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Equivariant extensions of $\mathbb{G}_a$-torsors over punctured surfaces

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Abstract. Motivated by the study of the structure of algebraic actions of the additive group on affine threefolds X, we consider a special class of such varieties whose algebraic quotient morphisms $X \to X/\!/\mathbb{G}_a$ restrict to principal homogeneous bundles over the complement of a smooth point of the quotient. We establish basic general properties of these varieties and construct families of examples illustrating their rich geometry. In particular, we give a complete classification of a natural subclass consisting of threefolds X endowed with proper $\mathbb{G}_a$-actions, whose algebraic quotient morphisms $\pi : X \to X/\!/\mathbb{G}_a$ are surjective with only isolated degenerate fibers, all isomorphic to the affine plane $\mathbb{A}^2$ when equipped with their reduced structures.

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1. Introduction

Algebraic actions of the complex additive group $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{C}}$ on normal complex affine surfaces S are essentially fully understood: the ring of invariants $\mathcal{O}(S)^{\mathbb{G}_{a,\mathbb{C}}}$ is a finitely generated algebra whose spectrum is a smooth affine curve $C = S/\!/\mathbb{G}_a$, and the inclusion $\mathcal{O}(S)^{\mathbb{G}_a} \subset \mathcal{O}(S)$ defines a surjective morphism $\pi : S \to C$ whose general fibers coincide with general orbits of the action, hence are isomorphic to the affine line $\mathbb{A}^1$ on which $\mathbb{G}_a$ acts by translations. The degenerate fibers of such $\mathbb{A}^1$-fibrations are known to consist of finite disjoint unions of smooth affine curves isomorphic to $\mathbb{A}^1$ when equipped with their reduced structure. A complete description of isomorphism classes of germs of invariant open neighborhoods of irreducible components of such fibers was established by Fieseler [8].

In contrast, very little is known so far about the structure of $\mathbb{G}_a$-actions on complex normal affine threefolds. For such a threefold X, the ring of invariants $\mathcal{O}(X)^{\mathbb{G}_a}$ is again finitely generated [13] and the morphism $\pi : X \to S$ induced

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by the inclusion $\mathcal{O}(X)^{\mathbb{G}_a} \subset \mathcal{O}(X)$ is an $\mathbb{A}^1$-fibration over a normal affine surface S. But in general, π is neither surjective nor equidimensional. Furthermore, it can have degenerate fibers over closed subsets of pure codimension 1 as well as of codimension 2. All of these possible degenerations are illustrated by the following example:

The restriction of the projection $\mathrm{pr}_{x,y}$ to the smooth threefold $X = \{x^2(x-1)v + yu^2 - x = 0\}$ in $\mathbb{A}^4$ is an $\mathbb{A}^1$-fibration $\pi : X \to \mathbb{A}^2$ which coincides with the algebraic quotient morphism of the $\mathbb{G}_a$-action on X associated to the locally nilpotent derivation $\partial = x^2(x-1)\partial_u - 2yu\partial_v$ of its coordinate ring. The restriction of π over the principal open subset $x^2(x-1) \neq 0$ of $\mathbb{A}^2$ is a trivial principal $\mathbb{G}_a$-bundle, but the fibers of π over the points $(1,0)$ and $(0,0)$ are respectively empty and isomorphic to $\mathbb{A}^2$. Furthermore, for every $y_0 \neq 0$, the inverse images under π of the points $(0, y_0)$ and $(1, y_0)$ are respectively isomorphic to $\mathbb{A}^1$ but with multiplicity 2, and to the disjoint union of two reduced copies of $\mathbb{A}^1$.

Partial results concerning the structure of one-dimensional degenerate fibers of $\mathbb{G}_a$-quotient $\mathbb{A}^1$-fibrations were obtained by Gurjar-Masuda-Miyanishi [9]. In the present article, as a step towards the understanding of the structure of two-dimensional degenerate fibers, we consider a particular type of non-equidimensional surjective $\mathbb{G}_a$-quotient $\mathbb{A}^1$-fibrations $\pi : X \to S$ which have the property that they restrict to $\mathbb{G}_a$-torsors[1] over the complement of a finite set of smooth points in S. These are simpler than the general case illustrated in the previous example since they do not admit additional degeneration of their fibers over curves in S passing through the given points. The local and global study of some classes of such fibrations was initiated by the second author [10]. He constructed in particular many examples of $\mathbb{G}_a$-quotient $\mathbb{A}^1$-fibrations on smooth affine threefolds X with image $\mathbb{A}^2$ whose restrictions over the complement of the origin are isomorphic to the geometric quotient $\mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a$ of SL_2 by the action of unitary upper triangular matrices.

One of the simplest examples of this type is the smooth threefold $X_0 \subset \mathbb{A}^5_{x,y,p,q,r}$ defined by the equations

$$X_0 : \begin{cases} xr - yq = 0 \\ yp - x(q-1) = 0 \\ pr - q(q-1) = 0 \end{cases}$$

and equipped with the $\mathbb{G}_a$-action associated to the locally nilpotent $\mathbb{C}[x,y]$-derivation $x^2\partial_p + xy\partial_q + y^2\partial_r$ of its coordinate ring. The equivariant open embedding $\mathrm{SL}_2 = \{xv - yu = 1\} \hookrightarrow X_0$ is given by $(x,y,u,v) \mapsto (x,y,xu,xv,yv)$. The $\mathbb{G}_a$-quotient morphism coincides with the surjective $\mathbb{A}^1$-fibration $\pi_0 = \mathrm{pr}_{x,y} : X_0 \to \mathbb{A}^2$. Its restriction over $\mathbb{A}^2 \setminus \{(0,0)\}$ is isomorphic to the quotient morphism $\mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a$, while its fiber over $(0,0)$ is the smooth quadric $\{pr - q(q-1) = 0\} \subset \mathbb{A}^3_{p,q,r}$, isomorphic to the quotient $\mathrm{SL}_2/\mathbb{G}_m$ of SL_2 by the action of its

[1] Sometimes also referred to as Zariski locally trivial principal $\mathbb{G}_a$-bundles.

diagonal torus (see Example 3.1). A noteworthy property of this example is that the $\mathbb{G}_a$-quotient morphism $\pi : X_0 \to \mathbb{A}^2$ factors through a locally trivial $\mathbb{A}^1$-bundle $\rho : X_0 \to \tilde{\mathbb{A}}^2$ over the the blow-up $\tau : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ of the origin.

It is a general fact that every irreducible component of a degenerate fiber of pure codimension one of a $\mathbb{G}_a$-quotient $\mathbb{A}^1$-fibration $\pi : X \to S$ on a smooth affine threefold is an $\mathbb{A}^1$-uniruled affine surface (see Proposition 2.3). We do not know whether every $\mathbb{A}^1$-uniruled surface can be realized as an irreducible component of the degenerate fiber of a $\mathbb{G}_a$-extension. But besides the smooth affine quadric $\mathrm{SL}_2/\mathbb{G}_m$ appearing in the previous example, the following one confirms that the affine plane $\mathbb{A}^2$ can also be realized (see also Examples 2.4 and 2.5 for other types of surfaces that can be realized): Let $X_1 \subset \mathbb{A}^5_{x,y,z_1z_2,w}$ be the smooth affine threefold defined by the equations

$$X_1 : \begin{cases} xw - y(yz_1 + 1) = 0 \\ xz_2 - z_1(yz_1 + 1) = 0 \\ z_1 w - yz_2 = 0, \end{cases}$$

equipped with the $\mathbb{G}_a$-action associated to the locally nilpotent $\mathbb{C}[x, y]$-derivation $x\partial_{z_1} + (2yz_1 + 1)\partial_{z_2} + y^2\partial_w$ of its coordinate ring. The morphism $\mathrm{SL}_2 \hookrightarrow X_1$ given by $(x, y, u, v) \mapsto (x, y, u, uv, yv)$ is an equivariant open embedding. The $\mathbb{G}_a$-quotient morphism coincides with the surjective $\mathbb{A}^1$-fibration $\pi_1 = \mathrm{pr}_{x,y} : X_1 \to \mathbb{A}^2$, whose fiber over the origin is the affine plane $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[z_2, w])$ and whose restriction over $\mathbb{A}^2 \setminus \{(0, 0)\}$ is again isomorphic to the quotient morphism $\mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a$. A special additional feature is that the $\mathbb{G}_a$-action on X_1 extending that on SL_2 is not only fixed point free but actually *proper*: its geometric quotient $X_1/\mathbb{G}_a$ is separated. One can indeed check that $X_1/\mathbb{G}_a$ is isomorphic to the complement $\tilde{\mathbb{A}}^2 \setminus \{o_1\}$ of a point o_1 supported on the exceptional divisor E of the blow-up $\tilde{\mathbb{A}}^2$ of $\mathbb{A}^2$ at the origin (see Example 4.2).

Relaxing the hypothesis that the $\mathbb{A}^1$-fibration $\pi : X \to S$ should arise as the quotient of a $\mathbb{G}_a$-action on an affine threefold X and considering the broader problem of describing the geometry of degeneration of $\mathbb{A}^1$-fibrations over irreducible closed subsets of pure codimension two of their base, we are led to the following more general notion:

Definition. Let (S, o) be a pair consisting of a normal separated 2-dimensional scheme S essentially of finite type over a field k of characteristic zero and of a closed point o contained in the smooth locus of S. A $\mathbb{G}_a$-*extension* of a $\mathbb{G}_a$-torsor $\rho : P \to S \setminus \{o\}$ is a $\mathbb{G}_a$-equivariant open embedding $j : P \hookrightarrow X$ into an integral scheme X equipped with a surjective morphism $\pi : X \to S$ of finite type and a $\mathbb{G}_{a,S}$-action, such that the commutative diagram

$$\begin{array}{ccc} P & \overset{j}{\hookrightarrow} & X \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ S \setminus \{o\} & \hookrightarrow & S \end{array}$$

is cartesian.

The examples X_0 and X_1 above provide motivation to study the following natural classes of $\mathbb{G}_a$-extensions $\pi : X \to S$ of a $\mathbb{G}_a$-torsor $\rho : P \to S \setminus \{o\}$, which are arguably the simplest possible types of $\mathbb{G}_a$-extensions from the viewpoints of their global geometry and of the properties of their $\mathbb{G}_a$-actions:

- (Type I) Extensions for which π factors through a locally trivial $\mathbb{A}^1$-bundle over the blow-up $\tau : \tilde{S} \to S$ of the point o, the fiber $\pi^{-1}(o)$ being then the total space of a locally trivial $\mathbb{A}^1$-bundle over the exceptional divisor of τ;
- (Type II) Extensions for which $\pi^{-1}(o)_{\mathrm{red}}$ is isomorphic to the affine plane $\mathbb{A}^2_\kappa$ over the residue field κ of S at o, X is smooth along $\pi^{-1}(o)$ and the $\mathbb{G}_{a,S}$-action on X is proper.

The first main result of this article, Proposition 3.3 and Theorem 3.7, is a complete description of $\mathbb{G}_a$-extensions of Type I together with an effective characterization of which among them have the additional property that the morphism $\pi : X \to S$ is affine. Our second main result, Theorem 4.8, consists of a classification of $\mathbb{G}_a$-extensions of Type II, under the additional assumption that the morphism $\pi : X \to S$ is quasi-projective. More precisely, given a $\mathbb{G}_a$-torsor $\rho : P \to S \setminus \{o\}$ and a $\mathbb{G}_a$-extension $\pi : X \to S$ with proper $\mathbb{G}_{a,S}$-action and reduced fiber $\pi^{-1}(o)_{\mathrm{red}}$ isomorphic to $\mathbb{A}^2_\kappa$, we establish that the possible geometric quotients $S' = X/\mathbb{G}_a$ belong to a very special class of surfaces isomorphic to open subsets of blow-ups of S with centers over o which we fully describe in Section 4.1. We show conversely that every such surface is indeed the geometric quotient of a $\mathbb{G}_a$-extension of $\rho : P \to S \setminus \{o\}$ with the desired properties.

In a second step, we tackle the question of existence of $\mathbb{G}_a$-extensions $\pi : X \to S$ of Type II for which the structure morphism π is not only quasi-projective but affine. Our method to produce extensions with this property is inspired by the observation that the threefolds X_0 and X_1 above are not only birational to each other due to the property that they both contain SL_2 as open subset, but in fact that the birational morphism

$$\eta : X_1 \to X_0, \quad (x, y, z_1, z_2, w) \mapsto (x, y, p, q, r) = (x, y, xz_1, yz_1 + 1, w)$$

expresses X_1 as a $\mathbb{G}_a$-equivariant affine modification of X_0 in the sense of Kaliman and Zaidenberg [11]. This suggests that extensions of Type II for which X is affine over S could be obtained as equivariant affine modification in a suitable generalized sense from extensions of Type I with the same property. Using this technique, we are able to show in Theorem 4.9 that for each possible geometric quotient S' above, there exist $\mathbb{G}_a$-extensions $\pi : X \to S$ of $\rho : P \to S \setminus \{o\}$ with geometric quotient $X/\mathbb{G}_a = S'$ such that π is an affine morphism.

As an application towards the initial question of the structure of $\mathbb{G}_a$-quotient $\mathbb{A}^1$-fibrations on affine threefolds, we in particular derive from this construction the existence of uncountably many pairwise non-isomorphic smooth affine threefolds X endowed with proper $\mathbb{G}_a$-actions, containing SL_2 as an invariant open subset with complement $\mathbb{A}^2$, whose geometric quotients are smooth quasi-projective surfaces which are not quasi-affine, and whose algebraic quotients are all isomorphic to $\mathbb{A}^2$.

The scheme of the article is the following. The Section 2 begins with a review of general properties of $\mathbb{G}_a$-extensions. We then set up the basic tools which will be used throughout the article: locally trivial $\mathbb{A}^1$-bundles with additive group actions and equivariant affine birational morphisms between these. In Section 3, we study $\mathbb{G}_a$-extensions of Type I. The last section is devoted to the classification of quasi-projective $\mathbb{G}_a$-extensions of Type II.

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2. Preliminaries

Notation 2.1. In the rest of the article, the term *surface* refers to a normal separated 2-dimensional scheme essentially of finite type over a field k of characteristic zero. A *punctured surface* $S_* = S \setminus \{o\}$ is the complement of a closed point o contained in the smooth locus of a surface S. We denote by κ the residue field of S at o.

Remark 2.2. We do not require that the residue field κ of S at o is an algebraic extension of k. For instance, S can very well be the spectrum of the local ring $\mathcal{O}_{X,Z}$ of an arbitrary smooth k-variety X at an irreducible closed subvariety Z of codimension two in X and o its unique closed point, in which case the residue field κ is isomorphic to the field of rational functions on Z.

In this section, we first review basic geometric properties of equivariant extensions of $\mathbb{G}_a$-torsors over punctured surfaces. We then collect various technical results on additive group actions on affine-linear bundles of rank one and their behavior under equivariant affine modifications.

2.1. Equivariant extensions of $\mathbb{G}_a$-torsors

A $\mathbb{G}_a$-torsor over a punctured surface $S_* = S \setminus \{o\}$ is an S_*-scheme $\rho : P \to S_*$ equipped with a $\mathbb{G}_a$-action $\mu : \mathbb{G}_{a,S_*} \times_{S_*} P \to P$ for which there exists a Zariski open cover $f : Y \to S_*$ of S_* such that $P \times_{S_*} Y$ is equivariantly isomorphic to $\mathbb{G}_{a,Y}$ acting on itself by translations. In the present article, we primarily focus on $\mathbb{G}_a$-torsors $\rho : P \to S_*$ whose restrictions $P \times_{S_*} U \to U \setminus \{o\}$ over every Zariski open neighborhood U of o in S are non-trivial. Since in this case the total space of P is affine over S (see, *e.g.*, [4, Proposition 1.2] whose proof carries over verbatim to our more general situation), it follows that for every $\mathbb{G}_a$-extension $j : P \hookrightarrow X$ the fiber $\pi^{-1}(o) \subset X$ of the surjective morphism $\pi : X \to S$ has pure codimension one in X. Two important families of examples of non-trivial normal $\mathbb{G}_a$-extensions $j : \mathrm{SL}_2 \to X$ of the $\mathbb{G}_a$-torsor $\rho : \mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a \simeq \mathbb{A}^2 \setminus \{(0,0)\}$, where $\mathbb{G}_a$ acts on SL_2 via left multiplication by upper triangular unipotent matrices, were constructed in [10, Sections 5 and 6]. Various other extensions were obtained from these by performing suitable equivariant affine modifications. One can observe that for all of these extensions, the fiber $\pi^{-1}(\{(0,0)\})$ is an $\mathbb{A}^1$-ruled surface, a property which is a consequence of the following more general fact:

Proposition 2.3. *Let $\rho : P \to S_*$ be a non-trivial $\mathbb{G}_a$-torsor over the punctured spectrum $S \setminus \{o\}$ of a regular local ring of dimension 2 over an algebraically closed field k and with residue field $\kappa(o) = k$, and let $\pi : X \to S$ be a $\mathbb{G}_a$-extension of P. If X is smooth along $\pi^{-1}(o)$, then every irreducible component F of $\pi^{-1}(o)_{\mathrm{red}}$ is a uniruled surface. Furthermore, if X is affine then F is $\mathbb{A}^1$-uniruled, hence $\mathbb{A}^1$-ruled when it is normal.*

Proof. Since $\pi^{-1}(o)$ has pure codimension one in X and X is smooth along $\pi^{-1}(o)$, every irreducible component of $\pi^{-1}(o)$ is a $\mathbb{G}_a$-invariant Cartier divisor on X. The complement X' in X of all but one irreducible component of $\pi^{-1}(o)$ is thus again a $\mathbb{G}_a$-extension of P, and we may therefore assume without loss of generality that $F = \pi^{-1}(o)_{\mathrm{red}}$ is irreducible. Let $x \in F$ be a closed point in the regular locus of F. Since F and X are smooth at x and X is connected, there exists a curve $C \subset X$, smooth at x and intersecting F transversally at x. The image $\pi(C)$ of C is a curve on S passing through o, and the closure B of $\pi^{-1}(\pi(C) \cap S_*)$ in X is a surface containing C. Since $\rho : P \to S_*$ is a $\mathbb{G}_a$-torsor, the restriction of π to $B \cap P$ is a trivial $\mathbb{G}_a$-torsor over the affine curve $\pi(C)$. So $\pi\mid_B : B \to \pi(C)$ is an $\mathbb{A}^1$-fibration. Let $\nu : \tilde{C} \to \pi(C)$ be the normalization of $\pi(C)$. Then $\pi\mid_B$ lifts to an $\mathbb{A}^1$-fibration $\theta : \tilde{B} \to \tilde{C}$ on the normalization $\tilde{B}$ of B. The fiber of θ over every point in $\nu^{-1}(o)$ is a union of rational curves. Since the normalization morphism $\mu : \tilde{B} \to B$ is surjective, one of the irreducible components of $\nu^{-1}(o)$ is mapped by μ onto a rational curve in F passing through x. This shows that for every smooth closed point x of F, there exists a non-constant rational map $h : \mathbb{P}^1 \dashrightarrow F$ such that $x \in h(\mathbb{P}^1)$. Thus F is uniruled. If X in addition is affine, then B and $\tilde{B}$ are affine surfaces, and the fibers of the $\mathbb{A}^1$-fibration $\theta : \tilde{B} \to \tilde{C}$ consist of the disjoint union of curves isomorphic to $\mathbb{A}^1$ when equipped with their reduced structure. This implies that F is not only uniruled but actually $\mathbb{A}^1$-uniruled. $\square$

Example 2.4. Let X be the smooth affine threefold in $\mathbb{A}^2 \times \mathbb{A}^4 = \mathrm{Spec}(k[x, y][c, d, e, f])$ defined by the equations

$$\begin{cases} xd - y(c+1) = 0 \\ xc^2 - y^2 e = 0 \\ yf - c(c+1) = 0 \\ xf^2 - (c+1)^2 e = 0 \\ de - cf = 0, \end{cases}$$

equipped with the $\mathbb{G}_a$-action induced by the locally nilpotent $k[x, y]$-derivation

$$xy\partial_c + y^2\partial_d + x(2c+1)\partial_f + (2x^2 f - 2xye)\partial_e$$

of its coordinate ring. The morphism $j : \mathrm{SL}_2 = \{xv - yu = 1\} \to X$ defined by $(x, y, u, v) \mapsto (x, y, yu, yv, xu^2, xuv)$ is an open embedding of SL_2 in X as the complement of the fiber over $o = (0, 0)$ of the projection $\pi = \mathrm{pr}_{x,y} : X \to \mathbb{A}^2$. So $j : \mathrm{SL}_2 \to X$ is an affine $\mathbb{G}_a$-extension of the $\mathbb{G}_a$-torsor $\rho : \mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a = \mathbb{A}^2 \setminus \{o\}$, for which $\pi^{-1}(o)$ consists of the disjoint union of two copies $D_1 = \{x = y = c = 0\} \simeq \mathrm{Spec}(k[d, f])$ and $D_2 = \{x = y = c + 1 = 0\} \simeq \mathrm{Spec}(k[d, e])$ of $\mathbb{A}^2$. Note that the induced $\mathbb{G}_a$-action on each of these is the trivial one.

Example 2.5. Let X be the affine $\mathbb{G}_a$-extension constructed in the previous example and let $C \subset D_1$ be any smooth affine curve. Let $\tau : \tilde{X} \to X$ be the blow-up of X along C, let $i : X' \hookrightarrow \tilde{X}$ be the open immersion of the complement of the proper transform of $D_1 \cup D_2$ in $\tilde{X}$ and let $\pi' = \pi \circ \tau \circ i : X' \to \mathbb{A}^2$. Since C and $D_1 \cup D_2$ are $\mathbb{G}_a$-invariant, the $\mathbb{G}_a$-action on X lifts to a $\mathbb{G}_a$-action on $\tilde{X}$ which restricts in turn to X'. By construction, π' is surjective, with fiber $\pi'^{-1}(o)$ isomorphic to $C \times \mathbb{A}^1$ and $\tau \circ i : X' \to X$ restricts to an equivariant isomorphism between $X' \setminus \pi'^{-1}(o)$ and $X \setminus \pi^{-1}(o) \simeq \mathrm{SL}_2$. So $\pi' : X' \to \mathbb{A}^2$ is a $\mathbb{G}_a$-extension of the $\mathbb{G}_a$-torsor $\rho : \mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a = \mathbb{A}^2 \setminus \{o\}$.

2.2. Recollection on affine-linear bundles

Affine-linear bundles of rank one over a scheme are natural generalizations of $\mathbb{G}_a$-torsors. To fix the notation, we briefly recall their basic definitions and properties.

By a line bundle on a scheme S, we mean the relative spectrum $p : M = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{M}^\vee) \to S$ of the symmetric algebra of the dual of an invertible sheaf of $\mathcal{O}_S$-modules $\mathcal{M}$. Such a line bundle M can be viewed as a locally constant group scheme over S for the group law $m : M \times_S M \to M$ whose comorphism

$$m^\sharp : \mathrm{Sym}^{\cdot}\mathcal{M}^\vee \to \mathrm{Sym}^{\cdot}\mathcal{M}^\vee \otimes \mathrm{Sym}^{\cdot}\mathcal{M}^\vee \simeq \mathrm{Sym}^{\cdot}\big(\mathcal{M}^\vee \oplus \mathcal{M}^\vee\big)$$

is induced by the diagonal homomorphism $\mathcal{M}^\vee \to \mathcal{M}^\vee \oplus \mathcal{M}^\vee$. An *$M$-torsor* is then an S-scheme $\theta : W \to S$ equipped with an action $\mu : M \times_S W \to W$ which is Zariski locally over S isomorphic to M acting on itself by translations.

This is the case precisely when there exists a Zariski open cover $f : Y \to S$ and an $\mathcal{O}_Y$-algebra isomorphism $\psi : f^*\mathcal{A} \to \mathrm{Sym}^{\cdot} f^*\mathcal{M}^\vee$ such that over $Y' = Y \times_S Y$ the automorphism $\mathrm{p}_1^*\psi \circ \mathrm{p}_2^*\psi^{-1} : \mathrm{Sym}^{\cdot}\mathcal{M}^\vee_{Y'} \to \mathrm{Sym}^{\cdot}\mathcal{M}^\vee_{Y'}$ of the symmetric algebra of $\mathcal{M}^\vee_{Y'} = \mathrm{p}_2^* f^*\mathcal{M}^\vee = \mathrm{p}_1^* f^*\mathcal{M}^\vee$ is *affine-linear*, *i.e.* induced by an $\mathcal{O}_{Y'}$-module homomorphism $\mathcal{M}^\vee_{Y'} \to \mathrm{Sym}^{\cdot}\mathcal{M}^\vee_{Y'}$ of the form

$$\beta \oplus \mathrm{id} : \mathcal{M}^\vee_{Y'} \to \mathcal{O}_{Y'} \oplus \mathcal{M}^\vee_{Y'} \hookrightarrow \bigoplus_{n \geq 0} \big(\mathcal{M}^\vee_{Y'}\big)^{\otimes n} = \mathrm{Sym}^{\cdot}\mathcal{M}^\vee_{Y'} \qquad (2.1)$$

for some $\beta \in \mathrm{Hom}_{Y'}(\mathcal{M}^\vee_{Y'}, \mathcal{O}_{Y'}) \simeq H^0(Y', \mathcal{M}_{Y'})$ which is a Čech 1-cocycle with values in $\mathcal{M}$ for the Zariski open cover $f : Y \to S$. Standard arguments show that the isomorphism class of $\theta : W \to S$ depends only on the class of β in the Čech cohomology group $\check{H}^1(S, \mathcal{M})$, and one eventually gets a one-to-one correspondence between isomorphism classes of M-torsors over S and elements of the cohomology group $H^1(S, M) = H^1(S, \mathcal{M}) \simeq \check{H}^1(S, \mathcal{M})$ with the zero element corresponding to the trivial torsor $p : M \to S$.

It is classical that every locally trivial $\mathbb{A}^1$-bundle $\theta : W \to S$ over a reduced scheme S can be equipped with the additional structure of a torsor under a uniquely determined line bundle M on S. The existence of this additional structure will be frequently used in the sequel, and we now quickly review its construction (see also, *e.g.*, [2, Section 2.3 and Section 2.4]). Letting $\mathcal{A} = \theta_* \mathcal{O}_W$, there exists by definition a Zariski open cover $f : Y \to S$ and a quasi-coherent $\mathcal{O}_Y$-algebra isomorphism $\varphi : f^*\mathcal{A} \to \mathcal{O}_Y[u]$. Over $Y' = Y \times_S Y$ equipped with the two projections p_1 and p_2 to Y, the $\mathcal{O}_{Y'}$-algebra isomorphism $\Phi = \mathrm{p}_1^*\varphi \circ \mathrm{p}_2^*\varphi^{-1}$ has the form

$$\Phi : \mathcal{O}_{Y'}[u] \to \mathcal{O}_{Y'}[u], \; u \mapsto au + b \tag{2.2}$$

for some $a \in \Gamma(Y', \mathcal{O}_{Y'}^*)$ and $b \in \Gamma(Y', \mathcal{O}_{Y'})$ whose pullbacks over $Y'' = Y \times_S Y \times_S Y$ by the three projections $\mathrm{p}_{12}, \mathrm{p}_{23}, \mathrm{p}_{13} : Y'' \to Y'$ satisfy the cocycle relations $\mathrm{p}_{13}^* a = \mathrm{p}_{23}^* a \cdot \mathrm{p}_{12}^* a$ and $\mathrm{p}_{13}^* b = \mathrm{p}_{23}^* a \cdot \mathrm{p}_{12}^* b + \mathrm{p}_{23}^* b$ in $\Gamma(Y'', \mathcal{O}_{Y''}^*)$ and $\Gamma(Y'', \mathcal{O}_{Y''})$ respectively. The first one says that a is a Čech 1-cocycle with values in $\mathcal{O}_S^*$ for the cover $f : Y \to S$, and thus it determines, via the isomorphism $H^1(S, \mathcal{O}_S^*) \simeq \mathrm{Pic}(S)$, a unique invertible sheaf $\mathcal{M}$ on S together with an $\mathcal{O}_Y$-module isomorphism $\alpha : f^*\mathcal{M}^\vee \to \mathcal{O}_Y$ such that $\mathrm{p}_1^*\alpha \circ \mathrm{p}_2^*\alpha^{-1} : \mathcal{O}_{Y'} \to \mathcal{O}_{Y'}$ is the multiplication by a. The second one can be equivalently reinterpreted as the fact that $\beta = \mathrm{p}_2^*({}^t\alpha)(b) \in \Gamma(Y', \mathcal{M}_{Y'})$ is a Čech 1-cocycle with values in $\mathcal{M}$ for the Zariski open cover $f : Y \to S$. Letting $\mathrm{Sym}^{\cdot}(\alpha) : \mathrm{Sym}^{\cdot} f^*\mathcal{M}^\vee \to \mathcal{O}_Y[u]$ be the graded $\mathcal{O}_Y$-algebra isomorphism induced by α, the isomorphism $\psi = \mathrm{Sym}^{\cdot}(\alpha^{-1}) \circ \varphi : f^*\mathcal{A} \to \mathrm{Sym}^{\cdot} f^*\mathcal{M}^\vee$ has the property that $\mathrm{p}_1^*\psi \circ \mathrm{p}_2^*\psi^{-1}$ is affine-linear, induced by the homomorphism $\beta \oplus \mathrm{id} : \mathcal{M}_{Y'}^\vee \to \mathcal{O}_{Y'} \oplus \mathcal{M}_{Y'}^\vee$. So $\theta : W \to S$ is a torsor under the line bundle $M = \mathrm{Spec}(\mathrm{Sym}^{\cdot} \mathcal{M}^\vee)$, with isomorphism class in $H^1(S, M)$ equal to the cohomology class of the cocycle β. Summing up, we obtain:

Proposition 2.6. *Let $\theta : W \to S$ be a locally trivial $\mathbb{A}^1$-bundle. Then there exists a unique pair (M, g) consisting of a line bundle M on S and a class $g \in H^1(S, M)$ such that $\theta : W \to S$ is an M-torsor with isomorphism class g.*

2.3. Additive group actions on affine-linear bundles of rank one

Given a locally trivial $\mathbb{A}^1$-bundle $\theta : W \to S$, which we view as an M-torsor for a line bundle $M = \mathrm{Spec}(\mathrm{Sym}^{\cdot} \mathcal{M}^\vee) \to S$ on S, with corresponding action $\mu : M \times_S W \to W$, every non-zero group scheme homomorphism $\xi : \mathbb{G}_{a,S} \to M$ induces a non-trivial $\mathbb{G}_{a,S}$-action $\nu = \mu \circ (\xi \times \mathrm{id}) : \mathbb{G}_{a,S} \times_S W \to W$ on W. A non-zero group scheme homomorphism $\xi : \mathbb{G}_{a,S} = \mathrm{Spec}(\mathcal{O}_S[t]) \to M = \mathrm{Spec}(\mathrm{Sym}^{\cdot} \mathcal{M}^\vee)$ is uniquely determined by a non-zero $\mathcal{O}_S$-module homomorphism $\mathcal{M}^\vee \to \mathcal{O}_S$, or equivalently by a non-zero global section $s \in \Gamma(S, \mathcal{M})$. The following proposition asserts conversely that every non-trivial $\mathbb{G}_{a,S}$-action on an M-torsor $\theta : W \to S$ uniquely arises from such a section.

Proposition 2.7 ([1, Chapter 3]). *Let $\theta : W \to S$ be a torsor under the action $\mu : M \times_S W \to W$ of a line bundle $M = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{M}^{\vee}) \to S$ on S and let $\nu : \mathbb{G}_{a,S} \times_S W \to W$ be a non-trivial $\mathbb{G}_{a,S}$-action on W. Then there exists a non-zero global section $s \in \Gamma(S, \mathcal{M})$ such that $\nu = \mu \circ (\xi \times \mathrm{id})$ where $\xi : \mathbb{G}_{a,S} \to M$ is the group scheme homomorphism induced by s.*

Proof. Let $\mathcal{A} = \theta_*\mathcal{O}_W$ and let $f : Y \to S$ be a Zariski open cover such that there exists an $\mathcal{O}_Y$-algebra isomorphism $\varphi : f^*\mathcal{A} \to \mathcal{O}_Y[u]$, and let

$$\Phi = \mathrm{p}_1^*\varphi \circ \mathrm{p}_2^*\varphi^{-1} : \mathcal{O}_{Y'}[u] \to \mathcal{O}_{Y'}[u], \quad u \mapsto au + b$$

be as in (2.2) above. Since $\theta : W \to S$ is an M-torsor, φ also determines an $\mathcal{O}_Y$-module isomorphism $\alpha : f^*\mathcal{M}^{\vee} \to \mathcal{O}_Y$ such that $\mathrm{p}_1^*\alpha \circ \mathrm{p}_2^*\alpha^{-1} : \mathcal{O}_{Y'} \to \mathcal{O}_{Y'}$ is the multiplication by a. The $\mathbb{G}_{a,S}$-action ν on W pulls back to a $\mathbb{G}_{a,Y}$-action $\nu \times \mathrm{id}$ on $W \times_S Y$. The comophism $\eta : \mathcal{O}_Y[u] \to \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t]$ of the non-trivial $\mathbb{G}_{a,Y}$-action $\varphi \circ (\nu \times \mathrm{id}) \circ (\mathrm{id} \times \varphi^{-1})$ on $\mathrm{Spec}(\mathcal{O}_Y[u])$ has the form $u \mapsto u \otimes 1 + 1 \otimes \gamma t$ for some non-zero $\gamma \in \Gamma(Y, \mathcal{O}_Y)$. Letting $\mathcal{I} = \gamma \cdot \mathcal{O}_Y$ be the ideal sheaf generated by γ, η factors as

$$\eta = (\mathrm{id} \otimes j) \circ \tilde{\eta} : \mathcal{O}_Y[u] \to \mathcal{O}_Y[u] \otimes \mathrm{Sym}^{\cdot}\mathcal{I} \to \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t]$$

where $\tilde{\eta}$ is the comorphism of an action of the line bundle $\mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{I}) \to Y$ on $\mathbb{A}^1_S \times_S Y \simeq W \times_S Y$ and $j : \mathrm{Sym}^{\cdot}\mathcal{I} \to \mathcal{O}_Y[t]$ is the homomorphism induced by the inclusion $\mathcal{I} \subset \mathcal{O}_Y$. Pulling back to Y', we find that $\mathrm{p}_2^*\gamma = a \cdot \mathrm{p}_1^*\gamma$, which implies that ${}^t\alpha(\gamma) \in \Gamma(Y, f^*\mathcal{M})$ is the pull-back f^*s to Y of a non-zero global section $s \in \Gamma(S, \mathcal{M})$. Letting $D = \mathrm{div}_0(s)$ be the divisors of zeros of s, we have $\mathcal{M}^{\vee} \simeq \mathcal{O}_S(-D) \subset \mathcal{O}_S$ and $f^*\mathcal{M}^{\vee} \simeq \mathcal{O}_Y(-f^*D) \subset \mathcal{O}_Y$ is equal to the ideal $\mathcal{I} = \gamma \cdot \mathcal{O}_Y$. The global section f^*s viewed as a homomorphism $f^*\mathcal{M}^{\vee} \to \mathcal{O}_Y$ coincides via these isomorphisms with the inclusion $\gamma \cdot \mathcal{O}_Y \hookrightarrow \mathcal{O}_Y$. We can thus rewrite η in the form

$$\eta = (\mathrm{id} \otimes \mathrm{Sym}^{\cdot} f^*s) \circ \tilde{\eta} : \mathcal{O}_Y[u] \to \mathcal{O}_Y[u] \otimes \mathrm{Sym}^{\cdot} f^*\mathcal{M}^{\vee} \to \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t].$$

By construction $\tilde{\eta} = (\varphi \otimes \mathrm{id}) \circ f^*\mu^{\sharp} \circ \varphi^{-1}$ where $f^*\mu^{\sharp}$ is the pullback of the comorphism $\mu^{\sharp} : \mathcal{A} \to \mathcal{A} \otimes \mathrm{Sym}^{\cdot}\mathcal{M}^{\vee}$ of the action $\mu : M \times_S W \to W$ of M on W. It follows that the pull-back $f^*\nu^{\sharp}$ of the comorphism of the action $\nu : \mathbb{G}_{a,S} \times W \to W$ factors as

$$f^*\nu^{\sharp} = (\mathrm{id} \otimes \mathrm{Sym}^{\cdot} f^*s) \circ f^*\mu^{\sharp} = f^*\mathcal{A} \to f^*\mathcal{A} \otimes \mathrm{Sym}^{\cdot} f^*\mathcal{M}^{\vee} \to f^*\mathcal{A} \otimes \mathcal{O}_Y[t].$$

This in turn implies that $\nu^{\sharp}$ factors as $(\mathrm{id} \otimes \mathrm{Sym}^{\cdot}s) \circ \mu^{\sharp} : \mathcal{A} \to \mathcal{A} \otimes \mathrm{Sym}^{\cdot}\mathcal{M}^{\vee} \to \mathcal{A} \otimes \mathcal{O}_Y[t]$ as desired. □

Remark 2.8. In the setting of Proposition 2.7, letting $U \subset S$ be the complement of the zero locus of s, the morphism ξ restricts to an isomorphism of group schemes $\xi|_U : \mathbb{G}_{a,U} \to M|_U$ for which $W|_U$ equipped with the $\mathbb{G}_{a,U}$-action $\nu|_U : \mathbb{G}_{a,U} \times_U$

$W|_U \to W|_U$ is a $\mathbb{G}_{a,U}$-torsor. This isomorphism class in $H^1(U, \mathcal{O}_U)$ of this $\mathbb{G}_{a,U}$-torsor coincides with the image of the isomorphism class $g \in H^1(S, \mathcal{M})$ of W by the composition of the restriction homomorphism res : $H^1(S, \mathcal{M}) \to H^1(U, \mathcal{M}|_U)$ with the inverse of the isomorphism $H^1(U, \mathcal{O}_U) \to H^1(U, \mathcal{M}|_U)$ induced by $s|_U$.

2.4. $\mathbb{G}_a$-equivariant affine modifications of affine-linear bundles of rank one

Recall [3] that given an integral scheme X with sheaf of rational functions $\mathcal{K}_X$, an effective Cartier divisor D on X and a closed subscheme $Z \subset X$ whose ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ contains $\mathcal{O}_X(-D)$, the *affine modification of X with center* $(\mathcal{I}, D)$ is the affine X-scheme $\sigma : X' = \operatorname{Spec}(\mathcal{O}_X[\mathcal{I}/D]) \to X$ where $\mathcal{O}_X[\mathcal{I}/D]$ denotes the quotient of the Rees algebra

$$\mathcal{O}_X[(\mathcal{I} \otimes \mathcal{O}_X(D))] = \bigoplus_{n \geq 0} (\mathcal{I} \otimes \mathcal{O}_X(D))^n t^n \subset \mathcal{K}_X[t]$$

of the fractional ideal $\mathcal{I} \otimes \mathcal{O}_X(D) \subset \mathcal{K}_X$ by the ideal generated by $1 - t$. If $X = \operatorname{Spec}(A)$ is affine, $D = \operatorname{div}(f)$ is principal and Z is defined by an ideal $I \subset A$ containing f then X' is isomorphic to the affine modification $\operatorname{Spec}(A[I/f])$ of X with center (I, f) in the sense of [11].

Now let S be an integral scheme and let $\theta : W \to S$ be a locally trivial $\mathbb{A}^1$-bundle. Let $C \subset S$ be an integral Cartier divisor, let $D = \theta^{-1}(C)$ be its inverse image in W and let $Z \subset D$ be a non-empty integral closed subscheme of D on which θ restricts to an open embedding $\theta|_Z : Z \hookrightarrow C$. Equivalently, Z is the closure in D of the image $\alpha(U)$ of a rational section $\alpha : C \to D$ of the locally trivial $\mathbb{A}^1$-bundle $\theta|_D : D \to C$ defined over a non empty open subset U of C. The complement F of $\theta|_Z(Z)$ in C is a closed subset of C and hence of S. Letting $i : S \setminus F \hookrightarrow S$ be the natural open embedding, we have the following result:

Lemma 2.9. *Let $\sigma : W' \to W$ be the affine modification of W with center $(\mathcal{I}_Z, D)$. Then the composition $\theta \circ \sigma : W' \to S$ factors through a locally trivial $\mathbb{A}^1$-bundle $\theta' : W' \to S \setminus F$ in such a way that we have a commutative diagram*

$$\begin{array}{ccc} W' & \xrightarrow{\sigma} & W \\ {\scriptstyle \theta'}\downarrow & & \downarrow{\scriptstyle \theta} \\ S \setminus F & \xrightarrow{i} & S. \end{array}$$

Proof. The question being local with respect to a Zariski open cover of S over which $\theta : W \to S$ becomes trivial, we can assume without loss of generality that $S = \operatorname{Spec}(A)$, $W = \operatorname{Spec}(A[x])$, $C = \operatorname{div}(f)$ for some non-zero element $f \in A$. The integral closed subscheme $Z \subset D$ is then defined by an ideal I of the form (f, g) where $g(x) \in A[x]$ is an element whose image in $(A/f)[x]$ is a polynomial of degree one in t. So $g(x) = a_0 + a_1 x + x^2 f R(x)$ where $a_0 \in A$, $a_1 \in A$ has non-zero residue class in A/f and $R(x) \in A[x]$. The condition that $\theta|_Z : Z \to C$ is an open embedding implies further that the residue classes $\overline{a}_0$ and $\overline{a}_1$ of a_0 and a_1 in A/f generate the unit ideal. The complement F of the image of $\theta|_Z(Z)$ in C is then equal to the closed subscheme of C with defining ideal $(\overline{a}_1) \subset A/f$, hence to the closed subscheme of S with defining ideal $(f, a_1) \subset A$. The algebra $A[t][I/f]$ is isomorphic to

$$\begin{aligned} A[x][u]/(g - fu) &= A[x]\big[u - x^2R(x)\big]/(a_0 + a_1x - f\big(u - t^2R(x)\big) \\ &\simeq A[x][v]/(a_0 + a_1x - fv). \end{aligned}$$

One deduces from this presentation that the morphism $\theta\circ\sigma : W' = \mathrm{Spec}(A[I/f]) \to \mathrm{Spec}(A)$ corresponding to the inclusion $A \to A[I/f]$ factors through a locally trivial $\mathbb{A}^1$-bundle $\theta' : W' \to S \setminus F$ over the complement of F. Namely, since $\overline{a}_0$ and $\overline{a}_1$ generate the unit ideal in A/f, it follows that a_1 and f generate the unit ideal in $A[x][u]/(g - fu)$. So W' is covered by the two principal affine open subsets

$$\begin{aligned} W'_{a_1} &\simeq \mathrm{Spec}(A_{a_1}[x][v]/(a_0 + a_1x - fv)) \simeq \mathrm{Spec}(A_{a_1}[v]) \simeq S_{a_1} \times \mathbb{A}^1 \\ W'_f &\simeq \mathrm{Spec}(A_f[x][v]/(a_0 + a_1x - fv)) \simeq \mathrm{Spec}(A_f[x]) \simeq S_f \times \mathbb{A}^1 \end{aligned}$$

on which θ' restricts to the projection onto the first factor. □

Remark 2.10. By construction, the restriction of the birational morphism $\sigma : W' \to W$ constructed in Lemma 2.9 over $S \setminus F$ is a morphism of locally trivial $\mathbb{A}^1$-bundles over $S \setminus F$, which restricts to an isomorphism over $S \setminus C$ but contracts ${\theta'}^{-1}(C)$ onto $Z \subset \theta^{-1}(C)$.

With the notation above, $\theta : W \to S$ and $\theta' : W' \to S \setminus F$ are torsors under the action of line bundles $M = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{M}^\vee)$ and $M' = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{M}'^\vee)$ for certain uniquely determined invertible sheaves $\mathcal{M}$ and $\mathcal{M}'$ on S and $S \setminus F$ respectively.

Lemma 2.11 ([1, Section 4.3]). *Let $\sigma : W' \to W$ be the affine modification of W with center $(\mathcal{I}_Z, D)$ as in Lemma* 2.9. *Then $\mathcal{M}' = \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_S(-C)|_{S\setminus F}$ and the commutative diagram of Lemma* 2.9 *is equivariant for the group scheme homomorphism $\xi : M' \to M$ induced by the homomorphism $\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_S(-C) \to \mathcal{M}$ obtained by tensoring the inclusion $\mathcal{O}_S(-C) \hookrightarrow \mathcal{O}_S$ by $\mathcal{M}$.*

Proof. Since M and M' are uniquely determined, the question is again local with respect to a Zariski open cover of S over which $\theta : W \to S$, hence M, becomes trivial. We can thus assume as in the proof of Lemma 2.9 that $S = \mathrm{Spec}(A)$, $W = \mathrm{Spec}(A[x])$, that $C = \mathrm{div}(f)$ for some non-zero element $f \in A$ and that $Z \subset D$ is defined by the ideal (f, g) for some $g = a_0 + a_1x + fx^2R(x) \in A[x]$. Furthermore, the action of $M \simeq \mathbb{G}_{a,S} = \mathrm{Spec}(A[t])$ on $W \simeq S \times \mathbb{A}^1$ is the one by translations $x \mapsto x + t$ on the second factor. Let $N = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{O}_S(C)) \simeq \mathrm{Spec}(\mathrm{Sym}^{\cdot} f^{-1}A)$ where $f^{-1}A$ denotes the free sub-A-module of the field of fractions $\mathrm{Frac}(A)$ of A generated by f^{-1}. As in the proof of Proposition 2.7, the inclusion $\mathcal{O}_S(-C) =$

$f\cdot\mathcal{O}_S\hookrightarrow\mathcal{O}_S$ induces a group-scheme homomorphism $\xi : N\to M$ whose comorphism $\xi^\sharp$ coincides with the inclusion $A[t]\subset \mathrm{Sym}^{\cdot} f^{-1}A = A[(f^{-1}t)]$. The comorphism of the corresponding action of N on W is given by

$$A[x]\to A[x]\otimes A\big[f^{-1}t\big],\ x\mapsto x\otimes 1+1\otimes t = x\otimes 1+f\otimes f^{-1}t.$$

This action lifts on $W'\simeq \mathrm{Spec}(A[x][v]/(a_0+a_1x-fv))$ to an action $\nu : N\times_S W'\to W'$ whose comorphism

$$A[x][v]/(a_0+a_1x-fv)\to A[x][v]/(a_0+a_1x-fv)\otimes A\big[f^{-1}t\big]$$

is given by $x\mapsto x\otimes 1+1\otimes t$ and $v\mapsto v\otimes 1+a_1\otimes f^{-1}t$. By construction, the principal open subsets $W'_{a_1}\simeq \mathrm{Spec}(A_{a_1}[v])\simeq \mathrm{Spec}(A_{a_1}[v/a_1])$ and $W'_f\simeq \mathrm{Spec}(A_f[x])\simeq \mathrm{Spec}(A_f[x/f])$ of W' equipped with the induced actions of $N|_{S_{a_1}}$ and $N|_{S_f}$ respectively are equivariantly isomorphic to $N|_{S_{a_1}}$ and $N|_{S_f}$ acting on themselves by translations. So $\theta' : W'\to S\setminus F$ is an $N|_{S\setminus F}$-torsor, showing that $\mathcal{M}'=\mathcal{M}\otimes_{\mathcal{O}_S}\mathcal{O}_S(-C)|_{S\setminus F}$ as desired. □

3. Extensions of $\mathbb{G}_a$-torsors of Type I: locally trivial bundles over the blow-up of a point

Given a surface S and a locally trivial $\mathbb{A}^1$-bundle $\theta : W\to\tilde{S}$ over the blow-up $\tau : \tilde{S}\to S$ of a closed point o in the smooth locus of S, the restriction of W over the complement $\tilde{S}\setminus E$ of the exceptional divisor E of τ is a locally trivial $\mathbb{A}^1$-bundle $\tau\circ\theta : W\mid_{\tilde{S}\setminus E}\to \tilde{S}\setminus E\overset{\simeq}{\to} S\setminus\{o\}$. This observation combined with the following re-interpretation of an example constructed in [10] suggests that locally trivial $\mathbb{A}^1$-bundles over the blow-up of a closed point o in the smooth locus of a surface S form a natural class of schemes in which to search for non-trivial $\mathbb{G}_a$-extension of $\mathbb{G}_a$-bundles over punctured surfaces.

Example 3.1. Let $o=V(x,y)$ be a global scheme-theoretic complete intersection closed point in the smooth locus of a surface S, where $x,y\in\Gamma(S,\mathcal{O}_S)$. Let $\rho : P\to S\setminus\{o\}$ and $\pi_0 : X_0\to S$ be the affine S-schemes in $S\times\mathbb{A}^2$ and $S\times\mathbb{A}^3$ with defining sheaves of ideals $(xv-yu-1)$ and $(xr-yq,\ yp-x(q-1),\ pr-q(q-1))$ in $\mathcal{O}_S[u,v]$ and $\mathcal{O}_S[p,q,r]$ respectively. The morphism of S-schemes $j_0 : P\to X_0$ defined by $(x,y,u,v)\mapsto(x,y,xu,xv,yv)$ is an open embedding, equivariant for the $\mathbb{G}_{a,S}$-actions on P and X_0 associated with the locally nilpotent $\mathcal{O}_S$-derivations $x\partial_u+y\partial_v$ and $x^2\partial_p+xy\partial_q+y^2\partial_r$ of $\rho_*\mathcal{O}_P$ and $(\pi_0)_*\mathcal{O}_{X_0}$ respectively. It is straightforward to check that $\rho : P\to S\setminus\{o\}$ is a $\mathbb{G}_{a,S_*}$-torsor and that $\pi_0 : X_0\to S$ is a $\mathbb{G}_a$-extension of P whose fiber over o is isomorphic to the smooth affine quadric $Q=\{pr-q(q-1)=0\}\subset\mathbb{A}^3_\kappa$. Viewing the blow-up $\tilde{S}$ of o as the closed subscheme of $S\times_k \mathrm{Proj}(k[u_0,u_1])$ with equation $xu_1-yu_0=0$, the morphism of S-schemes $\theta : X_0\to\tilde{S}$ defined by

$$(x,y,p,q,r)\mapsto((x,y),[x:y])=((x,y),[q:r])=((x,y),[p:q-1])$$

is a locally trivial $\mathbb{A}^1$-bundle. Note that since the $\mathbb{G}_{a,S}$-action on X_0 restricts to the trivial $\mathbb{G}_{a,\kappa}$-action on Q, $\theta : X_0 \to \tilde{S}$ is not a $\mathbb{G}_{a,\tilde{S}}$-torsor. Instead, letting $E \simeq \mathbb{P}^1_\kappa$ be the exceptional divisor of the blow-up, one can check that $\theta : X_0 \to \tilde{S}$ is a torsor under the line bundle corresponding to the invertible sheaf $\mathcal{O}_{\tilde{S}}(2E)$, and that its restriction over E is the non-trivial $\mathcal{O}_{\mathbb{P}^1_\kappa}(-2)$-torsor $Q \to \mathbb{P}^1_\kappa$, $(p, q, r) \mapsto [q : r] = [p : q-1]$.

Notation 3.2. Given a surface S and a closed point o in the smooth locus of S, with residue field κ, we denote by $\tau : \tilde{S} \to S$ the blow-up of o, with exceptional divisor $E \simeq \mathbb{P}^1_\kappa$. We identify $\tilde{S} \setminus E$ and $S_* = S \setminus \{o\}$ by the isomorphism induced by τ. For every $\ell \in \mathbb{Z}$, we denote by $M(\ell) = \mathrm{Spec}(\mathrm{Sym}^{\cdot} \mathcal{O}_{\tilde{S}}(-\ell E))$ the line bundle on $\tilde{S}$ corresponding to the invertible sheaf $\mathcal{O}_{\tilde{S}}(\ell E)$.

The aim of this section is to give a classification of all possible $\mathbb{G}_a$-equivariant extensions of Type I of a given $\mathbb{G}_a$-torsor $\rho : P \to S_*$, that is $\mathbb{G}_a$-extensions $\pi : W \to S$ that factor through locally trivial $\mathbb{A}^1$-bundles $\theta : W \to \tilde{S}$.

3.1. Existence of $\mathbb{G}_a$-extensions of Type I

By virtue of Propositions 2.6 and 2.7, there exists a one-to-one correspondence between $\mathbb{G}_a$-equivariant extensions of a $\mathbb{G}_a$-torsor $\rho : P \to S_*$ that factor through a locally trivial $\mathbb{A}^1$-bundle $\theta : W \to \tilde{S}$ and pairs (M, ξ) consisting of an M-torsor $\theta : W \to \tilde{S}$ for some line bundle M on $\tilde{S}$ and a group scheme homomorphism $\xi : \mathbb{G}_{a,\tilde{S}} \to M$ restricting to an isomorphism over $\tilde{S} \setminus E$, such that W equipped with the $\mathbb{G}_{a,\tilde{S}}$-action deduced by composition with ξ restricts on $S_* = \tilde{S} \setminus E$ to a $\mathbb{G}_{a,S_*}$-torsor $\theta|_{S_*} : W|_{S_*} \to S_*$ isomorphic to $\rho : P \to S_*$. The condition that $\xi : \mathbb{G}_{a,\tilde{S}} \to M$ restricts to an isomorphism outside E implies that $M \simeq M(\ell)$ for some ℓ, which is necessarily non-negative, and that ξ is induced by the canonical global section of $\mathcal{O}_{\tilde{S}}(\ell E)$ with divisor ℓE.

Proposition 3.3. *Let $\rho : P \to S_*$ be a $\mathbb{G}_{a,S_*}$-torsor. Then there exists an integer $\ell_0 \geq 0$ depending on P only such that for every $\ell \geq \ell_0$, P admits a $\mathbb{G}_a$-extension to a uniquely determined $M(\ell)$-torsor $\theta_\ell : W(P, \ell) \to \tilde{S}$ equipped with the $\mathbb{G}_{a,\tilde{S}}$-action induced by the canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ with divisor ℓE.*

Proof. The $\mathbb{G}_{a,S_*}$-torsor $\rho : P \to S_*$ is determined up to isomorphism by a cohomology class in $H^1(S_*, \mathcal{O}_{S_*})$, while an $M(\ell)$-torsor is determined up to isomorphism by a class in $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$. The assertion is thus equivalent to saying that the homomorphisms

$$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \to H^1(S_*, \mathcal{O}_{\tilde{S}}(nE)|_{S_*}) \simeq H^1(S_*, \mathcal{O}_{S_*}), \quad n \geq 0$$

induced by restriction are injective for all $n \geq 0$ and that their images exhaust $H^1(S_*, \mathcal{O}_{S_*})$. To see this, we will establish that the natural homomorphism

$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \to H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}((n+1)E)$ is injective for all $n \geq 0$ and that $H^1(S_*, \mathcal{O}_{S_*}) \simeq \operatorname{colim}_{n\geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$.

The invertible sheaves $\mathcal{O}_{\tilde{S}}(nE)$, $n \geq 0$, form an inductive system of sub-$\mathcal{O}_{\tilde{S}}$-modules of the sheaf $\mathcal{K}_{\tilde{S}}$ of rational functions on $\tilde{S}$, where for each n, the injective transition homomorphism $j_{n,n+1} : \mathcal{O}_{\tilde{S}}(nE) \hookrightarrow \mathcal{O}_{\tilde{S}}((n+1)E)$ is obtained by tensoring the canonical section $\mathcal{O}_{\tilde{S}} \to \mathcal{O}_{\tilde{S}}(E)$ with divisor E with $\mathcal{O}_{\tilde{S}}(nE)$. Let $i : S_* = \tilde{S} \setminus E \hookrightarrow \tilde{S}$ be the open inclusion. Since E is a Cartier divisor, it follows from [6, Théorème 9.3.1] that $i_*\mathcal{O}_{S_*} \simeq \operatorname{colim}_{n\geq 0}\mathcal{O}_{\tilde{S}}(nE)$. Furthermore, since $E \simeq \mathbb{P}^1_\kappa$ is the exceptional divisor of $\tau : \tilde{S} \to S$, we have $\mathcal{O}_{\tilde{S}}(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1_\kappa}(-1)$, and the long exact sequence of cohomology for the short exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(nE) \to \mathcal{O}_{\tilde{S}}((n+1)E) \to \mathcal{O}_{\tilde{S}}((n+1)E)|_E \to 0, \quad n \geq 0, \tag{3.1}$$

combined with the vanishing of $H^0(\mathbb{P}_\kappa{}^1, \mathcal{O}_{\mathbb{P}^1_\kappa}(-n-1))$ for every $n \geq 0$ implies that the transition homomorphisms

$$H^1(j_{n,n+1}) : H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \to H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}((n+1)E)), \ n \geq 0,$$

are all injective. By assumption, S whence $\tilde{S}$ is noetherian, and $i : S_* \to \tilde{S}$ is an affine morphism as E is a Cartier divisor on $\tilde{S}$. We thus deduce from [12, Theorem 8] and [7, Corollaire 1.3.3] that the canonical homomorphism

$$\psi : \operatorname{colim}_{n\geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \to H^1(S_*, \mathcal{O}_{S_*}) \tag{3.2}$$

obtained as the composition of the canonical homomorphisms

$$\operatorname{colim}_{n\geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \to H^1(\tilde{S}, \operatorname{colim}_{n\geq 0}\mathcal{O}_{\tilde{S}}(nE)) = H^1(\tilde{S}, i_*\mathcal{O}_{S_*})$$

and $H^1(\tilde{S}, i_*\mathcal{O}_{S_*}) \to H^1(S_*, \mathcal{O}_{S_*})$ is an isomorphism.

Let $g \in H^1(S_*, \mathcal{O}_{S_*})$ be the isomorphism class of the $\mathbb{G}_{a,S_*}$-torsor $\rho : P \to S_*$. If $g = 0$, then since ψ is an isomorphism, we have $\psi^{-1}(g) = 0$ and, since the homomorphisms $H^1(j_{n,n+1})$ are injective, it follows that $\psi^{-1}(g)$ is represented by the zero sequence $(0)_n \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$, $n \geq 0$. Consequently, the only $\mathbb{G}_a$-extensions of P are the line bundles $W(P, \ell) = M(\ell)$, $\ell \geq 0$, each equipped with the $\mathbb{G}_{a,\tilde{S}}$-action induced by its canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$.

Otherwise, if $g \neq 0$, then $h = \psi^{-1}(g) \neq 0$, and since the homomorphisms $H^1(j_{n,n+1})$, $n \geq 0$ are injective, it follows that there exists a unique minimal integer ℓ_0 such that h is represented by the sequence

$$h_n = H^1(j_{n-1,n}) \circ \cdots \circ H^1\big(j_{\ell_0,\ell_0+1}\big)\big(h_{\ell_0}\big) \in H^1\big(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)\big), \ n \geq \ell_0 \tag{3.3}$$

for some non-zero $h_{\ell_0} \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell_0 E))$. It then follows from Proposition 2.7 that for every $\ell \geq \ell_0$, the $M(\ell)$-torsor $\theta_\ell : W(P, \ell) \to \tilde{S}$ with isomorphism class h_ℓ equipped with the $\mathbb{G}_{a,\tilde{S}}$-action induced by the canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ is a $\mathbb{G}_a$-extension of P.

Conversely, for every $\mathbb{G}_a$-extension of P into an $M(\ell)$-torsor $\theta: W \to \tilde{S}$ equipped with the $\mathbb{G}_{a,\tilde{S}}$-action induced by the canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$, it follows from Proposition 2.7 again that the image of the isomorphism class $h_\ell \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ of W in $H^1(\tilde{S} \setminus E, \mathcal{O}_{\tilde{S}}(\ell E)|_{\tilde{S}\setminus E}) \simeq H^1(S_*, \mathcal{O}_{S_*})$ is equal to g. Letting $h \in \mathrm{colim}_{n\geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$ be the element represented by the sequence

$$h_n = \big(H^1(j_{n-1,n} \circ \cdots \circ j_{\ell,\ell+1})(h_\ell)\big)_{n\geq \ell} \in H^1\big(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)\big),\ n \geq \ell$$

we have $\psi(h) = g$ and since ψ is an isomorphism, we conclude that $W \simeq W(P, \ell)$ as $M(I)$-torsors. □

3.2. $\mathbb{G}_a$-extensions with affine total spaces

The extensions $\theta : W \to \tilde{S}$ we get from Proposition 3.3 are not necessarily affine over S. In this subsection we establish a criterion for affineness which we then use to characterize all extensions $\theta : W \to \tilde{S}$ of Type I of a $\mathbb{G}_a$-torsor $\rho : P \to S_*$ whose total spaces W are affine over S.

Lemma 3.4. *Let $S = \mathrm{Spec}(A)$ be an affine surface and let $o = V(x, y)$ be a global scheme-theoretic complete intersection point in the smooth locus of S. Let $\tau : \tilde{S} \to S$ be the blow-up of o with exceptional divisor E and let $\theta : W \to \tilde{S}$ be an $M(\ell)$-torsor for some $\ell \geq 0$. Then the following hold:*

a) $H^1(W, \mathcal{O}_W) = 0$;
b) *The scheme W is affine if and only if $H^1(W, \theta^*\mathcal{O}_{\tilde{S}}(\ell E)) = 0$ for some $\ell \geq 2$.*

Proof. Since o is a scheme-theoretic complete intersection, we can identify $\tilde{S}$ with the closed subvariety of $S \times_k \mathbb{P}^1_k = S \times_k \mathrm{Proj}(k[t_0, t_1])$ defined by the equation $xt_1 - yt_0 = 0$. The restriction $p : \tilde{S} \to \mathbb{P}^1_k$ of the projection to the second factor is an affine morphism. More precisely, letting $U_0 = \mathbb{P}^1_k \setminus \{[1 : 0]\} \simeq \mathrm{Spec}(k[z])$ and $U_\infty = \mathbb{P}^1_k \setminus \{[0 : 1]\} \simeq \mathrm{Spec}(k[z'])$ be the standard affine open cover of $\mathbb{P}^1_k$, we have $p^{-1}(U_0) \simeq \mathrm{Spec}(A[z]/(x - yz))$ and $p^{-1}(U_\infty) \simeq \mathrm{Spec}(A[z']/(y - xz'))$. The exceptional divisor $E \simeq \mathbb{P}^1_\kappa$ of $\tau : \tilde{S} \to S$ is a flat quasi-section of p with local equations $y = 0$ and $x = 0$ in the affine charts $p^{-1}(U_0)$ and $p^{-1}(U_\infty)$ respectively. Every $M(\ell)$-torsor $\theta : W \to \tilde{S}$ with $\ell \geq 0$ is isomorphic to the scheme obtained by gluing $W_0 = p^{-1}(U_0) \times \mathrm{Spec}(k[u])$ with $W_\infty = p^{-1}(U_\infty) \times \mathrm{Spec}(k[u'])$ over $U_0 \cap U_\infty$ by an isomorphism induced by a k-algebra isomorphism of the form

$$A\big[(z')^{\pm 1}\big]/\big(y - xz'\big)\big[u'\big] \ni \big(z', u'\big) \mapsto \big(z^{-1}, z^\ell u + p\big) \in A\big[z^{\pm 1}\big]/(x - yz)[u]$$

for some $p \in A[z^{\pm 1}]/(x - yz)$. Since $H^1(W, \mathcal{O}_W) \simeq \check{H}^1(W, \mathcal{O}_W) \simeq \check{H}^1(\{W_0, W_\infty\}, \mathcal{O}_W)$, it is enough in order to prove a) to check that every Čech 1-cocycle g with values in $\mathcal{O}_W$ for the covering of W by the affine open subsets W_0 and W_∞ is a coboundary. Viewing g as an element $g = g(z^{\pm 1}, u) \in A[z^{\pm 1}]/(x - yz)[u]$, it is enough to show that every monomial $g_s = hz^r u^s$ where $h \in A, r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$ is a coboundary, which is the case if and only if there exist $a(z, u) \in A[z]/(f - gz)[u]$ and $b(z', u') \in A[z']/(y - xz')[u']$ such that $g = b(z^{-1}, z^\ell u + p) - b(z, u)$. If $r \geq 0$ then $g \in A[z]/(x - yz)[u]$ is a coboundary. We thus assume from now on that $r < 0$. Suppose that $s > 0$. Then we can write $u^s = z^{-\ell s}(z^\ell u + p)^s - R(u)$ where $R \in A[z^{\pm 1}]/(x - yz)[u]$ is polynomial whose degree in u is strictly less than s. Then since $r < 0$,

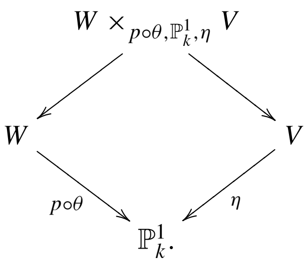
$$\begin{aligned} hz^r u^s &= hz^{r-\ell s}\big(z^\ell u + p\big)^s - hz^r R(u) \\ &= b\big(z^{-1}, z^\ell u + p\big) - hz^r R(u), \end{aligned}$$

where $b(z', u') = h(z')^{-r+\ell s}(u')^s \in A[z']/(y - xz')[u']$. So g_s is a coboundary if and only if $-hz^r R(u)$ is. By induction, we only need to check that every monomial $g_0 = hz^r \in A[z^{\pm 1}]/(x - yz)[u]$ of degree 0 in u is a coboundary. But such a cocycle is simply the pull-back to W of a Čech 1-cocycle h_0 with value in $\mathcal{O}_{\tilde{S}}$ for the covering of $\tilde{S}$ by the affine open subsets $p^{-1}(U_0)$ and $p^{-1}(U_\infty)$. Since the canonical homomorphism

$$H^1(S, \mathcal{O}_S) = H^1\big(S, \tau_* \mathcal{O}_{\tilde{S}}\big) \to H^1\big(\tilde{S}, \mathcal{O}_{\tilde{S}}\big) \simeq \check{H}^1\Big(\big\{p^{-1}(U_0), p^{-1}(U_\infty)\big\}, \mathcal{O}_{\tilde{S}}\Big)$$

is an isomorphism and $H^1(S, \mathcal{O}_S) = 0$ as S is affine, we conclude that h_0 is a coboundary, hence that g_0 is a coboundary too. This proves a).

Now suppose that $H^1(W, \theta^* \mathcal{O}_{\tilde{S}}(\ell E)) = 0$ for some $\ell \geq 2$. Let $\eta : V \to \mathbb{P}^1_k$ be a non-trivial $\mathcal{O}_{\mathbb{P}^1_k}(-\ell)$-torsor and consider the fiber product $W \times_{p \circ \theta, \mathbb{P}^1_k, \eta} V$:



By virtue of [5, Proposition 3.1], V is an affine surface. Since $p \circ \theta : W \to \mathbb{P}^1_k$ is an affine morphism, so is $\mathrm{pr}_V : W \times_{\mathbb{P}^1_k} V \to V$ and hence, $W \times_{\mathbb{P}^1_k} V$ is an affine scheme. On the other hand, since $p^* \mathcal{O}_{\mathbb{P}^1_k}(-1) \simeq \mathcal{O}_{\tilde{S}}(E)$, the projection $\mathrm{pr}_W : W \times_{\mathbb{P}^1_k} V \to W$ is a $\theta^* M(\ell)$-torsor, hence is isomorphic to the trivial one $q : \theta^* M(\ell) \to W$ by hypothesis. So W is isomorphic to the zero section of $\theta^* M(\ell)$, which is a closed subscheme of the affine scheme $W \times_{\mathbb{P}^1_k} V$, hence an affine scheme. □

Corollary 3.5. *With the notation of Lemma* 3.4, *for an $M(\ell)$-torsor $\theta : W \to \tilde{S}$, $\ell \geq 0$, the following are equivalent:*

1) *W is an affine scheme;*
2) *$W|_E$ is a non-trivial $M(\ell)|_E$-torsor;*
3) *The isomorphism class of W in $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ does not belong to the image of the injective homomorphism $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}((\ell-1)E)) \hookrightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$.*

Proof. Since the isomorphism class of $W|_E$ in $H^1(E, \mathcal{O}_{\tilde{S}}(\ell E)|_E)$ is the image of the isomorphism class of W in $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ by the restriction homomorphism $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E)) \to H^1(E, \mathcal{O}_{\tilde{S}}(\ell E)|_E)$, the equivalence of 2) and 3) simply follows from the long exact sequence of cohomology of the short exact sequence (3.1).

If $W|_E$ is a trivial torsor, then it is a line bundle over $E \simeq \mathbb{P}^1_\kappa$. Its zero section is then a proper curve in $W|_E$ hence in W, which prevents W from being affine. So 1) $\Rightarrow$2). Conversely, suppose that $D = W|_E$ is a non-trivial $M(\ell)|_E$-torsor. Then by virtue of [5, Proposition 3.1], D is an affine surface, and so $H^1(D, \mathcal{O}_W((n+1)D)|_D) = 0$ for every $n \in \mathbb{Z}$. By a) in Lemma 3.4, $H^1(W, \mathcal{O}_W) = 0$, and we deduce successively from the long exact sequence of cohomology for the short exact sequence

$$0 \to \mathcal{O}_W(nD) \to \mathcal{O}_W((n+1)D) \to \mathcal{O}_W((n+1)D)\mid_D \to 0$$

in the case $n = 0$ and then $n = 1$ that $H^1(W, \mathcal{O}_W(D)) = H^1(W, \mathcal{O}_W(2D)) = 0$. Since $\mathcal{O}_W(2D) \simeq \theta^*\mathcal{O}_{\tilde{S}}(2E)$, we conclude from b) in the same lemma that W is affine. □

Remark 3.6. Since $M(\ell)|_E \simeq \mathcal{O}_{\mathbb{P}^1_\kappa}(-\ell)$, we infer in particular from Corollary 3.5 that for $\ell = 0, 1$, there is no $M(\ell)$-torsor $\theta : W \to \tilde{S}$ with affine total space W.

We obtain the following characterization:

Theorem 3.7. *A $\mathbb{G}_{a,S_*}$-torsor $\rho : P \to S_*$ admits a $\mathbb{G}_a$-extension to a locally trivial $\mathbb{A}^1$-bundle whose total space is affine over S if and only if for every Zariski open neighborhood U of o, $P \times_{S_*} U \to U_* = U \setminus \{o\}$ is a non-trivial $\mathbb{G}_{a,U_*}$-torsor.*

When it exists, the corresponding locally trivial $\mathbb{A}^1$-bundle $\theta : W \to \tilde{S}$ is unique and is an $M(\ell_0)$-torsor for some $\ell_0 \geq 2$, whose restriction to $E \simeq \mathbb{P}^1_\kappa$ is a non-trivial $\mathcal{O}_{\mathbb{P}^1_\kappa}(-\ell_0)$-torsor.

Proof. Since by construction π restricts over S_* to $\rho : P \to S_*$ which is an affine morphism, π is affine if and only if there exists an open neighborhood U of o in S such that $\pi^{-1}(U)$ is affine. Replacing S by a suitable affine open neighborhood of o, we can therefore assume without loss of generality that $S = \mathrm{Spec}(A)$ is affine and that o is a scheme-theoretic complete intersection $o = V(x, y)$ for some elements $x, y \in A$.

If there exists a Zariski open neighborhood U of o such that the restriction of P over U_* is the trivial $\mathbb{G}_{a,U_*}$-torsor, then the image in $H^1(U_*, \mathcal{O}_{U_*})$ of the isomorphism class g of P is zero and so, arguing as in the proof of Proposition 3.3, every $\mathbb{G}_a$-extension $\theta : W \to \tilde{S}$ restricts on $\tau^{-1}(U)$ to the trivial $M(\ell)|_{\tau^{-1}(U)}$-torsor $M(\ell)|_{\tau^{-1}(U)} \to \tau^{-1}(U)$, hence to a trivial torsor on $E \subset \tau^{-1}(U)$. By virtue of Corollary 3.5, W is not affine, hence is not affine over S.

Now suppose that $\rho : P \to S_*$ is a $\mathbb{G}_{a,S_*}$-torsor with isomorphism class $g \in H^1(S_*, \mathcal{O}_{S_*})$ such that $P \times_{S_*} U \to U_*$ is non-trivial for every open neighborhood U of o. The inverse image $h = \psi^{-1}(g) \in \operatorname{colim}_{n\geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$ of g by the isomorphism (3.2) is represented by a sequence of non-zero elements $h_n \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$ as in (3.3) above. Since $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ and $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(E)) = 0$ as $\mathcal{O}_{\tilde{S}}(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1_\kappa}(-1)$, we deduce from Corollary 3.5 that there exists precisely one $\ell_0 \geq 2$ with the property that an $M(\ell_0)$-torsor $\theta_{\ell_0} : W_{\ell_0} \to \tilde{S}$ with isomorphism class $h_{\ell_0} \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell_0 E))$ has affine total space W_{ℓ_0}. □

3.3. Examples

In this subsection, we consider $\mathbb{G}_a$-torsors of the punctured affine plane. So $S = \mathbb{A}^2 = \operatorname{Spec}(k[x, y])$, $o = (0, 0)$ and $\mathbb{A}^2_* = \mathbb{A}^2 \setminus \{o\}$. We let $\tau : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ be the blow-up of o, with exceptional divisor $E \simeq \mathbb{P}^1$ and we let $i : \mathbb{A}^2_* \hookrightarrow \tilde{\mathbb{A}}^2$ be the immersion of $\mathbb{A}^2_*$ as the open subset $\tilde{\mathbb{A}}^2 \setminus E$. We further identify $\tilde{\mathbb{A}}^2$ with the total space $f : \tilde{\mathbb{A}}^2 \to \mathbb{P}^1$ of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ in such a way that E corresponds to the zero section of this line bundle.

3.3.1. *A simple case: homogeneous $\mathbb{G}_a$-torsors*

Following [4, Section 1.3], we say that a non-trivial $\mathbb{G}_{a,\mathbb{A}^2_*}$-torsor $\rho : P \to \mathbb{A}^2_*$ is homogeneous if it admits a lift of the $\mathbb{G}_m$-action $\lambda \cdot (x, y) = (\lambda x, \lambda y)$ on $\mathbb{A}^2_*$ which is locally linear on the fibers of ρ. By [4, Proposition 1.6], this is the case if and only if the isomorphism class g of P in $H^1(\mathbb{A}^2_*, \mathcal{O}_{\mathbb{A}^2_*})$ can be represented on the open covering of $\mathbb{A}^2_*$ by the principal open subsets $\mathbb{A}^2_x$ and $\mathbb{A}^2_y$ by a Čech 1-cocycle of the form $x^{-m}y^{-n}p(x, y)$ where $m, n \geq 0$ and $p(x, y) \in k[x, y]$ is a homogeneous polynomial of degree $r \leq m + n - 2$. Equivalently, P is isomorphic the $\mathbb{G}_{a,\mathbb{A}^2_*}$-torsor

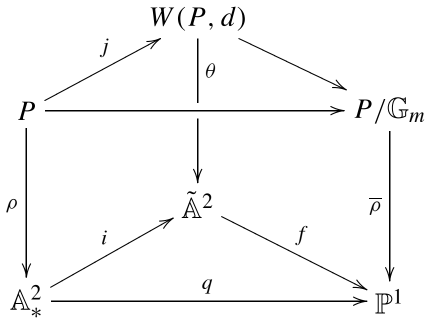
$$\rho = \operatorname{pr}_{x,y} : P_{m,n,p} = \{x^m v - y^n u = p(x, y)\} \setminus \{x = y = 0\} \to \mathbb{A}^2_*,$$

which admits an obvious lift $\lambda \cdot (x, y, u, v) = (\lambda x, \lambda y, \lambda^{m-d} u, \lambda^{n-d} v)$, where $d = m + n - r$, of the $\mathbb{G}_m$-action on $\mathbb{A}^2_*$. Let $q : \mathbb{A}^2_* \to \mathbb{A}^2_*/\mathbb{G}_m = \mathbb{P}^1$ be the quotient morphism of the aforementioned $\mathbb{G}_m$-action on $\mathbb{A}^2_*$. Then it follows from [4, Example 1.8] that the inverse image by the canonical isomorphism

$$\bigoplus_{k\in\mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(k)) \simeq H^1(\mathbb{P}^1, q_*\mathcal{O}_{\mathbb{A}^2_*}) \to H^1(\mathbb{A}^2_*, \mathcal{O}_{\mathbb{A}^2_*})$$

of the isomorphism class g of such an homogeneous torsor is an element h of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-d))$. Furthermore, the $\mathbb{G}_m$-equivariant morphism $\rho : P \to \mathbb{A}^2_*$ descends to a locally trivial $\mathbb{A}^1$-bundle $\overline{\rho} : P/\mathbb{G}_m \to \mathbb{P}^1 = \mathbb{A}^2_*/\mathbb{G}_m$ which is an $\mathcal{O}_{\mathbb{P}^1}(-d)$-torsor with isomorphism class $h \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-d))$.

Since $f^*\mathcal{O}_{\mathbb{P}^1}(-d) \simeq \mathcal{O}_{\tilde{\mathbb{A}}^2}(dE)$, the fiber product $W(P,d) = \tilde{\mathbb{A}}^2 \times_{\mathbb{P}^1} P/\mathbb{G}_m$ is equipped via the restriction of the first projection with the structure of an $M(d)$-torsor $\theta : W(P,d) \to \tilde{\mathbb{A}}^2$ with isomorphism class $f^*h \in H^1(\tilde{\mathbb{A}}^2, \mathcal{O}_{\tilde{\mathbb{A}}^2}(dE))$. On the other other hand, $W(P,d)$ is a line bundle over $P/\mathbb{G}_m$ via the second projection, hence is an affine threefold as $P/\mathbb{G}_m$ is affine. By construction, we have a commutative diagram



in which each square is cartesian. In other words, $W(P,d)$ is obtained from the $\mathbb{G}_m$-torsor $P \to P/\mathbb{G}_m$ by "adding the zero section". The open embedding $j : P \hookrightarrow W(P,d)$ is equivariant for the $\mathbb{G}_a$-action on $W(P,d)$ induced by the canonical global section of $\mathcal{O}_{\tilde{\mathbb{A}}^2}(dE)$ with divisor dE (see Proposition 2.7). By Theorem 3.7, $\theta : W(P,d) \to \tilde{\mathbb{A}}^2$ is the unique $\mathbb{G}_a$-extension of $\rho : P \to \mathbb{A}^2_*$ with affine total space.

In the simplest case $d = 2$, the unique homogeneous $\mathbb{G}_{a,\mathbb{A}^2_*}$-torsor is the geometric quotient $\mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a$ of the group SL_2 by the action of its subgroup of upper triangular unipotent matrices equipped with the diagonal $\mathbb{G}_m$-action, and we recover Example 3.1.

3.3.2. *General case*

Here, given an arbitrary non-trivial $\mathbb{G}_a$-torsor $\rho : P \to \mathbb{A}^2_*$, we describe a procedure to explicitly determine the unique $\mathbb{G}_a$-extension $\theta : W \to \tilde{\mathbb{A}}^2$ of P with affine total space W from a Čech 1-cocycle $x^{-m}y^{-n}p(x,y)$, where $m, n \geq 0$ and $p(x,y) \in k[x,y]$ is a non-zero polynomial of degree $s \leq m+n-2$, representing the isomorphism class $g \in H^1(\mathbb{A}^2_*, \mathcal{O}_{\mathbb{A}^2_*})$ of P on the open covering of $\mathbb{A}^2_*$ by the principal open subsets $\mathbb{A}^2_x$ and $\mathbb{A}^2_y$.

Write $p(x, y) = p_r + p_{r+1} + \cdots + p_s$ where the $p_i \in k[x, y]$ are the homogeneous components of p, and $p_r \neq 0$. In the decomposition

$$H^1\left(\mathbb{A}^2_*, \mathcal{O}_{\mathbb{A}^2_*}\right) \simeq H^1\left(\mathbb{P}^1, q_*\mathcal{O}_{\mathbb{A}^2_*}\right) \simeq \bigoplus_{s\in\mathbb{Z}} H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s)\right)$$

a non-zero homogeneous component $x^{-m}y^{-n}p_i$ of $x^{-m}y^{-n}p(x, y)$ corresponds to a non-zero element of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m-n+i))$. On the other hand, since for every $\ell \in \mathbb{Z}$, $\mathcal{O}_{\tilde{\mathbb{A}}^2}(\ell E) = f^*\mathcal{O}_{\mathbb{P}^1}(-\ell)$ and $f : \tilde{\mathbb{A}}^2 \to \mathbb{P}^1$ is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, it follows from the projection formula that

$$H^1\left(\tilde{\mathbb{A}}^2, \mathcal{O}_{\tilde{\mathbb{A}}^2}(\ell E)\right) \simeq H^1\left(\mathbb{P}^1, f_*\mathcal{O}_{\tilde{\mathbb{A}}^2} \otimes \mathcal{O}_{\mathbb{P}^1}(-\ell)\right) \simeq \bigoplus_{t\geq 0} H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell)\right).$$

The image of $x^{-m}y^{-n}p(x, y)$ in $\bigoplus_{s\in\mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s))$ belongs to $\bigoplus_{t\geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell))$ if and only if $\ell \geq \ell_0 = m+n-r \geq 2$. Given such an ℓ, the image $(h_t)_{t\geq 0} \in \bigoplus_{t\geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell))$ of $x^{-m}y^{-n}p(x, y)$ then defines a unique $M(\ell)$-torsor $\theta_\ell : W(P, \ell) \to \tilde{\mathbb{A}}^2$ whose restriction over the complement of E is isomorphic to $\rho : P \to \mathbb{A}^2_*$ when equipped with the action $\mathbb{G}_a$-action induced by the canonical section of $\mathcal{O}_{\tilde{\mathbb{A}}^2}(\ell E)$ with divisor ℓE. On the other hand, the restriction of $W|_E \to E$ over E is an $\mathcal{O}_{\mathbb{P}^1}(-\ell)$-torsor with isomorphism class $h_0 \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-\ell))$. By definition, h_0 is non-zero if and only if $\ell = \ell_0$, and we conclude from Theorem 3.7 that $\theta_{\ell_0} : W(P, \ell_0) \to \tilde{\mathbb{A}}^2$ is the unique $\mathbb{G}_a$-extension of $\rho : P \to \mathbb{A}^2_*$ with affine total space.

4. Quasi-projective $\mathbb{G}_a$-extensions of Type II

In this section we consider the following subclass of extensions of Type II of a $\mathbb{G}_a$-torsor over a punctured surface.

Definition 4.1. A $\mathbb{G}_a$-extension $\pi : X \to S$ of a $\mathbb{G}_a$-torsor $\rho : P \to S_*$ over a punctured surface $S_* = S \setminus \{o\}$ is said to be a *quasi-projective* extension of Type II if it satisfies the following properties:

i) X is quasi-projective over S and the $\mathbb{G}_{a,S}$-action on X is proper;
ii) X is smooth along $\pi^{-1}(o)$ and $\pi^{-1}(o)_{\mathrm{red}} \simeq \mathbb{A}^2_\kappa$.

Example 4.2. Let $o = V(x, y)$ be a global scheme-theoretic complete intersection closed point in the smooth locus of a surface S and let $\rho : P \to S \setminus \{o\}$ be the $\mathbb{G}_a$-torsor with defining sheaf of ideals $(xv - yu - 1) \subset \mathcal{O}_S[u, v]$ as in Example 3.1. Let $\pi_1 : X_1 \to S$ be the affine S-scheme with defining sheaf of ideals $(xw - y(yz_1+1), xz_2 - z_1(yz_1+1), z_1w - yz_2) \subset \mathcal{O}_S[z_1, z_2, w]$. The morphism of S-schemes $j_1 : P \to X_1$ defined by $(x, y, u, v) \mapsto (x, y, u, uv, yu)$ is an open embedding, equivariant for the $\mathbb{G}_{a,S}$-action on X_1 associated with the locally nilpotent

$\mathcal{O}_S$-derivation $x\partial_{z_1} + (2yz_1 + 1)\partial_{z_2} + y^2\partial_w$ of $\pi_*\mathcal{O}_{X_1}$. The fiber $\pi_1^{-1}(o)$ is isomorphic to $\mathbb{A}^2_\kappa = \mathrm{Spec}(\kappa[z_2, w])$ on which the $\mathbb{G}_{a,S}$-action restricts to $\mathbb{G}_{a,\kappa}$-action by translations associated to the derivation ∂_{z_2} of $\kappa[z_2, w]$. It is straightforward to check that X_1 is smooth along $\pi_1^{-1}(o)$. We claim that the geometric quotient of the $\mathbb{G}_{a,S}$-action on X_1 is isomorphic to the complement of a κ-rational point o_1 in the blow-up $\tau : \tilde{S} \to S$ of o. Such a surface being in particular separated, the $\mathbb{G}_{a,S}$-action on X_1 is proper, implying that $j_1 : P \hookrightarrow X_1$ is a quasi-projective extension of P of Type II.

Indeed, let us identify $\tilde{S}$ with the closed subvariety of $S \times_k \mathrm{Proj}(k[u_0, u_1])$ with equation $xu_1 - yu_0 = 0$ in such a way that τ coincides with the restriction of the first projection. The morphism $f : X_1 \to \tilde{S}$ defined by

$$(x, y, z, u, v) \mapsto ((x, y), [x : y]) = ((x, y), [yz_1 + 1 : w])$$

is $\mathbb{G}_a$-invariant and maps $\pi_1^{-1}(o)$ dominantly onto the exceptional divisor $E \simeq \mathrm{pr}_S^{-1}(o) \simeq \mathrm{Proj}(\kappa[u_0, u_1])$ of τ. The induced morphism

$$f|_{\pi^{-1}(o)} : \pi^{-1}(o) = \mathrm{Spec}(\kappa[z_2, w]) \to E, \quad (z_2, w) \mapsto [1 : w]$$

factors as the composition of the geometric quotient $\pi_1^{-1}(o) \to \pi_1^{-1}(o)/\mathbb{G}_{a,\kappa} \simeq \mathrm{Spec}(\kappa[w])$ with the open immersion $\pi_1^{-1}(o)/\mathbb{G}_{a,\kappa} \hookrightarrow E$ of $\pi_1^{-1}(o)/\mathbb{G}_{a,\kappa}$ as the complement of the κ-rational point $o_1 = ((0, 0), [0 : 1]) \in E$. On the other hand, the composition

$$\tau \circ f \circ j_1 : P \xrightarrow{\simeq} X_1 \setminus \pi_1^{-1}(o) \to \tilde{S} \setminus E \xrightarrow{\simeq} S \setminus \{o\}$$

coincides with the geometric quotient morphism $\rho : P \to S \setminus \{o\}$. So $f : X_1 \to \tilde{S}$ factors through a surjective morphism $q : X_1 \to \tilde{S} \setminus \{o_1\}$ whose fibers all consist of precisely one $\mathbb{G}_a$-orbit. Since q is a smooth morphism, q is a $\mathbb{G}_a$-torsor which implies that $X_1/\mathbb{G}_a \simeq \tilde{S} \setminus \{o_1\}$.

The scheme of the classification of quasi-projective extensions of Type II of a given $\mathbb{G}_a$-torsor $\rho : P \to S_*$ which we give below is as follows: we first construct in Section 4.1 families of such extensions, in the form of $\mathbb{G}_a$-torsors $q : X \to S'$ over quasi-projective S-schemes $\tau : S' \to S$ such that $\tau^{-1}(o)_{\mathrm{red}}$ is isomorphic to $\mathbb{A}^1_\kappa$, S' is smooth along $\tau^{-1}(o)$, and $\tau : S' \setminus \tau^{-1}(o) \to S_*$ is an isomorphism. We then show in Section 4.2 that for a quasi-projective $\mathbb{G}_a$-extension $\pi : X \to S$ of Type II of a given $\mathbb{G}_a$-torsor $\rho : P \to S_*$, the structure morphism $\pi : X \to S$ factors through a $\mathbb{G}_a$-torsor $q : X \to S'$ over one of these S-schemes S'. In the last subsection, we focus on the special case where $\pi : X \to S$ has the stronger property of being an affine morphism.

4.1. A family of $\mathbb{G}_a$-extensions over quasi-projective S-schemes

Let again (S, o) be a pair consisting of a surface and a closed point o contained in the smooth locus of S, with residue field κ. We let $\overline{\tau}_1 : \overline{S}_1 \to S$ be the blow-up of o, with exceptional divisor $\overline{E}_1 \simeq \mathbb{P}^1_\kappa$. Then for every $n \geq 2$, we let $\overline{\tau}_{n,1}$:

$\overline{S}_n = \overline{S}_n(o_1, \dots, o_{n-1}) \to \overline{S}_1$ be the scheme obtained from $\overline{S}_1$ by performing the following sequence of blow-ups of κ-rational points:

a) The first step $\overline{\tau}_{21} : \overline{S}_2(o_1) \to \overline{S}_1$ is the blow-up of a κ-rational point $o_1 \in \overline{E}_1$ with exceptional divisor $\overline{E}_2 \simeq \mathbb{P}^1_\kappa$;
b) Then for every $2 \le i \le n-2$, we let $\overline{\tau}_{i+1,i} : \overline{S}_{i+1}(o_1, \dots, o_i) \to \overline{S}_k(o_1, \dots, o_{i-1})$ be the blow-up of a κ-rational point $o_i \in \overline{E}_i$, with exceptional divisor $\overline{E}_{i+1} \simeq \mathbb{P}^1_\kappa$;
c) Finally, we let $\overline{\tau}_{n,n-1} : \overline{S}_n(o_1, \dots, o_{n-1}) \to \overline{S}_{n-1}(o_1, \dots, o_{n-2})$ be the blow-up of a κ-rational point $o_{n-1} \in \overline{E}_{n-1}$ which is a smooth point of the reduced total transform of $\overline{E}_1$ by $\overline{\tau}_1 \circ \cdots \circ \overline{\tau}_{n-1,n-2}$.

We let $\overline{E}_n \simeq \mathbb{P}^1_\kappa$ be the exceptional divisor of $\overline{\tau}_{n,n-1}$ and we let

$$\overline{\tau}_{n,1} = \overline{\tau}_{2,1} \circ \cdots \circ \overline{\tau}_{n,n-1} : \overline{S}_n(o_1, \dots, o_{n-1}) \to \overline{S}_1.$$

The inverse image of o in $\overline{S}_n(o_1, \dots, o_{n-1})$ by $\overline{\tau}_1 \circ \overline{\tau}_{n,1}$ is a tree of κ-rational curves in which $\overline{E}_n$ intersects the reduced proper transform of $\overline{E}_1 \cup \cdots \cup \overline{E}_{n-1}$ in $\overline{S}_n(o_1, \dots, o_{n-1})$ transversally in a unique κ-rational point.

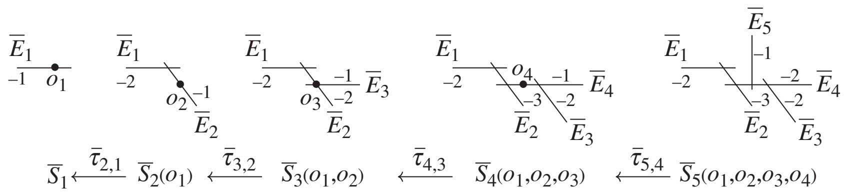


Figure 4.1. The successive total transforms of $\overline{E}_1$ in a possible construction of a surface of the form $\overline{S}_5(o_1, \dots, o_4)$ over a k-rational point o. The integers indicate the self-intersections of the corresponding curves.

Notation 4.3. For every κ-rational point $o_1 \in \overline{E}_1$, we let $S_1(o_1) = \overline{S}_1 \setminus \{o_1\}$, $E_1 = \overline{E}_1 \cap S_1 \simeq \mathbb{A}^1_\kappa$ and we let $\tau_1 : S_1(o_1) \to S$ be the restriction of $\overline{\tau}_1$.

For $n \ge 2$, we let $S_n(o_1, \dots, o_{n-1}) = \overline{S}_n(o_1, \dots, o_{n-1}) \setminus \overline{E}_1 \cup \cdots \cup \overline{E}_{n-1}$ and $E_n = S_n(o_1, \dots, o_{n-1}) \cap \overline{E}_n \simeq \mathbb{A}^1_\kappa$. We denote by $\tau_{n,1} : S_n(o_1, \dots, o_{n-1}) \to \overline{S}_1$ the birational morphism induced by $\overline{\tau}_{n,1}$ and we let $\tau_n = \overline{\tau}_1 \circ \tau_{n,1} : S_n(o_1, \dots, o_{n-1}) \to S$.

The following lemma summarizes some basic properties of the so-constructed S-schemes:

Lemma 4.4. *For every $n \ge 1$, the following hold for $S_n = S_n(o_1, \dots, o_{n-1})$:*

a) *$\tau_n : S_n \to S$ is quasi-projective and restricts to an isomorphism over S_* while $\tau_n^{-1}(o)_{\mathrm{red}} = E_n$;*

b) *S_n is smooth along $\tau_n^{-1}(o)$;*
c) *$\tau_n^* : \Gamma(S, \mathcal{O}_S) \to \Gamma(S_n, \mathcal{O}_{S_n})$ is an isomorphism.*

Moreover for $n \geq 2$, the morphism $\tau_{n,1} : S_n \to \overline{S}_1$ is affine.

Proof. Properties a)-c) are straightforward consequences of the construction. For the last assertion, let $D = \overline{E}_1 + \sum_{i=2}^{n-1} a_i \overline{E}_i$ where $(a_i)_{i=2,\ldots,n-1}$ is a sequence of positive rational numbers and let $m \geq 1$ be so that mD is a Cartier divisor on $\overline{S}_n$. Then a direct computation shows that the restriction of $\mathcal{O}_{\overline{S}_n}(mD)$ to $\overline{\tau}_{n,1}^{-1}(o_1)_{\mathrm{red}} = \bigcup_{i=2}^{n} \overline{E}_i$ is an ample invertible sheaf provided that the sequence $(a_i)_{i=2,\ldots,n-1}$ decreases rapidly enough with respect to the distance of $\overline{E}_i$ to $\overline{E}_1$ in the dual graph of $\overline{E}_1 \cup \cdots \cup \overline{E}_{n-1}$. Concretely, it suffices to choose the sequence $(a_i)_{i=2,\ldots,n-1}$ according to the following rule: If $\overline{E}_i$ has distance d to $\overline{E}_1$ and a_j is known for $\overline{E}_j$ closer to $\overline{E}_1$, then we pick $a_i \in \mathbb{Q}_{>0}$ such that $a_i \overline{E}_i^2 + a_k > 0$ where $\overline{E}_k$ is the unique curve intersecting $\overline{E}_i$ at distance $d-1$ from $\overline{E}_1$, and $a_1 = 1$. Since $\overline{\tau}_{n,1}$ restricts to an isomorphism over $\overline{S}_1 \setminus \{o_1\}$, it then follows from [7, Théorème 4.7.1] that $\mathcal{O}_{\overline{S}_n}(mD)$ is $\overline{\tau}_{n,1}$-ample on $\overline{S}_n$. Since by definition $\tau_{n,1}$ is the restriction of the projective morphism $\overline{\tau}_{n,1} : \overline{S}_n \to \overline{S}_1$ to $S_n = \overline{S}_n \setminus \overline{E}_1 \cup \cdots \cup \overline{E}_{n-1} = \overline{S}_n \setminus \mathrm{Supp}(D)$, we conclude that $\tau_{n,1}$ is an affine morphism. □

Remark 4.5. Blowing up the point o_{i-1} in E_{i-1}, the multiplicity of the new exceptional curve E_i as an irreducible component of $(\overline{\tau}_1 \circ \overline{\tau}_{i,1})^{-1}(o)$ equals the sum of the multiplicities of E_{i-1} and possibly E_{i-2} (if it contains o_{i-1}), while the multiplicities of the previous exceptional curves remain unchanged. By construction, $\tau_1^{-1}(o) = E_1$ in $S_1(o_1)$, but for $n \geq 2$, we have $\tau_n^{-1}(o) = mE_n$ for some integer $m \geq 1$ which depends on the sequence of κ-rational points $o_1, \ldots, o_{n-1}$ blown-up to construct $S_n(o_1, \ldots, o_{n-1})$. For instance, it is straightforward to check that $m = 1$ if and only if for every $i \geq 1, o_i \in \overline{E}_i$ is a smooth point of the reduced total transform of $\overline{E}_1$ in $\overline{S}_i(o_1, \ldots, o_{i-1})$.

The structure morphism of a $\mathbb{G}_a$-torsor being affine, hence quasi-projective, the total space of any $\mathbb{G}_a$-torsor $q : X \to S_n$ over an S-scheme $\tau_n : S_n = S_n(o_1, \ldots, o_n) \to S$ is a quasi-projective S-scheme $\pi = \tau_n \circ q : X \to S$ equipped with a proper $\mathbb{G}_{a,S}$-action. Furthermore $\pi^{-1}(o)_{\mathrm{red}} = q^{-1}(E_n) \simeq E_n \times \mathbb{A}^1_\kappa \simeq \mathbb{A}^2_\kappa$ and X is smooth along $\pi^{-1}(o)$ as S_n is smooth along E_n. On the other hand, $\pi : X \to S$ is by construction a $\mathbb{G}_a$-extension of its restriction $\rho : P \to S_n \setminus E_n \simeq S_*$ over $S_n \setminus E_n$, hence is a quasi-projective $\mathbb{G}_a$-extension of P of Type II. The following proposition shows conversely that every $\mathbb{G}_a$-torsor $\rho : P \to S_*$ admits a quasi-projective $\mathbb{G}_a$-extension of Type II into a $\mathbb{G}_a$-torsor $q : X \to S_n$.

Proposition 4.6. *Let $\rho : P \to S_*$ be a $\mathbb{G}_a$-torsor. Then for every $n \geq 1$ and every S-scheme $\tau_n : S_n(o_1, \ldots, o_{n-1}) \to S$ as in Notation 4.3 there exist a $\mathbb{G}_a$-torsor $q : X \to S_n(o_1, \ldots, o_{n-1})$ and an equivariant open embedding $j : P \hookrightarrow X$ such* *that in the following diagram*

$$\begin{array}{ccc} P & \overset{j}{\hookrightarrow} & X \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle q} \\ S_n(o_1, \dots, o_{n-1}) \setminus E_n & \hookrightarrow & S_n(o_1, \dots, o_{n-1}) \\ {\scriptstyle \tau_n}\downarrow\wr & & \downarrow{\scriptstyle \tau_n} \\ S_* & \hookrightarrow & S \end{array}$$

all squares are cartesian. In particular, $j : P \hookrightarrow X$ *is a quasi-projective* $\mathbb{G}_a$*-extension of* P *of Type II.*

Proof. Letting $S_n = S_n(o_1, \dots, o_{n-1})$, we have to prove that every $\mathbb{G}_a$-torsor $\rho : P \to S_n \setminus E_n \simeq S_*$ is the restriction of a $\mathbb{G}_a$-torsor $q : X \to S_n$. It is enough to show that there exists a Zariski open neighborhood U of E_n in S_n and a $\mathbb{G}_a$-torsor $q : Y \to U$ such that $Y \mid_{U \setminus E_n} \simeq P \mid_{U \setminus E_n}$. Indeed, if so then a $\mathbb{G}_a$-torsor $q : X \to S_n$ with the desired property is obtained by gluing P and Y over $U \setminus E_n$ by the isomorphism $Y \mid_{U \setminus E_n} \simeq P \mid_{U \setminus E_n}$. In particular, we can replace S_n by the inverse image by $\tau_n : S_n \to S$ of any Zariski open neighborhood of o in S. We can thus assume from the very beginning that $S = \mathrm{Spec}(A)$ is affine and that $o = V(f, g)$ is a scheme-theoretic intersection for some $f, g \in A$. Up to replacing f and g by other generators of the maximal ideal of o in A, we can assume that the proper transform L_1 in $\overline{\tau}_1 : \overline{S}_1 \to S$ of the curve $L = V(f) \subset S$ intersects $\overline{E}_1$ in o_1. We denote by $M_1 \subset \overline{S}_1$ the proper transform of the curve $M = V(g) \subset S$. By virtue of Lemma 4.6 below, it is enough to find an affine open subset U_n of S_n such that $U_n \setminus E_n = U_n \cap (S_n \setminus E_n)$ is affine and $S_n = U_n \cup (S_n \setminus E_n)$. In the case $n = 1$, $U_1 = \overline{S}_1 \setminus L_1 \subset S_1$ has the desired property since $U_1 \setminus E_1 = \overline{S}_1 \setminus \overline{\tau}_1^{-1}(L) \simeq S \setminus L$ is indeed affine. In the case where $n \geq 2$, the open subset $\overline{S}_1 \setminus M_1$ of $\overline{S}_1$ is affine and it contains o_1 because M_1 intersects $\overline{E}_1$ in a point distinct from o_1. Since $\tau_{n,1} : S_n \to \overline{S}_1$ is an affine morphism by Lemma 4.4, $U_n = \tau_{n,1}^{-1}(\overline{S}_1 \setminus M_1)$ is an affine open neighborhood of E_n in S_n with the property that $U_n \cap (S_n \setminus E_n) = U_n \setminus E_n = \tau_{n,1}^{-1}(\overline{S}_1 \setminus \overline{\tau}_1^{-1}(M))$ is affine. □

In the proof of Proposition 4.6, we used the following elementary extension result:

Lemma 4.7. *Let* $X = U \cup V$ *be a scheme with a cover by two Zariski open subsets* U *and* V*. Suppose that* U *and* $U \cap V$ *are affine. Then every* $\mathbb{G}_a$*-torsor on* V *is the restriction of a* $\mathbb{G}_a$*-torsor on* X*, possibly not unique.*

Proof. The assertion is equivalent to the surjectivity of the restriction homomorphism $H^1(X, \mathcal{O}_X) \to H^1(V, \mathcal{O}_V)$ which follows directly from the Mayer-Vietoris long exact sequence of cohomology of $\mathcal{O}_X$ for the covering of X by U and V. Indeed, this sequence reads

$$\begin{aligned} 0 \to & H^0(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_X) \oplus H^0(V, \mathcal{O}_X) \to H^0(U \cap V, \mathcal{O}_X) \to \cdots \\ & \to H^1(X, \mathcal{O}_X) \to H^1(U, \mathcal{O}_X) \oplus H^1(V, \mathcal{O}_V) \to H^1(U \cap V, \mathcal{O}_X) \to \cdots, \end{aligned}$$

and $H^1(U, \mathcal{O}_X) = H^1(U \cap V, \mathcal{O}_X) = 0$ as U and $U \cap V$ are affine. □

4.2. Classification

The following theorem shows that every quasi-projective $\mathbb{G}_a$-extension of Type II of a given $\mathbb{G}_a$-torsor $\rho : P \to S_*$ is isomorphic to one of the schemes $q : X \to S_n$ constructed in Section 4.1.

Theorem 4.8. *Let $\rho : P \to S_*$ be a $\mathbb{G}_a$-torsor and let*

$$\begin{array}{ccc} P & \xhookrightarrow{\;j\;} & X \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ S_* & \hookrightarrow & S \end{array}$$

be a quasi-projective $\mathbb{G}_a$-extension of P of Type II. Then there exists an integer $n \geq 1$ and a scheme $\tau_n : S_n(o_1, \ldots, o_{n-1}) \to S$ such that X is a $\mathbb{G}_a$-torsor $q : X \to S_n(o_1, \ldots, o_{n-1}) \simeq X/\mathbb{G}_a$ and $\rho : P \to S_*$ coincides with the restriction of q to $S_n(o_1, \ldots, o_{n-1}) \setminus E_n \simeq S_*$.

Proof. Since the $\mathbb{G}_{a,S}$-action on X is proper, the geometric quotient $X/\mathbb{G}_{a,S}$ exists in the form of a separated algebraic S-space $\delta : X/\mathbb{G}_{a,S} \to S$. Furthermore, since by definition of an extension $\pi^{-1}(S_*) \simeq P$, we have $\pi^{-1}(S_*)/\mathbb{G}_{a,S} \simeq P/\mathbb{G}_{a,S} \simeq S_*$ and so δ restricts to an isomorphism over S_*. On the other hand, $\pi^{-1}(o) \simeq \mathbb{A}^2_\kappa$ is equipped with the induced proper $\mathbb{G}_{a,\kappa}$-action, whose geometric quotient $\mathbb{A}^2_\kappa/\mathbb{G}_{a,\kappa}$ is isomorphic to $\mathbb{A}^1_\kappa$. It follows from the universal property of geometric quotients that $\delta^{-1}(o) = \mathbb{A}^2_\kappa/\mathbb{G}_{a,\kappa} = \mathbb{A}^1_\kappa$.

Since X is smooth in a neighborhood of $\pi^{-1}(o)$, $X/\mathbb{G}_{a,S}$ is smooth in neighborhood of $\delta^{-1}(o)$. Let $\overline{\tau}_1 : \overline{S}_1 \to S$ be the blow-up of o. Since $\delta : X/\mathbb{G}_a \to S$ contracts $\delta^{-1}(o)$ to the point o, it follows from the universal property of blow-ups for surfaces that δ lifts to a morphism $\delta_1 : X/\mathbb{G}_{a,S} \to \overline{S}_1$. Letting $\pi_1 : \pi_1 : X \to \overline{S}_1$ be the induced morphism, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;\pi_1\;} & \overline{S}_1 \\ \downarrow & \overset{\delta_1}{\nearrow} & \downarrow{\scriptstyle\overline{\tau}_1} \\ X/\mathbb{G}_a & \xrightarrow{\;\delta\;} & S. \end{array}$$

Furthermore, since $\delta : X/\mathbb{G}_{a,S} \to S$ and $\overline{\tau}_1 : \overline{S}_1 \to S$ are separated, it follows that $\delta_1 : X/\mathbb{G}_{a,S} \to \overline{S}_1$ is separated. By construction, the image of $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ by δ_1 is contained in $\overline{E}_1$.

If δ_1 is not constant on $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ then δ_1 is a separated quasi-finite birational morphism. Since $\overline{S}_1$ is normal, δ_1 is thus an open immersion by virtue of Zariski Main Theorem for algebraic spaces [14, Tag 05W7]. Since $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}^1_\kappa$, the only possibility is that $\overline{S}_1 \setminus \delta_1(X/\mathbb{G}_{a,S})$ consists of a unique κ-rational point $o_1 \in \overline{E}_1$ and $\delta_1 : X/\mathbb{G}_{a,S} \to S_1(o_1) = \overline{S}_1 \setminus \{o_1\}$ is an isomorphism. So $\pi_1 : X \to S_1(o_1)$ is a $\mathbb{G}_a$-torsor whose restriction to $S_1(o_1) \setminus E_1 \simeq S_*$ coincides with $\rho : P \to S_*$.

Otherwise, if δ_1 is constant on $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$, then its image consists of a unique κ-rational point $o_1 \in \overline{E}_1$. The same argument as above implies that $\pi_1 : X \to \overline{S}_1$ and $\delta_1 : X/\mathbb{G}_{a,S} \to \overline{S}_1$ lift to a $\mathbb{G}_{a,S}$-invariant morphism $\pi_2 : X \to \overline{S}_2(o_1)$ and a separated morphism $\delta_2 : X/\mathbb{G}_{a,S} \to \overline{S}_2(o_1)$ to the blow-up $\overline{\tau}_{2,1} : \overline{S}_2(o_1) \to \overline{S}_1$ of $\overline{S}_1$ at o_1, with exceptional divisor $\overline{E}_2$. If the restriction of δ_2 to $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ is not constant then δ_2 is an open immersion and the image of $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ is an open subset of $\overline{E}_2$ isomorphic to $\mathbb{A}^1_\kappa$. The only possibility is that $\delta_2(\pi^{-1}(o)/\mathbb{G}_{a,\kappa}) = \overline{E}_2 \setminus \overline{E}_1$. Indeed, otherwise $\overline{S}_2 \setminus \delta_2(X/\mathbb{G}_{a,S})$ would consist of the disjoint union of a point in $\overline{E}_2 \setminus (\overline{E}_1 \cap \overline{E}_2)$ and of the curve $\overline{E}_1 \setminus (\overline{E}_1 \cap \overline{E}_2)$ which is not closed in $\overline{S}_2$, in contradiction to the fact that δ_2 is an open immersion. Summing up, $\delta_2 : X/\mathbb{G}_{a,S} \to S_2(o_1) = \overline{S}_2(o_1) \setminus \overline{E}_1$ is an isomorphism mapping $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ isomorphically onto E_2. So $\pi_2 : X \to S_2(o_1)$ is a $\mathbb{G}_a$-torsor whose restriction to $S_2(o_1) \setminus E_2 \simeq S_*$ coincides with $\rho : P \to S_*$.

Otherwise, if δ_2 is constant on $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$, then $\delta_2(\pi^{-1}(o)/\mathbb{G}_{a,\kappa})$ is a κ-rational point $o_2 \in \overline{E}_2$, and there exists a unique minimal sequence of blow-ups $\overline{\tau}_{k+1,k} : \overline{S}_{k+1}(o_1, \dots, o_k) \to \overline{S}_k(o_1, \dots, o_{k-1})$, $k = 2, \dots, m-1$ of successive κ-rational points $o_k \in \overline{E}_k \subset \overline{S}_k(o_1, \dots, o_{k-1})$, with exceptional divisors $\overline{E}_{k+1} \subset \overline{S}_{k+1}(o_1, \dots, o_k)$ such that $\pi_2 : X \to \overline{S}_2(o_1)$ and $\delta_2 : X/\mathbb{G}_{a,S} \to \overline{S}_2(o_1)$ lift respectively to a $\mathbb{G}_{a,S}$-invariant morphism $\pi_m : X \to \overline{S}_m(o_1, \dots, o_{m-1})$ and a separated morphism $\delta_m : X/\mathbb{G}_{a,S} \to \overline{S}_m(o_1, \dots, o_{m-1})$ with the property that the restriction of δ_m to $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ is non-constant. By Zariski Main Theorem [14, Tag 05W7] again, we conclude that δ_m is an open immersion, mapping $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}^1_\kappa$ isomorphically onto an open subset of $\overline{E}_m \simeq \mathbb{P}^1_\kappa$. As in the previous case, the image of $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ in $\overline{E}_m$ must be equal to the complement of the intersection of $\overline{E}_m$ with the proper transform of $\overline{E}_1 \cup \cdots \cup \overline{E}_{m-1}$ in $\overline{S}_m(o_1, \dots, o_{m-1})$ since otherwise $\overline{S}_m(o_1, \dots, o_{m-1}) \setminus \delta_m(X/\mathbb{G}_{a,S})$ would not be closed in $\overline{S}_m(o_1, \dots, o_{m-1})$. Since $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}^1_\kappa$, it follows that $\overline{E}_m$ intersects the proper transform of $\overline{E}_1 \cup \cdots \cup \overline{E}_{m-1}$ in a unique κ-rational point, implying in turn that $o_{m-1} \in \overline{E}_{m-1}$ is a smooth κ-rational point of the reduced total transform $\overline{E}_1 \cup \cdots \cup \overline{E}_{m-1}$ of $\overline{E}_1$ in $\overline{S}_{m-1}(o_1, \dots, o_{m-2})$. Summing up,

$$\delta_m : X/\mathbb{G}_{a,S} \to \overline{S}_m(o_1, \dots, o_{m-1}) \setminus \overline{E}_1 \cup \cdots \cup \overline{E}_{m-1}$$

is an isomorphism with an S-scheme of the form $S_m(o_1,\ldots,o_{m-1})$ as constructed in Section 4.1, mapping $\pi^{-1}(o)_{\mathrm{red}}/\mathbb{G}_{a,\kappa}$ isomorphically onto $E_m = S_m(o_1,\ldots,o_{m-1}) \cap \overline{E}_m$. It follows in turn that $\pi_m : X \to S_m(o_1,\ldots,o_{m-1})$ is a $\mathbb{G}_a$-torsor whose restriction to $S_m(o_1,\ldots,o_{m-1}) \setminus E_m \simeq S_*$ coincides with $\rho : P \to S_*$. This completes the proof. □

4.3. Affine $\mathbb{G}_a$-extensions of Type II

In this subsection, given a $\mathbb{G}_a$-torsor $\rho : P \to S_*$, we consider the existence of quasi-projective $\mathbb{G}_a$-extensions of Type II

$$\begin{array}{ccc} P & \xhookrightarrow{j} & X \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ S_* & \hookrightarrow & S \end{array}$$

with the additional property that X is affine over S. As in the case of extension to $\mathbb{A}^1$-bundles over the blow-up of o treated in Section 3.2, a necessary condition for the existence of such extensions is that the restriction of P over every open neighborhood of the closed point o in S is non-trivial. Indeed, if there exists an affine open neighborhood U of o over which P is trivial, then $P \simeq U \setminus \{o\} \times \mathbb{A}^1_k$ is strictly quasi-affine, hence cannot be the complement of a Cartier divisor $\pi^{-1}(o)$ in any affine U-scheme $X|_U$. The next theorem shows that this condition is actually sufficient:

Theorem 4.9. *Let $\rho : P \to S_*$ be a $\mathbb{G}_a$-torsor such that for every open neighborhood U of o in S, the restriction $P \times_{S_*} U \to U \setminus \{o\}$ is non-trivial. Then for every $n \geq 1$ and every S-scheme $\tau_n : S_n(o_1,\ldots,o_{n-1}) \to S$ as in Notation* 4.3 *there exists a quasi-projective $\mathbb{G}_a$-extension of P of Type II into the total space of a $\mathbb{G}_a$-torsor $q : X \to S_n(o_1,\ldots,o_{n-1})$ for which $\pi = \tau_n \circ q : X \to S$ is an affine morphism.*

The following example illustrates the strategy of the proof given below, which consists in constructing such affine extensions $\pi : X \to S$ by performing a well-chosen equivariant affine modification of extensions of $\rho : P \to S_*$ into locally trivial $\mathbb{A}^1$-bundles $\theta : W(P) \to \tilde{S}$ over the blow-up $\tau : \tilde{S} \to S$ of the point o.

Example 4.10. Let again X_0 and X_1 be the $\mathbb{G}_a$-extensions of $\rho : P = \{xv - yu = 1\} \to S \setminus \{o\}$ considered in Example 3.1 and 4.2. Recall that X_0 and X_1 are the affine S-schemes in $\mathbb{A}^3_S$ defined respectively by the equations

$$X_0 : \quad \begin{cases} xr - yq = 0 \\ yp - x(q-1) = 0 \\ pr - q(q-1) = 0 \end{cases} \qquad \text{and} \qquad X_1 : \quad \begin{cases} xw - y(yz_1+1) = 0 \\ xz_2 - z_1(yz_1+1) = 0 \\ z_1 w - yz_2 = 0 \end{cases}$$

equipped with the $\mathbb{G}_{a,S}$-actions associated with the locally nilpotent $\mathcal{O}_S$-derivations $\partial_0 = x^2\partial_p + xy\partial_q + y^2\partial_r$ and $\partial_1 = x\partial_{z_1} + (2yz_1+1)\partial_{z_2} + y^2\partial_w$ respectively.

The morphism $\pi_0 : X_0 \to S$ factors through the structure morphism $\theta : X_0 \to \tilde{S}$ of a torsor under a line bundle on the blow-up $\tau : \tilde{S} \to S$ of the origin, with the property that the restriction of X_0 to exceptional divisor $E = \mathbb{P}^1_\kappa$ of τ is a non-trivial torsor under the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1_\kappa}(-2)$. The $\mathbb{G}_{a,S}$-action on X_0 restricts to the trivial one on $X_0|_E = \pi_0^{-1}(o)$. More precisely, ∂_0 is a global section of the sheaf $\mathcal{T}_{X_0} \otimes \mathcal{O}_{X_0}(-2X_0|_E)$ of vector fields on X_0 that vanish at order 2 along $X_0|_E$. One way to obtain from X_0 a $\mathbb{G}_a$-extension $\pi : X \to S$ of $\rho : P \to S \setminus \{o\}$ with fiber $\pi^{-1}(o)_{\mathrm{red}}$ isomorphic to $\mathbb{A}^2_\kappa$ and a fixed point free action is thus to perform an equivariant affine modification which simultaneously replaces $X_0|_E$ by a copy of $\mathbb{A}^2_\kappa$ and decreases the "fixed point order of ∂_0 along $X_0|_E$", typically a modification with divisor D equal to $X_0|_E$ and whose center $Z \subset X_0|_E$ is supported by a curve isomorphic to $\mathbb{A}^1_\kappa$ which is mapped isomorphically onto its image by the restriction of θ. The birational S-morphism

$$\eta : X_1 \to X_0, \quad (x, y, z_1, z_2, w) \mapsto (x, y, xz_1, yz_1 + 1, w)$$

is equivariant for the $\mathbb{G}_{a,S}$-actions on X_0 and X_1 and corresponds to an equivariant affine modification of this type: it restricts to an isomorphism outside the fibers of π_0 and π_1 over o, and it contracts $\pi_1^{-1}(o) = \mathrm{Spec}(\kappa[z_2, w])$ onto the curve $\{p = q - 1 = 0\} \subset \pi_0^{-1}(o) = \{pr - q(q-1) = 0\}$. This curve is isomorphic to $\mathbb{A}^1_\kappa = \mathrm{Spec}(\kappa[r])$ and it is mapped by the restriction

$$\theta|_{\pi_0^{-1}(o)} : \pi_0^{-1}(o) \simeq \{pr - q(q-1) = 0\} \to E = \mathbb{P}^1_\kappa, \; (p, q, r) \mapsto [p : q-1] = [q : r]$$

of θ isomorphically onto the complement of the κ-rational point $[0 : 1] \in \mathbb{P}^1_\kappa$.

Proof of Theorem 4.9. By virtue of Theorem 3.7, there exists a unique integer $\ell_0 \geq 2$ such that $\rho : P \to S_*$ is the restriction of a torsor $\theta_1 : W_1 \to \overline{S}_1$ under the line bundle $M_1(\ell_0) = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{O}_{\overline{S}_1}(-\ell_0\overline{E}_1)) \to \overline{S}_1$ whose total space W_1 is affine over $\overline{S}_1$. We now treat the case of $S_1(o_1)$ and $S_n(o_1, \ldots, o_{n-1}), n \geq 2$ separately.

Given a κ-rational point $o_1 \in \overline{E}_1$, the restriction of W_1 over $E_1 = \overline{E}_1 \setminus \{o_1\} \simeq \mathbb{A}^1_\kappa$ is the trivial $\mathbb{A}^1$-bundle $E_1 \times \mathbb{A}^1_\kappa$. Since on the other hand the restriction $\theta_1|_{\overline{E}_1} : W_1|_{\overline{E}_1} \to \overline{E}_1$ is a non-trivial $\mathcal{O}_{\mathbb{P}^1}(-\ell_0)$-torsor (see Theorem 3.7), it follows that for every section $s : E_1 \to W_1|_{E_1}$ the image Z_1 of E_1 in $W_1|_{\overline{E}_1}$ is a closed curve isomorphic to E_1. Indeed, otherwise if Z_1 is not closed in $W_1|_{\overline{E}_1}$ then its closure $\overline{Z}_1$ would be a section of $\theta_1|_{\overline{E}_1}$ in contradiction with the fact that $\theta_1|_{\overline{E}_1} : W_1|_{\overline{E}_1} \to \overline{E}_1$ is a non-trivial $\mathcal{O}_{\mathbb{P}^1}(-\ell_0)$-torsor. Let $D_1 = \theta_1^{-1}(\overline{E}_1)$ and let $\sigma_1 : W_1' \to W_1$ be the affine modification of W_1 with center $(\mathcal{I}_{Z_1}, D_1)$. By virtue of Lemmas 2.9 and

2.11, $\theta_1 \circ \sigma_1 : W_1' \to \overline{S}_1$ factors through a torsor $\theta_1' : W_1' \to \overline{S}_1 \setminus \{o_1\} = S_1(o_1)$ under the line bundle

$$M_1'(\ell_0 - 1) = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{O}_{S_1(o_1)}((-\ell_0 + 1)E_1)) \to S_1(o_1).$$

Now since $E_1 \simeq \mathbb{A}^1_\kappa$ is affine, the restriction of θ_1' over $E_1 \subset S_1(o_1)$ is the trivial $M_1'(\ell_0 - 1)|_{E_1}$-torsor. Letting $D_2 = \theta_1'^{-1}(E_1)$ and $Z_2 \subset D_2$ be any section of $\theta_1'|_{D_2} : D_2 \to E_1$, the affine modification $\sigma_2 : W_2' \to W_1'$ with center $(\mathcal{I}_{Z_2}, D_2)$ is then an $M_1'(\ell_0 - 2)$-torsor $\theta_2' : W_2' \to S_1(o_1)$. Iterating this construction $\ell_0 - 1$ times, we reach a $\mathbb{G}_{a,S_1(o_1)}$-torsor $q = \theta'_{\ell_0+1} : X = W'_{\ell_0+1} \to S_1(o_1)$. Since $\sigma_1 : W_1' \to W_1$ and each $\sigma_i : W_i' \to W_{i-1}'$, $i \geq 2$, restricts to an isomorphism over the complement of E_1, the restriction of $q : X \to S_1(o_1)$ over $S_1(o_1) \setminus E_1 \simeq S_*$ is isomorphic to $\rho : P \to S_*$. Furthermore, since the morphisms σ_i, $i = 1, \dots, \ell_0+1$ are affine and $\overline{\tau}_1 \circ \theta_1 : W_1 \to S$ is an affine morphism, it follows that

$$\tau_1 \circ q = \overline{\tau}_1 \circ \theta_1 \circ \sigma_1 \circ \cdots \sigma_{\ell_0+1} : X \to S$$

is an affine morphism. So $q : X \to S_1(o_1)$ is a $\mathbb{G}_a$-extension of $\rho : P \to S_*$ with the desired property.

Now suppose that $n \geq 2$. It follows from the construction of the morphism $\tau_{n,1} : S_n = S_n(o_1, \dots, o_{n-1}) \to \overline{S}_1$ given in subsection 4.1 that $\tau_{n,1}^*\mathcal{O}_{\overline{S}_1}(\ell_0\overline{E}_1) \simeq \mathcal{O}_{S_n}(mE_n)$ for some $m \geq 2$. The fiber product $W_n = W_1 \times_{\overline{S}_1} S_n$ is thus a torsor $\theta_n : W_n \to S_n$ under the line bundle

$$M_n(m) = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{O}_{S_n}(-mE_n)) \to S_n$$

whose restriction to $S_n \setminus E_n \simeq S_*$ is isomorphic to $\rho : P \to S_*$. Furthermore, since $\tau_{n,1}$ is an affine morphism by virtue of Lemma 4.4, so is the projection $\mathrm{pr}_{W_1} : W_n \to W_1$. Since $\overline{\tau}_1 \circ \theta_1 : W_1 \to S$ is an affine morphism, we conclude that $\tau_n \circ \theta_n = \overline{\tau}_1 \circ \tau_{n,1} \circ \theta_n = \overline{\tau}_1 \circ \theta \circ \mathrm{pr}_{W_1} : W_n \to S$ is an affine morphism as well. Since $E_n \simeq \mathbb{A}^1_\kappa$, the restriction of θ_n over E_n is the trivial $M_n(m)|_{E_n}$-torsor. The desired $\mathbb{G}_{a,S_n}$-torsor $q : X \to S_n$ extending $\rho : P \to S_*$ is then obtained from $\theta_n : W_n \to S_n$ by performing a sequence of m successive affine modifications similar to those applied in the previous case. □

Remark 4.11. In the case where S is affine, the total spaces X of the varieties $q : X \to S_n(o_1, \dots, o_{n-1})$ of Theorem 4.9 are all affine. To our knowledge, these are the first instances of smooth affine threefolds equipped with proper $\mathbb{G}_a$-actions whose geometric quotients are smooth quasi-projective surfaces which are not quasi-affine.

We do not know in general if under the conditions of Theorem 4.9 every quasi-projective $\mathbb{G}_a$-extensions of P of Type II into the total space of a $\mathbb{G}_a$-torsor $q : X \to S_n(o_1, \dots, o_{n-1})$ has the property that $\pi = \tau_n \circ q : X \to S$ is an affine morphism. In particular, we ask the following:

Question 4.12. Is the total space X of a quasi-projective $\mathbb{G}_a$-extension $\pi : X \to \mathbb{A}^2$ of $\rho = \mathrm{pr}_{x,y} : \mathrm{SL}_2 = \{xv - yu = 1\} \to \mathbb{A}^2_*$ of Type II always an affine variety?

4.4. Examples

In the next paragraphs, we construct two countable families of quasi-projective $\mathbb{G}_a$-extensions of the $\mathbb{G}_a$-torsor $\mathrm{SL}_2 \to \mathrm{SL}_2/\mathbb{G}_a \simeq \mathbb{A}^2 \setminus \{(0,0)\}$ of Type II with affine total spaces. As a consequence of [10, Section 3], for any non-trivial $\mathbb{G}_a$-torsor $\rho : P \to S_*$ over a local punctured surface S_*, these provide, by suitable base changes, families of examples of $\mathbb{G}_a$-extensions of P whose total spaces are all affine over S.

4.4.1. *A family of* $\mathbb{G}_a$*-extensions of* SL_2 *of "Type II-A"*

Let $S = \mathbb{A}^2 = \mathrm{Spec}(k[x, y_0])$ and let $X_n \subset \mathbb{A}^{n+2}_S = \mathrm{Spec}(k[x, y_0][z_1, z_2, y_1, \ldots, y_n])$, $n \geq 1$, be the smooth threefold defined by the system of equations

$$\begin{cases} y_i y_j - y_k y_\ell = 0 & i, j, k, \ell = 0, \ldots, n,\ i + j = k + \ell \\ z_2 y_i - z_1 y_{i+1} = 0 & i = 0, \ldots, n-1 \\ x y_{i+1} - y_i(y_0 z_1 + 1) = 0 & i = 0, \ldots, n-1 \\ x z_2 - z_1(y_0 z_1 + 1) = 0. & \end{cases}$$

The threefold X_n can be endowed with a fixed point free $\mathbb{G}_{a,S}$-action induced by the locally nilpotent $k[x, y_0]$-derivation

$$x\partial_{z_1} + (2y_0 z_1 + 1)\partial_{z_2} + \sum_{i=1}^{n} i y_0 y_{i-1} \partial_{y_i}$$

of its coordinate ring. The scheme-theoretic fiber over $o = \{(0,0)\}$ of the $\mathbb{G}_a$-invariant morphism $\pi_n = \mathrm{pr}_{x,y_0} : X_n \to S$ is isomorphic $\mathbb{A}^2 = \mathrm{Spec}(k[z_2, y_n])$, on which the induced $\mathbb{G}_a$-action is a translation induced by the derivation ∂_{z_2} of $k[z_2, y_n]$. On the other hand, the morphism $j : \mathrm{SL}_2 = \{xv - y_0 u = 1\} \to X_n$ defined by

$$(x, y, u, v) \mapsto (x, u, uv, y, yv, yv^2, \ldots, yv^n)$$

is an equivariant open embedding of SL_2 equipped with the $\mathbb{G}_a$-action induced by the locally nilpotent derivation $x\partial_u + y_0\partial_v$ of its coordinate ring into X_n with image equal to $\pi^{-1}(\mathbb{A}^2 \setminus \{o\})$. So $j : \mathrm{SL}_2 \hookrightarrow X_n$ is a quasi-projective $\mathbb{G}_a$-extension of SL_2 into the affine variety X_n, with $\pi_n^{-1}(o) \simeq \mathbb{A}^2_k$.

The restrictions of the projection $\mathbb{A}^{n+3}_S \to \mathbb{A}^{n+2}_S$ onto the first $n+2$ variables induce a sequence of $\mathbb{G}_a$-equivariant birational morphisms $\sigma_{n+1,n} : X_{n+1} \to X_n$. The threefolds X_n thus form a countable tower of $\mathbb{G}_a$-equivariant affine modifications of X_1. It follows from Example 4.2 that X_1 is a quasi-projective extension of SL_2 of Type II with geometric quotient isomorphic to a quasi-projective surface of the form $S_1(o_1)$. More generally, we have the following result.

Proposition 4.13. *For every $n \geq 2$, the morphism $j : \mathrm{SL}_2 \hookrightarrow X_n$ is a quasi-projective $\mathbb{G}_a$-extension of Type II. The geometric quotient $X_n/\mathbb{G}_a$ is isomorphic to a quasi-projective surface $S_n = S_n(o_1, \ldots, o_n)$ as in Section* 4.1 *for which*

$\overline{S}_n(o_1, \ldots, o_{n-1}) \setminus S_n$ consists of a chain of $n-1$ smooth rational curves with self-intersection -2, i.e. *the exceptional set of the minimal resolution of a surface singularity of type A_{n-1}.*

Proof. To see this, we consider the following sequence of blow-ups: the first one $\overline{\tau}_1 : \overline{S}_1 \to U_0 = \mathbb{A}^2$ is the blow-up of the origin, with exceptional divisor $\overline{E}_1$, and we let $U_1 \simeq \mathbb{A}^2 = \operatorname{Spec}(k[x, w_1])$ be the affine chart of $\overline{S}_1$ on which $\overline{\tau}_1 : \overline{S}_1 \to \mathbb{A}^2$ is given by $(x, w_1) \mapsto (x, xw_1)$. Then we let $\overline{\tau}_{2,1} : \overline{S}_2(o_1) \to \overline{S}_1$ be the blow-up of the point $o_1 = (0,0) \in U_1 \subset \overline{S}_1$ with exceptional divisor $\overline{E}_2$, and we let $U_2 \simeq \mathbb{A}^2 = \operatorname{Spec}(k[x, w_2])$ be the affine chart of $\overline{S}_2(o_1)$ on which the restriction of $\overline{\tau}_{2,1} : \overline{S}_2(o_1) \to \overline{S}_1$ coincides with the morphism $U_2 \to U_1$, $(x, w_2) \mapsto (x, xw_2)$. For every $2 < m \le n$, we define recursively the blow-up

$$\overline{\tau}_{m,m-1} : \overline{S}_m(o_1, \ldots, o_{m-1}) \to \overline{S}_{m-1}(o_1, \ldots, o_{m-2})$$

of the point $o_{m-1} = (0,0) \in U_{m-1} \subset \overline{S}_{m-1}(o_1, \ldots, o_{m-2})$ with exceptional divisor $\overline{E}_m$ and we let $U_m \simeq \mathbb{A}^2 = \operatorname{Spec}(k[x, w_m])$ be the affine chart of $\overline{S}_m(o_1, \ldots, o_{m-1})$ on which the restriction of $\overline{\tau}_{m,m-1}$ coincides with the morphism $U_m \to U_{m-1}$, $(x, w_m) \mapsto (x, xw_m)$. By construction, we have a commutative diagram

$$\begin{array}{ccccccccc}
\overline{S}_n(o_1,\ldots,o_{n-1}) & \xrightarrow{\overline{\tau}_{n,n-1}} & \overline{S}_{n-1}(o_1,\ldots,o_{n-2}) & \xrightarrow{\overline{\tau}_{n-1,n-2}} & \cdots & \xrightarrow{\overline{\tau}_{2,1}} & \overline{S}_1 & \xrightarrow{\overline{\tau}_1} & \mathbb{A}^2 \\
\uparrow & & \uparrow & & & & \uparrow & & \| \\
U_n & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U_1 & \longrightarrow & \mathbb{A}^2 = U_0.
\end{array}$$

The total transform of $\overline{E}_1$ in $\overline{S}_n(o_1, \ldots, o_{n-1})$ is a chain $\overline{E}_1 \cup \overline{E}_2 \cup \cdots \cup \overline{E}_{n-1} \cup \overline{E}_n$ formed by $n-1$ curves with self-intersection -2 and the curve $\overline{E}_n$ which has self-intersection -1.

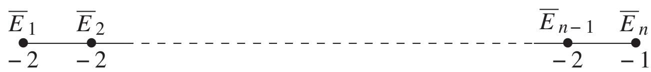


Figure 4.2. Dual graph of the total transform of $\overline{E}_1$ in $\overline{S}_n(o_1, \ldots, o_n)$.

The morphism $\pi : X_n \to S$ lifts to a morphism $\pi_1 : X_n \to \overline{S}_1$ defined by

$$(x, z_1, z_2, y_0, y_1, \ldots, y_n) \mapsto ((x, y_0), [x : y_0]) = ((x, y), [y_0 z_1 + 1 : y_1]).$$

This morphism contracts $\pi^{-1}(o)$ onto the point $o_1 = ((0,0), [1 : 0])$ of the exceptional divisor $\overline{E}_1$ of $\overline{\tau}_1$. The induced rational map $\pi_1 : X_n \dashrightarrow U_1$ is given by

$$(x, z_1, z_2, y_0, y_1, \ldots, y_n) \mapsto \left(x, \frac{y_1}{y_0 z_1 + 1}\right)$$

and it contracts $\pi^{-1}(o)$ onto the origin $o_1 = (0,0)$. So π_1 lifts to a morphism $\pi_2 : X_n \to \overline{S}_2(o_1)$, and with our choice of charts, the induced rational map

$\pi_2 : X_n \dashrightarrow U_2$ is given by

$$(x, z_1, z_2, y_0, y_1, \dots, y_n) \mapsto \left(x, \frac{y_2}{(y_0 z_1 + 1)^2}\right).$$

If $n = 2$ then the image of $\pi^{-1}(o) = \mathrm{Spec}(k[z_2, y_2])$ by π_2 is equal to $\overline{E}_2 \cap U_2$ and $\pi_2^{-1}(\overline{E}_2 \cap U_2)$ is equivariantly isomorphic to $(\overline{E}_2 \cap U_2) \times \mathrm{Spec}(k[z_2])$ on which $\mathbb{G}_a$ acts by translations on the second factor. So $\pi_2 : X_n \to \overline{S}_2(o_1)$ factors through a $\mathbb{G}_a$-bundle $q_2 : X_2 \to S_2(o_1) = \overline{S}_2(o_1) \setminus \overline{E}_1$ and $X_2/\mathbb{G}_a \simeq S_2(o_1)$. Otherwise, if $n > 2$ then π_2 contracts $\pi^{-1}(o)$ onto the point $o_2 = (0, 0) \in \overline{E}_2 \cap U_2 \subset \overline{S}_2(o_1)$. So $\pi_2 : X_n \to \overline{S}_2(o_1)$ lifts to a morphism $\pi_3 : X_n \to \overline{S}_3(o_1, o_2)$. With our choice of charts, for each $2 < m < n$, the induced rational map $\pi_m : X_n \dashrightarrow U_m$ is given by

$$(x, z_1, z_2, y_0, y_1, \dots, y_n) \mapsto \left(x, \frac{y_m}{(y_0 z_1 + 1)^m}\right),$$

hence contracts $\pi^{-1}(o)$ onto the point $o_m = (0, 0) \in U_m \subset \overline{S}_m(o_1, \dots, o_{m-1})$. It thus lifts to a morphism $\pi_m : X_n \to \overline{S}_m(o_1, \dots, o_{m-1})$. At the last step, the image of $\pi^{-1}(o) = \mathrm{Spec}(k[z_2, y_n])$ by the rational map $\pi_n : X_n \dashrightarrow U_n$ induced by $\pi_n : X_n \to \overline{S}_n(o_1, \dots, o_{n-1})$ is equal to $\overline{E}_n \cap U_n$, and we conclude as above that $\pi_n : X_n \to \overline{S}_n(o_1, \dots, o_{n-1})$ factors through a $\mathbb{G}_a$-bundle

$$q_n : X_n \to S_n(o_1, \dots, o_{n-1}) = \overline{S}_n(o_1, \dots, o_{n-1}) \setminus \big(\overline{E}_1 \cup \cdots \cup \overline{E}_{n-1}\big),$$

hence that $X_n/\mathbb{G}_a$ is isomorphic to the quasi-projective surface $S_n(o_1, \dots, o_{n-1})$. □

4.4.2. *A family of $\mathbb{G}_a$-extensions of* SL_2 *of "Type II-D"*

To conclude this section, we present as an illustration of the proof of Theorem 4.9 another countable family of quasi-projective $\mathbb{G}_a$-extensions of SL_2 of Type II with affine total spaces.

Let again $\overline{\tau}_1 : \overline{S}_1 \to S = \mathbb{A}^2$ be the blow-up of the origin $o = \{(0, 0)\}$ in $\mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ with exceptional divisor $\overline{E}_1 \simeq \mathbb{P}^1$, identified with the closed subvariety of $\mathbb{A}^2 \times \mathbb{P}^1_{[w_0:w_1]}$ with equation $xw_1 - yw_0 = 0$ in such a way that τ coincides with the restriction of the first projection. The second projection identifies $\overline{S}_1$ with the total space $p : \overline{S}_1 \to \mathbb{P}^1$ of the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$. We fix trivializations $p^{-1}(U_\infty) = \mathrm{Spec}(k[z_\infty][u_\infty])$ and $p^{-1}(U_0) = \mathrm{Spec}(k[z_0][u_0])$ over the open subsets $U_\infty = \mathbb{P}^1 \setminus \{[0 : 1]\} = \mathrm{Spec}(k[z_\infty])$ and $U_0 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \mathrm{Spec}(k[z_0])$ in such a way that the gluing of $p^{-1}(U_\infty)$ and $p^{-1}(U_0)$ over $U_0 \cap U_\infty$ is given by the isomorphism $(z_0, u_0) \mapsto (z_\infty, u_\infty) = (z_0^{-1}, z_0 u_0)$.

For every $n \geq 1$, we let $S_{2n+3,0} = \mathrm{Spec}(k[z_0, u_0^{\pm 1}])$,

$$S_{2n+3,\infty} = \mathrm{Spec}\left(k\big[z_\infty, u_\infty, v_\infty\big]/\big(u_\infty^n v_\infty - z_\infty^2 - u_\infty\big)\right),$$

and we let S_{2n+3} be the surface obtained by gluing $S_{2n+3,0}$ and $S_{2n+3,\infty}$ along the open subsets $S_{2n+3,0} \setminus \{z_0 = 0\}$ and $S_{2n+3,\infty} \setminus \{z_\infty = u_\infty = 0\}$ by the isomorphism

$$(z_0, u_0) \mapsto (z_\infty, u_\infty, v_\infty) = \left(z_0^{-1}, z_0u_0, (z_0u_0)^{-n}\left(z_0^{-2} + z_0u_0\right)\right).$$

The canonical open immersion $S_{2n+3,0} \hookrightarrow p^{-1}(U_0)$ and the projection $\mathrm{pr}_{z_\infty,u_\infty}$: $S_{2n+3,\infty} \to p^{-1}(U_\infty)$ glue to a global birational affine morphism $\tau_{2n+3,1} : S_{2n+3} \to \overline{S}_1$ restricting to an isomorphism $S_{2n+3} \setminus \{z_\infty = u_\infty = 0\} \to \overline{S}_1 \setminus \overline{E}_1$ where we identified the closed subset $E_{2n+3} = \{z_\infty = u_\infty = 0\} \simeq \mathrm{Spec}(k[v_\infty])$ of $S_{2n+3,\infty}$ with its image in S_{2n+3}. We leave to the reader to check that with the notation of Section 4.1, $S_{2n+3} = S_{2n+3}(o_1, \dots, o_{2n+2})$ for a surface $\overline{\tau}_{2n+3,1}$: $\overline{S}_{2n+3,1}(o_1, \dots, o_{2n+2}) \to \overline{S}_1$ obtained by first blowing-up the point $o_1 = (0, 0) \in p^{-1}(U_\infty)$ with exceptional divisor $\overline{E}_2$, then the point $o_2 = \overline{E}_1 \cap \overline{E}_2$ with exceptional divisor $\overline{E}_3$, then a point $o_3 \in \overline{E}_3 \setminus (\overline{E}_1 \cup \overline{E}_2)$ with exceptional divisor $\overline{E}_4$ and then a sequence of points $o_i \in \overline{E}_i \setminus \overline{E}_{i-1}$ with exceptional divisors $\overline{E}_{i+1}$, $i = 5, \dots, 2n + 2$ in such a way that the total transform of $\overline{E}_1$ in $\overline{S}_{2n+3,1}$ is a tree depicted in Figure 4.3. Letting $\tau_{2n+3} = \overline{\tau}_1 \circ \tau_{2n+3,1} : S_{2n+3} \to \mathbb{A}^2$, we have $\tau_{2n+3}^{-1}(o)_{\mathrm{red}} = E_{2n+3} \simeq \mathbb{A}^1$ and $\tau_{2n+3}^*(o) = 2E_{2n+3}$.

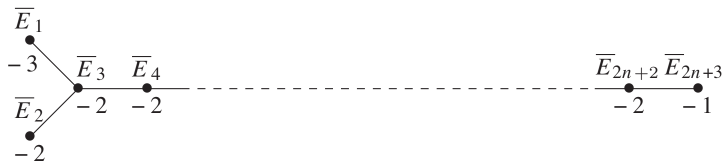


Figure 4.3. Dual graph of the total transform of $\overline{E}_1$ in $\overline{S}_{2n+3}(o_1, \dots, o_{2n+2})$.

Now we let $q : X_{2n+3} \to S_{2n+3}$ be the $\mathbb{G}_a$-bundle defined as the gluing of the trivial $\mathbb{G}_a$-bundles $X_{2n+3,0} = S_{2n+3,0} \times \mathrm{Spec}(k[t_0])$ and $X_{2n+3,\infty} = S_{2n+3,\infty} \times \mathrm{Spec}(k[t_\infty])$ over $S_{2n+3,0}$ and $S_{2n+3,\infty}$ respectively along the open subsets $X_{2n+3,0} \setminus \{z_0 = 0\}$ and $X_{2n+3,\infty} \setminus \{z_\infty = u_\infty = 0\}$ by the $\mathbb{G}_a$-equivariant isomorphism

$$(z_0, u_0, t_0) \mapsto (z_\infty, u_\infty, v_\infty, t_\infty) = \left(z_0^{-1}, z_0u_0, (z_0u_0)^{-n}\left(z_0^{-2} + z_0u_0\right), t_0 + z_0^{-1}u_0^{-2}\right).$$

Let $\pi_{2n+3} = \overline{\tau}_1 \circ \tau_{2n+3,1} \circ q : X_{2n+3} \to \mathbb{A}^2$.

Proposition 4.14. *For every $n \geq 1$, the variety X_{2n+3} is affine and there exists a $\mathbb{G}_a$-equivariant open embedding $j : \mathrm{SL}_2 \hookrightarrow X_{2n+3}$ which makes $\pi_{2n+3} : X_{2n+3} \to \mathbb{A}^2$ a quasi-projective $\mathbb{G}_a$-extension of SL_2 of Type II, with fiber $\pi_{2n+3}^{-1}(o)$ isomorphic to $\mathbb{A}^2$ of multiplicity two, and geometric quotient $X_{2n+3}/\mathbb{G}_a \simeq S_{2n+3}$.*

Proof. Let $j_1 : \mathrm{SL}_2 \hookrightarrow W = W(\mathrm{SL}_2, 2)$ be the $\mathbb{G}_a$-extension of SL_2 into a locally trivial $\mathbb{A}^1$-bundle $\theta : W \to \overline{S}_1$ with affine total space constructed in Example 3.1. Recall that the image of j_1 coincides with the restriction of θ to $\overline{S}_1 \setminus \overline{E}_1 = \mathbb{A}^2 \setminus \{o\}$.

With our choice of coordinates, the open subsets $W_0 = \theta^{-1}(q^{-1}(U_0))$ and $W_\infty = \theta^{-1}(q^{-1}(U_\infty))$ of W are respectively isomorphic to $p^{-1}(U_0) \times \mathrm{Spec}(k[w_0])$ and $p^{-1}(U_\infty) \times \mathrm{Spec}(k[w_\infty])$ glued over $U_0 \cap U_\infty$ by the isomorphism

$$(z_0, u_0, w_0) \mapsto (z_\infty, u_\infty, w_\infty) = \left(z_0^{-1}, z_0 u_0, z_0^2 w_0 + z_0\right).$$

The $\mathbb{G}_a$-action on W_0 and W_∞ are given respectively by $\alpha \cdot (z_0, u_0, w_0) = (z_0, u_0, w_0 + \alpha u_0^2)$ and $\alpha \cdot (z_\infty, u_\infty, w_\infty) = (z_\infty, u_\infty, w_\infty + \alpha u_\infty^2)$.

Let $W' = W \times_{\overline{S}_1} S_{2n+3}$, equipped with the natural lift of the $\mathbb{G}_a$-action on W. Since $\tau_{2n+3,1} : S_{2n+3} \to \overline{S}_1$ restricts to an isomorphism over $\overline{S}_1 \setminus \overline{E}_1$, the composition $j' = \tau_{2n+3,1}^{-1} \circ j_1 : \mathrm{SL}_2 \to W'$ is a $\mathbb{G}_a$-equivariant open embedding. Furthermore, since W is affine and $\tau_{2n+3,1}$ is an affine morphism, it follows that W' is affine. By construction, W' is covered by the two open subsets

$$\begin{cases} W_0' = W \times_{p^{-1}(U_0)} S_{2n+3,0} \simeq S_{2n+3,0} \times \mathrm{Spec}(k[w_0]) \\ W_\infty' = W \times_{p^{-1}(U_\infty)} S_{2n+3,\infty} \simeq S_{2n+3,\infty} \times \mathrm{Spec}(k[w_\infty]). \end{cases}$$

The local $\mathbb{G}_a$-equivariant morphisms

$$\begin{cases} \beta_0 : X_{2n+3,0} = S_{2n+3,0} \times \mathrm{Spec}(k[t_0]) \to W_0' \\ \beta_\infty : X_{2n+3,\infty} = S_{2n+3,\infty} \times \mathrm{Spec}(k[t_\infty]) \to W_\infty' \end{cases}$$

of schemes over $S_{2n+1,0}$ and $S_{2n+3,\infty}$ respectively defined by $t_0 \mapsto w_0 = u_0^2 t_0$ and $t_\infty \mapsto w_\infty = u_\infty^2 t_\infty$ glue to a global $\mathbb{G}_a$-equivariant birational affine morphism $\beta : X_{2n+3} \to W'$, restricting to an isomorphism over $S_{2n+3} \setminus E_{2n+3} \simeq \mathbb{A}^2 \setminus \{o\}$. Summing up, X_{2n+3} is affine over W' hence affine, and the composition $\beta^{-1} \circ j' : \mathrm{SL}_2 \hookrightarrow X_{2n+3}$ is a $\mathbb{G}_a$-equivariant open embedding which realizes $\pi : X_{2n+3} \to \mathbb{A}^2$ as a $\mathbb{G}_a$-extension of SL_2 of Type II with affine total space. By construction, $\pi_{2n+3}^{-1}(o) = q^{-1}(2E_{2n+3})$ is isomorphic to $\mathbb{A}^2$, with multiplicity two, while the geometric quotient $X_{2n+3}/\mathbb{G}_a$ is isomorphic to S_{2n+3}. □

Remark 4.15. For every $n \geq 1$, the birational morphism $S_{2(n+1)+3,\infty} \to S_{2n+3,\infty}$, $(z_\infty, u_\infty, v_\infty) \mapsto (z_\infty, u_\infty, u_\infty v_\infty)$ extends to a birational morphism $S_{2(n+1)+3} \to S_{2n+3}$ which lifts in turn in a unique way to a $\mathbb{G}_a$-equivariant birational morphism $\gamma_{n+1,n} : X_{2(n+1)+3} \to X_{2n+3}$. So in a similar way as for the family constructed in Section 4.4.1, the family of threefolds X_{2n+3}, $n \geq 1$, form a tower of $\mathbb{G}_a$-equivariant affine modifications of the initial one X_5.

References

[1] A. DUBOULOZ, "Sur une Classe de Schémas avec Actions de Fibrés en Droites", Ph.D Thesis, Université Joseph-Fourier-Grenoble I, https://tel.archives-ouvertes.fr/tel-00007733/, 2004.

[2] A. DUBOULOZ *Danielewski-Fieseler surfaces*, Transform. Groups **10** (2005), 139–162.
[3] A. DUBOULOZ, *Quelques remarques sur la notion de modification affine*, math. AG/0503142, (2005).
[4] A. DUBOULOZ and D. R. FINSTON, *On exotic affine* 3*-spheres*, J. Algebraic Geom. **23** (2014), 445–469.
[5] A. DUBOULOZ, *Complements of hyperplane sub-bundles in projective spaces bundles over* $\mathbb{P}^1$, Math. Ann. **361** (2015), 259–273.
[6] A. GROTHENDIECK, "Éléments de Géométrie Algébrique, I. Le Langage des Schémas", Inst. Hautes Études Sci. Publ. Math., Vol. 4, 1960.
[7] A. GROTHENDIECK, "Éléments de Géométrie Algébrique, III. Étude Cohomologique des Faisceaux Cohérents", Inst. Hautes Études Sci. Publ. Math., Vol. 11 and Vol. 17, 1961 and 1963.
[8] K-H. FIESELER, *On complex affine surfaces with* $\mathbb{C}_+$*-action*, Comment. Math. Helv. **69** (1994), 5–27.
[9] R. V. GURJAR, K. MASUDA and M. MIYANISHI, $\mathbb{A}^1$*-fibrations on affine threefolds*, J. Pure Appl. Algebra **216** (2012), 296–313.
[10] I. HEDÉN, *Affine extensions of principal additive bundles over a punctured surface*, Transform. Groups **21** (2016), 427–449.
[11] S. KALIMAN and M. ZAIDENBERG, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups **4** (1999), 53–95.
[12] G. KEMPF, *Some elementary proofs of basic theorems in the cohomology of quasi-coherent sheaves*, Rocky Mountain J. Math. **10** (1980), 637–646.
[13] M. NAGATA, "Lectures on the Fourteenth Problem of Hilbert", Lecture Notes, Tata Institute, Bombay, Vol. 31, 1959.
[14] THE STACKS PROJECT AUTHORS, *Stacks Project*, http://stacks.math.columbia.edu, 2017.

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