1}$$

holds for all integrable functions u on $\lambda B := B(x,\lambda r)$ and their p-weak upper gradients g. Here, $g : X \to [0,\infty]$ is a *p-weak upper gradient* of $u : X \to [-\infty,\infty]$ if for Mod_p-almost every curve γ in X,

$$|u(x)-u(y)| \le \int_\gamma g\,ds, \tag{2.2}$$

where x and y are the end points of γ and the left-hand side is interpreted as ∞ whenever at least one of the terms therein is infinite. By "holding for Mod_p-almost every curve γ" we mean that there is $\rho \in L^p(X)$ such that $\int_\gamma \rho\,ds = \infty$ for all γ, where (2.2) fails. All curves considered here are nonconstant, compact and rectifiable, and thus can be parameterized by arc length ds.

Having defined the p-weak upper gradients, the *Newtonian Sobolev space* $N^{1,p}(X)$ is defined as the collection of all $u \in L^p(X)$ having a p-weak upper gradient $g \in L^p(X)$. Every $u \in N^{1,p}(X)$ has a *minimal* p-weak upper gradient g_u (well-defined up to sets of measure zero) such that $g_u \le g$ a.e. for every p-weak upper gradient $g \in L^p(X)$ of u.

The space $N^{1,p}(U)$ is defined similarly for p-quasiopen $U \subset X$, but in that case (2.2) is only required for Mod_p-almost every curve γ within U. This is possible since p-quasiopen sets are measurable, by Björn-Björn [3, Lemma 9.3]. It was shown in [3, Proposition 3.5] that if U is p-quasiopen then p-weak upper gradients with respect to U coincide with those taken with respect to the whole space X.

Functions in $N^{1,p}(X)$ (and in $N^{1,p}(U)$ if U is p-quasiopen) are precisely defined up to sets of zero p-capacity, which in turn is defined for an arbitrary $E \subset X$ as

$$C_p(E) = \inf_u \int_X \bigl(|u|^p + g_u^p\bigr)\,d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

A set $V \subset X$ is *p-finely open* if $X \setminus V$ is p-thin at every $x \in V$, *i.e.*, the Wiener type integral

$$\int_0^1 \biggl(\frac{\operatorname{cap}_p(B(x,r)\setminus V, B(x,2r))}{\operatorname{cap}_p(B(x,r), B(x,2r))}\biggr)^{1/(p-1)} \frac{dr}{r} < \infty. \tag{2.3}$$

Here, the variational p-capacity cap_p is for bounded open A with $C_p(X \setminus A) > 0$ defined by

$$\text{cap}_p(E, A) = \inf_u \int_X g_u \, d\mu,$$

with the infimum taken over all $u \in N^{1,p}(X)$ such that $\chi_E \le u \le \chi_A$.

Since, under our assumptions, cap_p and C_p have the same zero sets, it follows that p-fine openness is preserved under removing sets of zero p-capacity, as such sets do not influence the Wiener type integral (2.3). This also shows that the complement of a set of zero p-capacity is not p-thin at any $x \in X$, so nonempty p-finely open sets must have positive p-capacity.

Adding sets of zero p-capacity to p-finely open sets does not necessarily preserve p-fine openness, but it produces p-quasiopen sets: By Theorem 1.4 (a) in Björn-Björn-Latvala [5], a set U is *p-quasiopen* if and only if $U = V \cup E$, where V is p-finely open and $C_p(E) = 0$. Typically, this decomposition is not unique.

The p-finely open sets define the p-fine topology, and a p-finely open set U is *p-finely connected* if it is connected in this topology, *i.e.*, it cannot be written as a disjoint union of nonempty p-finely open sets.

For p-quasiopen sets U, it is natural to define the *local Newtonian Sobolev space* $N^{1,p}_{\text{fine-loc}}(U)$, which consists of all functions $u : U \to [-\infty, \infty]$ such that $u \in N^{1,p}(V)$ for every p-finely open p-strict subset $V \Subset U$. Here, V is a *p-strict subset* of U if there is $v \in N^{1,p}(X)$ such that $\chi_V \le v \le \chi_U$, or equivalently, if $\text{cap}_p(V, U) < \infty$. The space $L^p_{\text{fine-loc}}(U)$ is defined similarly.

Every $u \in N^{1,p}_{\text{fine-loc}}(U)$ has a *minimal* p-weak upper gradient $g_u \in L^p_{\text{fine-loc}}(U)$ (well-defined up to sets of measure zero) such that $g_u \le g$ a.e. for every p-weak upper gradient $g \in L^p_{\text{fine-loc}}(U)$ of u, see Björn-Björn-Latvala [4, Section 5]. (The results in [4] are for the even larger space $N^{1,p}_{\text{quasi-loc}}(U)$, but can easily be adapted to $N^{1,p}_{\text{fine-loc}}(U)$.)

For p-quasiopen $U \subset \mathbb{R}^n$, the spaces $N^{1,p}(U)$ and $N^{1,p}_{\text{fine-loc}}(U)$ are essentially the Sobolev spaces $W^{1,p}(U)$ and $W^{1,p}_{\text{loc}}(U)$, defined by Kilpeläinen-Malý [12], see the discussion after Corollary 3.2 for more details. (We remark that the space $N^{1,p}_{\text{fine-loc}}(U)$ is more natural in fine potential theory than the smaller space $N^{1,p}_{\text{loc}}(U)$ consisting of those functions u such that for every $x \in U$ there is $r > 0$ such that $u \in N^{1,p}(U \cap B(x, r))$.)

Similarly to $N^{1,p}(U)$, also functions in $N^{1,p}_{\text{fine-loc}}(U)$ are precisely defined up to sets of p-capacity zero, as seen in the following lemma.

Lemma 2.1. *Let U be p-quasiopen and $u, v \in N^{1,p}_{\text{fine-loc}}(U)$. If $u = v$ a.e. in U, then $u = v$ p-q.e. in U,* i.e. $C_p(\{x \in U : u(x) \ne v(x)\}) = 0$.

Proof. Björn-Björn-Latvala [4, Theorem 4.4] shows that u and v are p-quasicontinuous, both with respect to C_p and C_p^U, where C_p^U is obtained by regarding U as a metric space in its own right. In Björn-Björn [2, Proposition 5.23] (applied to U)

then implies that $u = v$ C_p^U-q.e., and thus also p-q.e., since C_p and C_p^U have the same zero sets (by Björn-Björn-Malý [6, Proposition 4.2]). □

3. Proofs

To prove Theorem 1.1, we first obtain the following result in general metric spaces.

Theorem 3.1. *Let $U \subset X$ be a p-quasiopen set. Then the following are equivalent*:

(i) *If $u \in N^{1,p}_{\text{fine-loc}}(U)$ and $g_u = 0$ a.e., then there is a constant c such that $u = c$ a.e. in U;*
(ii) *U is p-quasiconnected;*
(iii) *If $U = V \cup E$, where V is p-finely open and $C_p(E) = 0$, then V is p-finely connected.*

Note that in (iii), the p-fine connectedness should hold for every decomposition of U. Latvala's result (Theorem 3.3 below) shows that in unweighted $\mathbb{R}^n$, it can equivalently be assumed only for some decomposition $U = V \cup E$; we do not know if this is true in metric spaces, nor in weighted $\mathbb{R}^n$.

It follows from the proof below that the space $N^{1,p}_{\text{fine-loc}}(U)$ in (i) can be replaced by the smaller space $N^{1,p}_{\text{loc}}(U)$, or even the smaller space consisting of those functions u such that $u \in N^{1,p}(U \cap B)$ for every ball $B \subset X$ (since $\chi_{U_1} \in N^{1,p}(U \cap B)$ in that case). If U is bounded, then the space $N^{1,p}(U)$ can be used instead, but this is not possible in general as can be seen by considering $U = \{(x_1, x_2) \in \mathbb{R}^n : x_1 \neq 0\}$, which is not p-quasiconnected and yet every $u \in N^{1,p}(U)$ with $g_u = 0$ a.e. must be a.e.-constant.

In (i) it is equivalent to require that $u = c$ p-q.e. in U by Lemma 2.1, while (iii) can equivalently be formulated as follows: If $C_p(E) = 0$ and $U \setminus E$ is p-finely open, then $U \setminus E$ is p-finely connected.

If a p-weak upper gradient of u is modified on a set of measure zero it remains a p-weak upper gradient of u. In particular, the condition $g_u = 0$ a.e. in (i) is equivalent to requiring that zero is a p-weak upper gradient of u. We will use this fact in the proof below.

Proof. $\neg$ (ii) $\Rightarrow \neg$ (iii) By assumption, there is a p-quasiopen set $U_1 \subset U$ such that $U_2 = U \setminus U_1$ is also p-quasiopen and in addition $C_p(U_j) > 0$, $j = 1, 2$. We can write $U_j = V_j \cup E_j$, where V_j is p-finely open and $C_p(E_j) = 0$. Then $C_p(V_j) = C_p(U_j) > 0$, and thus V_j is nonempty, $j = 1, 2$. Letting $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ shows that (iii) fails.

$\neg$ (iii) $\Rightarrow \neg$ (ii) Let $U = V \cup E$, where $C_p(E) = 0$ and V is a p-finely open set which is not p-finely connected. Then $V = V_1 \cup V_2$ for some nonempty disjoint p-finely open sets V_1 and V_2. Since $V_2 \cup E$ is p-quasiopen, $V_1 = U \setminus (V_2 \cup E)$ is relatively p-quasiclosed within U, as well as p-quasiopen. As $C_p(V_j) > 0$, $j = 1, 2$, it follows that U is not p-quasiconnected.

$\neg$ (i) $\Rightarrow \neg$ (ii) Let $u \in N^{1,p}_{\text{fine-loc}}(U)$ be a function with $g_u = 0$ a.e. and assume that there is $m \in \mathbb{R}$ such that $C_p(U_\pm) > 0$, where

$$U_+ = \{x \in U : u(x) > m\} \quad \text{and} \quad U_- = \{x \in U : u(x) \le m\}.$$

Since zero is a p-weak upper gradient of u, there are Mod_p-almost no rectifiable curves starting in U_+ and ending in U_-. Hence zero is a p-weak upper gradient of $v = \chi_{U_+}$ defined on U, and thus $v \in N^{1,p}_{\text{fine-loc}}(U)$.

By Björn-Björn-Latvala [4, Theorem 4.4], v is p-quasicontinuous. Hence the level sets

$$\left\{x \in U : v(x) > \tfrac{1}{2}\right\} = U_+ \quad \text{and} \quad \left\{x \in U : v(x) < \tfrac{1}{2}\right\} = U_-,$$

which together constitute U, are p-quasiopen, by Björn-Björn-Malý [6, Proposition 3.4]. Since $C_p(U_\pm) > 0$, (ii) fails.

$\neg$ (ii) $\Rightarrow \neg$ (i) As U is not p-quasiconnected, it can be written as a union of two disjoint p-quasiopen sets U_1 and U_2 with positive p-capacity. We shall show that the characteristic function $u = \chi_{U_1}$ has zero as a p-weak upper gradient (within U), and consequently belongs to $N^{1,p}_{\text{fine-loc}}(U)$. Since it is not p-q.e.-constant, and thus not a.e.-constant either (by Lemma 2.1), this violates (i).

To see that zero is a p-weak upper gradient of u, it suffices to show that there are Mod_p-almost no curves within U passing from U_1 to U_2. As U_1 and U_2 are p-quasiopen, Shanmugalingam [15, Remark 3.5] implies that they are p-path open (within U), *i.e.* for Mod_p-almost every curve $\gamma : [0, l_\gamma] \to U$, the preimages $\gamma^{-1}(U_j)$, $j = 1, 2$, are relatively open (and nonempty and disjoint) in

$$[0, l_\gamma] = \gamma^{-1}(U_1) \cup \gamma^{-1}(U_2).$$

But this is impossible, so there are no such curves. □

The following direct consequence of the equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 3.1 motivates our terminology.

Corollary 3.2. *Every p-quasiopen p-quasiconnected set $U \subset X$ is weakly p-quasiconnected.*

The following result about preserving p-fine connectedness was proved by Latvala [13] in unweighted $\mathbb{R}^n$, while it still remains open in more general situations (including $\mathbb{R}^n$ with p-admissible weights).

Theorem 3.3 (Latvala [13, Theorem 1.1]). *Let $V \subset \mathbb{R}^n$ (unweighted) be p-finely open and p-finely connected, $1 < p \le n$. If $C_p(E) = 0$ then $V \setminus E$ is also p-finely connected (and p-finely open).*

(A similar statement with $p > n$ is trivial, since in that case the p-fine topology is just the usual Euclidean one.) The converse implication is much easier and holds in general metric spaces satisfying our assumptions:

Lemma 3.4. *Let $V \subset X$ be p-finely open and assume that $V \setminus E$ is p-finely connected for some E with $C_p(E) = 0$. Then V is also p-finely connected.*

Proof. Assume that V is not p-finely connected, *i.e.* it can be written as $V = V_1 \cup V_2$, where V_1 and V_2 are nonempty disjoint p-finely open sets. In particular, they have positive p-capacity. Then $V_j \setminus E$ are also p-finely open and nonempty, so $V \setminus E = (V_1 \setminus E) \cup (V_2 \setminus E)$ cannot be p-finely connected. □

As already mentioned, the Newtonian Sobolev space $N^{1,p}$ is more precisely defined than the traditional Sobolev spaces $W^{1,p}$ on $\mathbb{R}^n$. For a p-quasiopen set U in unweighted $\mathbb{R}^n$, the space denoted $W^{1,p}_{\mathrm{loc}}(U)$ in Kilpeläinen-Malý [12] coincides with the space

$$\widehat{N}^{1,p}_{\text{fine-loc}}(U) = \Big\{u : u = v \text{ a.e. for some } v \in N^{1,p}_{\text{fine-loc}}(U)\Big\},$$

see Björn-Björn-Latvala [4, Theorem 5.7]. Similarly, functions in $W^{1,p}(U)$ are a.e. equal to functions from $N^{1,p}(U)$. Moreover, the modulus of the p-fine gradient ∇u, introduced in [12], coincides a.e. with the minimal p-weak upper gradient g_v of v, *i.e.* $g_v = |\nabla u|$ a.e., see [4, Theorems 5.3 and 5.7]. The situation is similar in weighted $\mathbb{R}^n$ with a p-admissible weight, provided that ∇u stands for the corresponding weighted p-fine gradient; *cf.* the discussion in Heinonen-Kilpeläinen-Martio [10, page 13]. Thus on unweighted and weighted $\mathbb{R}^n$ (with a p-admissible weight), (i) in Theorem 3.1 is equivalent to (a) in Theorem 1.1. We are now ready to prove our main result.

Proof of Theorem 1.1. (a) ⇔ (b) ⇔ (d) These equivalences follow from Theorem 3.1, in view of the discussion above.

(b) ⇒ (c) This follows from Corollary 3.2.

¬ (d) ⇒ ¬ (c) By assumption, $U = V \cup E$, where $C_p(E) = 0$ and V is p-finely open but not p-finely connected. Let $U = V' \cup E'$ be any other decomposition of U into a p-finely open set V' and a set E' with $C_p(E') = 0$. Lemma 3.4 then implies that $V' \setminus (E \cup E') = V \setminus (E \cup E')$ is not p-finely connected. An application of Latvala's theorem 3.3 then shows that V' is not p-finely connected either, *i.e.* (c) fails. □

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