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Quasiconformal and HQC mappings between Lyapunov Jordan domains

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Abstract. Let h be a quasiconformal (qc) mapping of the unit disk $\mathbb{U}$ onto a Lyapunov domain. We show that h maps subdomains of Lyapunov type of $\mathbb{U}$, which touch the boundary of $\mathbb{U}$, onto domains of similar type. In particular if h is a harmonic qc (hqc) mapping of $\mathbb{U}$ onto a Lyapunov domain, using it, we prove that h is co-Lipschitz (co-Lip) on $\mathbb{U}$. This settles an open intriguing problem.

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1. Introduction

Throughout the paper we consider the following setting ($\mathbb{U}_{\mathrm{qc}}$): Let $h : \mathbb{U} \to D$ be a K-qc map, where $\mathbb{U}$ is the unit disk and suppose that D is a Lyapunov domain(see Definition 1.1 below). If in addition h is harmonic we say that h satisfies the hypothesis ($\mathbb{U}_{\mathrm{hqc}}$). Under the hypothesis ($\mathbb{U}_{\mathrm{qc}}$) we prove that for every $a \in \mathbb{T} = \{|z| = 1\}$, there is a special Lyapunov domain U_a, of a fixed shape, in the unit disk $\mathbb{U}$ which touches a and a special, convex Lyapunov domain $\mathrm{lyp}(D)_b^-$(see the Subsection 3.2, in particular Definition 3.6, Proposition 3.8, and the definition (v) before the the proof of Theorem 3.9)[1], of a fixed shape, in D, which touches $b = h(a)$, such that $\mathrm{lyp}(D)_b^- \subset h(U_a) \subset H_b$, where H_b is a half-plane whose the boundary line contains b. We can regard this result as "a good local approximation of a qc mapping h by its restriction to a special Lyapunov domain so that its codomain is locally convex". In addition, if h is harmonic, using this result, we prove that h is co-Lip on $\mathbb{U}$. This settles an open intriguing problem in the subject and can be regarded as a version of the Kellogg- Warschawski theorem for hqc. In order to discuss the subject we first need a few basic definitions (see Section 2 for more details).

[1] $\hat{D}_0$ and D_0^- ($D_0^- \subset \hat{D}_0$) are defined in Definition 3.6 and Proposition 3.8, $\mathrm{lyp}(D)_b^- = \underline{T}_b(D_0^-)$.

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By $|z|$ we denote the modulus of complex number z and sometimes by e we denote Euclidean distance between complex numbers.

Definition 1.1 (Lyapunov curves).

(i) Throughout the paper by $\varepsilon, \epsilon, c, c_1, \varepsilon_1, \epsilon_1, \kappa, \kappa_1$ etc. we denote positive constants and by μ, μ_1 etc. constants in the interval (0, 1);

(ii) Suppose that γ is a rectifiable, oriented, differentiable planar curve given by its arc-length parameterization g. If

$$l_1 = \mathrm{lyp}(\gamma) = \mathrm{lyp}(\gamma, \mu) := \sup_{t,s\in[0,l]} \frac{|g'(t) - g'(s)|}{|t - s|^{\mu}} < \infty,$$

we say that γ is a $C^{1,\mu}$ curve. $C^{1,\mu}$ curves are also known as Lyapunov (we say also more precisely μ-Lyapunov) curves. We call lyp(γ) the Lyapunov multiplicative constant. In this setting we say that γ is (μ, l_1)-Lyap (of order μ with multiplicative constant l_1). We say that a bounded planar domain D is μ-Lyapunov (respectively (μ, l_1)-Lyap), $0 < \mu < 1$, if it is bounded by μ-Lyapunov((μ, l_1)-Lyap) curve γ. In this setting it is convenient occasionally to use $l_1 = l_1(D)$ instead of lyp(γ).

For a complex valued function defined on a domain in the complex plane $\mathbb{C}$, we use the notation $\lambda_f = l_f(z) = |\partial f(z)| - |\bar{\partial} f(z)|$ and $\Lambda_f(z) = |\partial f(z)| + |\bar{\partial} f(z)|$, if $\partial f(z)$ and $\bar{\partial} f(z)$ exist.

Note that Lyp(ε, c) is a special domain of Lyapunov type with two cusps and vertex at 0.

Definition 1.2 (Elementary Lyapunov curves and special Lyapunov domains). The curve $\gamma(c, \mu) = \gamma(c, \mu, r_0)$ is defined, in polar coordinates (r, φ), by joining the curves $\varphi = cr^{\mu}$ and $\pi - \varphi = cr^{\mu}$, $0 \le r < r_0$, which share the origin (see Example 2.7 for more details). An arc L, which is isometric to the curve $\gamma(c, \mu)$ we call an elementary Lyapunov (more precisely μ-Lyapunov) curve. If A is the isometry we call $b = A(0)$ the vertex of L. If an arc C is a circle arc or elementary μ-Lyapunov for some $0 < \mu < 1$ we call it an elementary Lyapunov arc.

For $\varepsilon, c > 0$ and $c|\varepsilon|^{\mu} < \pi/2$, we use the notation:

(i) $L_0 = L(\varepsilon) = \mathrm{Lyp}(\varepsilon, c, \mu) = \{w : c|w|^{\mu} < \arg(w) < \pi - c|w|^{\mu}, |w| < \varepsilon\}$.

If this set is subset of H, D it seems convenient to denote it shortly by H_0, D_0 respectively.

A special domain of Lyapunov type (with possible two cusps) is a convex domain whose the boundary consists of two elementary Lyapunov curves. If the part of boundary of a Lyapunov (μ-Lyapunov) domain is an elementary Lyapunov curve with vertex at b, we call it special Lyapunov(μ-Lyapunov with elementary arc) domain with vertex at b.

Note that the curve $\gamma(c,\mu)$ is $C^{1,\mu}$ but it is not C^{1,μ_1} for $\mu_1 > \mu$ (at the origin), and $\mathrm{Lyp}(\varepsilon, c, \mu)$ is a special domain of Lyapunov type with two cusps and vertex at 0.

As an application of the Gehring-Osgood inequality [6,32] concerning qc mappings and quasi-hyperbolic distances, in the particular case of punctured planes, we prove Proposition 2.8 (we refer to this result as (GeOs)), which roughly stated says that:

- If f is a K-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $z_1, z_2 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, then the measures of the convex angles between $f(z_1)$, $f(z_2)$ and z_1, z_2 can be compared.

Using this we prove the part (IV) of Theorem 3.5 (we shortly refer this result as (S-0)), which can be considered as our main result, and Theorem 3.9 which is a global version of (S-0).

Theorem 3.9 gives an approximation of Lyapunov domains by special Lyapunov domains and it is a crucial result for the application to hqc mappings, stated here as:

(S-1) Suppose that D is a Lyapunov domain and $h : \mathbb{U} \to D$ is a qc homeomorphism. Then for every $a \in \mathbb{T} = \{|z| = 1\}$, there is a special Lyapunov domain U_a, of a fixed shape, in the unit disk $\mathbb{U}$ which touches a and a special, convex Lyapunov domain $\mathrm{lyp}(D)_b^-$, of a fixed shape, in D, which touches $b = h(a)$, such that $\mathrm{lyp}(D)_b^- \subset h(U_a) \subset H_b$, where H_b is a half-plane whose the boundary line contains b. Using this we reduce the proof of co-Lip property, stated here as:

(L0) If h satisfies the hypothesis ($\mathbb{U}_{\mathrm{hqc}}$) then it is co-Lip;

to what we call locally convex case. In order to avoid confusion, note that in addition Theorem 3.9 states that there is a special, convex Lyapunov domain $\mathrm{lyp}(D)_b$ (see Definition 3.6 in which we suggest also simple notation $\hat{D}_b$ instead of it and define $\hat{D}_b$ as image of $\hat{D}_0$ under T_b), of a fixed shape, in D, which touches $b = h(a)$, such that $h(U_a) \subset \mathrm{lyp}(D)_b \subset H_b$ (see Figure 3.1). But we do not use this part in the proof of (L0). In the first versions of the manuscript we use notation $\mathrm{lyp}(D)_b^-$ in order to indicate that it is an elementary Lyapunov domain. But in order to simplify notion we further use D_b^- frequently instead of it. Set $d_b(w) = \mathrm{dist}(w, \partial D_b^-)$.

By an elementary argument one can prove:

(L1) If $w - b$ is in the direction of the normal vector n_b of ∂D at b, then $d_b(w) \approx |w - b|$ if $|w - b|$ is small enough.

Note that the subject of hqc mappings has been intensively studied by the participants of the Belgrade Analysis Seminar (see Section 2 for more details), in particular by Kalaj, who proved that if h is a hqc mapping of the unit disk onto a Lyapunov domain, then h is Lipschitz [13]. Kalaj also probably first posed the problem whether h is, in fact, bi-Lipschitz and proved if the codomain of h is $C^{1,1}$ then h is bi-Lipschitz [15]). Since there is a conformal mapping of the unit disk $\mathbb{U}$ onto a C^1 domain which is not Lipschitz, Kalaj's result from [13] is nearly optimal. In [18], it is shown that a harmonic diffeomorphism h between two C^2 Jordan domains is a (K, K') quasiconformal mapping for some constants $K \geq 1$ and $K' \geq 0$ if and only if h is bi-Lipschitz continuous (note that $(K, 0)$ qc is K-qc). These results naturally lead to the following question (conjecture):

Question 1. If $h : \mathbb{U} \to D$ is a hqc homeomorphism, where D is a Lyapunov domain, is h co-Lipschitz (shortly co-Lip)?

In Theorem 4.1 we give an affirmative answer to Question 1.

The following simple statements play an important role in the proof of Theorem 4.1 (co-Lip).

[26, Proposition 5] states that if h is a harmonic univalent orientation preserving K-qc mapping of domain D onto D', then $d(z)\Lambda_h(z) \approx d_h(z), z \in D$. We need only a corollary of this:

(S-2) $d(z)\Lambda_h(z) \succeq d_h(z), z \in D$.

Using a slightly modification of the [26, proof of Theorem 1.1] (planar case) and Kellogg's theorem we can derive:

(S-3) Suppose that h is a Euclidean harmonic mapping from a Lyapunov domain G into a domain D and there is a half space H_b which touches a point $b \in \partial D$ such that $D = h(G) \subset H_b$. Then $e(h(z), b) \succeq d_G(z), z \in G$, where e here denotes the Euclidean distance.

We say that a domain D is locally convex at a point $b \in \partial D$ if there is a half space H_b such that $D \subset H_b$.

For the convenience of the reader we summarize that (S-1), (S-2) and (S-3), are the main ingredients in the proof of Theorem 4.1 stated here as

Theorem 1.3. *Suppose $h : \mathbb{U} \to D$ is a hqc homeomorphism, where D is a Lyapunov domain with $C^{1,\mu}$ boundary. Then h is co-Lipschitz.*

Remark 1.4. Note that, in general, $h(U_a)$ is not convex and we can not apply our consideration [26] (see the proof of Theorem 1.2 there) directly; but $h(U_a) \subset H_b$ is locally convex at b and we can apply (S-3) (note that we do not use the fact that $\text{lyp}(D)_b^-$ is a convex Lyapunov domain).

Recall that a mapping h which is $(\mathbb{U}_{qc})$ satisfies (S-1). If h is in addition harmonic then we can apply (S-3). This is crucial for the proof of theorem and it reduces the proof to the locally-convex case.

Note that in order to apply (S-3), we introduce several definitions and prove several properties which are mainly of technical character in the Subsection 3.2 (Global approximation).

Definition 1.5 (Hypothesis (Sp0), ($\mathbb{H}_{\mathbf{qc}}$), ($\mathbb{H}_{\mathbf{hqc}}$), (Lyp-0), ($\mathbb{H}^{\mathbf{0}}_{\mathbf{qc}}$), (H-0), ($\mathbb{U}_{\mathbf{qc}}$), ($\mathbb{U}_{\mathbf{hqc}}$), ($\mathbb{U}^{\mathbf{0}}_{\mathbf{qc}}$), and (U-1)). It is convenient to consider the following definitions:

(Sp0) If D is a Jordan bounded planar domain, $0 \in \partial D$ and D has the real axis as a tangent at 0, with inner normal pointing upwards, we say that D has (Sp0) property;

($\mathbb{H}_{\text{qc}}$) If $h : \mathbb{H} \xrightarrow{\text{onto}} D$ is K-qc map, where $\mathbb{H}$ is the upper -half plane and D is a Lyapunov domain with the boundary ∂D positively oriented we say that h satisfies ($\mathbb{H}_{\text{qc}}$) property (with respect to D).

If D is a Lyapunov domain using rotation and translation if it is necessary we can suppose that:

(Lyp-0) D is a Lyapunov domain and satisfies (Sp0);

($\mathbb{H}_{\text{hqc}}$) If, in addition to ($\mathbb{H}_{\text{qc}}$), h is harmonic, we say that h satisfies the hypothesis ($\mathbb{H}_{\text{hqc}}$);

($\mathbb{H}^0_{\text{qc}}$) If h satisfies the hypothesis ($\mathbb{H}_{\text{qc}}$), D satisfies the hypothesis (Lyp-0) and $h(0) = 0$,we say that h satisfies the hypothesis ($\mathbb{H}^0_{\text{qc}}$) (with respect to D);

(H-0) If h satisfies the hypothesis ($\mathbb{H}^0_{\text{qc}}$), and in addition h is harmonic on $\mathbb{H}$, we say that h satisfies the hypothesis ($\mathbb{H}^0_{\text{hqc}}$) (shortly (H-0);

($\mathbb{U}_{\text{qc}}$) If $\mathbb{H}$ is replaced by $\mathbb{U}$ in the hypothesis $\mathbb{H}_{\text{qc}}$ (respectively $\mathbb{H}_{\text{hqc}}$) we denote the corresponding hypotheses by ($\mathbb{U}_{\text{qc}}$) (respectively ($\mathbb{U}_{\text{hqc}}$));

($\mathbb{U}^0_{\text{qc}}$) If h satisfies the hypothesis ($\mathbb{U}_{\text{qc}}$), D satisfies the hypothesis (Lyp-0) and $h(1) = 0$,we say that h satisfies the hypothesis ($\mathbb{U}^0_{\text{qc}}$);

(U-1) If, in addition to ($\mathbb{U}^0_{\text{qc}}$), h is harmonic, we say that h satisfies the hypothesis (U-1).

Let T_a be the translation defined by $T_a(z) = z + a$. If h satisfies ($\mathbb{H}_{\text{qc}}$) property with respect to a domain G, $a \in \mathbb{R}$ and $b = h(a)$, then there is a mapping $R_b = R_0 \circ T_{-b}$, where R_0 is a rotation around 0, such that $R_b(G)$ satisfies the hypothesis (Lyp-0) and therefore $R_b \circ h \circ T_a$ satisfies the hypothesis ($\mathbb{H}^0_{\text{qc}}$).[2]

Note that in this paper we consider only the planar case. The plan of the exposition is as follows: In Section 2, we consider the background, definitions and basic properties of Lyapunov domains and we prove Proposition 2.8, which may be considered to be a version of the Gehring-Osgood inequality related to the measures of the corresponding angles. In Section 3, we prove Theorem 3.5 and Theorem 3.9. In Section 4 we give the proof of Theorem 4.1(co-Lip).

The second author communicated the main result of this paper at CMFT 2017.[3]

[2] R_b is short notation for the mapping $\underline{T}_b^{-1}$ which appears in Definition 3.6.

[3] See Cmft2017, Jule 10-15, Lublin, Poland (see http://cmft2017.umcs.lublin.pl), plenary speakers.

We also suggest to the interested reader to make rough picture and scheme with corresponding notations in order to follow the manuscript; and first to read Section 3 without proofs[4] and and then Section 4 with all details and finally to consider complete proofs and technical details in Section 3. For some basic definitions see Subsection 2.1 and 3.2, Definitions 3.6 and 4.5.

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2. Background

The next example which is shortly discussed in [8, 27], see also [20], shows that there is a conformal map of unit disk onto C^1 domain which is not bi-Lipschitz.

Example 2.1. Set

$$w = A(z) = \frac{z}{\ln \frac{1}{z}}, \quad w(0) = 0\,.$$

Note $\ln \frac{1}{z} = -\ln z$, $w'(z) = -(\ln z)^{-1} + (\ln z)^{-2}$ and $w'(z) \to 0$ if $z \to 0$ throughout $\mathbb{H}$. For r small enough A is univalent in $U_r^+ = \{z : \operatorname{Im} z > 0, |z| < r\}$. We can check that there is a smooth domain $D \subset U_r^+$ such that interval $(-r_0, r_0)$, $r_0 > 0$, is a part of the boundary of D, $D^* = A(D)$ is C^1 domain and A is not co-Lipschitz on D.

For basic properties of qc mappings the reader can consult Ahlfors's lovely book [4]. Let γ be a Jordan curve. By the Riemann mapping theorem there exists a Riemann conformal mapping of the unit disk onto the Jordan domain $G = \operatorname{int}\gamma$. By Caratheodory's theorem it has a continuous extension to the boundary. Moreover, if $\gamma \in C^{n,\alpha}$, $n \in \mathbb{N}$, $0 \le \alpha < 1$, then the Riemann conformal mapping has a $C^{n,\alpha}$ extension to the boundary (this result is known as Kellogg's theorem), see [34]. Conformal mappings are quasiconformal and harmonic. Hence quasiconformal harmonic (abbreviated by HQC) mappings are a natural generalization of conformal mappings.

Remark 2.2. Note that:

a) The proof of Kellogg's theorem for conformal mapping is not elementary and it is based on some techniques which we can not adapt for hqc;
b) Since there is a conformal map of unit disk onto C^1 domain which is not bi-Lipschitz (Example 2.1 above), it seems that the hypothesis that domains are Lyapunov is essential.

[4] Pay attention to Theorem 3.5(IV).

By a) and b) in mind, it seems that we need new approaches to study hqc mappings.

Recall that HQC mappings are now a very active area of investigation and some new methods have been developed for studying this subject (see for example [28] and literature cited there). Concerning the background we mention only a few results which are closely related to our results.

It seems that O. Martio [24] was the first one who considered HQC mapping of the unit disk. The author of this paper started considering distortion property of hqc mappings in 1988 /89 [5], see [27, Appendix 3]. Later M. Pavlović proved in [31] that HQC mappings of the unit disk are Lipschitz. An asymptotically sharp variant have been obtained by Partuka and Sakan [29]. Among other things Knežević and the second author in [19] showed that a K-qc harmonic mapping of the unit disk onto itself is a $(1/K, K)$ quasi-isometry with respect to the Poincaré and Euclidean metrics. For bi-lipschitz approximations of quasiconformal maps see Bishop [5]. M. Mateljević [26] and V. Manojlović [21] showed that hqc mappings are Bi-Lipschitz with respect to quasi hyperbolic metrics. Since the composition of a harmonic mapping and a conformal mapping is itself harmonic, using the case of the unit disk and Kellogg's theorem, these theorems can be generalized to the class of mappings from arbitrary Jordan domains with Lyapunov boundary onto the unit disk. However the composition of a conformal and a harmonic mapping is not, in general, a harmonic mapping. This means in particular, that results of this kind for arbitrary image domains do not follow directly from the case in which the codomain is the unit disk or the upper half-plane and Kellogg's theorem. In [17], Kalaj and the second author show how to combine Kellogg's theorem with the so called inner type estimate and that the simple proof in the case of the upper half-plane has an analogue for C^2 domains; namely, they proved a version of the "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain estimate concerning their Laplacian. As an application of this estimate, it is shown that quasi-conformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz. The discussion in [17] includes harmonic mappings with respect to (a) spherical and Euclidean metrics (which are approximately analytic) as well as (b) the metric induced by the holomorphic quadratic differential.

Although the following two statements did not get attention immediately after their publications, it turns out, surprisingly, that they play an important role in the proof of Theorem 4.1 (co-Lip).

Proposition 2.3 ([26, Corollary 1, Proposition 5]; see also [21]). *Every e-harmonic quasi-conformal mapping of the unit disc* (*more generally of a strongly hyperbolic domain*) *is a quasi-isometry with respect to the hyperbolic distance.*

[5] During the visiting position at Wayne State University, Detroit, 1988/89.

Theorem 2.4 ([25]). *Suppose that* $h = f + \overline{g}$ *is a Euclidean orientation preserving harmonic mapping from* $\mathbb{U}$ *onto the bounded convex domain* $D = h(\mathbb{U})$*, which contains a disc* $B(h(0); R_0)$.

(I) *Then* $|f'| \geq R_0/4$ *on* $\mathbb{U}$*;*
(II) *Suppose, in addition, that h is qc. Then* $l_h \geq (1-k)|f'| \geq (1-k)R_0/4$ *on* $\mathbb{U}$*;*
(III) *In particular,* h^{-1} *is Lipschitz.*

See also D. Kalaj doctoral thesis [12, Corollary 1.3.11], and Partyka and Sakan [30]. Concerning the Lipschitz and bi-Lipschitz properties of hqc, Kalaj [13, 15] proved:

Theorem 2.5. *Suppose* $h : D_1 \to D_2$ *is a hqc homeomorphism, where* D_1 *and* D_2 *are domains with* $C^{1,\mu}$, $0 < \mu < 1$*, boundary.*

(I) *Then h is Lipschitz;*
(II) *If, in addition,* D_2 *is convex, then h is bi-Lipschitz;*
(III) *If* $\mu = 1$*, then h is bi-Lipschitz.*

With this theorem in mind Question 1 is natural. The proof of part (a) of [13, Theorem 2.5] is based on an application of Mori's theorem on quasiconformal mappings, which has also been used in [31] in the case $D_1 = D_2 = \mathbb{U}$, and a geometric lemma related to Lyapunov domains.

2.1. Notation

Here we give a few basic definitions.

Definition 2.6 (qc).

(i) By $\mathbb{C}$ we denote the the complex plane and by $\mathbb{T}$ the unit circle. For $r > 0$ and $w \in \mathbb{C}$, we denote by $B(w, r)$ and the $C(w, r)$ the disk and circle of radius r with center at w;
(ii) By $\mathbb{C}^*$ we denote the punctured complex plane $\mathbb{C} \setminus \{0\}$, by $\mathbb{H}^*$ the lower half plane $\{z : \operatorname{Im} z < 0\}$ and by $\mathbb{U}^+$ the upper half disk $\{z : \operatorname{Im} z > 0, |z| < 1\}$;
(iii) Recall that, for a complex valued function h defined on a domain in the complex plane $\mathbb{C}$, we use the notation

$$\lambda_h = l_h(z) = |\partial h(z)| - |\bar{\partial} h(z)| \quad \text{and} \quad \Lambda_h(z) = |\partial h(z)| + |\bar{\partial} h(z)|,$$

if $\partial h(z)$ and $\bar{\partial} h(z)$ exist. A homeomorphism $h\colon D \to G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be K-quasiconformal (K-qc or k-qc), $K \geq 1$, if f is absolutely continuous on a.e. horizontal and a.e. vertical line in D and there is $k \in [0, 1)$ such that

$$|h_{\bar{z}}| \leq k|h_z| \quad \text{a.e. on } D, \tag{2.1}$$

where $K = \frac{1+k}{1-k}$, *i.e.* $k = \frac{K-1}{K+1}$. Note that the condition (2.1) can be written as

$$D_h := \frac{\Lambda_h}{\lambda_h} = \frac{|h_z| + |h_{\bar{z}}|}{|h_z| - |h_{\bar{z}}|} \leq K, \tag{2.2}$$

where $K = \frac{1+k}{1-k}$, *i.e.* $k = \frac{K-1}{K+1}$;

(iv) Let $\Omega \subset \mathbb{R}^n$ and $\mathbb{R}^+ = [0, \infty)$ and $f,\ g : \Omega \to \mathbb{R}^+$. If there is a positive constant c such that $f(x) \le c\, g(x)\,,\ x \in \Omega$, we write $f \preceq g$ on Ω. If there is a positive constant c such that

$$\frac{1}{c}\, g(x) \le f(x) \le c\, g(x)\,, \quad x \in \Omega\,,$$

we write $f \approx g$ (or $f \approx g$) on Ω.

To gain some intuition about Lyapunov curves we give a basic example:

Example 2.7. For $c > 0, 0 < \mu < 1$, and $x_0 > 0$, the curve $f(c, \mu) = f(c, \mu, x_0)$ in the xy-plane which is defined by

(i) $y = c|x|^{1+\mu},\ |x| < x_0$

is $C^{1,\mu}$ at the origin but is not C^{1,μ_1} for $\mu_1 > \mu$. It is convenient to write this equation using polar coordinates $z = re^{i\varphi}$ in the form: $r \sin\varphi = cr^{1+\mu}(\cos\varphi)^{1+\mu}$. Next, if $0 \le \varphi \le \pi/2$, we have $\sin\varphi = cr^{\mu}(\cos\varphi)^{1+\mu}$, $0 \le r < r_0$, where r_0 is a positive number. Since $\sin\varphi = \varphi + o(\varphi)$ and $\cos\varphi = 1 + o(1)$, we find $\varphi = cr^{\mu} + o(1)$ when $\varphi \to 0$. If $\pi/2 \le \varphi \le \pi$, we have $\sin(\pi - \varphi) = \sin\varphi = cr^{\mu}(\cos\varphi)^{1+\mu}$, $0 \le r < r_0$, where r_0 is a positive number. Since $\sin(\pi - \varphi) = \pi - \varphi + o(\pi - \varphi)$ and $\cos\varphi = -1 + o(1)$, we find $\pi - \varphi = cr^{\mu} + o(1)$ when $\varphi \to \pi$. The curve $\gamma(c, \mu) = \gamma(c, \mu, r_0)$ defined by joining the curves $\varphi = cr^{\mu}$ and $\pi - \varphi = cr^{\mu}$, $0 \le r < r_0$, which share the origin, has similar properties near the origin to the curve defined by (i). The reader can check that the curves $f(c, \mu)$ and $\gamma(c, \mu)$ are $C^{1,\mu}$ at the origin but are not C^{1,μ_1} for $\mu_1 > \mu$.

Note that if a curve satisfies $\varphi \le cr^{\mu}$, then it is is below the curve $\gamma(c, \mu)$.

2.2. Gehring-Osgood inequality

We can compute the quasihyperbolic metric k on $\mathbb{C}^*$ by using the covering $\exp : \mathbb{C} \to \mathbb{C}^*$, where exp is the exponential function. Let $z_1, z_2 \in \mathbb{C}^*$, $z_1 = r_1 e^{it_1}$, $z_2 = r_2 e^{it_2}$ and $\theta = \theta(z_1, z_2) \in [0, \pi]$ the measure of the convex angle between z_1, z_2. We use

$$k(z_1, z_2) = \sqrt{\left|\ln \frac{r_2}{r_1}\right|^2 + \theta^2}.$$

This well-known formula is due to Martin and Osgood.

Let $\ell = \ell(z_1)$ be the line defined by 0 and z_1. Then z_2 belongs to one half-plane, say M, on which $\ell = \ell(z_1)$ divides $\mathbb{C}$.

Locally, denote by ln a branch of Log on M. Note that ln maps M conformally onto a horizontal strip of width π. Since $w = \ln z$, we find that the quasi-hyperbolic metric

$$|\mathrm{d}w| = \frac{|\mathrm{d}z|}{|z|}.$$

Note that $\rho(z) = \frac{1}{|z|}$ is the quasi-hyperbolic density for $z \in \mathbb{C}^*$ and therefore

$$k(z_1, z_2) = |w_1 - w_2| = |\ln z_1 - \ln z_2|.$$

Let $z_1, z_2 \in \mathbb{C}^*$, $w_1 = \ln z_1 = \ln r_1 + it_1$. Then $z_1 = r_1 e^{it_1}$ and there is $t_2 \in [t_1, t_1 + \pi)$ or $t_2 \in [t_1 - \pi, t_1)$ such that $w_2 = \ln z_2 = \ln r_2 + it_2$. Hence

$$k(z_1, z_2) = \sqrt{\left|\ln \frac{r_2}{r_1}\right|^2 + (t_2 - t_1)^2}\,.$$

Now using the quasi-hyperbolic distance k as a corollary of the Gehring-Osgood inequality, we can prove the following result which we will need.

Proposition 2.8. *Let f be a K-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $\alpha = K^{-1}$. If $z_1, z_2 \in \mathbb{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$* (respectively $\theta^* \in [0, \pi]$) is the measure of the convex angle between z_1, z_2 (respectively $f(z_1)$, $f(z_2)$), then

$$\theta^* \le c \max\{\theta^\alpha, \theta\},$$

where $c = c(K)$. In particular, if $\theta \le 1$, then $\theta^* \le c\theta^\alpha$.

Proof. By the Gehring-Osgood inequality,

$$k\Big(f(z_1), f(z_2)\Big) \le c \max\{k(z_1, z_2)^\alpha, k(z_1, z_2)\},$$

where $c = c(K)$. It is clear that $\theta^* \le k\Big(f(z_1), f(z_2)\Big)$. Since $|z_1| = |z_2|$ and $k(z_1, z_2) = t_2 - t_1 = \theta$, we get the desired result. □

3. Main result

We first need some definitions.

Elementary Lyapunov domains, Arc-chord constant b_γ and the second Lyapunov constant $l_D^2 = l_\gamma^2$.

Definition 3.1 (Elementary Lyapunov domains).

(i) Recall for $r > 0$ and $w \in \mathbb{C}$, we denote by $B(w, r)$ and the $C(w, r)$ the disk and circle of radius r with center at w. In particular, we use notation $B(r)$ and the $C(r)$ for the disk and circle of radius r with center at 0 and we denote by $C^+(r)$ the half circle in the upper half plane;

(ii) Definition of $L_b^-(\varepsilon)$. Further for $v > 0$ let the circle $C(iv, r)$ touch the curve $\gamma = \gamma(\mu, c)$ at points w_1 and w_2 (say that $u_1 < u_2$) and let l^+ be the upper half arc of the circle $C(iv, r)$ joining w_1 and w_2 and γ_1 be the part of γ over $[u_1, u_2]$, where $u_k = \operatorname{Re} w_k$, $k = 1, 2$. Then the domain enclosed by l^+ and γ_1 we denote by $\mathrm{Lyp}(r, c)$. If ϵ^0 is maximum of $r > 0$ for which $\mathrm{Lyp}(r, c)$ belongs to $L(\varepsilon)$, we denote the domain $\mathrm{Lyp}(\epsilon^0, c)$ by $\mathrm{Lyp}^-(\varepsilon, c)$. If A is an Euclidean isometry and $A(0) = b$, we denote the domain $A(\mathrm{Lyp}^-(\varepsilon, c))$ by $L_b^-(\varepsilon) = \mathrm{Lyp}_b^-(\varepsilon, c)$ and call it an elementary μ- Lyapunov domain.

Although the boundary of an elementary Lyapunov domain consists of an elementary μ- Lyapunov arc γ_0 and a circle arc C_0 with common end points, say a_0 and b_0 note that it has no cusps because γ_0 and C_0 have common tangents at points a_0 and b_0.

Definition 3.2 (arc-chord condition). Let C be a rectifiable Jordan closed curve and z_1, z_2 finite points of C. They divide C into two arc, and we consider one with smaller Euclidean length and denote its length with $d_C(z_1, z_2)$.

(a) The curve C is said to satisfy the arc-chord condition if the ratio of this length to the distance $|z_1 - z_2|$ is bounded by a fixed number $b_C = b_C^{arc}$ (which we call arc-chord constant of C) for all finite $z_1, z_2 \in C$;
(b) The curve C is said to satisfy the arc-chord condition at a fixed point $z_1 \in C$ if the ratio of the length $d_C(z_1, z)$ to the distance $|z_1 - z|$ is bounded by a fixed number $b_C(z_1) = b_C^{arc}(z_1)$ for all finite $z \in C$;
(c) If D is a μ-Lyapunov domain bounded by a curve γ, we define $l_2 = l_2(D) = l_D^2 = l_\gamma^2 = \frac{\pi}{2} l_1 b_\gamma^{1+\mu}$, and we call it the second Lyp-constant, where $l_1 = lyp(\gamma, \mu)$.

3.1. Auxiliary results

Suppose that D satisfies the hypothesis (Lyp-0). Further, one can prove:

(c1) It is known that that a C^1 curve satisfies the arc-chord condition;
(d) There is $r_1 > 0$ such $\partial D \cap B(r_1)$ is graph of a function F, $v = F(u)$, $-u_1 < u < u_2$, where $u_1, u_2 > 0$, and that the set $V = \{(u, v) : -u_1 < u < u_2, F(u) < v\} \cap B(r_1)$ belongs D, where u, v are the cartesian coordinates in w-plane and $w = u + iv$;
(e) Let D be a bounded Lyapunov domain, $a_0 \in D$ and let ψ be a conformal mapping of D onto $\mathbb{U}$ with $\psi(a_0) = 0$. Then there are constants $k_1 = \underline{k}_1(D, a_0)$ and $k_2 = \underline{k}_2(D, a_0)$ (which we call the lower and upper Kellogg multiplicative constants of D with respect to a_0 respectively) such that $k_1|z_1 - z_2| \le |\psi(z_1) - \psi(z_2)| \le k_2|z_1 - z_2|$, $z_1, z_2 \in D$;
(f) Mori's theorem. Let $f : \mathbb{U} \to \mathbb{U}$ be a surjective K-qc mapping with $f(0) = 0$ and $\alpha = 1/K$. Then $|f(z_1) - f(z_2)| \le 16|z_1 - z_2|^\alpha$, $z_1, z_2 \in \mathbb{U}$, *i.e.* f is α-Holder continuous;
(g) Let the mapping A is given by $A(z) = i\frac{z-i}{z+i} + i$. Then $A(i) = i$ and A maps $\mathbb{H}$ onto $B_1 = B(i, 1)$. Since $A'(z) = \frac{2}{|z+i|^2}$, we first find $|A'(z)| \le 2$ and therefore $|A(z)| \le 2|z|$, $z \in \mathbb{H}$;
(h) Suppose that D is a bounded convex planar domain, $f : \overline{D} \to \mathbb{C}$ is holomorphic mapping and $z_0 \in D$. Then there is a constant $c > 0$ such that $|f(z) - f(z_0)| \le c|z - z_0|$, $z \in D$. The proof is straightforward;
(i) Let f be a K-qc mapping of the half -plane $\mathbb{H}$ on a domain D such that $f(0) = 0$, and suppose that ∂D is a K-quasi-circle and $\alpha = K^{-1}$. Then f has a K_1-qc extension to a map $\tilde{f}$ of the complex plane, which by abuse of notation we denote sometimes again by f if there is no possibility of confusion.

Definition 3.3.

(i) If $z_1, z_2 \in \mathbb{C}^*$ by $\theta(z_1, z_2)$ we denote the measure of the convex angle between z_1, z_2;

(ii) For $p \in \mathbb{C}$, set

$$X(z) = X_p(z) = \frac{pz}{z-p}, z \in \overline{\mathbb{C}} \quad \text{and} \quad Y = Y_p = X^{-1};$$

(iii) If f is homeomorphism of $\overline{\mathbb{C}}$ onto itself, we define $p = p(f) = f^{-1}(\infty)$;

(iv) If γ is an arc in $\mathbb{C}$ and $Z : \gamma \to \mathbb{C}^*$ continuous map by $\Delta_\gamma \text{Arg} Z$ we denote the variation of $\text{Arg} Z$ along γ.

Note that X and Y are Möbius automorphisms of $\overline{\mathbb{C}}$ with the following properties: $Y(z) = -\frac{pz}{z-p}$, $X(0) = Y(0) = 0$, $X(p) = \infty$, $X(\infty) = p$, $Y(p) = \infty$ and $Y(\infty) = -p$. If we set $\breve{f} = f \circ X$, then $f = \breve{f} \circ Y$. X_p and Y_p map lines $l_\beta = \{re^{i\beta} : r \in \mathbb{R}\}$ onto the circles which contain 0 and p. Since Y_p map the circle $C(0, |p|)$ onto line L which does not contain 0. If $z_n = e^{-i/n}p$ and $z_n' = e^{i/n}p$, then $\theta(z_n', z_n) \to 0$ and $\theta(Xz_n', Xz_n) \to \theta_0, \theta_0 \neq 0$, if $n \to \infty$. This example shows that we need to adapt a version of Proposition 2.8 to hold for the mappings X_p.

Proposition 3.4. *Let f be a K-qc mapping of the plane $\overline{\mathbb{C}}$ onto itself, $f(0) = 0$, $p = f^{-1}(\infty)$, $\alpha = K^{-1}$ and $r_0 = |p|/2$.*

(I) (a) *Then $f = \breve{f} \circ Y$, where $Y = Y_p$, $\breve{f}$ is K-qc mapping of the plane $\overline{\mathbb{C}}$ onto itself, with $\breve{f}(0) = 0$ and $\breve{f}(\infty) = \infty$;*

(b) *If $z_1, z_2 \in \mathbb{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$ (respectively $\theta^* \in [0, \pi]$) is the measure of the convex angle between z_1, z_2 (respectively $\breve{f}(z_1), \breve{f}(z_2)$), then $\theta^* \leq c \max\{\theta^\alpha, \theta\}$, where $c = c(K)$. In particular, if $\theta \leq 1$, then $\theta^* \leq c\theta^\alpha$;*

(II) *If $z_1, z_2 \in \mathbb{C}^* \cap B(0, r_0)$, $|z_1| = |z_2|$, then*

$$\theta(Xz_1, Xz_2) \leq \left(1 + r_0^{-1}\right)\theta(z_1, z_2);$$

(III) *For given $H_0' = \text{Lyp}(\varepsilon, c, \mu)$, $\varepsilon < r_0$, there is $H_0 = \text{Lyp}(\varepsilon_1, c_1, \mu_1)$ such that $Y(H_0) \subset H_0'$.*

Proof. (I) Set $\breve{f} = f \circ X$. Since $X(p) = \infty$ and $p = f^{-1}(\infty)$, we have $\breve{f}(\infty) = \infty$. Since X is Möbius automorphism of $\overline{\mathbb{C}}$, $\breve{f}$ is K-qc. By (a) and an application of Proposition 2.8 to $\breve{f}$, (b) follows.

(II) Let $|z_1| = |z_2| = R < r_0$. If necessary we can re-numerate points such that $z_k = Re^{it_k}$, $k = 1, 2$, $t_1 \leq t_2 \leq t_1 + \pi$ and $l = l(z_1, z_2)$ be the circular arc defined by $l(t) = Re^{it}$, $t_1 \leq t \leq t_2$. We are going to estimate the variation $\Delta_l \text{Arg} T$ and $\Delta_l \text{Arg}\, X$. Since $X(z) = X_p(z) = \frac{pz}{z-p}$, we can write:

(i) $\arg X = \arg z - \arg T + \arg p$, where $T = z - p$; hence

(ii) $\Delta_l \text{Arg} X \leq \Delta_l \text{Arg}\, Id + \Delta_l \text{Arg}\, T$, where Id is the identity map.

Since $T'/T = 1/T$, for $z = re^{it}$,

$$(\arg T)_t = \operatorname{Im}\left(\frac{T'}{T} i r e^{it}\right).$$

For $z \in B(0, r_0)$, we have $|(\arg T)_t| \le \frac{1}{|z-p|} \le 1/r_0$, and therefore

$$\theta(Tz_1, Tz_2) \le \Delta_l \mathrm{Arg} T \le r_0^{-1} |t_2 - t_1|.$$

Hence, by the item (ii), for $|z_1| = |z_2| = R < r_0$,

$$\theta(Xz_1, Xz_2) \le \Delta_l \mathrm{Arg} X \le \left(1 + r_0^{-1}\right)|t_2 - t_1|.$$

(III) We only outline a proof. Set $\zeta = Y(z)$, $\theta = \arg z$, $\varphi = \arg \zeta$, $z = x + iy = re^{i\theta}$, $\zeta = \xi + i\eta = \rho e^{i\varphi}$. So we define the functions $\theta = \theta(\rho, \varphi)$, $\varphi = \varphi(r, \theta)$, $\rho = \rho(r, \theta)$ and $r = r(\rho, \varphi)$.

Since Y is conformal mapping on $\overline{B(0, r_0)}$ and $Y(0) = 0$, by the item (h) in Subsection 3.1, we find $\rho(r, \theta) \approx r$ and $r = r(\rho, \varphi) \approx \rho$.

Let $\zeta \in \gamma := \gamma(\varepsilon, c, \mu)$, $\rho = |\zeta|$ and $z' = X(\rho) = r(\rho, 0)e^{i\theta'}$, where $\theta' = \theta(\rho, 0)$.

Case 1. Suppose that $p = p_1 + ip_2$, $p_2 > 0$.

Since $X(\infty) = p$, X maps the coordinate axis $\eta = 0$ (in the ζ -plane) onto the circle $K = C(iR_0, R_0)$ which contains p, where $R_0 = R_0(p) = \frac{|p|^2}{2|p_2|}$ depends only on p. Y_p maps $B = B(iR_0, R_0)$ onto $\mathbb{H}$. Let K' be semi circle $y = R_0 - \sqrt{R_0^2 - x^2}$. Then $\theta' \approx x(\rho) \preceq r(\rho) \approx r$. By the part (II) of the Proposition, $\theta(z, z') \preceq \varphi \preceq \rho^\mu \preceq r^\mu$. Hence, since $\theta \le \theta(z, z') + \theta'$, we find $\theta \preceq r^\mu$ (thus we can choose $\mu_1 = \mu$).

In a similar way we consider:

Case 2. Suppose that $p = p_0 = p_1 + ip_2$, $p_2 < 0$. In this case Y_p maps $B = B(-iR_0, R_0)$ onto $\mathbb{H}^*$.

Case 3. $p_0 \in \mathbb{R}$. In this case Y_p maps $\mathbb{H}$ onto itself. □

Theorem 3.5.

(I) *Suppose:*

(i) *h is a K-qc map from $\mathbb{H}$ onto a Lyapunov domain D. Then h has a K_1-qc extension to a map $\tilde{h}$ of the complex plane;*

(II) *If h satisfies the hypothesis $\mathbb{H}^0_{qc}$, then there is a constant $l_0 = 16\,2^\alpha k_1^{-1}$ which depends on K_1 and the Kellogg multiplicative constant of D (with respect to $a_0 = h(i)$) $k_1 = \underline{k}_1(D, a_0)$; such that:*

(ii) $|h(z)| \le l_0 |z|^{1/K_1}$ *if $z \in \mathbb{H}$ and $|z| \le 1$;*

(III) *If D satisfies the hypothesis* (Lyp-0)*, then there are constants $\varepsilon > 0$ and $c > 0$ such that for $|w| < \varepsilon$ (here $c\varepsilon^{\mu} < \pi$) and $w \in \partial D$, either $|\arg(w)| < c|w|^{\mu}$ or $|\pi - \arg(w)| < c|w|^{\mu}$ where* arg *is the branch of the argument determined by $-\pi/2 < \arg(w) < 3\pi/2$ and moreover that set*

$$D_0 = D_0(\varepsilon) = \mathrm{Lyp}(\varepsilon, c, \mu) = \left\{w : c|w|^{\mu} < \arg(w) < \pi - c|w|^{\mu}, |w| < \varepsilon\right\}$$

satisfies $D_0 \subset D$, where c depends only on the Lyapunov multiplicative constant of ∂D;

(iii) *We can choose $c = l_2 = l_D^2 = \frac{\pi}{2} l_1 b_{\gamma}^{1+\mu}$, where $l_1 = lyp(\gamma)$ and $b_{\gamma} = b_{\gamma}^{arc}$ is the arc-chord constant of γ;*

(IV) *Then there is a constant $c_1 = c_1(\mu, \varepsilon, c, K_1, l_2, |p|)$ such that the region*

$$H_0 = H_0(\varepsilon) = \left\{z : c_1|z|^{\mu/K_1^2} < \arg(z) < \pi - c_1|z|^{\mu/K_1^2}, |z| < (\varepsilon/l_0)^{K_1}\right\}$$

satisfies:

(a) *$h(H_0) \subset D_0$;*
(b) *There are constants $\varepsilon_2, c_2, \mu_2$ such that $D_0' \subset h(H_0)$, where $D_0' = \mathrm{Lyp}(\varepsilon_2, c_2, \mu_2)$.*

Note that $H_0 = H_0(\varepsilon) = \mathrm{Lyp}(\varepsilon_1, c_1, \mu_1)$, where $\mu_1 = \mu/K_1^2$ and $\varepsilon_1 = (\varepsilon/l_0)^{K_1}$.

Recall that the hypothesis (i) (together with some technical requirements $h(0) = 0$ and that D satisfies (Sp0)) in the theorem is essentially equivalent to the hypothesis ($\mathbb{H}^0_{\mathrm{qc}}$). From the proof below it is clear that the hypothesis (i) implies:

(i1): h is a qc mapping of $\mathbb{H}$ onto the quasidisk D (which is much weaker then (i)), and that the statement (I) holds under the hypothesis (i1).

If in addition to (i1), $h(0) = 0$ and $0 \in \partial D$, we leave to the interested reader to state and prove a corresponding version of the statement (II). Since h^{-1} is also qc the proof of (IV) of Theorem 3.5 shows that the following holds:

(IV′): for each special domain of Lyapunov type X_0 with vertex at 0, there is a special domain of Lyapunov type Y_0 with vertex at 0 such that $Y_0 \subset h(X_0)$. In particular, we can choose $X_0 = H_0$ and Y_0 to be an elementary Lypunov domain D_0^- such that $D_0^- \subset h(H_0)$.

On the Figure 3.1 the domains D, D_0, D_0^-, H_0 and $h(H_0)$ are enclosed by lines whose colors are red, blue, green, violet (on the left) and violet (on the right) respectively.

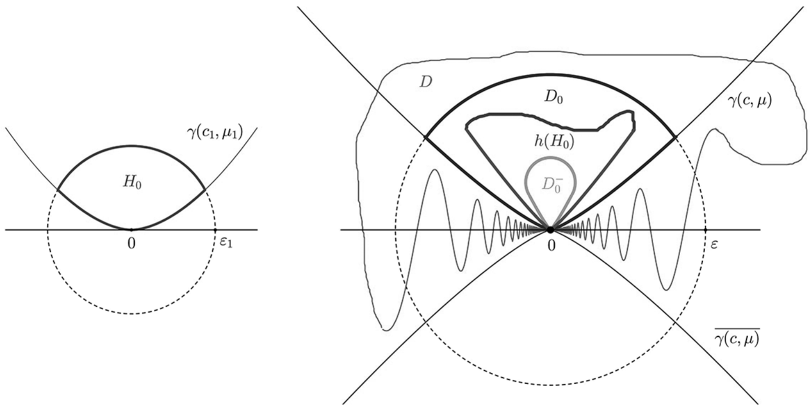

Figure 3.1.

3.1.1. *Proofs*

Proof of (I). Since D is $C^{1,\mu}$, D is a quasi-circle and therefore by the item (i) from Subsection 3.1, the statement (I) follows. □

Proof of (II). We suppose that $a_0 = h(i)$ is given. Let $B_1 = B(i; 1)$ and R_1 a conformal mapping of B_1 onto $\mathbb{H}$ such that R_1 fixes 0 and i and R_2 a conformal mapping of D onto B_1 such that $R_2(0) = 0$ and $R_2(a_0) = i$. Set $\underline{h} = R_2 \circ h \circ R_1$. Then $\underline{h}(0) = 0$, $\underline{h}(i) = i$, $\underline{h}$ maps B_1 onto itself, $h = R_2^{-1} \circ \underline{h} \circ R_1^{-1}$. Set $w = h(z)$, $\zeta = R_1^{-1}(z)$, $\zeta' = R_2^{-1}(w)$ and $\zeta' = \underline{h}(\zeta)$. Note that $R_1^{-1}(\mathbb{H})$ is a disk. By the item (g) in Subsection 3.1, we find $|R_1^{-1}(z)| \le 2|z|$, $z \in \mathbb{H}$. Since D is a Lyapunov bounded domain, $|R_2^{-1}(\zeta')| \le k_3|\zeta'|$, $\zeta' \in B_1$, where $k_3 = \underline{k}_1(D, a_0)^{-1}$. It is clear that $\underline{h}$ is K_1-qc with $\underline{h}(i) = i$. Now an application of Mori's theorem to $\underline{h}$ on B_1, shows that $\underline{h}$ is α -Hölder continuous on B_1 with a multiplicative constant 16 , where $\alpha = 1/K_1$ and in particular $|\zeta'| = |\underline{h}(\zeta)| \le 16|\zeta|^\alpha$. Since $h(0) = 0$, there is a constant $l_0 = l_0(K_1)$ such that the part (ii) of the theorem holds: $|h(z)| \le l_0|z|^{1/K_1}$ if $z \in \mathbb{H}$ and $|z| \le 1$. □

Proof of (III). Set $l_1 = lyp(\gamma)$. Since D satisfies the hypothesis (Lyp-0), the item (d) in Subsection 3.1 holds. By the item (d), there is $\epsilon_1 > 0$ such that the trace of the path γ_1 which is defined in (d) by $v = F(u), u \in [-\epsilon_1, \epsilon_1]$, where F is C^1, is on ∂D. Let $\gamma_2 = \partial D \backslash tr(\gamma_1)$. Then there is a constant $\epsilon_2 > 0$ such that γ_2 has no points in the disk $B(\epsilon_2)$. Let L be the length of ∂D and $\hat{\gamma}$ a parametrization of the positively oriented boundary ∂D by the arc-length parameter s, where $s \in [0, L]$ and $s(0) = 0$ (we need arc-length parameter only around 0). Set $\hat{\gamma}(s) = u(s) + iv(s)$ and $w = Re^{i\Psi}$. Then

$$|v(s)| = |v(s) - v(0)| = \left|v'\big(s^1\big)\right|s \le c_1 s^{1+\mu},$$

where $c_1 = l_1$. Since a quasicircle satisfies the arc-chord condition, we have the following:

(a1) If a C^1 curve is a quasicircle, then $s \le c_2 R$, where $c_2 = b_\gamma = b_\gamma^{arc}$.

Also we can prove the following version of (a1).[6]

(a2) By definition of Lyapunov curve $\hat{\gamma}'(t) = 1 + \epsilon(t)$, where $|\epsilon(t)| \le ct^\mu$. Hence $\hat{\gamma}(s) = \int_0^s \hat{\gamma}'(t)dt$ and therefore there is $\tau = \tau(c,\mu)$ such that $s \le 2|\hat{\gamma}(s)|$ for $s \le \tau$.

Hence, there is $\epsilon > 0$ such that:

(iv) $|\arg \hat{\gamma}(s)| < \frac{\pi}{2}$ for $s \in (0, \epsilon]$;
(v) $\frac{\pi}{2} < \arg\, \hat{\gamma}(s) < 3\frac{\pi}{2}$ for $s \in [L - \epsilon, L)$.

We can choose $\epsilon < \epsilon_2$. Hence, for s small ($0 < s < \epsilon$), we find

$$R|\Psi| \le \frac{2}{\pi} R|\sin(\Psi)| = \frac{2}{\pi} v(s) \le c_3 s^{1+\mu}, \quad \text{where} \quad c_3 = \frac{2}{\pi} c_1,$$

and therefore there is a constant c such that:

(vi) $|\Psi| \le cR^\mu$, where c is given by the item (iii) in Theorem 3.5.

Using the mapping $A(w) = -w$ and (vi), we find that:

(vi') $|\pi - \Psi| \le cR^\mu$ for $s \in [L - \epsilon, L)$.

From (vi) and (vi'), (III) follows. □

Proof of (IV). We use the notation from Proposition 3.4. Set $\breve{h} = h \circ Y_p$, where $p = h^{-1}(\infty) \in \mathbb{C}$. By the statement (I) and easy part of Proposition 3.4, $h = \breve{h} \circ Y_p$, and h is K_1-qc mapping of the plane $\overline{\mathbb{C}}$ onto itself, with $\breve{h}(0) = 0$ and $\breve{h}(\infty) = \infty$. We use notation: polar coordinates $\zeta = \rho e^{i\varphi}$ in the ζ-plane and $w = Re^{i\Psi}$ in the w-plane. Recall by (III) there is a curve $\gamma = \gamma(c, \mu, R_0)$ in D. Set γ_0: $\Psi = \Psi_0(R) = cR^\mu, 0 \le R \le R_0$. Hence there is a part C_1 of the boundary of D around 0 (say the right half part) which is below γ_0 and which defines the curve γ_1.

Case 1. We first prove for $\breve{h}$. Let $w = Re^{i\Psi} \in \gamma_0$ and let $w' = Re^{i\Psi'}$ be the intersection of the circle T_R with γ_1. Then $\theta(w, w') \le (|\Psi| + |\Psi'|) \le 2|\Psi|$. Set $\zeta = \breve{h}^{-1}(w)$ and $\zeta' = \breve{h}^{-1}(w')$. Since γ_0 is the right half of $\gamma = \gamma(c, \mu)$, $\zeta' > 0$ and $\varphi = \theta(z, z')$. Hence using the quasihyperbolic metric k on $\mathbb{C}^*$ (Proposition 2.8), we have $\varphi \le \kappa_2 \Psi^\alpha$, where $\alpha = 1/K_1$. Since, by (II), $R \le c\rho^\alpha$, we find $\varphi \le \kappa_2 \big(\Psi_0(c\rho^\alpha)\big)^\alpha$ and therefore we get $\varphi \le \kappa_3 r^{\alpha^2 \mu}$. Thus we find:

(vii) The curve $\breve{h}^{-1}(\gamma_0)$ is below the curve $\gamma(\kappa_3, \alpha^2\mu)$.

[6] We observed it after writting a revision of the manuscript.

Note that if a curve satisfies $\varphi \leq c\rho^{\mu}$, then it is is below the curve $\gamma(c, \mu)$. Recall that we set $\mu_1 = \mu/K_1^2$ and $\varepsilon_1 = (\varepsilon/l_0)^{K_1}$. Note that γ_0 is the right half of $\gamma = \gamma(c, \mu)$ and that in a similar way as above we conclude that:

(viii) $\breve{h}^{-1}(\gamma(c, \mu))$ is below the curve $\gamma(c_1, \mu_1)$, $\rho < \varepsilon_1$.

By the part (ii) of the theorem, $h(B(\varepsilon_1)) \subset B(\varepsilon)$ and it is readable that it yields (a). Since $\breve{h}^{-1}$ is also qc (a) implies (b). Thus we have proved (IV) for $\breve{h}$ with $c_1 = \kappa_3$.

Case 2. Proof for h. By Case 1, there is H_0' such that $\breve{h}(H_0') \subset D_0$. By Proposition 3.4 there is H_0 such that $Y(H_0) \subset H_0'$ and it completes proof. Thus we have proved (a).

Let us prove that (a) implies (b). Namely, since h^{-1} is also qc, by (a) there is D_0' such that $h^{-1}(D_0') \subset H_0$ and therefore $D_0' \subset h(H_0)$. □

3.2. Global approximation

Concerning the previous theorem, note that $\mu_1 \leq \mu$ and $\varepsilon_1 \leq \varepsilon$, and in particular, one can derive (see (IV')):

(a) There is $\underline{\epsilon} = \underline{\epsilon}(\varepsilon, c) < \varepsilon_1$ such that $h(L^1) \subset L' \subset D_0$, where $L' = \mathrm{Lyp}^-(\varepsilon, c)$ is μ-Lyapunov and $L^1 = \mathrm{Lyp}^-(\underline{\epsilon}, c)$ is μ_1-Lyapunov and $L^1 \subset L'$. Hence, since h^{-1} is also qc, $h^{-1}(L^1) \subset L'$ and therefore $L^1 \subset h(L')$;

(b) In a similar way, there is $\underline{\epsilon}^1 \leq \underline{\epsilon}$ and $\mu_2 \leq \mu_1$ such that $h^{-1}(L_-^1) \subset L^1$, where $L_-^1 = \mathrm{Lyp}^-(\underline{\epsilon}^1, c)$ is μ_2-Lyapunov.

Hence we derive:

(IVa) If h satisfies the hypothesis $\mathbb{H}^0_{\mathrm{qc}}$, then $L_-^1 \subset h(L^1) \subset D_0$.

Note that it is easy to transfer Theorem 3.5 to the setting of the unit disk. Now we show that the corresponding version of it holds with $\mathbb{U}$ instead of $\mathbb{H}$.

We first need a version of (IV′) for $\mathbb{U}$ with special Lyapunov convex domains. Note that H_0 has two cusps. In this subsection by D_0 we denote the set defined in Theorem 3.5.

Definition 3.6 (lyp$(D)_b$). Here we define $\underline{A}_0$, R_a, T_b and h_a.

(i) Consider the conformal mapping $A_0 = \underline{A}_0$ defined by $\underline{A}_0(z) = \frac{4i-z}{4i+z}$; $\underline{A}_0$ maps $\mathbb{H}$ onto $\mathbb{U}$ such that $\underline{A}_0(0) = 1$ and $\underline{A}_0(-4i) = \infty$;

(ii) For $a = e^{i\alpha} \in \mathbb{T}$ define $R_a(z) = e^{i\alpha}z$, and for $b \in \partial D$ if the unit inner normal $n_b = e^{i\beta}$ at b exists, we define $T_b(w) = \underline{T}_b(w) = -ie^{i\beta}w + b$;

(iii) If D satisfies the hypothesis (Lyp-0), we define $\hat{D}_b = T_b(\hat{D}_0)$, where $\hat{D}_0$ is defined in the item (A) below. If we wish to indicate that $\hat{D}_b$ is an elementary Lyapunov domain we use notation lyp$(D)_b$;

(iv) For $a \in \mathbb{T}$, set $h_a = h_a^b := \underline{T}_b^{-1} \circ h \circ R_a$, $a \in \mathbb{T}$, where $b = h(a)$, and let $\hat{h} = h \circ A_0$ and $\hat{h}_a = h_a \circ A_0$.

It is clear that $\underline{T}_b^{-1}$ is defined by $\underline{T}_b^{-1}(z) = ie^{i\beta}(z-b)$, $\underline{T}_b^{-1}(b) = 0$, and if D has the tangent at $b \in \partial D$ then $\underline{T}_b^{-1}(D)$ has (Sp0) property.

(v) Next suppose that D satisfies the hypothesis (Lyp-0) (see Definition 1).

We will prove that(see also Proposition 3.8):

(A) There is $\varepsilon_2 > 0$ such that $\hat{D}_0 \subset T_b^{-1}(D)$ for every $b \in \partial D$, where $\hat{D}_0 = \mathrm{Lyp}^-(\epsilon_2, c)$;

(vi) In addition to (v) suppose that h is a qc mapping of $\mathbb{U}$ onto D and $h(1) = 0$ (that is h satisfies the hypothesis $\mathbb{U}^0_{\mathrm{qc}}$).

Then $\hat{h}$ is a qc mapping of $\mathbb{H}$ onto D with $\hat{h}(0) = 0$.

It seems useful to consider the following properties, which is an immediate corollary of (IVa):

(B) If h satisfies the hypothesis $\mathbb{U}^0_{\mathrm{qc}}$, there are corresponding elementary Lyapunov domains $H^1 \subset \mathbb{H}$ and $D^1_- \subset D$ with vertex at 0 such that $D^1_- \subset \hat{h}(H^1) \subset \hat{D}_0$. See Proposition 3.8 for a stronger result;

(vii) Further set $U_1 = A_0(H^1)$ and $U^1_- = A_0(D^1_-)$ and set $U_a = R_a(U_1)$.

Note that H^1 is a special μ_1-Lyapunov. Thus we have:

(C) $U^1_- \subset h(U_1) \subset \hat{D}_0$.

In order to state a corresponding form of (IVa) for $\mathbb{U}$, it is convenient to call $V = A_0(L)$ an elementary domain if L is elementary (see also Proposition 3.4). Now, by (IVa), it is clear that we have:

(IVb) If h satisfies the hypothesis $\mathbb{U}^0_{\mathrm{qc}}$ and L is an elementary Lyapunov domain with vertex at 0 in $\mathbb{H}$, then there are elementary Lyapunov domains V_1 and V^1_- in $\mathbb{U}$ with vertex at 1 such that $V^1_- \subset h(V_1) \subset L$;

(D) Now, we also suppose that h satisfies the hypothesis $\mathbb{U}^0_{\mathrm{qc}}$. Recall by the item (i) from Subsection 3.1, then h has a K_1-qc extension to a map $\tilde{h}$ of the complex plane.

We can choose $p = \tilde{p}$ such that $h(\tilde{p}) = \infty$ and $|\tilde{p}| \geq 3$. Set $\hat{p} = A_0^{-1}(\tilde{p})$, $\tilde{p}_\alpha = h_a^{-1}(\infty)$ and $\hat{p}_\alpha = A_0^{-1}(\tilde{p}_\alpha)$. Check that $\tilde{p} \notin h(B(0,2))$ and therefore $|\hat{p}| \geq 2$.Hence, since $\tilde{p}_\alpha = h_a^{-1}(\infty) = e^{-i\alpha}\tilde{p}$, we find:

(D0) $h_a^{-1}(\infty) \notin h(B(0,2))$ and therefore, $|\hat{p}_\alpha| \geq 2$ for every $a = e^{i\alpha} \in \mathbb{T}$.

Note that $T_b \circ h_a = h \circ R_a$ and that h satisfies the hypothesis $\mathbb{U}^0_{\mathrm{qc}}$, $(U\text{-}1)$ if and only if $\hat{h}$ satisfies $\mathbb{H}^0_{\mathrm{qc}}$, $(H\text{-}0)$ respectively.

In addition, we need a property of C^1 domains. Suppose that domain D is uv-plane.

Lemma 3.7.

(i) *Suppose that a* C^1 *domain D satisfies the hypothesis* (Sp0)*;*
(I) *Then there is* $r > 0$ *such that for each* $w \in \partial D$, $\underline{T}_w^{-1}(\partial D) \cap B(0, r)$ *is a graph with respect to* uv*-coordinates.*

Proof. Let L be the length of ∂D and $\hat{\gamma} : [0, L] \to \partial D$ a parametrization of the positively oriented boundary ∂D by the arc-length parameter s. We also write $w = \hat{\gamma}(s)$, where $s \in [0, L]$ and $s(0) = 0$. Here there is the function $s = s(w)$ which is the inverse of the function $w = \hat{\gamma}(s)$ and which maps ∂D onto $[0, L]$. Since $\hat{\gamma}'$ is continuous on $[0, L]$, it is uniformly continuous on $[0, L]$. The function $s = s(w)$ is continuous on ∂D and hence $C(w) = \hat{\gamma}'(s(w))$ is continuous on ∂D and uniformly continuous on ∂D. Therefore there is $r_2 > 0$ such that:

(1) $|\arg \hat{\gamma}'(s_2) - \arg \hat{\gamma}'(s_1)| < \pi/8$ for $|w_2 - w_1| \le r_2$, where $w_k = \hat{\gamma}(s_k)$, $k = 1, 2$.

Let us prove (I) for $r = r_2/2$. Contrary, suppose that (I) is not true. Then for some $w_0 \in \partial D$, $D(w_0, r) := (\partial D) \cap B(w_0, r)$ is not a graph with respect to coordinates determined by unit vectors $\hat{\gamma}'(w)$ and n_w. Hence there are two points w_1 and w_2 in this set such that $w_1 w_2$ is parallel to the normal n_{w_0} of ∂D at w_0. Therefore there is w_3 in this set such that $\gamma'(s)$ at w_3 is parallel to the normal n_{w_0}. This contradicts (1). Thus we have (I). □

Using the approach in the proof of statement (III), (IV), (IV′) of Theorem 3.5, (IVb) and (iii), we can prove:

Proposition 3.8.

(a): *Suppose that D satisfies the hypothesis* (Lyp-0)*;*
(I): *There is an elementary Lyapunov domains* $\hat{D}_0$ *in D with vertex at* 0 *such that* $\hat{D}_0 \subset \underline{T}_b^{-1}(D)$*;*[7]
(b): *In addition to* (a) *suppose that* $h : \mathbb{U} \to D$ *is a qc homeomorphism;*
(IIa): *Then there is a Lyapunov domain* $\hat{U}_1$ *in* $\mathbb{U}$ *with vertex at* 1 *such that for every* $a \in \mathbb{T}$, $h_a(\hat{U}_1) \subset \hat{D}_0$*;*
(IIb): *In addition, there is an elementary Lyapunov domain* $D_0^- = \mathrm{Lyp}(\varepsilon_0^-, c_0, \mu)$ *in D with vertex at* 0 *such that* $D_0^- \subset h_a(\hat{U}_1)$ *for every* $a \in \mathbb{T}$*;*
(III): $D_0^- \subset h_a(\hat{U}_1) \subset \hat{D}_0$.

Note that in general $h_a(\mathbb{U})$ is not a fixed domain for $a \in \mathbb{T}$ and therefore we need first to consider the part (I) and then the part (II).

Proof. (I). D_b satisfies the hypothesis (Lyp-0) for every $b \in \partial D$. Consider the family $\underline{\mathcal{D}} := \{D_b = \underline{T}_b^{-1}(D) : b \in \partial D\}$. For $b \in \partial D$ define $\varepsilon(b)$ to be maximum

[7] Note that we use the domains with ˆ-notation if we have uniform estimates.

of ε for which $\mathrm{Lyp}(\varepsilon, l_2(D)) \subset \overline{D_b}$, where $l_2(D)$ is the second Lyp-constant. By Lemma 3.7, there is $r > 0$ such that $D_b \cap B(r)$ is a graph with respect to uv-coordinates for each $b \in \partial D$. Since all domains D_b, $b \in \partial D$, have the same Lyapunov multiplicative constants, using (iii) and the approach in the proof (III) of Theorem 3.5, we can prove that there is $\varepsilon_0 > 0$ such that $\varepsilon(b) \geq \varepsilon_0$ for all $b \in \partial D$, and therefore an elementary Lyapunov domain $\hat{D}_0$ such that $\hat{D}_0 \subset D_b$ for every $b \in \partial D$ and (I) follows.

In addition, it seems that we can prove that the function $\varepsilon(b) = \varepsilon_D(b)$ is continuous with respect to b.

(IIa): Recall that we use the notation $a_0 = h(0)$ and $\hat{h}_a = h_a \circ \underline{A}_0$. Let ω_b be a conformal mapping of D_b onto $\mathbb{U}$ such that $\omega_b(\underline{T}_b^{-1}(a_0)) = 0$ and $\omega^b = \omega_b \circ \underline{T}_b^{-1}$. Since $\omega^b(0) = 0$, by the Kellogg-Warshawski theorem there are two positive constants l_1 and l_2 such that $l_1 \leq |\omega'(w)| \leq l_2$, $w \in D$. Since $\underline{T}_b^{-1}$ is a Euclidean isometry, we have $|(\underline{T}_b^{-1})'| = 1$ on D, and therefore $l_1 \leq |\omega_b'(w)| \leq l_2$, $w \in D_b$. Hence, since $\omega_b \circ h_a$ maps $\mathbb{U}$ onto itself, $\omega_b \circ h_a(0) = 0$ and $h_a(1) = 0$ there is a constant l^0 which depends on K and the Kellogg multiplicative constant of D(with respect to a_0), such that for all $a \in \mathbb{T}$, $|h_a(z)| \leq l^0 |z-1|^{1/K}$ if $|z| \leq 1$.

Using $\underline{A}_0$ we can get the corresponding result for $\hat{h}_a$: there is a constant l_0 which depends on K_1 and the Kellogg multiplicative constant of D, such that for all $a \in \mathbb{T}$,

(iv): $|\hat{h}_a(z)| \leq l_0 |z|^{1/K_1}$ if $|z| \leq 1$, $\mathrm{Im} z \geq 0$.

The functions $\hat{h}_a$, $a \in \mathbb{T}$, are K_1-qc. By (iv), an application of (IVb) to the functions $\hat{h}_a$, $a \in \mathbb{T}$, and the statement ($D0$) (from Subsection 3.2), and the item (iii) of Proposition 3.4 with $r_0 = 1$ to the functions $\hat{h}_a$, $a \in \mathbb{T}$, show that there is a Lyapunov domain $\hat{H}_0$ in $\mathbb{H}$ with vertex at 0 such that $\hat{h}_a(\hat{H}_0) \subset \hat{D}_0$. Set $\hat{U}_1 = A_0(\hat{H}_0)$. It yields the proof of (II).

(IIb) Using (IVb), since the corresponding parameters are the same for h and h_a, one can get (III);

(III) It is clear that (IIa) and (IIb) can be stated as (III). □

It is convenient to introduce the following notation:

(v) For $b \in \partial D$, set $D_b^- = \mathrm{lyp}(D)_b^- = \underline{T}_b(D_0^-)$ and for $a \in \mathbb{T}$, $\hat{U}_a = R_a(\hat{U}_1)$. If we wish to indicate that D_b^- is an elementary Lyapunov domain we use notation $\mathrm{lyp}(D)_b^-$.

Now using Euclidean isometry $\underline{T}_b$ it is easy to get the corresponding results of the property (III) of Proposition 3.8 for domains with vertexes at b. Namely, by the property (III) of Proposition 3.8 we have $\underline{T}_b(D_0^-) \subset \underline{T}_b(h_a(\hat{U}_1)) \subset \underline{T}_b(\hat{D}_0)$. By the definitions $\underline{T}_b \circ h_a(\hat{U}_1) = h \circ R_a(\hat{U}_1) = h(\hat{U}_a)$ and therefore the part (I) of the next theorem follows. By (L1) (see the introduction) we get the part (II). So we have the crucial result:

Theorem 3.9. *Suppose that D is a Lyapunov domain and $h : \mathbb{U} \xrightarrow{\text{onto}} D$ is a* qc *homeomorphism. Then:*

(I) *For every $a \in \mathbb{T}$, $\mathrm{lyp}(D)_b^- \subset h(\hat{U}_a) \subset \mathrm{lyp}(D)_b$, where $b = h(a)$;*
(II) *If $w - b$ is in the direction of the normal vector n_b then, $d_b(w) \approx |w - b|$ if $\varepsilon_2 = \varepsilon_2(c_2, \mu_2)$ is a small enough constant.*

4. Proof that *h* is co-Lipschitz

Here we give a proof of the co-Lip property:

Theorem 4.1. *Suppose $h : \mathbb{U} \xrightarrow{\text{onto}} D$ is a hqc homeomorphism, where D is a Lyapunov domain with $C^{1,\mu}$ boundary,* i.e. *belonging to $\mathcal{D}_1$. Then h is co-Lipschitz.*

We first need a few results mentioned in the introduction which are of auxiliary character.

Theorem 4.2 ([26, Theorem 1.3]). *Suppose that h is a Euclidean harmonic complex valued mapping from the unit ball $\mathbb{B} \subset \mathbb{R}^n$ onto a bounded domain $D = h(\mathbb{B})$, which contains the ball $B(h(0); R_0)$ and there is a half space H_b which touches the point $b \in \partial D$ such that $D = h(\mathbb{B}) \subset H_b$. Then:*

(I) *$e(h(z), b) \geq (1 - |z|)\overline{c}_n R_0$, $z \in \mathbb{B}$, where $\overline{c}_n = \frac{1}{2^{n-1}}$.*[8]

Sometimes, we refer to this result as a version of Harnack's lemma.

In [26] we stated this result under the condition that the domain $D = h(\mathbb{B})$ is convex. But, a slight modification of the [26, proof of Theorem 1.1] (planar case) shows that the theorem holds under the hypothesis (a).

Proof of (I). We only outline an argument. To $b \in \partial D$ we associate a nonnegative harmonic function $u = u_b$. Let Λ_b be the boundary of H_b and let $n = n_b \in T_b\mathbb{R}^n$ be a unit vector such that Λ_b is defined by $(w - b, n_b) = 0$. By hypothesis, Λ_b is a supporting hyper-plane such that $(w - b, n_b) \geq 0$ for every $w \in \overline{D}$. Define $u(z) = (h(z) - b, n_b)$ and $d_b = d(h(0), \Lambda_a)$. Then $u(0) = (h(0) - b, n_b) = d(h(0), \Lambda_a)$. Let $b_0 \in \Lambda_a$ be the point such that $d_b = |h(0) - b_0|$. Then from the geometry it is clear that $d_a \geq R_0$, etc. (one can follow the proof from [26]). ☐

Proposition 4.3. *Suppose that h is a Euclidean harmonic mapping from the Lyapunov domain G into a domain Ω and* (i) *there is a half space H_b which touches a point $b \in \partial\Omega$ such that $h(G) \subset H_b$.*

Then $e(h(z), b) \succeq d_G(z)$, $z \in G$.

[8] Recall e here denotes the Euclidean distance.

We refer to this result as Harnak type estimate. Note if $h : \mathbb{U} \to D$ satisfies hypothesis U-1, in general a point $b \in \partial D$ does not satisfy the hypothesis (i). We use elementary Lyapunov domain described in Proposition 3.8 to apply this proposition.

Proof. Let $\phi : \mathbb{U} \to G$ be a conformal mapping and $h_1 = h \circ \phi$. Application of Koebe's theorem to ϕ and Theorem 4.2 on $h_1 : \mathbb{U} \to G$ yield the result. □

Now we illustrate relation between the circles and special Lyapunov curves and then prove Lemma 4.4.

If $M(0, d), d > 0$, then the circle C with center at M and radius d is given by the equation $x^2 + (y - d)^2 = d^2$ and the half -circle C^- with $y = d - (d^2 - x^2)^{1/2}$. Hence

$$d - y = \big(d^2 - x^2\big)^{1/2} = d\big(1 - x^2/d^2\big)^{1/2} = d\big(1 - x^2/2d^2 + o\big(x^2\big) = x^2/2d + o\big(x^2\big)$$

and therefore $y = \frac{1}{2d}x^2 + o(x^2)$. The graph of the curve $\gamma(c, \mu; \epsilon)$, where $c = 1/d, \epsilon = d$ is above the half -circle C.

Lemma 4.4. *For $c > 0, 0 < \mu < 1$, and $x_0 > 0$, let the curve C be defined by* (the curve C is defined in Example 2.7 and denoted by $f(c, \mu)$):

(1) $y = C(x) = cx^{1+\mu}, |x| < x_0$;
(2) Let $M(0, d), d > 0$, be a point and d' the distance from M to the graph of the curve (1);
(I) Then $d' \succeq d$;
(a) There is an $\epsilon^0 > 0$ such that if $d \le C(\epsilon^0)$, then $d \le 2d'$;
(b) For ϵ^0 we can choose the positive solution of the equation $c^2(1 + \mu)x_1^{2\mu} = 1$ with respect to x_1.

Proof. Let $d' = |M - M'|$, where $M'(x_1, y_1)$. Since $y'(x) = c(1 + \mu)x^{\mu}$, we find that

$$k = \frac{d - y_1}{x_1} = \big(c(1 + \mu)x_1^{\mu}\big)^{-1}.$$

Hence, $d - y_1 = \frac{1}{c(1+\mu)}x_1^{1-\mu}$ and

$$d = \frac{1}{c(1 + \mu)}x_1^{1-\mu} + y_1 = \frac{1}{c(1 + \mu)}x_1^{1-\mu}\left(1 + c^2(1 + \mu)x_1^{2\mu}\right).$$

Hence, $d \le 2d'$ if $(1 + c^2(1 + \mu)x_1^{2\mu}) \le 2$ (that is if $\varepsilon_2 = \varepsilon_2(c_2, \mu_2)$ is a small enough constant). □

We are now ready to finish the proof of the theorem.

4.1. Proof of Theorem 4.1

We will apply Proposition 3.8 and notation used there, and Theorem 4.2. Further chose a fixed positive real number $x_0 \in \hat{U}_1$. (E) Set $H(a) = T_b^{-1}\big(h(x_0 a)\big)$, $w' = T_b^{-1}(w)$, $d_0(w') = \operatorname{dist}(w', \partial D_0^-)$ and $d^0(a) = d_0\big(T_b^{-1}\big(h(x_0 a)\big)\big)$, where $b = h(a)$.

It is straightforward to check that H and d^0 are continuous function with respect to $a \in \mathbb{T}$. Hence there is $s_0 > 0$ such that $B(H(a), s_0) \subset D_0^-$ and therefore we conclude.

(E0) If h satisfies the hypothesis $\mathbb{U}^0_{qc}$, there is a constant $s_0 > 0$ which does not depend of a such that $B(h(x_0 a), s_0) \subset D_b^-$, $a \in \mathbb{T}$.

Definition 4.5. (d1) For $a = e^{i\alpha} \in \mathbb{T}$ let ϕ_0 be the conformal mapping of $\mathbb{U}$ onto $\hat{U}_1$ such that $\phi_0(0) = x_0$ and set $\hat{U}_a = e^{i\alpha}\hat{U}_1$, $\phi_a = e^{i\alpha}\phi_0$ and $F = F(h) = F_a = h \circ \phi_a$.

Thus for $a = e^{i\alpha} \in \mathbb{T}$, $\phi = \phi_a$ is the conformal mapping of $\mathbb{U}$ onto U_a such that $\phi_a(0) = x_0 e^{i\alpha}$.

Let $b \in \partial D$, $w - b = \epsilon n_b$, $\epsilon \le \epsilon_0$. Then $w \in D_b^-$.

(d2) Set $a = h^{-1}(b)$, $z = h^{-1}(w)$, $z' = \phi_a^{-1}(z)$, $d(z') = 1 - |z'|$, $r' = |z'|$, $\hat{d}_a(z) = \operatorname{dist}(z, \partial\hat{U}_a)$, $d_b(w) = \operatorname{dist}(w, \partial D_b^-)$. Recall that in particular for $b = 0$, $d_0(w') = \operatorname{dist}(w', \partial D_0^-)$.

(d3) $d_b'(w) = \operatorname{dist}(w, \partial Y_a)$, where $Y_a := h(\hat{U}_a)$.

Let H_b be the half plane which contains $\hat{D}_b$ and touches D at b. Since F_a maps $\mathbb{U}$ into H_b, $(F_a(z') - b, n_b)$ is non-negative in $z' \in \mathbb{U}$, and by a version of Harnack's estimate (planar version of Theorem 4.2 applied with $R_0 = s_0$), $|F_a(z') - b| \ge s^0(1 - r')$, *i.e.* $|w - b| \ge s^0 d(z')$, where $s^0 = s_0/2$. Hence

$$|w - b| \succeq d(z'), \quad z' \in \mathbb{U}. \tag{4.1}$$

Now we apply Lemma 4.4 (the geometric property of the domain D_b^-).[9] More precisely we apply the property (L1) from the introduction which is a corollary of Lemma 4.4.[10]

Set $\epsilon_0 := \min\{\varepsilon_0^-, \epsilon^0\}$ and $D(\epsilon_0) := \{w \in D : d(w, \partial D) \ge \epsilon_0\}$. Then there is $r_0, r_1 \in (0,1)$ such that $D(\epsilon_0) \subset h(B(r_0))$ and $r' = |z'| \ge r_1$ implies $|z| \ge r_0$.

By (L1) and (E0), we conclude:

(E1) There is $s_1 > 0$ which is independent of z' such that $d_b(w) \ge s_1 d(z')$, $r' = |z'| \ge r_1$.

[9] Hence roughly speaking we find $|w - b| \preceq d_b(w)$ and therefore

$$d_b(w) \succeq d(z'). \tag{4.2}$$

[10] Note here that we use a version of Harnack's lemma, more precisely Theorem 4.2, and that the estimate in this theorem depends on R_0 and it is independent of h.

We now estimate $\Lambda_h(z)$. Using the fact that $h(\hat{U}_a) = Y_a \supset D_b^-$, we find first that $d'_b(w) \geq d_b(w)$ and therefore by the property (S-2) from the introduction [11]

$$\Lambda_h(z) \succeq \frac{d'_b(w)}{\hat{d}_a(z)} \geq \frac{d_b(w)}{\hat{d}_a(z)}.$$

Since $\phi_a(\mathbb{U}) = \hat{U}_a$ and $\hat{U}_a$ is a Lyapunov domain of a fixed shape, $d_a(z) \approx d(z')$.

Hence, using (E1), we conclude:

(F) $\lambda_h(z) \approx \Lambda_h(z) \geq s_2 > 0$, $|z| \geq r_0$, where $s_2 > 0$ is a constant independent of z.

It is clear that there is a constant $s_3 > 0$ such that:

(F1) $\lambda_h(z) \approx \Lambda_h(z) \geq s_3 > 0, z \in B(r_0)$.

By (F) and (F1), there is a constant $s_4 > 0$ such that $\lambda_h(z) \approx \Lambda_h(z) \geq s_4 > 0$, $z \in \mathbb{U}$.

Hence it is readable that h is co-Lip on $\mathbb{U}$.

5. Further comments and related results

We briefly discus the connection with the Radó-Kneser-Choquet theorem (shortly RKC-Theorem) and hyperbolic-harmonic mappings; it will be the subject of further investigations.

Quasiconformal Euclidean-harmonic mappings are bi-Lipschitz with respect to the quasi-hyperbolic metric, *cf.* [21,26] (Proposition 2.3 here). It turns out that, as in the Euclidean case, quasiconformal hyperbolic-harmonic mappings are bi-Lipschitz with respect to the hyperbolic metric, *cf.* Wan [33] and of Markovic [22].

Very recently, concerning the initial Schoen conjecture (and more generally the Schoen-Li-Wang conjecture) Markovic made a major breakthrough. In [23], Markovic used the result of Li and Tam that every diffeomorphism of $\mathbb{S}^2$ admits a harmonic quasiisometric extension to show that every quasisymmetric homeomorphism of the circle $\partial\mathbb{H}^2$ admits a harmonic quasiconformal extension to the hyperbolic plane $\mathbb{H}^2$. This proves the initial Schoen conjecture.

In particular, concerning complex valued harmonic functions, Kalaj and the second author, shortly KM-approach, study lower bounds of the Jacobian, *cf.* [27, 28] and references cited there. The corresponding results for harmonic maps between surfaces were previously obtained by Jost and Jost-Karcher [10, 11]. We refer to this result as the JK- result (approach). G. Alessandrini and V. Nesi prove necessary and sufficient criteria of invertibility for planar harmonic mappings which generalize a classical result of H. Kneser, also known as the Radó-Kneser-Choquet

[11] Recall that $d'_b(w) = \text{dist}(w, \partial Y_a)$, where $Y_a := h(\hat{U}_a)$.

theorem (RKC-Theorem), *cf.* [1]. Note only here that in the planar case the JK-result is reduced to Theorem RKC. Kalaj [14] also has extended the Rado-Choquet-Kneser theorem to mappings between the unit circle and Lyapunov closed curves with Lipschitz boundary data and essentially positive Jacobian at the boundary (but without restriction on the convexity of the image domain). The proof is based on the extension of the Rado-Choquet-Kneser theorem by Alessandrini and Nesi [2] and an approximation scheme is used in it. Motivated by an approach described in Kalaj's Studia paper [14] and using the continuity of so called E-function, the second author found a new proof of Kalaj's result, *cf.* [27,28].

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