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Peak functions and boundary behaviour of holomorphically invariant distances and metrics on strictly pseudoconvex domains

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Abstract. We give a parameter version of Graham-Kerzman approximation theorem for bounded holomorphic functions on strictly pseudoconvex domains. Also, we present some stability results for the localization of Carathéodory-Reiffen and Kobayashi-Royden pseudometrics and some uniform estimates for the boundary behaviour of the Kobayashi and Carathéodory pseudodistences on such domains.

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1. Introduction

For a bounded domain $G \subset \mathbb{C}^n$, its boundary point ζ is called a *peak point* with respect to $\mathcal{O}(\overline{G})$, the family of functions which are holomorphic in a neighborhood of $\overline{G}$, if there exists a function $f \in \mathcal{O}(\overline{G})$ such that $f(\zeta) = 1$ and $f(\overline{G} \setminus \{\zeta\}) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Such a function is a *peak function for G at ζ.* The peak functions turned out to be an important and fruitful concept in complex analysis, which has been used for instance to show the existence of (complete) proper holomorphic embeddings of strictly pseudoconvex domains into the unit ball $\mathbb{B}^N$ with large N (see [2,4]), to estimate the boundary behavior of Carathéodory and Kobayashi metrics [1,6], or to construct the solution operators for $\overline{\partial}$ problem with L^∞ or Hölder estimates [3,11].

It is well known that if G is strictly pseudoconvex, then its every boundary point allows a peak function. It was Graham, who showed in [6] that in this situation there exists an open neighborhood $\widehat{G}$ of $\overline{G}$, and a continuous function $h : \widehat{G} \times \partial G \to \mathbb{C}$ such that for $\zeta \in \partial G$, the function $h(\cdot; \zeta)$ is a peak function for G at ζ.

Let us consider the following:

Situation 1.1. Let $(G_t)_{t \in T}$ be a family of bounded strictly pseudoconvex domains with $\mathcal{C}^2$-smooth boundaries, where T is a compact metric space with associated

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metric d. Let $U \Subset \mathbb{C}^n$ be a domain such that:

(i) $\bigcup_{t\in T} \partial G_t \Subset U$;
(ii) For each $t \in T$ there exists a defining function $r_t \in \mathcal{C}^2(U)$ for G_t such that its Levi form $\mathcal{L}_{r_t}$ is positive on $U \times (\mathbb{C}^n \setminus \{0\})$;
(iii) The mapping $T \ni t \mapsto r_t \in \mathcal{C}^2(U)$ is uniformly continuous.

Remark 1.2. Observe that for the family $(G_t)_{t\in T}$ of bounded strictly pseudoconvex domains as in Situation 1.1 for sufficiently small $R > 0$ the set $G_t \cap \mathbb{B}(\zeta, R)$ is connected for any $t \in T$ and $\zeta \in \partial G_t$.

Recently we have proved the following parameter version of Graham's result on peak functions (see [9]):

Theorem 1.3. *Let $(G_t)_{t\in T}$ be a family of strictly pseudoconvex domains as in Situation* 1.1. *Then there exists an $\varepsilon > 0$ such that for any $\eta_1 < \varepsilon$ there exist an $\eta_2 > 0$ and positive constants d_1, d_2 such that for any $t \in T$ there exist a domain $\widehat{G_t}$ containing $\overline{G_t}$, and functions $h_t(\cdot;\zeta) \in \mathcal{O}(\widehat{G_t})$, $\zeta \in \partial G_t$ fulfilling the following conditions:*

(a) $h_t(\zeta;\zeta) = 1, |h_t(\cdot;\zeta)| < 1$ *on* $\overline{G_t} \setminus \{\zeta\}$ *(in particular, $h_t(\cdot;\zeta)$ is a peak function for G_t at ζ);*
(b) $|1 - h_t(z;\zeta)| \le d_1\|z - \zeta\|, z \in \widehat{G_t} \cap \mathbb{B}(\zeta, \eta_2)$;
(c) $|h_t(z;\zeta)| \le d_2 < 1, z \in \overline{G_t}, \|z - \zeta\| \ge \eta_1$.

Moreover, the constants $\varepsilon, \eta_2, d_1, d_2$, domains $\widehat{G_t}$, and functions $h_t(\cdot;\zeta)$ may be chosen in such a way that for any $\alpha > 0$ and any fixed triple (t_0, ζ_0, z_0), where $t_0 \in T, \zeta_0 \in \partial G_{t_0}$, and $z_0 \in \widehat{G_{t_0}}$, there exists a $\delta > 0$ such that whenever the triple (s, ξ, w) satisfies $s \in T, \xi \in \partial G_s, w \in \widehat{G_s}$, and $\max\{d(s, t_0), \|\xi - \zeta_0\|, \|w - z_0\|\} < \delta$, then $|h_{t_0}(z_0;\zeta_0) - h_s(w;\xi)| < \alpha$.

Remark 1.4. The principal strength of Theorem 1.3 lies in the continuity property and in the uniformity of the estimates given there: namely, all of the constants $\varepsilon, \eta_2, d_1, d_2$ can be chosen independently of t.

Remark 1.5. The crucial point of the proof of Theorem 1.3 is the setting of certain continuously varying $\bar\partial$ problems on some domains $\widetilde{G_t}$ with $\overline{G_t} \subset \widetilde{G_t}, t \in T$, and solving those problems in a subtle way, with uniform estimate C, given by [12, Theorems V.2.7 and V.3.6] and not depending on the domains G_t, to warrant that the solutions will vary in a continuous way. Namely, we use the following result (we give the formulation which best fits our purposes); for a bounded function f on a set G, we from now on denote its sup-norm on G by $\|f\|_G$:

Theorem 1.6. *Given a strictly pseudoconvex domain $G \subset \mathbb{C}^n$, there exist a neighbourhood $\mathfrak{U}$ of G in $\mathcal{C}^2$ topology on domains and a positive constant C such that for any strictly pseudoconvex domain $D \in \mathfrak{U}$ and any $\bar\partial$-closed $(0,1)$-form $\alpha = \sum_{j=1}^n \alpha_j d\bar z_j$ of class $\mathcal{C}^\infty$ on D such that $\|\alpha\|_D := \sum_{j=1}^n \|\alpha_j\|_D < \infty$, there exists a function $v \in \mathcal{C}^\infty(D)$ satisfying $\bar\partial v = \alpha$ and $\|v\|_D \le C\|\alpha\|_D$.*

Then the compactness of T gives a constant C as above, good for all domains G_t.

The technique mentioned in the above remark, together with Theorem 1.3 itself, is also a vital ingredient of the proof of the first result given in the hereby paper. This is the following approximation result for bounded holomorphic functions defined near the boundary points of strictly pseudoconvex domains. The family of such functions will be from now on denoted by $\mathcal{H}^\infty$.

Theorem 1.7. *Let $(G_t)_{t\in T}$ be a family of strictly pseudoconvex domains as in Situation 1.1 and let $R > 0$ be such that the set $G_t \cap \mathbb{B}(\zeta, R)$ is connected for any $t \in T$ and any $\zeta \in \partial G_t$. Then, if R is taken to be sufficiently small, there exists a $\rho < R$ such that for any $\varepsilon > 0$ there exists an $L = L(\varepsilon, R) > 0$ with the property that for any $t \in T$, any choice of points $\zeta_t \in \partial G_t$, $w^t \in G_t \cap \mathbb{B}(\zeta_t, \rho)$, and any function $f_t \in \mathcal{H}^\infty(G_t \cap \mathbb{B}(\zeta_t, R))$, there exist an $\hat{f}_t \in \mathcal{H}^\infty(G_t)$ such that:*

(A) $D^\alpha \hat{f}_t(w^t) = D^\alpha f_t(w^t)$ *for* $|\alpha| \le 1$*;*
(B) $\|\hat{f}_t\|_{G_t} \le L\|f_t\|_{G_t\cap\mathbb{B}(\zeta_t,R)}$*;*
(C) $\|\hat{f}_t - f_t\|_{G_t\cap\mathbb{B}(\zeta_t,\rho)} < \varepsilon\|f\|_{G_t\cap\mathbb{B}(\zeta_t,R)}$.

Remark 1.8. Notice that the estimate in (B) depends only on ε and R. In particular, it is independent of t. Also, the "size" of domains of definition of the functions we approximate is uniform, as the constant R do not depend on t. Therefore, Theorem 1.7 amplifies Graham-Kerzman theorem, *i.e.* [6, Theorem 2] (see also [7, Theorem 19.1.3]).

If in addition we are interested in interpolation problem at more than one point, we have the following variant of Theorem 1.7:

Theorem 1.9. *Let the family of strictly pseudoconvex domains $(G_t)_{t\in T}$ and number $R > 0$ be as in Theorem 1.7. Then, if R is taken to be sufficiently small, there exists a $\rho < R$ such that for any $\varepsilon > 0$ and any $m \in \mathbb{N}$, $m \ge 2$, there exists an $L = L(m, \varepsilon, R) > 0$ with the property that for any $t \in T$, any $\zeta_t \in \partial G_t$, any choice of pairwise different points $\mathcal{W}_{m,t} = \{w_1^t, \dots, w_m^t\} \subset G_t \cap \mathbb{B}(\zeta_t, \rho)$, and any function $f_t \in \mathcal{H}^\infty(G_t \cap \mathbb{B}(\zeta_t, R))$, there exist an $\hat{f}_t \in \mathcal{H}^\infty(G_t)$ such that:*

(A′) $D^\alpha \hat{f}_t(w_j^t) = D^\alpha f_t(w_j^t)$ *for* $|\alpha| \le 1$*, and* $j = 1, \dots, m$*;*
(B′) *There exists an* $N = N(\varepsilon, R, \mathcal{W}_{m,t})$ *such that*

$$\|\hat{f}_t\|_{G_t} \le (L + N)\|f_t\|_{G_t\cap\mathbb{B}(\zeta_t,R)};$$

(C′) $\|\hat{f}_t - f_t\|_{G_t\cap\mathbb{B}(\zeta_t,\rho)} < \varepsilon\|f\|_{G_t\cap\mathbb{B}(\zeta_t,R)}$.

Remark 1.10. Note that for $m \ge 2$ the situation is totally different than in Theorem 1.7, as the estimate in (B′) is not any more independent on t. In fact, it even depends on the choice of system of points $\mathcal{W}_{m,t}$. The reason for this discrepancy is that for $m = 1$ (*i.e.* in the situation from Theorem 1.7) the constant L may be chosen independently of $w^t \in G_t \cap \mathbb{B}(\zeta_t, \rho)$ because it comes from the uniform (independent of t) estimates for solutions of certain $\bar{\partial}$ problems stated on some small modifications G^t of domains G_t (*cf.* Remark 1.5) with the property that $G_t \cap \mathbb{B}(\zeta_t, \rho) \Subset G^t$ with uniform distance to the boundary, which allows us to use the Cauchy inequalities with constant also independent of t. On the other hand, for $m \geq 2$ this estimate is aberrated, since we have in addition to take care of the distances between points from $\mathcal{W}_{m,t}$ while solving the interpolation problem - see Section 2 for the details. Observe however, that if the systems $\mathcal{W}_{m,t}$ of pairwise different points $\{w_1^t, \dots, w_m^t\} \subset G_t \cap \mathbb{B}(\zeta_t, \rho)$ are chosen in such a way that the function $T \ni t \mapsto \min\{|w_{j,l}^t - w_{k,l}^t| : j, k \in \{1, \dots, m\}, j \neq k, l \in \{1, \dots, n\}\}$ is continuous, then N in (B$'$) may be chosen to be zero (it falls under L) and therefore the estimate given there once again becomes independent of t in this exceptional case.

We also consider the stability problems of the boundary behaviour and localization of holomorphically contractible systems. We say that a system of functions

$$\boldsymbol{d}_G : G \times G \to [0, +\infty),$$

where G runs over all domains in $\mathbb{C}^n$ with any n, is *holomorphically contractible* if $\boldsymbol{d}_{\mathbb{D}} = \boldsymbol{p}$, the Poincaré distance on $\mathbb{D}$, and if all holomorphic mappings are contractions with respect to the family $(\boldsymbol{d}_G)$, that is, for any two domains $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ and any mapping $f \in \mathcal{O}(D, G)$ we have

$$\boldsymbol{d}_G(f(z), f(w)) \leq \boldsymbol{d}_D(z, w), \quad z, w \in D.$$

If all functions are additionally pseudodistances, we say that $(\boldsymbol{d}_G)$ is a *holomorphically contractible system of pseudodistances*. This definition have an infinitesimal counterpart: we say that a system of pseudometrics

$$\boldsymbol{\delta}_G : G \times \mathbb{C}^n \to [0, +\infty)$$

where G runs over all domains in $\mathbb{C}^n$ with any n, is *holomorphically contractible* if $\boldsymbol{\delta}_{\mathbb{D}}(z; X) = \sup\{|f'(z)X| : f \in \mathcal{O}(\mathbb{D}, \mathbb{D}), f(z) = 0\}$ and if for any two domains $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ and any mapping $f \in \mathcal{O}(D, G)$ we have

$$\boldsymbol{\delta}_G(f(z); f'(z)X) \leq \boldsymbol{\delta}_D(z; X), \quad z \in D, X \in \mathbb{C}^n,$$

with $f'(z)$ abbreviating the $\mathbb{C}$-differential of f at z. For a good exposition on the topic of holomorphically contractible objects, we refer the Reader to the monograph [7].

With Theorems 1.3 and 1.7 at hand, we are able to deliver some uniform localization results for Carathéodory-Reiffen pseudometric $\boldsymbol{\gamma}_G$ (Proposition 3.1) and for Kobayashi-Royden pseudometric $\boldsymbol{\kappa}_G$ (Proposition 3.2): these are the parameter versions of [6, Proposition 6] (see also [7, Theorems 19.3.1 and 19.3.2]).

In [5], some upper and lower estimates for the boundary behaviour of the Kobayashi pseudodistance $\boldsymbol{k}_G$ on strictly pseudoconvex domain G are given. It is showed there that for such domain G, for any couple of distinct points $\zeta, \xi \in \partial G$ there exist constants K and C such that

$$\boldsymbol{k}_G(z, w) \geq -\frac{1}{2} \log \operatorname{dist}(z, \partial G) - \frac{1}{2} \log \operatorname{dist}(w, \partial G) - K$$

whenever $z, w \in G$ are such that z is close to ζ and w is close to ξ, and

$$\begin{aligned} \boldsymbol{k}_G(z, w) \leq & -\frac{1}{2}(\log \operatorname{dist}(z, \partial G) + \log \operatorname{dist}(w, \partial G)) \\ & + \frac{1}{2}(\log(\operatorname{dist}(z, \partial G) + \|z - w\|) + \log(\operatorname{dist}(w, \partial G) + \|z - w\|)) + C, \end{aligned}$$

whenever $z, w \in G$ are close to ζ (*cf.* [5, Corollary 2.4 and Proposition 2.5]; observe that for the upper estimate the strict pseudoconvexity is not needed – in [5] the domain G is only assumed to have $\mathcal{C}^{1+\varepsilon}$ boundary). We prove that given $(G_t)_{t \in T}$, a family of strictly pseudoconvex domains as in Situation 1.1, the estimates as above are uniform with respect to $t \in T$ and $\zeta, \xi \in \partial G_t$, *i.e.* the bounds K and C given there can be taken to be independent of $t \in T$ and of $\zeta, \xi \in \partial G_t$ - in the case of lower estimate, depending only on $\|\zeta - \xi\|$ (and when it comes to the upper estimate, the domains G_t do not necessarily have to be strictly pseudoconvex), see Propositions 3.5 and 3.7 below. These results are inspired by [10, Propositions 9.1 and 9.2]. In correspondence to that paper, note that the role of the set of parameters T is there played by a convergent sequence of numbers with its limit added.

We also give some estimates in this spirit for the Carathéodory pseudodistance $\boldsymbol{c}_D$ – see Proposition 3.3 and Corollary 3.4, and Proposition 3.9 (compare with [7, Theorem 19.2.1, Corollary 19.2.2, and Proposition 19.2.4]). Note that Proposition 3.3 is the consequence of Theorem 1.3.

The proof of Theorems 1.7 and 1.9 is presented in Section 2, while the stability results for the localization of Carathéodory-Reiffen and Kobayashi-Royden pseudometrics and the uniform estimates for the boundary behaviour of Kobayashi and Carathéodory pseudodistances come in Section 3.

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2. Proof of Theorems 1.7 and 1.9

Recall that a bounded domain $G \subset \mathbb{C}^n$ is called a *strictly pseudoconvex* one if there exist a neighborhood U of ∂G and a *defining function* $r : U \to \mathbb{R}$ of class $\mathcal{C}^2$ on U and such that:

(I) $G \cap U = \{z \in U : r(z) < 0\}$;
(II) $(\mathbb{C}^n \setminus \overline{G}) \cap U = \{z \in U : r(z) > 0\}$;
(III) $\nabla r(z) \neq 0$ for $z \in \partial G$, where $\nabla r(z) := \big(\frac{\partial r}{\partial \bar{z}_1}(z), \cdots, \frac{\partial r}{\partial \bar{z}_n}(z)\big)$;

together with

$$\mathcal{L}_r(z;X) > 0 \text{ for } z \in \partial G \text{ and nonzero } X \in T_z^{\mathbb{C}}(\partial G),$$

where $\mathcal{L}_r$ denotes the Levi form of r and $T_z^{\mathbb{C}}(\partial G)$ is the complex tangent space to ∂G at z.

It is known that U and r can be chosen to satisfy (I)-(III) and, additionally:

(IV) $\mathcal{L}_r(z;X) > 0$ for $z \in U$ and all nonzero $X \in \mathbb{C}^n$;

cf. [8].

The beginning of the proof is common for both Theorems 1.7 and 1.9 and it consists of stating and solving (with uniform estimates) certain family of $\bar\partial$-problems on domains being deformations of the domains G_t, and producing, with the aid of the solutions of those $\bar\partial$-problems, the family of holomorphic functions on domains G_t that would satisfy the assertions of the Theorems, except those concerning the interpolation. In this place the argument splits into two parts, different for both Theorems, as we have to consider different interpolation problems. As we have already mentioned in the Introduction, this is the source of the fact that the estimates in (B) and in (B′) are of different nature.

Beginning of proof of Theorems 1.7 *and* 1.9. Set $\eta_2 < \eta_1, d_1, d_2 < 1, \widehat{G_t}$, and $h_t(\cdot;\zeta)$ for $t \in T, \zeta \in \partial G_t$ according to Theorem 1.3, where η_1 is small enough to assure that the set $G_t \cap \mathbb{B}(\zeta, R)$ is connected for every $t \in T$ and $\zeta \in \partial G_t$, where $R := 2\eta_1$. Replacing h_t with $\frac{h_t+3}{4}$ we may assume that

$$|h_t(z;\zeta)| \geq \frac{1}{2}, \quad z \in \overline{G_t}, \zeta \in \partial G_t. \tag{2.1}$$

Let $d_3 \in (d_2, 1)$ and choose $0 < \eta \leq \eta_2$ such that for any $t \in T$ we have $\mathbb{B}(\zeta; 2\eta) \subset \widehat{G_t}$ for all $\zeta \in \partial G_t$ as well as $|h_t(z;\zeta)| \geq d_3$ whenever $\zeta \in \partial G_t$ and $\|z-\zeta\| \leq \eta$ (this is possible because of the uniform choice of d_1 in Theorem 1.3). Define $\rho := \min\{\frac{\eta}{2}, \frac{\eta_1}{5}\}$.

For a fixed $t \in T$ there are points $\zeta_1^t, \dots, \zeta_{N_t}^t \in \partial G_t$ such that

$$\partial G_t \subset \bigcup_{j=1}^{N_t} \mathbb{B}\Big(\zeta_j^t, \rho\Big).$$

For any $j \in \{1, \dots, N_t\}$ we modify the domain G_t near the boundary point ζ_j^t in order to get a strictly pseudoconvex domain G_j^t satisfying:

(1′) $G_t \subset G_j^t \subset \widehat{G_t} \cap G_t^{(\eta)}$ (where $G_t^{(\eta)}$ denotes the η-hull of G_t);

(2′) $\overline{G_t} \cap \overline{\mathbb{B}(\zeta_j^t, 2\rho)} \Subset G_j^t$ and $\operatorname{dist}(\overline{G_t} \cap \overline{\mathbb{B}(\zeta_j^t, 2\rho)}, \partial G_j^t) \geq \beta > 0$ with β independent of j;

(3′) $G_t \setminus \mathbb{B}(\zeta_j^t, \frac{7}{2}\rho) = G_j^t \setminus \mathbb{B}(\zeta_j^t, \frac{7}{2}\rho)$;

(4′) The estimate C for the solution of $\bar\partial$-problem for G_t given by Theorem 1.6 is good for G_j^t;

see Figure 2.1 below.

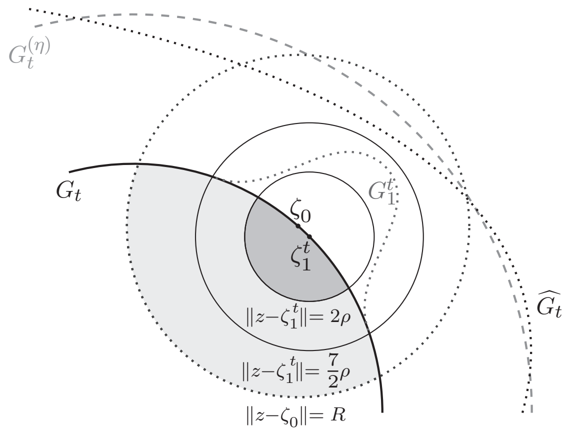

Figure 2.1. Geometric situation as described in (1′)-(4′): with t fixed, the black thick arc represents a part of the boundary of the domain G_t near ζ_1^t, gray dashed one – a part of the boundary of $G_t^{(\eta)}$, and the black dotted one – a part of the boundary of $\widehat{G_t}$. The gray dotted curve illustrates the required modification G_1^t of the domain G_t near ζ_1^t. Note we want to perform this kind of modification near finite family of suitably chosen boundary points $\zeta_1^t, \dots, \zeta_{N_t}$ (*i.e.* such that $\partial G_t \subset \bigcup_{j=1}^{N_t} \mathbb{B}(\zeta_j^t, \rho)$) and then to allow all the data vary uniformly in t in such a way that suitably chosen ρ and the estimate from below β of the distance of the closure of the domain colored in deeper gray tint are good for all $t \in T$ and all $j \in \{1, \dots, N_t\}$, and so is the estimate C, given by Theorem 1.6, of the norm of the solution of $\bar\partial$-problem for all domains G_t, G_j^t, $j = 1, \dots, N_t$, $t \in T$, (this is (1)-(4)). The next part of the proof begins with taking arbitrary point $\zeta_0 \in \partial G_t$ which then must be contained in some of the balls $\mathbb{B}(\zeta_1, \rho), \dots, \mathbb{B}(\zeta_{N_t}, \rho)$, so that $\overline{G_t} \cap \mathbb{B}(\zeta_0, \rho) \Subset G_1^t$, which will be used later in the proof. Finally, the area filled in gray (both tints) is the domain of definition of bounded holomorphic function we want to approximate.

Observe that for s close enough to t we may choose points $\zeta_1^s, \dots, \zeta_{N_s}^s \in \partial G_s$ such that $N_s = N_t$, ζ_j^s is close to ζ_j^t (with arbitrarily prescribed distance), $\partial G_s \subset \bigcup_{j=1}^{N_s} \mathbb{B}(\zeta_j^s, \rho)$, and with the property that for any $j \in \{1, \dots, N_s\}$ we can find strictly pseudoconvex deformation G_j^s of G_s near ζ_j^s such that:

(1) $G_s \subset G_j^s \subset \widehat{G_s} \cap G_s^{(\eta)}$;

(2) $\overline{G_s} \cap \overline{\mathbb{B}(\zeta_j^s, 2\rho)} \Subset G_j^s$ and $\operatorname{dist}(\overline{G_t} \cap \overline{\mathbb{B}(\zeta_j^t, 2\rho)}, \partial G_j^t) \geq \frac{\beta}{2} > 0$;

(3) $G_s \setminus \mathbb{B}(\zeta_j^s, 4\rho) = G_j^s \setminus \mathbb{B}(\zeta_j^s, 4\rho)$;

(4) The estimate C for the solution of $\bar{\partial}$-problem given by Theorem 1.6 is good for G_j^s.

Using the compactness of T, we see that the constants C and β may be chosen indepenedently of t and j.

Fix now $t = t_0 \in T$ and $\zeta_0 \in \partial G_t$. Let $f \in \mathcal{H}^\infty(G_t \cap \mathbb{B}(\zeta_0, R))$.

There exists a $j_0 \in \{1, \dots N_t\}$ such that $\zeta_0 \in \mathbb{B}(\zeta_{j_0}^t, \rho)$. To simplify the notation, let us assume without loss of generality that $j_0 = 1$.

Choose a $\chi \in \mathcal{C}^\infty(\mathbb{C}^n, [0, 1])$ such that $\chi \equiv 1$ on $\mathbb{B}(\zeta_0, \frac{6\eta_1}{5})$ and $\chi \equiv 0$ outside $\mathbb{B}(\zeta_0, \frac{9\eta_1}{5})$ and define $\alpha_t := (\bar{\partial}\chi)f$ on $G_t \cap \mathbb{B}(\zeta_0, R) = G_t \cap \mathbb{B}(\zeta_0, 2\eta_1)$ and $\alpha_t := 0$ on $G_t \setminus \mathbb{B}(\zeta_0, 2\eta_1)$. Note that in view of the fact that $\alpha_t \equiv 0$ on $(G_t \cap \mathbb{B}(\zeta_0, \frac{6\eta_1}{5})) \cup (G_t \setminus \mathbb{B}(\zeta_0, \frac{9\eta_1}{5}))$, after trivial extension by zero, it can be treated as a $\bar{\partial}$-closed $(0, 1)$-form of class $\mathcal{C}^\infty$ on G_1^t.

For $k \in \mathbb{N}$ (this will be specified later) consider the equation

$$\bar{\partial} v_k^t = (h_t(\cdot; \zeta_0))^k \alpha_t. \tag{2.2}$$

Then we have a solution $v_k^t \in \mathcal{C}^\infty(G_1^t)$ of the problem (2.2) such that

$$\left\| v_k^t \right\|_{G_1^t} \leq C \| (h_t(\cdot; \zeta_0))^k \|_{\mathrm{spt}\alpha_t} \| \alpha_t \|_{G_1^t}.$$

Recall that the constant C is independent of t and of $j \in \{1, \dots, N_t\}$. Further estimation gives

$$\left\| v_k^t \right\|_{G_1^t} \leq C C_1 d_2^k \| f \|_{G_t \cap \mathbb{B}(\zeta_0, R)},$$

with the constant C_1 depending only on η_1 (in particular, not depending on t).

Define the function $f_k := \chi f - h_t(\cdot; \zeta_0)^{-k} v_k^t$ and observe it is holomorphic on G_t (recall that at the beginning of the construction we have normalized h_t to satisfy (2.1)). Consequently, the function $h_t(\cdot; \zeta_0)^{-k} v_k^t$ is holomorphic on $G_1^t \cap \mathbb{B}(\zeta_0, \eta)$ (on the set $G_t \cap \mathbb{B}(\zeta_0, \eta)$ it follows from the holomorphicity of f_k, and on the remaining part - from the triviality of extension of α_t by zero and from the choice of η). Furthermore

$$\left\| h_t(\cdot; \zeta_0)^{-k} v_k^t \right\|_{G_1^t \cap \mathbb{B}(\zeta_0, \eta)} \leq C C_1 \left(\frac{d_2}{d_3} \right)^k \| f \|_{G_t \cap \mathbb{B}(\zeta_0, R)}.$$

Note that for $z \in \overline{G_t} \cap \mathbb{B}(\zeta_0, \rho)$ we have $\|z - \zeta_1^t\| \leq \|z - \zeta_0\| + \|\zeta_0 - \zeta_1^t\| \leq 2\rho$. Therefore, $\overline{G_t} \cap \mathbb{B}(\zeta_0, \rho) \subset \overline{G_t} \cap \mathbb{B}(\zeta_1^t, 2\rho) \Subset G_1^t$.

On the set $G_t \cap \mathbb{B}(\zeta_0, \eta)$ we have the equality $f_k - \chi f = -h_t(\cdot; \zeta_0)^{-k} v_k$, and the latter function is holomorphic on bigger set $G_1^t \cap \mathbb{B}(\zeta_0, \eta)$. Therefore, for $z \in G_t \cap \mathbb{B}(\zeta_0, \rho)$ we have

$$\left| \frac{\partial f_k}{\partial z_j}(z) - \frac{\partial f}{\partial z_j}(z) \right| = \left| \frac{\partial}{\partial z_j} \Big(h_t(\cdot; \zeta_0)^{-k} v_k \Big)(z) \right| \leq \frac{C C_1}{L_1} \left(\frac{d_2}{d_3} \right)^k \| f \|_{G_t \cap \mathbb{B}(\zeta_0, R)},$$

where the last inequality is a consequence of the Cauchy inequalities and in virtue of (2), the constant L_1 may be chosen independently of $t, \zeta_0 \in \partial G_t$, and $z \in G_t \cap \mathbb{B}(\zeta_0, \rho)$. The same argument gives

$$\|f_k - f\|_{G_t \cap \mathbb{B}(\zeta_0, \eta)} \leq \frac{CC_1}{L_1}\left(\frac{d_2}{d_3}\right)^k \|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)}. \qquad \square$$

The remaining part of proof is different for both Theorems.

End of proof of Theorem 1.7. Set $\varepsilon > 0$ and let $w = w^t \in G_t \cap \mathbb{B}(\zeta_0, \rho)$ ($t = t_0 \in T$ and $\zeta_0 \in \partial G_t$ are as above). Define $\hat{f}_k \in \mathcal{O}(G_t)$ by $\hat{f}_k(z) := f_k(z) + p(z)$, where

$$p(z) := f(w) - f_k(w) + \sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}(w) - \frac{\partial f_k}{\partial z_j}(w)\right)\left(z_j - w_j\right).$$

It can be easily checked that $\hat{f}_k(w) = f(w)$, as well as $\frac{\partial \hat{f}_k}{\partial z_j}(w) = \frac{\partial f}{\partial z_j}(w)$. Furthermore,

$$\begin{aligned}\|\hat{f}_k - f\|_{G_t \cap \mathbb{B}(\zeta_0, \rho)} &\leq \|f_k - f\|_{G_t \cap \mathbb{B}(\zeta_0, \rho)} + |f(w) - f_k(w)| \\ &\quad + n\mathrm{diam}U\left|\frac{\partial f}{\partial z_j}(w) - \frac{\partial f_k}{\partial z_j}(w)\right| \\ &\leq (2 + n\mathrm{diam}U)\frac{CC_1}{L_1}\left(\frac{d_2}{d_3}\right)^k \|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)}.\end{aligned}$$

Let finally $k_0 \in \mathbb{N}$ so large that $(2 + n\mathrm{diam}U)\frac{CC_1}{L_1}\left(\frac{d_2}{d_3}\right)^{k_0} \leq \varepsilon$ and define $\hat{f} := \hat{f}_{k_0}$. Observe that k_0 depends only on ε and η_1. It is left to estimate the norm of the latter function:

$$\begin{aligned}\|\hat{f}\|_{G_t} &\leq \left\|f_{k_0}\right\|_{G_t} + |f(w) - f_{k_0}(w)| + n\mathrm{diam}U\left|\frac{\partial f}{\partial z_j}(w) - \frac{\partial f_{k_0}}{\partial z_j}(w)\right| \\ &\leq \left\|f_{k_0}\right\|_{G_t} + \varepsilon\|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)}.\end{aligned}$$

This, together with the estimate

$$\left\|f_{k_0}\right\|_{G_t} \leq \|\chi f\|_{G_t} + \left\|(h_t(\cdot;\zeta_0))^{-k_0} v_k^t\right\|_{G_t} \leq \left(1 + 2^{k_0} CC_1 d_2^{k_0}\right)\|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)} \quad (2.3)$$

gives the conclusion with $L := 1 + 2^{k_0} CC_1 d_2^{k_0} + \varepsilon$, depending only on ε and R. $\square$

End of proof of Theorem 1.9. Set $\varepsilon > 0$ and let a system of pairwise different points $\mathcal{W}_{m,t} = \{w_1^t, \ldots, w_t^m\} \subset G_t \cap \mathbb{B}(\zeta_0, \rho)$ ($t = t_0 \in T$ and $\zeta_0 \in \partial G_t$ are as above). Put $w^i := w_i^t, i = 1, \ldots, m$. We introduce some useful notation: for pairwise distinct complex numbers $x^1, \dots, x^m$, the basis Lagrange polynomials are defined as

$$l_i(z) = l_i^m(z) := \prod_{j=1, j\neq i}^{m} \frac{z - x^j}{x^i - x^j}, \quad i = 1, \dots, m.$$

Given a $z = (z_1, \dots, z_n)$ and $w^1 = (w_1^1, \dots, w_n^1), \dots, w^m = (w_1^m, \dots, w_n^m) \in \mathbb{C}^n$ let us put

$$l_{i,j}(z_j) := \prod_{k=1,k\neq i}^{m} \frac{z_j - w_j^k}{w_j^i - w_j^k}, \quad i = 1, \dots, m,\ j = 1, \dots, n.$$

Assume first that $w_j^{i_1} \neq w_j^{i_2}$ for $j = 1, \dots, n$ whenever $i_1 \neq i_2$. Define $\hat{f}_k \in \mathcal{O}(G_t)$ by $\hat{f}_k(z) := f_k(z) + p(z)$, where

$$\begin{aligned} p(z) := \sum_{i=1}^{m} \Big(f\big(w^i\big) - f_k\big(w^i\big)\Big) \Bigg[\Bigg(1 + \frac{2}{n} \sum_{j=1}^{n} \frac{\partial l_{i,j}}{\partial z_j}\big(w_j^i\big)\big(w_j^i - z_j\big)\Bigg) \frac{1}{n} \sum_{j=1}^{n} \big(l_{i,j}(z_j)\big)^2 \Bigg] \\ + \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j}\big(w^i\big) - \frac{\partial f_k}{\partial z_j}\big(w^i\big)\right) \big(z_j - w_j^i\big)\big(l_{i,j}(z_j)\big)^2 \right). \end{aligned}$$

One can verify that $\hat{f}_k(w^i) = f(w^i)$ and $\frac{\partial f_k}{\partial z_j}(w^i) = \frac{\partial f}{\partial z_j}(w^i)$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. We estimate

$$\begin{aligned} &\|\hat{f}_k - f\|_{G_t \cap \mathbb{B}(\zeta_0, \rho)} \\ &\le \|f_k - f\|_{G_t \cap \mathbb{B}(\zeta_0, \rho)} + (M(1 + 2M \operatorname{diam} U)) \sum_{i=1}^{m} \Big| f\big(w^i\big) - f_k\big(w^i\big)\Big| \\ &\quad + M \operatorname{diam} U \sum_{i=1}^{m} \sum_{i=1}^{n} \left| \frac{\partial f}{\partial z_j}\big(w^i\big) - \frac{\partial f_k}{\partial z_j}\big(w^i\big) \right| \\ &\le \frac{CC_1}{L_1} \left(\frac{d_2}{d_3}\right)^k (1 + Mm(1 + M(2+n)\operatorname{diam} U) \|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)}, \end{aligned}$$

where $M = M(\mathcal{W}_{m,t})$. The last term is smaller than $\varepsilon \|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)}$, provided that $k = k_0$ is sufficiently large (observe this choice of k is independent of f). Then, performing similar computations, for $\hat{f} := \hat{f}_{k_0}$, because of (2.3), we get

$$\begin{aligned} \|\hat{f}\|_{G_t} &\le \big\|f_{k_0}\big\|_{G_t} + \frac{CC_1}{L_1} \left(\frac{d_2}{d_3}\right)^{k_0} (Mm(1 + M(2+n)\operatorname{diam} U) \|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)} \\ &\le \left(1 + 2^{k_0} CC_1 d_2^{k_0} + \frac{CC_1}{L_1} \left(\frac{d_2}{d_3}\right)^{k_0} (Mm(1 + M(2+n)\operatorname{diam} U) \right) \|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)} \\ &=: (L + N) \|f\|_{G_t \cap \mathbb{B}(\zeta_0, R)}. \end{aligned}$$

Observe that $L = 1 + 2^{k_0} C C_1 d_2^{k_0}$ depends only on ε and R, while

$$N = \frac{CC_1}{L_1}\left(\frac{d_2}{d_3}\right)^{k_0}(Mm(1 + M(2 + n)\mathrm{diam}U)$$

do also depend on the choice of $\mathcal{W}_{m,t}$. See however, that according to what we have said in Remark 1.10, if the systems $\mathcal{W}_{m,t}$ were chosen to be continuously dependent on t, then N would fall into L and therefore would become independent of t.

Let us pass to the remaining case, namely: the one where we assume that some of points $w^1, \dots, w^m$ have at least one common coordinate. For $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ define

$$\tilde{l}_{i,j}(z_j) \prod_{k=1, k\neq i, w_j^k \neq w_j^i}^{m} \frac{z_j - w_j^k}{w_j^i - w_j^k} \quad \text{if } \exists k \neq i : w_j^i \neq w_j^k,$$

and

$$\tilde{l}_{i,j}(z_j) \equiv 0 \quad \text{otherwise.}$$

Observe that for a fixed i not all of $\tilde{l}_{i,j}$ are zero (since the points $w^1, \dots, w^m$ are pairwise different). Therefore, we can define

$$A_i := \left\{ j \in \{1, \dots, n\} : \tilde{l}_{i,j} \text{ is nonzero} \right\} \neq \varnothing, \quad n_i := |A_i|, \quad 1 = 1, \dots, m.$$

Observe that if $j \notin A_i$, then

$$B_{i,j} := \left\{ (k_i, m_i) \in \{1, \dots, m\} \times (\{1, \dots, n\} \setminus \{j\}) : w_{m_i}^i \neq w_{m_i}^{k_i} \right\} \neq \varnothing.$$

Define

$$\begin{aligned}
&p(z)\\
&:= \sum_{i=1}^{m}\Big(f(w^i) - f_k(w^i)\Big)\left[\left(1 + \frac{2}{n_i}\sum_{j\in A_i}\frac{\partial \tilde{l}_{i,j}}{\partial z_j}\big(w_j^i\big)\big(w_j^i - z_j\big)\right)\frac{1}{n_i}\sum_{j\in A_i}\Big(\tilde{l}_{i,j}(z_j)\Big)^2\right]\\
&\quad + \sum_{i=1}^{m}\left(\sum_{j\in A_i}\left(\frac{\partial f}{\partial z_j}(w^i) - \frac{\partial f_k}{\partial z_j}(w^i)\right)\big(z_j - w_j^i\big)\Big(\tilde{l}_{i,j}(z_j)\Big)^2\right)\\
&\quad + \sum_{i=1}^{m}\left(\sum_{j\notin A_i}\left(\frac{\partial f}{\partial z_j}(w^i) - \frac{\partial f_k}{\partial z_j}(w^i)\right)\big(z_j - w_j^i\big)\prod_{(k_i,m_i)\in B_{i,j}}\left(\frac{z_{m_i} - w_{m_i}^{k_i}}{w_{m_i}^i - w_{m_i}^{k_i}}\right)^2\right)
\end{aligned}$$

and $\hat{f}(z) = \hat{f}_k(z) := f(z) + p(z)$ with k sufficiently large, and we end the proof carrying out similar computations as before. □

Remark 2.1. In Theorem 1.7 one can require in the conclusion that

$$\|\hat{f}_t - f_t\|_{G_t \cap \mathbb{B}(\zeta_t, \rho)} < \varepsilon.$$

Analyzing the proof of our result, we see that it is possible to get this kind of estimate. There is, however, a price we have to pay - the constant L is not any more independent of f.

Remark 2.2. In [6] it is stated that the estimate in (C) holds true also for the derivatives of $\hat{f}$ and f up to previously prescribed order. The same result is possible to get here - with L still independent of t.

3. Boundary behaviour of Kobayashi and Carathéodory pseudodistances and pseudometrics

Let us recall the definitions of the holomorphically contractible pseudodistances and pseudometrics we are interested in.

For arbitrary domain G the Carathéodory pseudodistance is defined as

$$\boldsymbol{c}_G(z, w) := \sup\{\boldsymbol{p}(0, f(w)) : f \in \mathcal{O}(G, \mathbb{D}), f(z) = 0\}, \quad z, w \in G,$$

the Carathéodory-Reiffen pseudometric is given by

$$\boldsymbol{\gamma}_G(z; X) := \sup\left\{\left|\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) X_j\right| : f \in \mathcal{O}(G, \mathbb{D}), f(z) = 0\right\}$$
$$z \in G, X = (X_1, \dots, X_n) \in \mathbb{C}^n,$$

the Kobayashi pseudodistance may be expressed as

$$\boldsymbol{k}_G(z, w) := \inf\left\{\sum_{j=1}^{N} \boldsymbol{p}(\xi_j, \zeta_j) : N \in \mathbb{N}, \xi_j, \zeta_j \in \mathbb{D}, \text{ and } \exists p_0, \dots, p_N \in G : \right.$$
$$p_0 = z, p_N = w, \exists f_j \in \mathcal{O}(\mathbb{D}, G) : f_j(\xi_j) = p_{j-1}, f_j(\zeta_j)$$
$$\left. = p_j, j = 1, \dots, N\right\}, \qquad z, w \in G,$$

and the Kobayashi-Royden pseudometric is defined as

$$\boldsymbol{\kappa}_G(z; X) := \inf\{\alpha > 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, G) : \varphi(0) = z, \alpha\varphi'(0) = X\},$$
$$z \in G, X = (X_1, \dots, X_n) \in \mathbb{C}^n.$$

We start with localization results with parameter for Carathéodory-Reiffen (Proposition 3.1) and Kobayashi-Royden (Proposition 3.2) pseudometrics. The proofs of these results are similar to the [6, proof of Proposition 6]. However, we include them for both the convenience of the Reader and in order to illustrate how Theorems 1.3 and 1.7 work in the proofs. Note that Theorems 1.3 and 1.7 play a crucial role here.

Proposition 3.1. **(*cf.* [6, Proposition 6] and the remark following it, see also [7, Theorem 19.3.1]).** *Let $(G_t)_{t\in T}$ be a family of strictly pseudoconvex domains as in Situation* 1.1. *Then for all sufficiently small $R > 0$ the set $G_t \cap \mathbb{B}(\zeta, R)$ is connected for any $t \in T$ and $\zeta \in \partial G_t$, and for all $X \in \mathbb{C}^n \setminus \{0\}$ we have*

$$\lim_{G_t\cap\mathbb{B}(\zeta,R)\ni z\to\zeta} \frac{\boldsymbol{\gamma}_{G_t\cap\mathbb{B}(\zeta,R)}(z; X)}{\boldsymbol{\gamma}_{G_t}(z; X)} = 1.$$

The convergence is uniform in t, ζ and $X \in \mathbb{C}^n \setminus \{0\}$ in the sense that for any $\alpha > 0$ there exists a $\beta \in (0, R)$ such that for any $t \in T$, $\zeta \in \partial G_t$, $w \in G_t \cap \mathbb{B}(\zeta, \beta)$, and $X \in \mathbb{C}^n \setminus \{0\}$ we have

$$\frac{\boldsymbol{\gamma}_{G_t\cap\mathbb{B}(\zeta,R)}(w; X)}{\boldsymbol{\gamma}_{G_t}(w; X)} \le 1 + \alpha.$$

Proof. Fix $\alpha > 0$ and let R be as in Remark 1.2 and smaller than ε from Theorem 1.3, and so small that the conclusion of Theorem 1.7 holds for it. Let $\rho < R$ be as in Theorem 1.7. For any $t \in T$, $\zeta \in \partial G_t$, $w \in G_t \cap \mathbb{B}(\zeta, \rho)$, and any $X \in \mathbb{C}^n$ let $f_t \in \mathcal{O}(G_t \cap \mathbb{B}(\zeta, R), \mathbb{D})$ be such that $f_t(w) = 0$ and

$$\boldsymbol{\gamma}_{G_t\cap\mathbb{B}(\zeta,R)}(w; X) = \left|\sum_{j=1}^{n} \frac{\partial f_t}{\partial z_j}(w) X_j\right|.$$

Take $0 < \mu \le \frac{\alpha}{2+\alpha}$. By Theorem 1.7, for any $t \in T$, $\zeta \in \partial G_t$, and $w \in G_t \cap \mathbb{B}(\zeta, \rho)$ there exists an $\hat{f}_t \in \mathcal{H}^\infty(G_t)$ such that $\|\hat{f}_t - f_t\|_{G_t\cap\mathbb{B}(\zeta,\rho)} < \mu$, $\frac{\partial \hat{f}_t}{\partial z_j}(w) = \frac{\partial f_t}{\partial z_j}(w)$, $j = 1, \dots, n$, $\hat{f}_t(w) = f_t(w) = 0$, and $\|\hat{f}_t\|_{G_t} \le L$.

Let $h_t(\cdot; \zeta)$ be a family of peak functions at the points $\zeta \in \partial G_t$, $t \in T$, given by Theorem 1.3 with constants $\eta_1 < \rho$, η_2, d_1, and d_2. Then, by estimates from Theorem 1.3, there exists a $\theta \in (0, 1)$ such that for any $t \in T$ and $\zeta \in \partial G_t$ we have

$$\{z \in G_t : |h_t(z; \zeta)| > \theta\} \subset G_t \cap \mathbb{B}(\zeta, \rho).$$

Pick a $k \in \mathbb{N}$ such that $\theta^k L < 1$. Take a $\beta < \rho$ with $|h_t(w; \zeta)|^k > 1 - \mu$ for each $t \in T$, $\zeta \in G_t$, and $w \in G_t \cap \mathbb{B}(\zeta, \beta)$ (again use estimates from Theorem 1.3).

Let $\widetilde{f}_t := \frac{1}{1+\mu}(h_t(\cdot; \zeta))^k \hat{f}_t$ for $t \in T$, $\zeta \in \partial G_t$. Fix $w \in G_t \cap \mathbb{B}(\zeta, \beta)$. Then $\widetilde{f}_t \in \mathcal{O}(G_t, \mathbb{D})$, $\widetilde{f}_t(w) = \hat{f}_t(w) = 0$, and

$$\frac{\partial \widetilde{f}_t}{\partial z_j}(w) = \frac{1}{1+\mu}(h_t(w; \zeta))^k \frac{\partial \hat{f}_t}{\partial z_j}(w), \quad j = 1, \dots, n.$$

Therefore, for $t \in T$, $\zeta \in \partial G_t$, $w \in G_t \cap \mathbb{B}(\zeta, \beta)$, and $X \in \mathbb{C}^n$,

$$\left|\sum_{j=1}^{n} \frac{\partial \widetilde{f}_t}{\partial z_j}(w) X_j\right| \ge \frac{1-\mu}{1+\mu} \boldsymbol{\gamma}_{G_t\cap\mathbb{B}(\zeta,R)}(w; X),$$

which yields

$$\boldsymbol{\gamma}_{G_t}(w; X) \geq \frac{1-\mu}{1+\mu} \boldsymbol{\gamma}_{G_t \cap \mathbb{B}(\zeta, R)}(w; X),$$

and this concludes the proof. □

Proposition 3.2. **(*cf.* [6, Proposition 6] and remark following it, see also [7, Theorem 19.3.2]).** *Let $(G_t)_{t \in T}$ be a family of strictly pseudoconvex domains as in Situation* 1.1. *Then there exists an $R > 0$ such that for any $t \in T$ and any $\zeta \in \partial G_t$ the set $G_t \cap \mathbb{B}(\zeta, R)$ is connected and for any $X \in \mathbb{C}^n \setminus \{0\}$ we have*

$$\lim_{G_t \cap \mathbb{B}(\zeta, R) \ni z \to \zeta} \frac{\boldsymbol{\kappa}_{G_t \cap \mathbb{B}(\zeta, R)}(z; X)}{\boldsymbol{\kappa}_{G_t}(z; X)} = 1.$$

The convergence is uniform in $\zeta \in \partial G_t$ and $X \in \mathbb{C}^n \setminus \{0\}$ in the sense that for any $\alpha > 0$ there exists a $\beta \in (0, R)$ such that for any $t \in T$, $\zeta \in \partial G_t$, $w \in G_t \cap \mathbb{B}(\zeta, \beta)$, and $X \in \mathbb{C}^n \setminus \{0\}$ there is

$$\frac{\boldsymbol{\kappa}_{G_t \cap \mathbb{B}(\zeta, R)}(w; X)}{\boldsymbol{\kappa}_{G_t}(w; X)} \leq 1 + \alpha.$$

Proof. Let R be as in Remark 1.2 and smaller than ε from Theorem 1.3. By [6, Lemma 4] (see also [7, Proposition 13.2.10]) we have

$$\boldsymbol{\kappa}_{G_t \cap \mathbb{B}(\zeta, R)}(z; X) \leq \coth \boldsymbol{k}_{G_t, \zeta, R}(z) \boldsymbol{\kappa}_{G_t}(z; X), \tag{3.1}$$

for $t \in T$, $\zeta \in \partial G_t$, $z \in G_t \cap \mathbb{B}(\zeta, R)$, and $X \in \mathbb{C}^n$, where

$$\boldsymbol{k}_{G_t, \zeta, R}(z) := \inf \left\{ \boldsymbol{k}_{G_t}(z, w) : w \in G_t \setminus \mathbb{B}(\zeta, R) \right\}.$$

Fix $\alpha > 0$ and let $h_t(\cdot; \zeta)$ be a family of peak functions at the points $\zeta \in \partial G_t$, $t \in T$ with constants $\eta_1 < R$, η_2, d_1, and d_2, given by Theorem 1.3. Let

$$\theta := \inf\{|1 - h_t(w; \zeta)| : t \in T, \zeta \in \partial G_t, w \in \overline{G_t} \setminus \mathbb{B}(\zeta; R)\}.$$

By Theorem 1.3, $\theta > 0$. Using now the estimates from Theorem 1.3 again, for $\lambda \in (0, \theta)$, sufficiently close to 0, we find a $\beta < R$ such that for any $t \in T$, $\zeta \in \partial G_t$, and $z \in G_t \cap \mathbb{B}(\zeta, \beta)$, there is $|1 - h_t(z; \zeta)| < \lambda$. Then for $t \in T$, $\zeta \in \partial G_t$, $z \in G_t \cap \mathbb{B}(\zeta, \beta)$, and $w \in G_t \setminus \mathbb{B}(\zeta, R)$ we get

$$\boldsymbol{k}_{G_t}(z, w) \geq \boldsymbol{c}_{G_t}(z, w) \geq \frac{1}{2} \log \frac{\theta}{\lambda}.$$

This, together with (3.1) and for λ sufficiently close to 0, gives

$$\boldsymbol{\kappa}_{G_t \cap \mathbb{B}(\zeta, R)}(z; X) \leq (1 + \alpha) \boldsymbol{\kappa}_{G_t}(z; X),$$

for $t \in T$, $\zeta \in \partial G_t$, $z \in G_t \cap \mathbb{B}(\zeta, \beta)$, and $X \in \mathbb{C}^n$, which finishes the proof. □

Theorem 1.7 is the main ingredient of the proof of the uniform estimates for the boundary behaviour of the Carathéodory pseudodistance, which reads as follows:

Proposition 3.3. *Let* $(G_t)_{t\in T}$ *be a family of strictly pseudoconvex domains as in Situation* 1.1. *Let* $K \subset \mathbb{C}^n$ *be a compact set such that* $K \subset G_t$ *for any* $t \in T$. *Then there exists a constant* $C > 0$ *such that for every* $t \in T$

$$\boldsymbol{c}_{G_t}(z, w) \geq -\frac{1}{2}\log \operatorname{dist}(w, \partial G_t) - C$$

whenever $z \in K$, $w \in G_t$.

Proof. This is a consequence of Theorem 1.3. Observe that the proof may be carried out along the lines of the [7, proof of Theorem 19.2.1], however, we give the argument here, for the convenience of the Reader. Let $\varepsilon > 0$ be such that for any $t \in T$ and $z \in G_t$ with $\operatorname{dist}(z, \partial G_t) < \varepsilon$ there exists exactly one point $\zeta_t(z) \in \partial G_t$ with

$$z = \zeta_t(z) - \operatorname{dist}(z, \partial G_t) \cdot \boldsymbol{n}(\zeta_t(z)),$$

where $\boldsymbol{n}(\zeta_t(z))$ is the unit outer normal vector to ∂G_t at $\zeta_t(z)$. Let $h_t(z; \zeta)$ be a family of peak functions given by Theorem 1.3 with constants η_1, η_2, d_1, and d_2, chosen so that

$$\eta_1 < \min\left\{\frac{\varepsilon}{2}, \inf\{\operatorname{dist}(K, \partial G_t) : t \in T\}\right\}.$$

Observe that in virtue of the assumptions on family $(G_t)_{t\in T}$, the latter quantity is positive.

For $z \in K$, $t \in T$, and $w \in G_t$ with $\operatorname{dist}(w, \partial G_t) < \min\left\{\eta_2, \frac{1-d_2}{d_1}\right\}$ we have

$$\begin{aligned}\tanh \boldsymbol{c}_{G_t}(z, w) &\geq \tanh \frac{|h_t(w, \zeta_t(w)) - h_t(z, \zeta_t(w))|}{|1 - \overline{h_t(z, \zeta_t(w))}h_t(w, \zeta_t(w))|}\\ &\geq \tanh \frac{1 - d_2 - d_1\|w - \zeta_t(w)\|}{1 - d_2 + d_1\|w - \zeta_t(w)\|} > 0,\end{aligned}$$

which implies

$$\boldsymbol{c}_{G_t}(z, w) = \frac{1}{2}\log\frac{1 - d_2}{d_1} - \frac{1}{2}\log \operatorname{dist}(w, \partial G_t).$$

If $\operatorname{dist}(w, \partial G_t) \geq \min\left\{\eta_2, \frac{1-d_2}{d_1}\right\}$, then

$$\boldsymbol{c}_{G_t}(z, w) \geq 0 \geq -\frac{1}{2}\log \operatorname{dist}(w, \partial G_t) - \widetilde{C}$$

where $\widetilde{C} > 0$ is independent of $t \in T$ and $z \in K$. □

Corollary 3.4. *Let $(G_t)_{t\in T}$ be a family of strictly pseudoconvex domains as in Situation* 1.1. *Let $\varepsilon > 0$. Then there exist positive constants $\rho_2 < \rho_1 < \varepsilon$ and $C > 0$ such that for every $t \in T$ and every $\zeta \in \partial G_t$ we have*

$$\boldsymbol{c}_{G_t}(z, w) \geq -\frac{1}{2} \log \operatorname{dist}(z, \partial G_t) - C$$

whenever $z \in G_t \cap \mathbb{B}(\zeta, \rho_2)$ and $w \in G_t \setminus \mathbb{B}(\zeta, \rho_1)$.

We include the next two results, as they generalize [10, Propositions 9.1 and 9.2] (see Remarks 3.6 and 3.8), which, we believe, might be of some interest. Proposition 3.5 relies on Corollary 3.4, and it gives the uniform lower estimate for the boundary behaviour of the Kobayashi distance. Proposition 3.7 deals with the uniform upper estimate for the boundary behaviour of the Kobayashi distance. It is independent of our main results.

Proposition 3.5. *Let $(G_t)_{t\in T}$ be a family of strictly pseudoconvex domains as in Situation 1.1. Let $\varepsilon > 0$. Then there exists a constant $C > 0$ such that for every $t \in T$ and every ζ, ξ, different points from ∂G_t such that $\|\zeta - \xi\| \geq \varepsilon$, we have*

$$\boldsymbol{k}_{G_t}(z, w) \geq -\frac{1}{2} \log \operatorname{dist}(z, \partial G_t) - \frac{1}{2} \log \operatorname{dist}(w, \partial G_t) - C$$

whenever $z, w \in G_t$ are such that z is close to ζ and w is close to ξ (with uniform size of the respective neighborhoods).

Proof. The proof goes similarly to the [7, proof of Proposition 19.2.7]. Only, one has to use our Corollary 3.4 instead of [7, Theorem 19.2.2]. □

Remark 3.6. Observe that [10, Proposition 9.1] can be deduced from Proposition 3.5.

Proposition 3.7. *Let $(G_t)_{t\in T}$ be a family of bounded domains with $\mathcal{C}^2$-smooth boundaries, where T is a compact metric space with associated metric d. Suppose we have a domain $U \Subset \mathbb{C}^n$ such that $\bigcup_{t\in T} \partial G_t \Subset U$ and with the property that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $s, t \in T$ with $d(s, t) \leq \delta$ there is $\|r_t - r_s\|_{\mathcal{C}^2(U)} < \varepsilon$, where r_t denotes a defining function for G_t, defined on U for any $t \in T$. Then there exists a constant $C > 0$ such that for any $t \in T$ and any $\zeta \in G_t$ there exists a neighborhood $V_{\zeta,t}$ of ζ, of uniform size, with the property that*

$$\begin{aligned}\boldsymbol{k}_{G_t}(z, w) \leq &-\frac{1}{2}(\log \operatorname{dist}(z, \partial G_t) + \log \operatorname{dist}(w, \partial G_t)) \\ &+\frac{1}{2}(\log(\operatorname{dist}(z, \partial G_t) + \|z-w\|) + \log(\operatorname{dist}(w, \partial G_t) + \|z - w\|)) + C,\end{aligned}$$

whenever $z, w \in G_t \cap V_{\zeta,t}$.

Proof. The proof follows the lines of the [7, proof of Proposition 19.2.9] with necessary modifications. Observe that R therein can be taken to be independent of $t \in T$ and $\zeta \in G_t$ (see also the proof of [10, Proposition 9.2]). Also, the final constant C, given explicitly by $\log 2 + \boldsymbol{k}_{\mathbb{B}(0,\frac{3}{5})\cup\mathbb{B}(1,\frac{3}{5})}(0,1)$ depends neither on $t \in T$ nor on $\zeta \in \partial G_t$. □

Remark 3.8. Observe that [10, Proposition 9.2] can be deduced from Proposition 3.7.

We also give the uniform upper estimate for both Carathéodory and Kobayashi distances, in the spirit of Proposition 3.3. This is independent of our main results, but we contain it here for the sake of completeness.

Proposition 3.9. *Let $(G_t)_{t\in T}$ be a family of bounded domains with $\mathcal{C}^2$-smooth boundaries, where T is a compact metric space with associated metric d. Suppose we have a domain $U \Subset \mathbb{C}^n$ such that $\bigcup_{t\in T} \partial G_t \Subset U$ and with the property that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $s, t \in T$ with $d(s,t) \le \delta$ there is $\|r_t - r_s\|_{\mathcal{C}^2(U)} < \varepsilon$, where r_t denotes a defining function for G_t, defined on U for any $t \in T$. Let $K \subset \mathbb{C}^n$ be a compact set such that $K \subset G_t$ for any $t \in T$. Then there exists a constant $C > 0$, such that for any $t \in T$ we have*

$$\boldsymbol{k}_{G_t}(z,w) \le -\frac{1}{2}\log \operatorname{dist}(w, \partial G_t) + C,$$

whenever $z \in K$ and $w \in G_t$.

Remark 3.10. In comparison with Situation 1.1, here we do not assume the strict pseudoconvexity of the domains, only the boundary regularity.

Proof. The proof goes along the lines of the proof of [7, Proposition 19.2.4]. We only discuss the necessary modifications.

Observe that ε_0 as in the proof of the mentioned result may be taken to be independent of $t \in T$.

For fixed $t \in T$ let $\delta > 0$ be such that for any $s \in T$ with $d(s,t) \le \delta$ the set

$$K_t := \overline{\bigcup_{s\in T: d(s,t)\le\delta} \{z \in G_s : \operatorname{dist}(z, \partial G_s) \ge \varepsilon_0\}}$$

is compact in G_s and, moreover, $\operatorname{dist}(K_t, \partial G_s) \ge \frac{\varepsilon_0}{2}$. Let $G_{t,\delta}$ be a bounded domain with $\mathcal{C}^2$-smooth boundary such that

$$K \cup K_t \Subset G_{t,\delta} \subset \bigcap_{s\in T: d(s,t)\le\delta} G_s.$$

For s as above and $z \in K$, $w \in G_s$, with $\operatorname{dist}(w, \partial G_s) \le \varepsilon_0$, using the same argument as in [7], we get the estimate

$$\boldsymbol{k}_{G_s}(z,w) \le -\frac{1}{2}\log \operatorname{dist}(w, \partial G_s) + \frac{1}{2}\log(2\varepsilon_0) + C_{t,\delta}$$

with $C_{t,\delta} := \sup\{\boldsymbol{k}_{G_{t,\delta}}(a,b) : a, b \in K \cup K_t\}$. By the compactness of T, the latter constant may be chosen independently of t. We end the proof as in [7]. □

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