

A fresh look at the notion of normality

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Abstract. Let G be a countably infinite cancellative amenable semigroup and let (F_n) be a (left) Følner sequence in G . We introduce the notion of an (F_n) -normal set in G and an (F_n) -normal element of $\{0, 1\}^G$. When $G = (\mathbb{N}, +)$ and $F_n = \{1, 2, \dots, n\}$, the (F_n) -normality coincides with the classical notion. We prove several results about (F_n) -normality, for example:

- If (F_n) is a Følner sequence in G , such that for every $\alpha \in (0, 1)$ we have $\sum_n \alpha^{|F_n|} < \infty$, then almost every (in the sense of the uniform product measure $(\frac{1}{2}, \frac{1}{2})^G$) $x \in \{0, 1\}^G$ is (F_n) -normal.
- For any Følner sequence (F_n) in G , there exists an effectively defined Champernowne-like (F_n) -normal set.
- There is a rather natural and sufficiently wide class of Følner sequences (F_n) in (\mathbb{N}, \times) , which we call “nice”, for which the Champernowne-like construction can be done in an algorithmic way. Moreover, there exists a Champernowne-like set which is (F_n) -normal for every nice Følner sequence (F_n) .

We also investigate and juxtapose combinatorial and Diophantine properties of normal sets in semigroups $(\mathbb{N}, +)$ and (\mathbb{N}, \times) . Below is a sample of results that we obtain:

- Let $A \subset \mathbb{N}$ be a classical normal set. Then, for any Følner sequence (K_n) in (\mathbb{N}, \times) there exists a set E of (K_n) -density 1, such that for any finite subset $\{n_1, n_2, \dots, n_k\} \subset E$, the intersection $A/n_1 \cap A/n_2 \cap \dots \cap A/n_k$ has positive upper density in $(\mathbb{N}, +)$. As a consequence, A contains arbitrarily long geometric progressions, and, more generally, arbitrarily long “geo-arithmetic” configurations of the form $\{a(b + ic)^j, 0 \leq i, j \leq k\}$.
- For any Følner sequence (F_n) in $(\mathbb{N}, +)$ there exist uncountably many (F_n) -normal Liouville numbers.
- For any nice Følner sequence (F_n) in (\mathbb{N}, \times) there exist uncountably many (F_n) -normal Liouville numbers.

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1. Introduction

It follows from the classical law of large numbers [12]¹ that given a fair coin whose sides are labeled 0 and 1, the infinite binary sequence (x_n) , obtained by independent tossing of the coin, is almost surely *normal*, meaning that, for any $k \in \mathbb{N} = \{1, 2, \dots\}$, any 0-1 word of length k , $w = \langle w_1, w_2, \dots, w_k \rangle \in \{0, 1\}^k$, appears in (x_n) with frequency 2^{-k} . This provides a proof of existence of normal sequences (note that *a priori* it is not even clear whether normal sequences exist!). There are also numerous explicit constructions of normal sequences (see for instance [16, 19, 38]). For example, the *Champernowne sequence* 1 10 11 100 101 110 \dots , which is formed by the sequence 1, 2, 3, 4, 5, 6, \dots written in base 2, is a normal sequence.

Any 0-1 sequence $(x_n) \in \{0, 1\}^{\mathbb{N}}$ may be viewed as the sequence of digits in the binary expansion of the real number $x = \sum_{n=1}^{\infty} x_n 2^{-n} \in [0, 1]$, which leads to an equivalent formulation of the above fact: almost every $x \in [0, 1]$ is normal in base 2. Similarly, due to the natural bijection between 0-1 sequences and subsets of \mathbb{N} (any subset of \mathbb{N} is identified with its indicator function which is a 0-1 sequence), one can talk about *normal sets* in \mathbb{N} (more accurately, in $(\mathbb{N}, +)$; see the discussion below).

The peculiar combinatorial and Diophantine properties of normal sequences/sets/numbers, together with the fact that they are “typical” (in the sense of measure), make them a natural object of interest and a source of various generalizations, see [2, 20, 25, 45].

The classical definition of normality of a 0-1 sequence $x = (x_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ is formulated as follows.²

Definition 1.1. For $n, k \in \mathbb{N}$ ($k \leq n$), and a 0-1 word $w \in \{0, 1\}^k$, we let $\mathbf{N}(w, x, n)$ be the number of times the word w occurs as a subword of the word $\langle x_1, x_2, \dots, x_n \rangle \in \{0, 1\}^n$:

$$\mathbf{N}(w, x, n) = \left| \left\{ m \in \{1, \dots, n - k + 1\} : \langle x_m, x_{m+1}, \dots, x_{m+k-1} \rangle = w \right\} \right|$$

(here $|\cdot|$ denotes the cardinality of a set). A sequence $x \in \{0, 1\}^{\mathbb{N}}$ is *normal* if for every $k \in \mathbb{N}$ and every $w \in \{0, 1\}^k$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{N}(w, x, n) = 2^{-k}. \quad (1.1)$$

One may ask a naive but in some sense natural question whether replacing the sequence of “averaging intervals” $\{1, 2, \dots, n\}$ (which are implicit in the above definition because one can write $\langle x_1, x_2, \dots, x_n \rangle = x|_{\{1, 2, \dots, n\}}$) by a more general

¹ See Appendix for historical notes.

² In [12], this formulation appears not as the definition but as a “characterization” of normality, see the Appendix for more details.

sequence (F_n) of (a priori arbitrary) finite subsets of \mathbb{N} leads to a meaningful generalization of the notion of normality. More precisely, one would like to count the number of times the word w occurs as a subword of $x|_{F_n}$:

$$\begin{aligned} & \mathbf{N}(w, x, F_n) \\ &= \left| \left\{ m \in \mathbb{N} : \{m, m+1, \dots, m+k-1\} \subset F_n \text{ and } \langle x_m, x_{m+1}, \dots, x_{m+k-1} \rangle = w \right\} \right| \end{aligned}$$

and call a sequence $x \in \{0, 1\}^{\mathbb{N}}$ (F_n) -normal if, for every $k \in \mathbb{N}$ and any $w \in \{0, 1\}^k$, one has

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \mathbf{N}(w, x, F_n) = 2^{-k}. \quad (1.2)$$

It turns out that in order for the above notion of (F_n) -normality to be nonvoid, the sequence of sets (F_n) has to be a *Følner sequence*, i.e., satisfy the so-called *Følner condition*:

$$\forall k \in \mathbb{N} \quad \lim_{n \rightarrow \infty} \frac{|F_n \cap (F_n - k)|}{|F_n|} = 1 \quad (1.3)$$

(in particular, it must hold that $|F_n| \rightarrow \infty$). As a matter of fact, the Følner condition is implied by a rather mild requirement that there exists an $x \in \{0, 1\}^{\mathbb{N}}$ such that, for each $k \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{w \in \{0, 1\}^k} \mathbf{N}(w, x, F_n) = 1. \quad (1.4)$$

The proof will be given later (see Theorem 2.10 below) in a more general context.

Our next observation is that if $x \in \{0, 1\}^{\mathbb{N}}$ is (F_n) -normal then not only words, but in fact all *0-1 blocks*, occur in x with “correct frequencies”, by which we mean the following. Let K be a nonempty finite subset of \mathbb{N} . Any element (function) $B \in \{0, 1\}^K$ will be called a *block*. We will say that a *shift of a block* $B \in \{0, 1\}^K$ *occurs* in the block $x|_{F_n} \in \{0, 1\}^{F_n}$ at a position $m \in \mathbb{N} \cup \{0\}$ if

$$(\forall i \in K) \quad i + m \in F_n \text{ and } x_{i+m} = B(i).$$

We let $\mathbf{N}(B, x, F_n)$ be the number of shifts of the block B occurring in $x|_{F_n}$, i.e.,

$$\mathbf{N}(B, x, F_n) = |\{m \in \mathbb{N} \cup \{0\} : (\forall i \in K) \quad i + m \in F_n \text{ and } x_{i+m} = B(i)\}|.$$

Then (F_n) -normality of x implies

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \mathbf{N}(B, x, F_n) = 2^{-|K|}, \quad (1.5)$$

for any nonempty finite set K and every block $B \in \{0, 1\}^K$. We will prove this implication in Section 2 using the language of dynamics (see Lemma 2.11).

Once we are driven into considering Følner sequences of the “averaging sets”, the natural context for continuing our discussion of normality becomes that of

countably infinite amenable cancellative semigroups³ G (it is known that such semigroups admit (left) Følner sequences see [39, Theorem 3.5 and Corollary 4.3], see also Definition 2.1). In order to adapt the definition of (F_n) -normality to this context, we pick a Følner sequence (F_n) in G , fix a 0-1-valued function $x = (x_g)_{g \in G} \in \{0, 1\}^G$, and, for each finite set $K \subset G$ and a block $B \in \{0, 1\}^K$, denote

$$\mathbf{N}(B, x, F_n) = \{g \in G \cup \{e\} : (\forall h \in K) \, hg \in F_n \text{ and } x_{hg} = B(h)\}, \quad (1.6)$$

where e is the formal identity element added to G in case G lacks an identity. We will say that x is (F_n) -normal if for any nonempty finite $K \subset G$ and every $B \in \{0, 1\}^K$, one has (as in the case of $(\mathbb{N}, +)$),

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \mathbf{N}(B, x, F_n) = 2^{-|K|}. \quad (1.7)$$

Let us now examine closer the dynamical underpinnings of the notion of normality. Let G be a countably infinite amenable cancellative semigroup. The semigroup G acts naturally on the symbolic space $\{0, 1\}^G$ by shifts, as follows: for $g \in G$ and $x = (x_h)_{h \in G}$, $\sigma_g(x) = (x_{hg})_{h \in G}$.⁴

For any nonempty finite set $K \subset G$, each block $B \in \{0, 1\}^K$ determines a *cylinder*

$$[B] = \{x \in \{0, 1\}^G : x|_K = B\}.$$

As we will explain later (see Theorem 2.8), if (F_n) is a Følner sequence then (F_n) -normality can be expressed in terms of the shift action and cylinder sets in the following way:

- An element $x \in \{0, 1\}^G$ is (F_n) -normal if and only if for every nonempty finite set K and every block $B \in \{0, 1\}^K$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} |\{g \in F_n : \sigma_g(x) \in [B]\}| = 2^{-|K|}.$$

When dealing with a general amenable semigroup G and a Følner sequence (F_n) , it is not a priori obvious whether (F_n) -normal elements $x \in \{0, 1\}^G$ exist. We solve this problem in the affirmative by showing, in Theorem 4.2 below, that for any countably infinite cancellative amenable semigroup G and any Følner sequence (F_n) in G , with $|F_n|$ strictly increasing, λ -almost every $x \in \{0, 1\}^G$ is (F_n) -normal, where λ is the uniform product measure $(\frac{1}{2}, \frac{1}{2})^G$ on $\{0, 1\}^G$. In an equivalent form (see Theorem 4.4), our result can be interpreted as a sort of pointwise ergodic theorem for Bernoulli shifts. Namely, for any Følner sequence (F_n) with $|F_n|$ strictly

³ A semigroup G is (two-sided) *cancellative* if, for any $a, b, c \in G$, $ab = ac \implies b = c$ and $ba = ca \implies b = c$.

⁴ In the classical case $G = (\mathbb{N}, +)$, the action is given by $\sigma^m(x) = (x_{n+m})_{n \in \mathbb{N}}$ (where $m \in \mathbb{N}$ and $x = (x_n)_{n \in \mathbb{N}}$).

increasing⁵, any *continuous* function f on $\{0, 1\}^G$ and λ -almost every $x \in \{0, 1\}^G$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(\sigma_g x) = \int f d\lambda.$$

We emphasize that the pointwise ergodic theorem for general actions of amenable groups (and measurable functions) holds only for *tempered* Følner sequences which satisfy the so-called Shulman's condition:

$$\left| \bigcup_{i=1}^n F_i^{-1} F_{n+1} \right| \leq C |F_{n+1}| \quad (1.8)$$

(see [37, page 83], see also [1] for the necessity of Shulman's condition).

The set \mathbb{N} of natural numbers has two natural semigroup operations: addition and multiplication. This leads to two parallel notions of normality of subsets of \mathbb{N} , which will be referred to as *additive* and *multiplicative* normality, respectively. The possibility of juxtaposing the Diophantine and combinatorial properties of additively and multiplicatively normal subsets of \mathbb{N} served as the initial motivation for this paper.

The notion of (F_n) -normality in $(\mathbb{N}, +)$ and (\mathbb{N}, \times) allows one to reconsider, from the more general point of view, the classical results dealing with the existence of normal Liouville numbers⁶ (see [14]⁷). While it is true that the set of Liouville numbers is residual (*i.e.*, topologically large, see [31, Theorem 5]), the set of (F_n) -normal numbers is, as we will show in subsection 4.2, of the first category (*i.e.*, topologically small). This holds for any Følner sequence (F_n) in either $(\mathbb{N}, +)$ or (\mathbb{N}, \times) . As for the largeness in the sense of measure, the situation is reversed: as we have already mentioned, the set of (F_n) -normal numbers is (for any Følner sequence (F_n) , in either $(\mathbb{N}, +)$ or (\mathbb{N}, \times) , with $|F_n|$ strictly increasing) of full Lebesgue measure, while it is well known that the set of Liouville numbers has Lebesgue measure zero (see for example [43]). So, using just the criteria of topological or measure-theoretic largeness it is impossible to decide whether the sets of Liouville numbers and of (F_n) -normal numbers have nonempty intersection.

Below is a brief description of results obtained in this paper.

- Section 2 is devoted to reviewing or establishing basic facts about amenable groups and semigroups, which are needed in the sequel. In particular we prove an auxiliary theorem which shows that in many situations one can deal, without loss of generality, with amenable groups rather than semigroups.

⁵ Actually, our assumption in Theorems 4.2 and 4.4 on the Følner sequence (F_n) is even weaker: for any $\alpha \in (0, 1)$, $\sum_{n \in \mathbb{N}} \alpha^{|F_n|} < \infty$.

⁶ Let us recall that an irrational number x is called a Liouville number if for every natural k there exists a rational number $\frac{p}{q}$ such that $|x - \frac{p}{q}| < \frac{1}{q^k}$.

⁷ In fact, in [14] Bugeaud proves the existence of *absolutely normal* (*i.e.*, classical normal with respect to any base) Liouville numbers. We are interested in (F_n) -normality in base 2, but for a general Følner sequence (F_n) in $(\mathbb{N}, +)$, as well as in (\mathbb{N}, \times) .

• In Section 3 we establish left invariance of the class of (F_n) -normal sets in countably infinite amenable cancellative semigroups.

• Section 4 contains our “ergodic theorem for Bernoulli shifts and continuous functions” which says that the set $\mathcal{N}((F_n))$ of (F_n) -normal elements of $\{0, 1\}^G$ is large in the sense of measure. By the way of contrast, we also prove that the set $\mathcal{N}((F_n))$ is small in the sense of topology (is of first category).

• In Section 5 we give a general Champernowne-like construction of an (F_n) -normal element $x \in \{0, 1\}^G$ for any countably infinite amenable cancellative semigroup G and any Følner sequence (F_n) in G .

• Section 6 focuses on the notion of normality in the semigroup (\mathbb{N}, \times) of multiplicative positive integers. We introduce a natural class of Følner sequences which we call “nice”. For any nice Følner sequence (F_n) we construct a Champernowne-like (F_n) -normal element $x \in \{0, 1\}^{\mathbb{N}}$. Due to monotileability of the semigroup (\mathbb{N}, \times) and properties of a nice Følner sequence, the construction resembles that of the classical Champernowne number and is much more transparent than the one described in the preceding section.

We also study the class of elements $x \in \{0, 1\}^{\mathbb{N}}$ which are normal with respect to all nice Følner sequences in (\mathbb{N}, \times) . We call these elements *net-normal*. We prove that the set of net-normal elements has measure zero but is nonempty (to this end we use a modification of the Champernowne-like construction from the preceding section).

• Section 7 is devoted to the study of combinatorial and Diophantine properties of additively and multiplicatively normal subsets of \mathbb{N} . In particular, we prove the following results:

- Let (F_n) be a Følner sequence in $(\mathbb{N}, +)$. Then any (F_n) -normal set S contains solutions of any partition-regular system of linear equations.⁸
- Let (F_n) be a Følner sequence in (\mathbb{N}, \times) . Then any (F_n) -normal set S contains solutions of any homogeneous system of polynomial equations which has solutions in \mathbb{N} .
- Let S be any classical normal set in $(\mathbb{N}, +)$. Then
 - (i) S contains solutions a, b, c of any equation $ia + jb = kc$, where i, j, k are arbitrary positive integers,
 - (ii) S contains pairs $\{n + m, nm\}$ with arbitrary large n, m ,
 - (iii) S contains arbitrarily long geometric progressions, and, more generally, arbitrarily long “geo-arithmetic” configurations of the form $\{a(b + ic)^j, 0 \leq i, j \leq k\}$.

• In Section 8 we show that for any Følner sequence (F_n) in $(\mathbb{N}, +)$ there exists an (F_n) -normal Liouville number (actually, we construct a Cantor set of such

⁸ A system of equations is called *partition-regular* if for any finite coloring of \mathbb{N} there exists a monochromatic solution.

numbers) and, likewise, for any nice Følner sequence (F_n) in (\mathbb{N}, \times) there exists an (F_n) -normal Liouville number (and indeed a Cantor set of (F_n) -normal Liouville numbers).

2. Preliminaries

We now present some background material concerning properties of Følner sequences in countably infinite amenable cancellative semigroups, and tilings in countably infinite amenable groups.

Let G be a cancellative semigroup. Recall that given $g \in G$ and a finite subset $F \subset G$, $g^{-1}F$ stands for $\{h \in G : gh \in F\}$.

Definition 2.1. A sequence (F_n) of finite subsets of G is a *Følner sequence* if it satisfies the *Følner condition*:

$$\forall g \in G \quad \lim_{n \rightarrow \infty} \frac{|F_n \cap g^{-1}F_n|}{|F_n|} = 1.$$

We have $g^{-1}F \cap F = \{f \in F : gf \in F\}$. By cancellativity, $f \in g^{-1}F \cap F \iff gf \in gF \cap F$, and thus

$$|g^{-1}F \cap F| = |gF \cap F|. \quad (2.1)$$

It follows that the Følner condition is equivalent to

$$\forall g \in G \quad \lim_{n \rightarrow \infty} \frac{|gF_n \cap F_n|}{|F_n|} = 1.$$

Another useful equivalent form of the Følner condition utilizes the notion of a (K, ε) -invariant set.

Definition 2.2. Given a nonempty finite set $K \subset G$ and $\varepsilon > 0$ we will say that a finite set $F \subset G$ is (K, ε) -invariant if

$$\frac{|KF \Delta F|}{|F|} \leq \varepsilon$$

(Δ stands for the symmetric difference of sets).

It is not hard to see that a sequence of finite sets (F_n) is Følner if and only if for any nonempty finite $K \subset G$ and $\varepsilon > 0$, the sets F_n are eventually (K, ε) -invariant.

We remark that a general Følner sequence need not be increasing with respect to inclusion (in particular, it can consist of disjoint sets), the cardinalities $|F_n|$ need not increase (but, of course $|F_n| \rightarrow \infty$), and the union $\bigcup_{n \geq 1} F_n$ need not equal the whole semigroup.

Given a Følner sequence (F_n) in G and a set $V \subset G$, one defines the *upper and lower (F_n) -densities* of V by the formulas

$$\begin{aligned}\overline{d}_{(F_n)}(V) &= \limsup_{n \rightarrow \infty} \frac{|F_n \cap V|}{|F_n|}, \\ \underline{d}_{(F_n)}(V) &= \liminf_{n \rightarrow \infty} \frac{|F_n \cap V|}{|F_n|}.\end{aligned}$$

If $\overline{d}_{(F_n)}(V) = \underline{d}_{(F_n)}(V)$, then we denote the common value by $d_{(F_n)}(V)$ and call it the (F_n) -density of V . The Følner property of (F_n) and cancellativity immediately imply that for any $V \subset G$ and any $g \in G$,

$$\overline{d}_{(F_n)}(V) = \overline{d}_{(F_n)}(gV) = \overline{d}_{(F_n)}(g^{-1}V) \quad (2.2)$$

(analogous equalities hold for $\underline{d}_{(F_n)}(\cdot)$ and $d_{(F_n)}(\cdot)$).

Definition 2.3. Let K and F be nonempty finite subsets of G .

- (1) The K -core of F is the set $F_K = \{h \in G : Kh \subset F\} = \bigcap_{g \in K} g^{-1}F$.
- (2) K is called an ε -modification of F if $\frac{|K \Delta F|}{|F|} \leq \varepsilon$, ($\varepsilon > 0$).

The following elementary lemma is a slightly more general form of Lemma 2.6 in [23]. We include the proof for the reader's convenience.

Lemma 2.4. For any $\varepsilon > 0$ and any nonempty finite subset K of an amenable cancellative semigroup G , there exists $\delta > 0$ (in fact $\delta = \frac{\varepsilon}{2|K|}$), such that if $F \subset G$ is finite and (K, δ) -invariant then the K -core of F is an ε -modification of F .

Proof. Note that (K, δ) -invariance of F implies that

$$(\forall g \in K) \quad |gF \setminus F| \leq \delta|F|,$$

i.e., using (2.1),

$$(\forall g \in K) \quad |g^{-1}F \cap F| = |gF \cap F| \geq (1 - \delta)|F|,$$

in particular, $|g^{-1}F \setminus F| \leq \delta|F|$. Using the above, we get

$$|F_K \cap F| = \left| \bigcap_{g \in K} (g^{-1}F \cap F) \right| \geq (1 - |K|\delta)|F|,$$

while $|F_K \cup F| \leq \sum_{g \in K} |g^{-1}F \setminus F| + |F| \leq (1 + \delta|K|)|F|$. Combining the two estimates above, we obtain $|F_K \Delta F| = |F_K \cup F| - |F_K \cap F| \leq 2\delta|K||F| = \varepsilon|F|$. \square

Definition 2.5. We will say that two Følner sequences (F_n) and (F'_n) in an amenable semigroup G are *equivalent* if $\frac{|F'_n \Delta F_n|}{|F_n|} \rightarrow 0$ (equivalently, $\frac{|F'_n \Delta F_n|}{|F'_n|} \rightarrow 0$).

Note that if (F_n) is a Følner sequence in G and, for each n , F'_n is an ε_n -modification of F_n , where $\varepsilon_n \rightarrow 0$, then (F'_n) is a Følner sequence equivalent to (F_n) .

Remark 2.6. It is not hard to see that if (F_n) and (F'_n) are equivalent Følner sequences then:

- (i) the notions of (upper/lower) (F_n) -density and (F'_n) -density coincide;
- (ii) the notions of (F_n) -normality and (F'_n) -normality coincide.

Invoking Lemma 2.4, we obtain the following lemma.

Lemma 2.7. *If (F_n) is a Følner sequence in a countably infinite cancellative semigroup G and $K \subset G$ is nonempty finite then the sequence $(F_{n,K})$ (of the K -cores of F_n), is a Følner sequence equivalent to (F_n) .*

In particular, for any $g \in G$, $(g^{-1}F_n)$ is a Følner sequence equivalent to (F_n) . By (2.1), the same holds for (gF_n) .⁹

We can now rephrase slightly the definition of (F_n) -normality using a “dynamical” modification $\tilde{N}(B, x, F_n)$ of the quantity $N(B, x, F_n)$ introduced in (1.6). For two nonempty finite sets $F, K \subset G$, a 0-1-valued function $x \in \{0, 1\}^G$, and a block $B \in \{0, 1\}^K$ let us denote by $\tilde{N}(B, x, F)$ the number of visits of the orbit of x to the cylinder $[B]$ at “times” belonging to F :

$$\tilde{N}(B, x, F) = |\{g \in F : \sigma_g(x) \in [B]\}|.$$

For comparison, as easily verified, $N(B, x, F)$ (see (1.6)) counts the visits of the orbit of x in $[B]$ at “times” belonging to the K -core of F in the extended semigroup $G \cup \{e\}$:

$$N(B, x, F) = |\{g \in F_K^o : \sigma_g(x) \in [B]\}|,$$

where

$$F_K^o = \{g \in G \cup \{e\} : Kg \subset F\}.$$

(Clearly, if G has an identity element then $F_K^o = F_K$. In any case, $F_K \subset F_K^o \subset F_K \cup \{e\}$). The difference between $\tilde{N}(B, x, F)$ and $N(B, x, F)$ is best seen in the classical case of $(\mathbb{N}, +)$. Let $F = \{1, 2, \dots, n\}$ and let B be a word w of length k . Then we have

$$\begin{aligned} \tilde{N}(w, x, F) &= |\{m \in \{1, \dots, n\} : \langle x_m, x_{m+1}, \dots, x_{m+k-1} \rangle = w\}|, \\ N(w, x, F) &= |\{m \in \{1, \dots, n-k+1\} : \langle x_m, x_{m+1}, \dots, x_{m+k-1} \rangle = w\}|. \end{aligned}$$

Here is now a reformulation of the definition of normality in terms of $\tilde{N}(B, x, F_n)$.

⁹ It is easy to see that if (F_n) is a Følner sequence in a countably infinite cancellative semigroup G and $g \in G$ then the sequence $(F_n g)$ satisfies the Følner condition. However, unless G is commutative, $(F_n g)$ need not be equivalent to (F_n) .

Theorem 2.8. *Let G be a countably infinite amenable cancellative semigroup and let (F_n) be a Følner sequence in G . An element $x \in \{0, 1\}^G$ is (F_n) -normal if and only if, for any nonempty finite $K \subset G$ and each $B \in \{0, 1\}^K$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \tilde{\mathbf{N}}(B, x, F_n) = 2^{-|K|}. \quad (2.3)$$

Proof. Notice that $\mathbf{N}(B, x, F_n)$ equals either $\tilde{\mathbf{N}}(B, x, F_{n,K})$ (if G contains an identity element) or at most $\tilde{\mathbf{N}}(B, x, F_{n,K}) + 1$ otherwise. Now apply Remark 2.6 (ii) and Lemma 2.7 above. \square

One of two main advantages of using the function $B \mapsto \tilde{\mathbf{N}}(B, x, F)$ over $B \mapsto \mathbf{N}(B, x, F)$ is its finite additivity on cylinders (even when the domains of the blocks involved in the summation do not necessarily coincide)¹⁰.

Lemma 2.9. *Let K_0, K_1, \dots, K_r be nonempty finite subsets of a countably infinite semigroup G and let $B_i \in \{0, 1\}^{K_i}$, $i = 0, 1, \dots, r$, be blocks such that $[B_0] = [B_1] \cup [B_2] \cup \dots \cup [B_r]$ is a disjoint union. Let $F \subset G$ be finite and let $x \in \{0, 1\}^G$. Then*

$$\tilde{\mathbf{N}}(B_0, x, F) = \sum_{i=1}^r \tilde{\mathbf{N}}(B_i, x, F).$$

Proof. Recall that for $i = 0, 1, 2, \dots, r$ we have $\tilde{\mathbf{N}}(B_i, x, F) = |\{g \in F : \sigma_g(x) \in [B_i]\}|$, and notice that by the assumption

$$\{g \in F : \sigma_g(x) \in [B_0]\} = \bigcup_{i=1}^r \{g \in F : \sigma_g(x) \in [B_i]\}$$

is a disjoint union. Since cardinality is a finitely additive function, we are done. \square

Another advantage of working with $\tilde{\mathbf{N}}(B, x, F)$ (rather than $\mathbf{N}(B, x, F)$) is that it can be represented as an *ergodic sum* of the indicator function of $[B]$:

$$\tilde{\mathbf{N}}(B, x, F) = \sum_{g \in F_n} 1_{[B]}(\sigma_g(x)). \quad (2.4)$$

We will be referring to this interpretation later.

We shall now fulfill the promise made in the introduction and prove that the rather mild condition (1.4) forces the sequence (F_n) to be Følner.

¹⁰ To see that the function $B \mapsto \mathbf{N}(B, x, F)$ is not finitely additive, recall that a word is an element of $\{0, 1\}^{\{1, 2, \dots, k\}}$, where k is its length, and consider the words $u = \langle 0 \rangle$, $v = \langle 0, 1 \rangle$, $w = \langle 0, 0, 0 \rangle$, $y = \langle 0, 0, 1 \rangle$. Notice that $[u] = [v] \cup [w] \cup [y]$ is a disjoint union of cylinders corresponding to words of lengths 2 and 3. Let $x = (0, 0, 0, 0, 0, \dots) \in \{0, 1\}^{\mathbb{N}}$ and consider the set $F = \{1, 2, 3\}$. We have $\mathbf{N}(u, x, F) = 3$, $\mathbf{N}(v, x, F) = 0$, $\mathbf{N}(w, x, F) = 1$, $\mathbf{N}(y, x, F) = 0$, and $\mathbf{N}(u, x, F) \neq \mathbf{N}(v, x, F) + \mathbf{N}(w, x, F) + \mathbf{N}(y, x, F)$.

Theorem 2.10. *Let (F_n) be a sequence of arbitrary finite subsets of a countably infinite semigroup G . Suppose that there exists a 0-1-valued function $x \in \{0, 1\}^G$ such that for every two-element set $K \subset G$ we have*

$$\lim_n \frac{1}{|F_n|} \sum_{B \in \{0,1\}^K} \mathbf{N}(B, x, F_n) = 1 \quad (2.5)$$

(this assumption weakens and generalizes (1.4)). Then (F_n) is a Følner sequence in G .

Proof. Let $x \in \{0, 1\}^G$ satisfy (2.5) for every nonempty finite $K \subset G$. Notice that, for any nonempty finite sets K and F , we have

$$\sum_{B \in \{0,1\}^K} \mathbf{N}(B, x, F) = |F_K|.$$

Now, let $g \in G$ be arbitrary. We fix some $h \in G$ and we let $K = \{h, gh\}$. By assumption, given any $\varepsilon > 0$, for n large enough we have $|F_{n,K}| \geq (1 - \varepsilon)|F_n|$. For our particular K , we have $F_{n,K} = h^{-1}F_n \cap h^{-1}g^{-1}F_n$, so, we obtain

$$(1 - \varepsilon)|F_n| \leq |h^{-1}F_n \cap h^{-1}g^{-1}F_n| = |h^{-1}(F_n \cap g^{-1}F_n)| \leq |F_n \cap g^{-1}F_n|.$$

This implies the Følner condition $\frac{|F_n \cap g^{-1}F_n|}{|F_n|} \rightarrow 1$. □

We remark that, without assuming the Følner condition, Theorem 2.8 may fail very badly, i.e., the notion of (F_n) -normality via the sets $\mathbf{N}(B, x, F_n)$ may differ drastically from the notion involving the sets $\tilde{\mathbf{N}}(B, x, F_n)$. By Theorem 2.10, if (F_n) is not Følner, the set of (F_n) -normal elements $x \in \{0, 1\}^G$ (i.e., elements satisfying (1.7)) is empty. On the other hand, taking for example $F_n = \{2, 4, \dots, 2n\}$ in $(\mathbb{N}, +)$ we see that even though (F_n) is not a Følner sequence, almost every element $x \in \{0, 1\}^{\mathbb{N}}$ satisfies, for every word w , the condition (2.3): $\lim_n \frac{1}{|F_n|} \tilde{\mathbf{N}}(w, x, F_n) = 2^{-|w|}$.

The following lemma shows that the formulas (1.2) and (1.5) in the Introduction lead to the same notion of (F_n) -normality in $(\mathbb{N}, +)$.

Lemma 2.11. *Let (F_n) be a Følner sequence in $(\mathbb{N}, +)$. If $x \in \{0, 1\}^{\mathbb{N}}$ is (F_n) -normal, i.e., satisfies (1.2) (for words) then it satisfies (1.5) (for blocks), i.e., for every nonempty finite $K \subset \mathbb{N}$ and every block $B \in \{0, 1\}^K$, we have*

$$\lim_n \frac{1}{|F_n|} \mathbf{N}(B, x, F_n) = 2^{-|K|}.$$

Proof. By Theorem 2.8, in (1.2), we can replace $\mathbf{N}(w, x, F_n)$ by $\tilde{\mathbf{N}}(w, x, F_n)$, and in (1.5) we can replace $\mathbf{N}(B, x, F_n)$ by $\tilde{\mathbf{N}}(B, x, F_n)$. If $B \in \{0, 1\}^K$ then, letting I

be the shortest interval in \mathbb{N} of the form $\{1, 2, \dots, r\}$, $r \in \mathbb{N}$, which contains K , we have the disjoint union representation of the cylinder $[B]$:

$$[B] = \bigcup_{w \in \{0,1\}^I, w|_K=B} [w].$$

By Lemma 2.9, the function $[B] \mapsto \lim_n \frac{1}{|F_n|} \tilde{\mathbf{N}}(B, x, F_n)$ is finitely additive on cylinders for which the limits exist. By (1.2), for each $w \in \{0, 1\}^I$, we have

$$\lim_n \frac{1}{|F_n|} \tilde{\mathbf{N}}(w, x, F_n) = \lim_n \frac{1}{|F_n|} \mathbf{N}(w, x, F_n) = 2^{-|I|}.$$

Thus,

$$\begin{aligned} & \lim_n \frac{1}{|F_n|} \mathbf{N}(B, x, F_n) \\ &= \lim_n \frac{1}{|F_n|} \tilde{\mathbf{N}}(B, x, F_n) \\ &= \sum_{w \in \{0,1\}^I, w|_K=B} \lim_n \frac{1}{|F_n|} \tilde{\mathbf{N}}(w, x, F_n) = 2^{|I|-|K|} \cdot 2^{-|I|} = 2^{-|K|}. \quad \square \end{aligned}$$

The next theorem will allow us to reduce the proofs of some results pertaining to countably infinite amenable cancellative semigroups to the setup of countably infinite amenable groups (in particular, we will be able to use the machinery of tilings). The fact given below appears independently in [21, Corollary 2.12].

Theorem 2.12. *Let G be a countably infinite amenable cancellative semigroup. Then there exists an amenable group \tilde{G} containing G as a subsemigroup, such that any Følner sequence (F_n) in G is a Følner sequence in \tilde{G} .*

Proof. First of all, any amenable cancellative semigroup is embeddable in a group H (see [44]). Then each element $g \in G$ has an inverse $g^{-1} \in H$. Let $\tilde{G} \subset H$ be the set of all finite products $g_1 g_2^{-1} \cdots g_{2k-1} g_{2k}^{-1}$, $k \in \mathbb{N}$, where all the terms g_i belong to $G \cup \{e\}$ (e is the identity element of H and must be added only in case G does not have an identity element). Clearly, \tilde{G} is a subgroup of H and it contains G (alternatively, \tilde{G} can be defined as the smallest subgroup of H containing G). Let (F_n) be a Følner sequence in G . Fix an element $\tilde{g} \in G$ and write it as a product $g_1 g_2^{-1} \cdots g_{2k-1} g_{2k}^{-1}$. Also fix an $\varepsilon > 0$ and denote $\varepsilon' = \frac{\varepsilon}{2k}$. For large n , the set F_n is (g_i, ε') -invariant for each $i = 1, 2, \dots, 2k$. Then

$$|F_n \triangle g_i F_n| = |F_n \triangle g_i^{-1} F_n| \leq \varepsilon' |F_n|$$

(the set $g_i^{-1} F_n$ is understood in \tilde{G}). Multiplying both sets in $F_n \triangle g_i^{-1} F_n$ by g_j on the left (for some $j \in \{1, 2, \dots, 2k\}$), we obtain

$$|g_j F_n \triangle g_j g_i^{-1} F_n| \leq \varepsilon' |F_n|.$$

Now, by the triangle inequality for the metric $|\cdot \Delta \cdot|$ (on finite sets), we have

$$|F_n \Delta g_j g_i^{-1} F_n| \leq |F_n \Delta g_j F_n| + |g_j F_n \Delta g_j g_i^{-1} F_n| \leq 2\varepsilon' |F_n|.$$

Repeating this argument k times (with the appropriate order of indices) we get

$$|F_n \Delta \bar{g} F_n| = |F_n \Delta g_1 g_2^{-1} \cdots g_{2k-1} g_{2k}^{-1} F_n| \leq 2k\varepsilon' |F_n| = \varepsilon |F_n|,$$

which means that F_n is (\bar{g}, ε) -invariant. We have shown that (F_n) is a Følner sequence in \tilde{G} . This also implies that the group \tilde{G} is amenable. \square

In this context we have the following fact.

Lemma 2.13. *Let (F_n) be a Følner sequence in a countably infinite amenable semi-group G which is embeddable in a group \tilde{G} such that (F_n) is a Følner sequence in \tilde{G} . A subset $A \subset G$ is (F_n) -normal in G if and only if it is (F_n) -normal, viewed as a subset of the group \tilde{G} (in other words, $1_A \subset \{0, 1\}^{\tilde{G}}$ is (F_n) -normal if and only if $1_A \subset \{0, 1\}^{\tilde{G}}$ is (F_n) -normal).*

Proof. In the proof we will use Theorem 3.1 which will be proved later and is independent from Lemma 2.13. If A is (F_n) -normal in \tilde{G} then clearly it is (F_n) -normal in G (because every nonempty finite set $K \subset G$ is also a subset of \tilde{G}). Suppose A is (F_n) -normal in G and let K be a nonempty finite subset of \tilde{G} . For large enough n_0 , the intersection $F_{n_0} \cap F_{n_0, K}$ is nonempty, i.e., there exists $g \in F_n \subset G$ and a bijection $h \mapsto f_h$ from K onto some $K' \subset F_{n_0}$, such that $hg = f_h$ for each $h \in K$. Then, in the group \tilde{G} , we have

$$\begin{aligned} d_{(F_n)} \left(\bigcap_{h \in K} h^{-1} A \right) &= d_{(F_n)} \left(g^{-1} \bigcap_{h \in K} h^{-1} A \right) = d_{(F_n)} \left(\bigcap_{h \in K} f_h^{-1} A \right) \\ &= d_{(F_n)} \left(\bigcap_{f \in K'} f^{-1} A \right). \end{aligned}$$

The meaning of $f^{-1}A$ is different in \tilde{G} and in G (in G it means $f^{-1}A \cap G$), however, since the sets F_n are contained in G , the value of $d_{(F_n)}(\bigcap_{f \in K'} f^{-1}A)$ does not depend on whether it is considered in \tilde{G} or in G . Since $K' \subset G$ and A is (F_n) -normal as a subset of G , the equivalence (1) \iff (3) in Theorem 3.1 and formula (3.2) yield that $d_{(F_n)}(\bigcap_{h \in K} h^{-1}A) = 2^{-|K|} = 2^{-|K'|}$. By invoking Theorem 3.1 again, we obtain (F_n) -normality of A as a subset of \tilde{G} . \square

Throughout the remainder of this section we assume that G is a countably infinite amenable group. Our key tool for handling (F_n) -normality in G is a special system of tilings $(\mathcal{T}_k)_{k \geq 1}$ of G which was constructed in [23]. (We could employ instead an older concept of quasi-tilings introduced in [42], but the system (\mathcal{T}_k) is a more convenient tool for our purposes.)

Let \mathcal{S} be a collection of finite subsets of G , each containing the identity element, which we will call *shapes*. To each $S \in \mathcal{S}$ we associate a *set of translates* (of S), $C_S \subset G$. We require that the sets C_S be pairwise disjoint and write $\mathcal{C} = \{C_S : S \in \mathcal{S}\}$. If the family

$$\mathcal{T} = \{Sc : S \in \mathcal{S}, c \in C_S\},$$

is a partition of G , we call it the *tiling* of G associated with the pair $(\mathcal{S}, \mathcal{C})$.

An element Sc of this partition will be called a *tile of shape S centered at c* . By disjointness of the tiles, the assignment $(S, c) \mapsto Sc$ is a bijection from $\{(S, c) : S \in \mathcal{S}, c \in C_S\}$ to \mathcal{T} , *i.e.*, each tile has a uniquely determined center and shape.

Given a tiling \mathcal{T} and a set $F \subset G$, the \mathcal{T} -*saturation* of F is defined as

$$F^{(\mathcal{T})} = \bigcup \{Sc \in \mathcal{T} : Sc \cap F \neq \emptyset\}.$$

Let K be the union of the sets SS^{-1} over all shapes S of those tiles of \mathcal{T} which have nonempty intersections with F . Formally,

$$K = \bigcup \{SS^{-1} : S \in \mathcal{S}, (\exists c \in C_S) Sc \cap F \neq \emptyset\}.$$

It is an easy observation that if F is a finite and (K, ε) -invariant set, then $|F^{(\mathcal{T})} \setminus F| \leq \varepsilon|F|$.

A tiling whose set of shapes is finite will be called *proper*.

A sequence of proper tilings $(\mathcal{T}_k)_{k \geq 1}$ is called a *congruent system of tilings* if for each k every tile of \mathcal{T}_{k+1} is a union of some tiles of \mathcal{T}_k .

A congruent system of tilings is *deterministic*, if, for each $k \geq 1$, all tiles of \mathcal{T}_{k+1} having the same shape are partitioned into the tiles of \mathcal{T}_k the same way. More precisely, we require that whenever $T'_1 = S'c_1$ and $T'_2 = S'c_2$ are two tiles of \mathcal{T}_{k+1} of the same shape S' (note that then $c_1, c_2 \in C_{S'}$) and $T'_1 = \bigcup_{i=1}^l T_{1,i}$ is the partition of T'_1 into the tiles of \mathcal{T}_k , then the sets $T_{2,i} = T_{1,i}c_1^{-1}c_2$ (with $i = 1, 2, \dots, l$) are also tiles of \mathcal{T}_k (and clearly they partition T'_2). It follows that in the deterministic case, the tiling \mathcal{T}_{k+1} *determines* all the tilings $\mathcal{T}_1, \dots, \mathcal{T}_k$. Also note that, with the above notation, the family $\{T_{1,i}c_1^{-1} : i = 1, 2, \dots, l\}$ (which is the same as $\{T_{2,i}c_2^{-1} : i = 1, 2, \dots, l\}$) is a partition of the shape S' into shifted shapes of the tiling \mathcal{T}_k . We will call this partition the *standard tiling of S' by the tiles of \mathcal{T}_k* (although formally, the sets $T_{1,i}c_1^{-1}$ need not be tiles of \mathcal{T}_k).

We will say that a system of proper tilings $(\mathcal{T}_k)_{k \geq 1}$ is *Følner* if for every nonempty finite set $K \subset G$ and every $\varepsilon > 0$, for large enough k , all shapes of \mathcal{T}_k (and thus also all tiles) are (K, ε) -invariant (in other words, if $(S_j)_{j \in \mathbb{N}}$ is obtained by enumerating the collection $\bigcup_k \mathcal{S}_k$ of all shapes used in the system of tilings, then (S_j) is a Følner sequence).

A proper tiling is called *syndetic* if for every shape S the set of translates C_S is (left) syndetic, *i.e.*, such that $KC_S = G$ for some finite set K (depending on S).

It is proved in [23] that every countably infinite amenable group G admits a congruent, deterministic, Følner system of proper tilings $(\mathcal{T}_k)_{k \geq 1}$. One can actually obtain a system of *syndetic* tilings with all the above properties, as follows. First, as noted in [23], any proper tiling \mathcal{T} of G can be represented as an element of the symbolic space $(\mathcal{S} \cup \{0\})^G$ (where the role of the alphabet is played by the finite collection \mathcal{S} of shapes of \mathcal{T} with an additional symbol 0). Now, a system of proper tilings $(\mathcal{T}_k)_{k \geq 1}$ becomes an element \mathbf{T} of the space $\prod_{k \geq 1} (\mathcal{S}_k \cup \{0\})^G = (\prod_{k \geq 1} (\mathcal{S}_k \cup \{0\}))^G$, on which G acts by shifts. Let $\overline{\mathcal{O}}(\mathbf{T})$ denote the orbit closure of \mathbf{T} with respect to the shift action. Every element $\mathbf{T}' \in \overline{\mathcal{O}}(\mathbf{T})$ is again a system of proper tilings $(\mathcal{T}'_k)_{k \geq 1}$ with the respective sets of shapes \mathcal{S}'_k satisfying $\mathcal{S}'_k \subset \mathcal{S}_k$ for each k .¹¹ Note that if \mathbf{T} is a Følner system of tilings, so is every $\mathbf{T}' \in \overline{\mathcal{O}}(\mathbf{T})$. Also, the properties of being congruent and deterministic pass from \mathbf{T} to all members of $\overline{\mathcal{O}}(\mathbf{T})$. The system $\overline{\mathcal{O}}(\mathbf{T})$ (with the shift action) has a minimal subsystem. Any element of this minimal subsystem, in addition to the preceding properties, is a system of syndetic tilings, which follows by a standard characterization of minimality in symbolic dynamics.

We define \mathcal{T}_0 to be the tiling all tiles of which are singletons (\mathcal{T}_0 has one shape $S = \{e\}$ and the corresponding set of translates C_S is the whole group).

3. Left invariance of (F_n) -normality

We call a subset $A \subset G$ (F_n) -normal if its indicator function 1_A , viewed as an element of $\{0, 1\}^G$, is (F_n) -normal. The goal of this section is to prove that if G is a countably infinite amenable cancellative semigroup and (F_n) is a Følner sequence in G then a set $A \subset G$ is (F_n) -normal if and only if so is gA , and also if and only if so is $g^{-1}A$.

The following theorem provides a characterization of normal sets in terms of “combinatorial independence”.

Theorem 3.1. *Let G be a countably infinite amenable cancellative semigroup and let (F_n) be a Følner sequence in G . Let $A \subset G$. We will use the following notation: $A^1 = A$, $A^0 = G \setminus A$. Consider the following five conditions:*

- (1) A is (F_n) -normal,
- (2) for any nonempty finite set K and any 0-1 block $B \in \{0, 1\}^K$ we have

$$d_{(F_n)} \left(\bigcap_{h \in K} h^{-1} A^{B(h)} \right) = 2^{-|K|}, \quad (3.1)$$

- (3) for any nonempty finite set K we have

$$d_{(F_n)} \left(\bigcap_{h \in K} h^{-1} A \right) = 2^{-|K|}, \quad (3.2)$$

¹¹ In fact, if for every element $\mathcal{T}'_k \in \overline{\mathcal{O}}(\mathcal{T}_k)$ we have $\mathcal{S}'_k = \mathcal{S}_k$ then \mathcal{T}_k is already syndetic.

(4) for any nonempty finite set K and any 0-1 block $B \in \{0, 1\}^K$ we have

$$d_{(F_n)} \left(\bigcap_{h \in K} h A^{B(h)} \right) = 2^{-|K|}, \quad (3.3)$$

(5) for any nonempty finite set K we have

$$d_{(F_n)} \left(\bigcap_{h \in K} h A \right) = 2^{-|K|}. \quad (3.4)$$

Then (1) \iff (2) \iff (3) \implies (4) \iff (5). If G is a group or G is commutative then all conditions (1)–(5) are equivalent.

Remark 3.2. If we enumerate the set K as $\{h_1, h_2, \dots, h_k\}$ then the blocks $B \in \{0, 1\}^K$ stand in 1-1 correspondence to 0-1 words of length k . Then, we can rewrite conditions (2)–(5) as follows:

(2) for any nonempty finite set $K = \{h_1, h_2, \dots, h_k\}$ and any 0-1 word w of length k , we have

$$d_{(F_n)}(h_1^{-1} A^{w_1} \cap h_2^{-1} A^{w_2} \cap \dots \cap h_k^{-1} A^{w_k}) = 2^{-k};$$

(3) for any nonempty finite set $K = \{h_1, h_2, \dots, h_k\}$ we have

$$d_{(F_n)}(h_1^{-1} A \cap h_2^{-1} A \cap \dots \cap h_k^{-1} A) = 2^{-k};$$

(4) for any nonempty finite set $K = \{h_1, h_2, \dots, h_k\}$ and any 0-1 word w of length k , we have

$$d_{(F_n)}(h_1 A^{w_1} \cap h_2 A^{w_2} \cap \dots \cap h_k A^{w_k}) = 2^{-k};$$

(5) for any nonempty finite set $K = \{h_1, h_2, \dots, h_k\}$ we have

$$d_{(F_n)}(h_1 A \cap h_2 A \cap \dots \cap h_k A) = 2^{-k}.$$

Proof of Theorem 3.1. In view of Theorem 2.8, (F_n) -normality can be defined via the condition (2.3). Observe that

$$g \in \bigcap_{h \in K} h^{-1} A^{B(h)} \iff (\forall h \in K) hg \in A^{B(h)} \iff \sigma_g(1_A) \in [B].$$

Thus (3.1) is just (2.3) written in terms of (F_n) -density, which immediately gives the equivalence (1) \iff (2). Next, (2) implies (3) because (3.2) is the particular case of (3.1) for the block B equal to the constant function 1 on K . By the same argument (4) implies (5).

We pass to proving that (3) \implies (2). Suppose that some two sets $A_1, A_2 \subset G$ have well defined (F_n) -densities and satisfy the “independence condition”:

$$d_{(F_n)}(A_1 \cap A_2) = d_{(F_n)}(A_1) \cdot d_{(F_n)}(A_2).$$

Then, by finite additivity of (F_n) -density, we have

$$\begin{aligned} d_{(F_n)}(A_1 \cap A_2^0) &= d_{(F_n)}(A_1) - d_{(F_n)}(A_1 \cap A_2) \\ &= d_{(F_n)}(A_1) - d_{(F_n)}(A_1) \cdot d_{(F_n)}(A_2) \\ &= d_{(F_n)}(A_1)(1 - d_{(F_n)}(A_2)) \\ &= d_{(F_n)}(A_1) \cdot d_{(F_n)}(A_2^0). \end{aligned}$$

Iterating the above calculation one shows that if a finite family $\{A_1, A_2, \dots, A_k\}$ of subsets of G satisfies the “independence condition”:

- for any subset $E \subset \{1, 2, \dots, k\}$ one has $d_{(F_n)}\left(\bigcap_{i \in E} A_i\right) = \prod_{i \in E} d_{(F_n)}(A_i)$,

then, for any 0-1-word $w \in \{0, 1\}^k$, the family $\{A_1^{w_1}, A_2^{w_2}, \dots, A_k^{w_k}\}$ also satisfies the independence condition. Next, notice that condition (3) applied to all possible nonempty subsets of K is precisely the independence condition for the family $\{h^{-1}A : h \in K\}$. This, combined with the preceding observation, implies (2).

The same argument proves the implication (5) \implies (4).

To prove the implication (3) \implies (5), we note that for large n the “ K^{-1} -core” of F_n , i.e., the set $\bigcap_{h \in K} hF_n$ is nonempty (like the K -core, it is eventually an ε -modification of F_n). Thus there exists an $n_0 \in \mathbb{N}$, a $g \in G$ and a bijection $h \mapsto f_h$ from K onto some $K' \subset F_{n_0}$, such that $g = hf_h$ for each $h \in K$. By (2.2), the (F_n) -density of $\bigcap_{h \in K} hA$ is the same as that of $g^{-1} \bigcap_{h \in K} hA = \bigcap_{h \in K} g^{-1}hA = \bigcap_{h \in K} f_h^{-1}A = \bigcap_{f \in K'} f^{-1}A$. By (3), this density equals $2^{-|K'|} = 2^{-|K|}$, as needed.

If G is a group then the equivalence (3) \iff (5) is obvious: the family $\{hA : h \in K\}$ is the same as $\{h^{-1}A : h \in K^{-1}\}$.

Suppose G is commutative and assume (4). Let K be a nonempty finite subset of G . As before, there exists $g \in G$ and a bijection $h \mapsto f_h$ from K onto some $K' \subset G$, such that $g = hf_h$ for each $h \in K$. By (2.2) we have

$$d_{(F_n)}\left(\bigcap_{h \in K} h^{-1}A^{B(h)}\right) = d_{(F_n)}\left(\bigcap_{h \in K} gh^{-1}A^{B(h)}\right).$$

We would like to replace gh^{-1} by f_h (using commutativity), however, in general $gh^{-1}A$ is only a subset of $h^{-1}gA = f_hA$ (an analogous inclusion holds for $A^0 = G \setminus A$). Thus

$$\frac{1}{|F_n|} \left| F_n \cap \bigcap_{h \in K} h^{-1}A^{B(h)} \right| \leq \frac{1}{|F_n|} \left| F_n \cap \bigcap_{h \in K} f_hA^{B(h)} \right| = \frac{1}{|F_n|} \left| F_n \cap \bigcap_{f \in K'} fA^{B'(f)} \right|,$$

where B' is defined on K' by $B'(f_h) = B(h)$. By (4), the right-hand side tends to $2^{-|K'|} = 2^{-|K|}$. But since the sum of the left-hand sides over all blocks $B \in \{0, 1\}^K$ equals 1, we have convergence of the left-hand side to $2^{-|K|}$ for every block, *i.e.* (3.1). We have proved that (4) \implies (2). \square

Theorem 3.3. *Let G be a countably infinite amenable cancellative semigroup and let (F_n) be a Følner sequence in G . For any $A \subset G$ and $g \in G$ we have the equivalences*

$$gA \text{ is } (F_n)\text{-normal} \iff A \text{ is } (F_n)\text{-normal} \iff g^{-1}A \text{ is } (F_n)\text{-normal}. \quad (3.5)$$

Proof. The first half of the proof relies on the equivalence (1) \iff (3) in Theorem 3.1. Assume that gA is (F_n) -normal and let K be a nonempty finite subset of G . Condition (3.2) applied to $K' = gK$ and the set gA reads

$$2^{-|K|} = d_{(F_n)} \left(\bigcap_{h \in K} (gh)^{-1} gA \right) = d_{(F_n)} \left(\bigcap_{h \in K} h^{-1} A \right),$$

i.e., we have obtained (3.2) for K and A . Now assume that A is (F_n) -normal and let K be a nonempty finite subset of G . Condition (3.2) applied to $K' = gK$ and the set A is

$$2^{-|K|} = d_{(F_n)} \left(\bigcap_{h \in K} (gh)^{-1} A \right) = d_{(F_n)} \left(\bigcap_{h \in K} h^{-1} g^{-1} A \right),$$

which gives (3.2) for K and $g^{-1}A$.

If G is a group or is commutative then we can use the equivalence (1) \iff (4) to reverse the implications: Assume that $g^{-1}A$ is (F_n) -normal and let K be a nonempty finite subset of G and $B \in \{0, 1\}^K$. We have

$$\frac{1}{|F_n|} \left| F_n \cap \bigcap_{h \in K} hg^2 g^{-1} A^{B(h)} \right| \leq \frac{1}{|F_n|} \left| F_n \cap \bigcap_{h \in K} hg A^{B(h)} \right|.$$

The condition (3.3) applied to $K' = Kg^2$, the block $B' \in \{0, 1\}^{Kg^2}$ defined by $B'(hg^2) = B(h)$, and the set $g^{-1}A$ implies that the left-hand side tends to $2^{-|K|}$. But since the sum of the right-hand sides over all blocks $B \in \{0, 1\}^K$ equals 1, we have convergence of the right-hand side to $2^{-|K|}$ for every block, *i.e.*, (3.3) holds for gA . Since (4) \implies (1) we have proved (F_n) -normality of gA .

Finally, we can use the fact that G is cancellative. By Theorem 2.12, it can be embedded in a group \tilde{G} such that (F_n) is a Følner sequence in \tilde{G} . Suppose $g^{-1}A$ is (F_n) -normal as a subset of G . By definition, the set $g^{-1}A$ regarded as a subset of G is equal, in \tilde{G} , to $g^{-1}A \cap G$. By Lemma 2.13, $g^{-1}A \cap G$ is (F_n) -normal as a subset of \tilde{G} . Since all sets F_n are contained in G , also the set $g^{-1}A$ is (F_n) -normal in \tilde{G} . In the group \tilde{G} , (F_n) -normality of $g^{-1}A$ implies (F_n) -normality of gA . Finally, by the trivial direction of Lemma 2.13, gA is also (F_n) -normal when viewed as a subset of G . \square

Remark 3.4. We were unable to prove the implication (4) \implies (2) in Theorem 3.1 for semigroups embeddable in groups.

Remark 3.5. In general, even if G is a group, (F_n) -normality is **not** right invariant: if A is (F_n) -normal then Ag is not guaranteed to be (F_n) -normal¹². For this reason, (F_n) -normality of the elements of $\{0, 1\}^G$ is **not** preserved by the shift-action: $\sigma_g(x)$ need not be (F_n) -normal if x is (if x is the indicator function of a set A then $\sigma_g(x)$ is the indicator function of Ag^{-1}). Nevertheless, under very mild assumptions on (F_n) , this may happen only with probability zero, see Corollary 4.3 below.

4. Properties of the family of (F_n) -normal sets

4.1. Ergodic interpretation of normality

Fix a countably infinite amenable cancellative semigroup G and a Følner sequence (F_n) in G . Suppose that G acts by continuous maps T_g on a compact metric space X , preserving a Borel probability measure μ . We will tacitly assume that, when convenient or necessary, the identity element (always denoted by e) is attached to the semigroup, and T_e is the identity mapping. A point $x \in X$ is called (F_n) -generic for μ if for any continuous function $f \in C(X)$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x) = \int f d\mu, \quad (4.1)$$

in other words, if the measures $\frac{1}{|F_n|} \sum_{g \in F_n} \delta_{T_g x}$ converge to μ in the weak-star topology.

Note that the shift action on the symbolic space $X = \{0, 1\}^G$ preserves (among many other measures) the product measure m^G , where m is the $(\frac{1}{2}, \frac{1}{2})$ -measure on $\{0, 1\}$. The measure m^G will be henceforth denoted by λ and called the (uniform) Bernoulli measure.

We have the following equivalent formulation of normality of a set $A \subset G$, in dynamical terms.

¹² For instance, a counterexample can be constructed in the group $G = \langle \sigma, \tau \rangle$ of transformations of the symbolic space $\{0, 1\}^{\mathbb{Z}}$, generated by the shift σ and the flip τ of the zero-coordinate symbol (note that $\tau^{-1} = \tau$). This group is solvable: the subset $H = \langle \sigma^{-k} \tau \sigma^k : k \in \mathbb{Z} \rangle$ (consisting of flips at finitely many coordinates, with no shift) is a normal subgroup of G and $G/H = \langle \sigma \rangle$ is Abelian. In particular, G is amenable. Each $g \in G$ is representable in a unique way as $\sigma^{k_g} h_g$ with $h_g \in H$. For each $h \in H$ denote by $m_h \in \mathbb{Z}$ the rightmost coordinate on which h applies the flip. Let (F'_n) be a Følner sequence in G . Let $m_n = \max\{m_{h_g} : g \in F'_n\}$. Now we create a new Følner sequence (F_n) by setting $F_n = F'_n \sigma^{m_n+1}$. Notice that any $g \in F_n$ does not flip the zero-coordinate symbol (but perhaps shifts it). This implies that F_{n_1} and $F_{n_2} \tau$ are disjoint for any $n_1, n_2 \in \mathbb{N}$. As we know, there exist an (F_n) -normal set $A' \subset G$ and its intersection with the union $A = \bigcup_n F_n$ is also (F_n) -normal. The set A is disjoint from $F_n \tau$ for all $n \geq 1$, which implies that $A\tau$ has (F_n) -density zero and hence cannot be (F_n) -normal.

Proposition 4.1. *A set $A \subset G$ is (F_n) -normal if and only if its indicator function 1_A is (F_n) -generic for the Bernoulli measure λ on $\{0, 1\}^G$.*

Proof. First of all, note that for any nonempty finite set $K \subset G$ and any $B \in \{0, 1\}^K$, we have $2^{-|K|} = \lambda([B])$. Thus, using (2.3) and (2.4), we can see that the (F_n) -normality of x can be equivalently expressed by the condition (4.1) (with $\mu = \lambda$ and $T_g = \sigma_g$) for all functions of the form $f = 1_{[B]}$. Finally, observe that the indicator functions of cylinders are linearly dense in the space $C(\{0, 1\}^G)$ of continuous functions on $\{0, 1\}^G$, which clearly ends the proof. \square

As was already mentioned in the Introduction, the existence of (F_n) -normal 0-1 sequences (and the fact that the set of such sequences has full measure) is often derived with the help of the pointwise ergodic theorem, which, in general, holds only along rather special (tempered) Følner sequences. However, in the specific case of the Bernoulli measure and continuous functions, the conventional pointwise ergodic theorem can be replaced by Theorem 4.2 below (more precisely, by its equivalent version Theorem 4.4), which is valid under much weaker restrictions on Følner sequences.

Theorem 4.2. *Let G be a countably infinite amenable cancellative semigroup. Let $(F_n)_{n \geq 1}$ be a Følner sequence in G such that for any $\alpha \in (0, 1)$ we have $\sum_{n=1}^{\infty} \alpha^{|F_n|} < \infty$. Then λ -almost every $x \in \{0, 1\}^G$ is (F_n) -normal, i.e., for any nonempty finite set $K \subset G$ and any block $B \in \{0, 1\}^K$, one has*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} |\{g \in F_n : \sigma_g(x) \in [B]\}| = 2^{-|K|}. \quad (4.2)$$

Proof. By Theorem 2.12 and Lemma 2.13, it suffices to consider the case where G is a group. Because there are countably many blocks over finite subsets of G , it suffices to prove that for any nonempty finite set $K \subset G$ and any block $B \in \{0, 1\}^K$, (4.2) holds for λ -almost every $x \in \{0, 1\}^G$.

Given $\varepsilon > 0$, we will partition the group G into finitely many sets D_0, D_1, \dots, D_r , such that $\underline{d}_{(F_n)}(D_0) \leq \varepsilon$ (the set D_0 may be empty), and for every $i > 0$, we have:

- (1) $\underline{d}_{(F_n)}(D_i) > 0$;
- (2) for all distinct $g_1, g_2 \in D_i$, $Kg_1 \cap Kg_2 = \emptyset$.

We start by showing that the existence of the sets D_0, D_1, \dots, D_r as above implies the assertion of the theorem. Choose a positive $\beta < \min\{\underline{d}_{(F_n)}(D_i), i = 1, 2, \dots, r\}$. Let n_0 be such that for every $n \geq n_0$,

$$\frac{|F_n \cap D_0|}{|F_n|} < 2\varepsilon, \quad \text{and, for each } i \in \{1, 2, \dots, r\}, \quad \frac{|F_n \cap D_i|}{|F_n|} > \beta.$$

Let $\Omega = (\{0, 1\}^G, \mathcal{B}, \lambda)$ where \mathcal{B} denotes the Borel σ -algebra in $\{0, 1\}^G$. Fix an $n \geq n_0$ and consider the finite sequence of $\{0, 1\}$ -valued random variables defined on Ω by

$$Y_g(x) = 1_{[B]}(\sigma_g(x)), \quad g \in F_n.$$

Also, for each $i = 0, 1, \dots, r$ define

$$\bar{Y}_i = \frac{1}{|F_n \cap D_i|} \sum_{g \in F_n \cap D_i} Y_g.$$

By (2), for each $i > 0$ the variable \bar{Y}_i is the average of finitely many independent random variables Y_g , each assuming the value 1 with probability $2^{-|K|}$. Clearly, the expected value of \bar{Y}_i equals $2^{-|K|}$. Now, the classical Bernstein's inequality (see, e.g., [11]) implies that

$$\lambda \left(\left\{ x : |\bar{Y}_i(x) - 2^{-|K|}| > \varepsilon \right\} \right) \leq \gamma^{|F_n \cap D_i|} < \gamma^{\beta|F_n|},$$

where $\gamma \in (0, 1)$ is some constant (not depending on n). Then, denoting by $X_\varepsilon = \{x : \exists i = 1, 2, \dots, r : |\bar{Y}_i(x) - 2^{-|K|}| > \varepsilon\}$, we have

$$\begin{aligned} \lambda(X_\varepsilon) &= \lambda \left(\bigcup_{i=1,2,\dots,r} \{x : |\bar{Y}_i(x) - 2^{-|K|}| > \varepsilon\} \right) \\ &\leq \sum_{i=1,2,\dots,r} \lambda \left(\{x : |\bar{Y}_i(x) - 2^{-|K|}| > \varepsilon\} \right) \leq r \gamma^{\beta|F_n|}. \end{aligned}$$

On the complementary set $\{0, 1\}^G \setminus X_\varepsilon$, for each $i = 1, 2, \dots, r$, we have the inequality $|\bar{Y}_i(x) - 2^{-|K|}| \leq \varepsilon$, i.e.,

$$\frac{1}{|F_n \cap D_i|} \sum_{g \in F_n \cap D_i} Y_g(x) \in [2^{-|K|} - \varepsilon, 2^{-|K|} + \varepsilon]. \quad (4.3)$$

For $i = 0$, recall that $\frac{|F_n \cap D_0|}{|F_n|} < 2\varepsilon$, and we have the trivial estimate

$$\frac{1}{|F_n \cap D_0|} \sum_{g \in F_n \cap D_0} Y_g(x) \in [0, 1]. \quad (4.4)$$

Averaging the left-hand sides of (4.3) and (4.4) over $i = 0, 1, 2, \dots, r$ (with weights $\frac{|F_n \cap D_i|}{|F_n|}$) we obtain

$$\frac{1}{|F_n|} \sum_{g \in F_n} Y_g(x) \in [2^{-|K|} - 3\varepsilon, 2^{-|K|} + 3\varepsilon].$$

So, the set on which the inequality

$$\left| \frac{1}{|F_n|} \sum_{g \in F_n} Y_g(x) - 2^{-|K|} \right| > 3\varepsilon$$

holds is contained in X_ε , thus has measure at most $r\gamma^{\beta|F_n|}$. Summarizing, we have shown that

$$\lambda \left\{ \left| \frac{|\{g \in F_n : \sigma_g(x) \in [B]\}|}{|F_n|} - 2^{-|K|} \right| > 3\varepsilon \right\} \leq r\gamma^{\beta|F_n|}.$$

Let $\alpha = \gamma^\beta$ and note that $\alpha \in (0, 1)$. By the assumption, $\sum_n \alpha^{|F_n|} < \infty$. The Borel-Cantelli Lemma now yields that for λ -almost every x , the numbers

$$\frac{|\{g \in F_n : \sigma_g(x) \in [B]\}|}{|F_n|} = \frac{1}{|F_n|} \tilde{N}(B, x, F_n)$$

eventually remain within 3ε from $2^{-|K|}$. Since ε is arbitrary, we have proved the desired almost everywhere convergence.

It remains to define the sets D_i . We will do that with the help of tilings. As we have mentioned earlier, G admits a congruent, deterministic, Følner system of proper, syndetic tilings $(\mathcal{T}_k)_{k \geq 1}$. Let k be such that all shapes $S \in \mathcal{S}$ of the tiling $\mathcal{T} = \mathcal{T}_k = (\mathcal{S}, C)$ are (K, δ) -invariant, where $\delta = \frac{\varepsilon}{2|K|}$. Then, for each $S \in \mathcal{S}$, the K -core of S , i.e., the set $S_K = \{g \in G : Kg \subset S\}$ satisfies

$$\frac{|S_K|}{|S|} \geq 1 - \varepsilon$$

(see Lemma 2.4 and notice since that K contains the identity element, we have $S_K \subset S$). Also, if T is any tile of \mathcal{T} and T_K denotes the K -core of T then

$$\frac{|T_K|}{|T|} \geq 1 - \varepsilon$$

(recall that $T = Sc$ where $S \in \mathcal{S}$, $c \in C_S$, in which case $T_K = S_K c$). Let now

$$D_0 = \bigcup_{T \in \mathcal{T}} T \setminus T_K.$$

We claim that $\bar{d}_{(F_n)}(D_0) \leq \varepsilon$. Indeed, this inequality is obvious if the Følner sequence (F_n) is replaced by the sequence $(F_n^{(\mathcal{T})})$ of the \mathcal{T} -saturation of the sets F_n . But the Følner sequences (F_n) and $(F_n^{(\mathcal{T})})$ are equivalent (see Definition 2.5) and hence they define the same upper densities of sets. For $S \in \mathcal{S}$ and $g \in S_K$, let $D_{(S,g)} = gC_S$. Since for any such pair (S, g) we have $Kg \subset S$ (and hence $Kgc \subset Sc$) and the sets Sc , $c \in C_S$, are tiles (and thus are pairwise disjoint), the sets Kh are pairwise disjoint when $h = gc$ varies over $D_{(S,g)}$. By syndeticity of the tiling, each set C_S is syndetic, and so is each of the sets $D_{(S,g)}$. It follows immediately from finite subadditivity of $\underline{d}_{(F_n)}(\cdot)$ and the fact that for any $D \subset G$ and $g \in G$, $\underline{d}_{(F_n)}(gD) = \underline{d}_{(F_n)}(D)$, that syndetic sets have positive lower (F_n) -density. In particular, $\underline{d}_{(F_n)}(D_{(S,g)}) > 0$. Finally, since there are finitely many pairs (S, g) , the sets $D_{(S,g)}$ can be enumerated as D_1, D_2, \dots, D_r ($r \in \mathbb{N}$). By construction, the family $\{D_i : i = 0, 1, 2, \dots, r\}$, is a partition of G . This ends the proof. \square

Recall (see Remark 3.5) that if the semigroup G is not commutative then generally speaking, the action σ_g need not preserve (F_n) -normality. Nevertheless, by a straightforward application of shift-invariance of λ , the following holds.

Corollary 4.3. *Let G be an infinitely countable amenable cancellative semigroup. If a Følner sequence (F_n) in G satisfies, for each $\alpha \in (0, 1)$, the summability condition $\sum_{n \in \mathbb{N}} \alpha^{|F_n|} < \infty$, then λ -almost every element $x \in \{0, 1\}^G$ has the property that all the images $\sigma_g(x)$ ($g \in G$) are (F_n) -normal.*

Proposition 4.1 allows us to formulate now an (ostensibly stronger) equivalent version of Theorem 4.2 as follows.

Theorem 4.4. *Under the assumptions of Theorem 4.2, λ -almost every $x \in \{0, 1\}^G$ is (F_n) -generic for λ , i.e., the convergence*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(\sigma_g x) = \int f d\lambda$$

holds for any continuous function f on $\{0, 1\}^G$.

Remark 4.5.

- (i) The requirement $\sum_{n=1}^{\infty} \alpha^{|F_n|} < \infty$ for each $\alpha \in (0, 1)$ is much weaker than (1.8) (of (F_n) being tempered), and is satisfied, for example, by any Følner sequence (F_n) such that $|F_n|$ strictly increases as $n \rightarrow \infty$;
- (ii) On the other hand, some condition on the growth of $|F_n|$ is necessary. For example, for $G = \mathbb{Z}$ consider the Følner sequence consisting of pairwise disjoint intervals: n_1 intervals of length 1 followed by n_2 intervals of length 2, followed by n_3 intervals of length 3, etc. If a number n_k is very large compared with k , then there exists a set $X_k \subset \{0, 1\}^{\mathbb{Z}}$ with $\lambda(X_k)$ close to 1 and such that for every $x \in X_k$ the restriction of x to at least one of the intervals F_n of length k will be filled entirely by 0's. We can thus arrange the sequence n_k so that the measures of the complements of the sets X_k are summable over k . Then, by the Borel-Cantelli Lemma, for almost every x there will be arbitrarily far Følner sets filled entirely with 0's (instead of being filled nearly half-half by 0's and 1's), contradicting (F_n) -genericity of x already on cylinders of length 1. In fact, in this example the set of (F_n) -generic elements has measure zero.

Remark 4.6. It is worth mentioning that the method used in the proof of Theorem 4.2 fails in proving the pointwise ergodic theorem (even for the Bernoulli measure) for discontinuous L^∞ functions. In [1], Akcoglu and del Junco proved that for any ergodic (and aperiodic) $(\mathbb{Z}, +)$ -action the pointwise ergodic theorem along the Følner sequence of intervals $[n, n + \lfloor \sqrt{n} \rfloor]$ fails for the indicator function of some measurable set A . Note that, for any $\alpha \in (0, 1)$, the sequence $\alpha^{|F_n|} = \alpha^{\lfloor \sqrt{n} \rfloor}$ is summable, thus in the case of the uniform Bernoulli measure, according to Theorem 4.2, the set A cannot be clopen in $\{0, 1\}^{\mathbb{Z}}$.

4.2. Normal elements form a first category set

In contrast to the measure-theoretic largeness established in Theorem 4.2, the following simple proposition demonstrates that the set of (F_n) -normal elements in $\{0, 1\}^G$ is always topologically small (*i.e.*, is of first category), without any assumptions on the Følner sequence.

Proposition 4.7. *Let G be a countably infinite cancellative amenable semigroup and let (F_n) be a Følner sequence in G . Then the set $\mathcal{N}((F_n))$ of (F_n) -normal elements is of first Baire category in $\{0, 1\}^G$.*

Proof. For $n \in \mathbb{N}$, a nonempty finite set $K \in G$, $B \in \{0, 1\}^K$ and $\varepsilon \in (0, 2^{-|K|})$, let $W(F_n, B, \varepsilon)$ be the union of all cylinders corresponding to the blocks C over the Følner set F_n , such that

$$2^{-|K|} - \varepsilon \leq \frac{1}{|F_n|} |\{g \in F_n : (\forall h \in K) hg \in F_n \text{ and } C(hg) = B(h)\}| \leq 2^{-|K|} + \varepsilon.$$

Since $W(F_n, B, \varepsilon)$ is a finite union of cylinders, it is clopen. The set $\mathcal{N}((F_n))$ can be written as

$$\bigcap_{K \subset G, K \text{ nonempty finite}} \bigcap_{B \in \{0, 1\}^K} \bigcap_{0 < \varepsilon < 2^{-|K|}} \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} W(F_n, B, \varepsilon).$$

Note that for each B, ε and n_0 as above, the closed set $\bigcap_{n \geq n_0} W(F_n, B, \varepsilon)$ has empty interior, because the set of the elements of $\{0, 1\}^G$ which are constant on complements of finite sets is dense in $\{0, 1\}^G$. Thus by the Baire theorem, the set $\bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} W(F_n, B, \varepsilon)$ is of first category and contains the set $\mathcal{N}((F_n))$, which ends the proof. \square

Corollary 4.8. *Let G be either $(\mathbb{N}, +)$ or (\mathbb{N}, \times) and let (F_n) be an arbitrary Følner sequence in G . Then the set of (F_n) -normal numbers in $[0, 1]$ (*i.e.*, numbers which have (F_n) -normal binary expansions) is of first category.*

Proof. For any countably infinite semigroup G in which we have a fixed enumeration, *i.e.*, a bijection between G and \mathbb{N} , $n \mapsto g_n$, the formula

$$\psi(x) = \sum_{n \in \mathbb{N}} 2^{-n} x(g_n), \text{ where } x = (x(g))_{g \in G} \in \{0, 1\}^G,$$

establishes a continuous map from $\{0, 1\}^G$ onto $[0, 1]$. This map is injective except on a countable set, on which it is two-to-one. Note that every continuous map ϕ on a compact domain, such that all but countably many fibers (preimages of points) are singletons and all other fibers are of first category, preserves the first category. Indeed, let A be a first category subset of the domain, *i.e.*, $A \subset B = \bigcup_n B_n$, where each B_n is compact and has empty interior. The set $C = \bigcup_n \phi^{-1}(\phi(B_n))$ contains the first category set B and differs from it by at most a countable union of fibers

(which is of first category), so C is also a first category set. By continuity, for each n the set $\phi(B_n)$ is compact and, moreover, it has empty interior (otherwise $\phi^{-1}(\phi(B_n))$ would have nonempty interior, which is impossible since C is of first category). Thus $\phi(B) = \bigcup_n \phi(B_n)$ is of first category, and so is its subset $\phi(A)$. We conclude that the binary expansion map ψ preserves the first category. Now it remains to apply this fact to $(\mathbb{N}, +)$ or (\mathbb{N}, \times) and invoke Proposition 4.7. \square

5. An effectively defined normal set

Although Lebesgue-almost every number is normal (in the classical sense) in any base $b \in \mathbb{N}$, the set of *computable* numbers (*i.e.*, the numbers whose b -ary expansion can be computed with the help of a Turing machine, like, for example, the Champernowne number) has Lebesgue measure zero. It is so, because the asymptotic Kolmogorov complexity of such expansions is zero, while, as shown by A. A. Brudno [13], a typical expansion has Kolmogorov complexity $\log b$. Hence computable normal numbers are highly exceptional among normal numbers.

Let G be a countably infinite amenable group and let (F_n) be an arbitrary Følner sequence in G . In this subsection we describe an “effective” construction of an (F_n) -normal Champernowne-like set (viewed, when convenient, as an element of $\{0, 1\}^G$). We use the term *effective* to indicate that our construction is given by an inductive algorithm which allows to determine, for every $g \in G$, whether it belongs to the set or not, in finitely many inductive steps. We cannot claim that our construction gives a computable set, since we make no assumptions on computability of the group G or the Følner sequence (F_n) .

In the construction of the classical binary Champernowne number three types of 0-1 words are involved:

- (1) The binary words which are expansions of natural numbers. We will call these words “bricks”. Notice that a brick never starts (on the left) with the symbol 0, and there are exactly 2^{k-1} bricks of length k ;
- (2) The “packages”. For each k , the k th “package” is the concatenation (in the lexicographical order) of all bricks of length k . The length of the k th package is $k2^{k-1}$;
- (3) Finally, the “chains”. For each k , the k th “chain” is formed by the packages, from the first to the k th, concatenated together (by increase of k). The k th chain stretches from the coordinate 1 to the coordinate $\sum_{i=1}^k i2^{i-1}$.

Once the chains are defined, the sequence representing the binary Champernowne number is obtained by taking the coordinatewise limit in $\{0, 1\}^{\mathbb{N}}$ of the chains (extended to infinite 0-1 words by adding zeros).

In the construction of the binary Champernowne number described above one can introduce the following three modifications which do not destroy the normality:

- (1) one can include as bricks also the words starting with the symbol 0 (the reason why they are not used is purely aesthetic) so that there are 2^k (rather than 2^{k-1}) bricks of length k ;

- (2) the package of order k may contain every brick of order k repeated more than once, as long as the number of repetitions is the same (or nearly the same) for every brick; then the length of the package of order k is $m_k k 2^k$ for some sequence m_k ;
- (3) in the chain, one may repeat each package of order k more than once, say n_k times (then the length of the k th chain equals $\sum_{i=1}^k n_i m_i i 2^{i-1}$).

While the modifications described above are not necessary in the construction of the classical Champernowne number, they contain an idea instrumental for the proof of the following theorem.

Theorem 5.1. *Let G be a countably infinite amenable group and let (F_n) be an arbitrary Følner sequence in G . Then there exists an effectively defined (F_n) -normal element $x \in \{0, 1\}^G$.*

Remark 5.2. The theorem provides (F_n) -normal elements even when the cardinalities $|F_n|$ do not strictly increase, in which case Theorem 4.2 does not necessarily apply.

Proof of Theorem 5.1. The construction involves a congruent, deterministic, Følner system $(\mathcal{T}_k)_{k \geq 0}$ of proper, syndetic tilings of G , starting with the tiling \mathcal{T}_0 comprised of singletons (see Section 2). We can choose the system (\mathcal{T}_k) independently of the Følner sequence (F_n) ; any such system of tilings will lead to an (F_n) -normal element.

For each $k \geq 1$ and each shape S of \mathcal{T}_k let $\mathbb{B}_S = \{0, 1\}^S$ be the set of all possible 0-1 blocks over S (clearly, $|\mathbb{B}_S| = 2^{|S|}$) and let $\bigcup_{S \in \mathcal{S}_k} \mathbb{B}_S$ be the set of “bricks” of order k . Syndeticity of the sets C_S together with the fact that (\mathcal{T}_k) is a Følner and deterministic system of tilings imply that for each $k \geq 1$ there exists an index $r(k)$ such that the standard tiling of each shape S' of the tiling $\mathcal{T}_{r(k)}$, by the tiles of \mathcal{T}_k , contains, for each shape S of \mathcal{T}_k , at least $2k2^{|S|}$ tiles of shape S . Let $\ell(S', S) \geq 2k2^{|S|}$ denote the number of tiles of \mathcal{T}_k , having the shape S , in the standard tiling of S' . We are now in a position to associate with each shape S' of $\mathcal{T}_{r(k)}$ a package of order k , $P(S') \in \{0, 1\}^{S'}$. Since for each $S \in \mathcal{S}_k$ we have $\ell(S', S) \geq 2k2^{|S|}$, one can divide the collection of all tiles T of shape S , occurring in the standard tiling of S' , into $2^{|S|}$ nonempty and disjoint families $\mathbb{T}_B^{(S, S')}$ indexed bijectively by the bricks $B \in \mathbb{B}_S$, and having roughly equal cardinalities. More precisely we can arrange that, for each $B \in \mathbb{B}_S$, $|\mathbb{T}_B^{(S, S')}| \in [\ell(S', S)2^{-|S|} - 1, \ell(S', S)2^{-|S|} + 1]$ (since $2 \leq \frac{1}{k}\ell(S', S)2^{-|S|}$, the above cardinalities differ by at most $\frac{100}{k}$ percent). Then, for each tile T of shape S occurring in the standard tiling of S' we define the restriction of $P(S')$ to T as the unique brick B such that $T \in \mathbb{T}_B^{(S, S')}$. This concludes the definition of the packages $P(S')$ of order $k \geq 1$. For completeness, we let $r(0) = 0$ and define the package of order 0 as the single symbol 0. This is consistent with the previous conventions: the package of order 0 has a shape corresponding to the tiling $\mathcal{T}_{r(0)} = \mathcal{T}_0$. Since \mathcal{T}_0 has only one shape (the singleton), the 0th package is a block over a singleton (i.e., a single symbol).

At this point we need to introduce some additional terminology. For a nonempty finite set $K \subset G$ and $\varepsilon > 0$, a block $C \in \{0, 1\}^F$ over another finite set $F \subset G$ is (K, ε) -normal if for every block $B \in \{0, 1\}^K$ one has

$$2^{-|K|} - \varepsilon \leq \frac{1}{|F|} \left| \{g \in F : (\forall h \in K) hg \in F \text{ and } C(hg) = B(h)\} \right| \leq 2^{-|K|} + \varepsilon.$$

Summing over all blocks $B \in \{0, 1\}^K$ one obtains that in order for C to be (K, ε) -normal, F must be $(K, 2|K|\varepsilon)$ -invariant.

The following fact is now easily verified:

- (1) For any nonempty finite set $K \subset G$ and any $\varepsilon > 0$, if k is sufficiently large then every package of order k is (K, ε) -normal. So is every concatenation of such (shifted) packages.

In order to define the (F_n) -normal element $x \in \{0, 1\}^G$ we first create a (not proper) *mixed tiling* Θ , i.e., a partition of G into tiles belonging to different tilings from the subsequence $(\mathcal{T}_{r(k)})_{k \geq 0}$ (this will be possible due to the fact that we are working with a congruent system of tilings). Then we will define x as follows: x restricted to a tile T of Θ equals the (appropriately shifted) package associated to the shape of T (if T belongs to the tiling $\mathcal{T}_{r(k)}$ then the order of the package is k). In this manner x becomes an infinite concatenation of packages of various orders. We remark that working with a mixed tiling is equivalent to working with chains: one can define the k th chain as the part of x covered by the tiles of Θ belonging to the tilings $\mathcal{T}_{r(0)}, \mathcal{T}_{r(1)}, \dots, \mathcal{T}_{r(k)}$. Conversely, whenever a Champernowne set is defined via the concept of chains, as a concatenation of packages of different orders, then the tiles of Θ are simply the domains of these packages.

It remains to describe how we define the mixed tiling Θ . The procedure will depend on the a priori given Følner sequence (F_n) (which so far was not involved in the construction).

For each $k \geq 1$ let n_k be such that the Følner sets F_n with $n > n_k$ are $(\mathbf{S}_k, \frac{1}{k})$ -invariant, where $\mathbf{S}_k = \bigcup_{S' \in \mathcal{T}_{r(k)}} S' S'^{-1}$ (then F_n is also $(\mathbf{S}_i, \frac{1}{k})$ -invariant for all $i \leq k$). We begin by defining Θ on the $\mathcal{T}_{r(1)}$ -saturation (denoted by \mathbf{F}_1) of the union $F_1 \cup F_2 \dots \cup F_{n_1}$ simply as \mathcal{T}_0 . Notice that Θ remains undefined on the complement of \mathbf{F}_1 which is a union of complete tiles of $\mathcal{T}_{r(1)}$. Inductively, let $k \geq 2$ and suppose that, after step $k-1$, Θ remains undefined on a union of complete tiles of $\mathcal{T}_{r(k-1)}$. In the k th step we define Θ as $\mathcal{T}_{r(k-1)}$ on the yet untiled part of the $\mathcal{T}_{r(k)}$ -saturation \mathbf{F}_k of the union $F_1 \cup F_2 \dots \cup F_{n_k}$. Note that Θ remains undefined on a union of complete tiles of $\mathcal{T}_{r(k)}$. Continuing in this way we will define the mixed tiling Θ on a set containing the union of all Følner sets F_n . If any part of the group remains untiled, we define Θ on that part as \mathcal{T}_0 . This concludes the construction of the mixed tiling Θ .

Observe that the mixed tiling Θ has the following properties:

- (2) Each F_n is covered only by tiles of those shapes S' for which F_n is $(S' S'^{-1}, \frac{1}{k})$ -invariant (where k is the largest index such that $n > n_k$). This implies that F_n differs from its Θ -saturation by at most $\frac{1}{k}|F_n|$ elements;

- (3) For each $k \geq 1$, Θ uses only finitely many tiles belonging to $\mathcal{T}_{r(k)}$.

As we have already explained earlier, Θ determines some $x \in \{0, 1\}^G$. It remains to verify the (F_n) -normality of x . Let $K \subset G$ be a nonempty finite set and let us fix some $\varepsilon > 0$. It is enough to show that, for n sufficiently large, $x|_{F_n}$ is $(K, 3\varepsilon)$ -normal. Pick $k \geq \frac{1}{\varepsilon}$ so large that all packages of orders larger than or equal to k are (K, ε) -normal (see (1) above). Choose $n \geq n_k$. In order to determine the parameter δ for which $x|_{F_n}$ is (K, δ) -normal we first replace F_n by its Θ -saturation. By (2), this affects the estimation of δ by at most ε . Next, we remove from this saturation all tiles of orders smaller than k (there are finitely many such tiles). If n is large enough, this last step also affects the estimation of δ by at most ε . Now it remains to examine the restriction of x to a set on which it is a concatenation of packages of orders at least k . By the choice of k , this restriction is (K, ε) -normal. It follows that x restricted to F_n is $(K, 3\varepsilon)$ -normal, as required. This concludes the proof. \square

Via Theorem 2.12 and Lemma 2.13, the above construction applies also to cancellative semigroups.

Corollary 5.3. *Let G be a countably infinite amenable cancellative semigroup and let (F_n) be a Følner sequence in G . Then there exists an effectively defined (F_n) -normal subset of G .*

6. Multiplicative normality

In this section we will focus on the action of (\mathbb{N}, \times) on the symbolic space $\{0, 1\}^{\mathbb{N}}$. In this case the shift action, henceforth called the *multiplicative shift* and denoted by $(\rho_n)_{n \in \mathbb{N}}$ is defined on $\{0, 1\}^{\mathbb{N}}$ as follows:

$$\text{if } x = (x_j)_{j \in \mathbb{N}} \text{ then } \rho_n(x) = (x_{jn})_{j \in \mathbb{N}}.$$

In other words, ρ_n maps each binary sequence to its subsequence obtained by reading its every n th term. Clearly, the classical $(\frac{1}{2}, \frac{1}{2})$ -Bernoulli measure on the symbolic space $\{0, 1\}^{\mathbb{N}}$ is invariant under both the additive and multiplicative shift actions, and in both cases it is the unique measure of maximal entropy. In fact we are dealing here with the case of a sequence of independent identically distributed random variables, which corresponds to the Bernoulli process regardless of the applied action, as long as the action “permutes” the indices (we use quotation marks, because our “permutations” are not surjective). Note that both shift actions are ergodic (in fact mixing) and their Kolmogorov-Sinai entropies equal the entropy of the generating partition $\{[0], [1]\}$, i.e., to $\log 2$, and so are the topological entropies of both shift actions. For a treatment of entropy for actions of amenable groups see for example [41].

6.1. Følner sequences in (\mathbb{N}, \times)

The semigroup (\mathbb{N}, \times) is a free Abelian semigroup generated by the set of primes. We will denote the set of primes by \mathbb{P} and view (\mathbb{N}, \times) as the direct sum¹³

$$\mathbb{G} = \bigoplus_{p \in \mathbb{P}} \mathbb{N}_p,$$

where, for each $p \in \mathbb{P}$, \mathbb{N}_p is the same additive semigroup $(\mathbb{N} \cup \{0\}, +)$. The isomorphism is given by

$$(k_1, k_2, \dots, k_r) \mapsto p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r},$$

where $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. are consecutive prime numbers. Notice that \mathbb{G} is *additive*, i.e., in this representation multiplication of natural numbers is interpreted as addition of vectors.

In order to deal with the actions of (\mathbb{N}, \times) it is crucial to identify convenient choices of Følner sequences in this semigroup. A natural choice of a Følner sequence in \mathbb{G} is given by *anchored* (i.e., containing the origin) rectangular boxes of the form

$$F = \{0, 1, \dots, k_1\} \times \{0, 1, \dots, k_2\} \times \cdots \times \{0, 1, \dots, k_d\} \times \{0\} \times \{0\} \times \cdots. \quad (6.1)$$

The parameter d (i.e., the largest index i such that $k_i > 0$) will be referred to as the *dimension* of F . The number k_i will be called the *size of F in the i th direction*. Let now

$$F_n = \{0, 1, \dots, k_1^{(n)}\} \times \{0, 1, \dots, k_2^{(n)}\} \times \cdots \times \{0, 1, \dots, k_{d(n)}^{(n)}\} \times \{0\} \times \{0\} \times \cdots \quad (6.2)$$

be a sequence of anchored rectangular boxes. With this notation, (F_n) is a Følner sequence in \mathbb{G} if and only if $\lim_n d_n = \infty$ and $\lim_n k_i^{(n)} = \infty$ for each $i \in \mathbb{N}$. Any such Følner sequence will be called *anchored rectangular*. The verification of the Følner property is straightforward. Preferably, the Følner sets should increase with respect to inclusion, which means that the sequences (d_n) and $(k_i^{(n)})$ for each i should be nondecreasing and the sum $k_1^{(n)} + k_2^{(n)} + \cdots + k_{d_n}^{(n)}$ should be strictly increasing. Such increasing Følner sequences will be called *nice*. Not every nice Følner sequence (F_n) is tempered. However, since the cardinalities $|F_n|$ strictly increase, Theorem 4.2 applies.

Every nice Følner sequence occurs as a subsequence of a specific *nice and slow* Følner sequence, such that at each step the sum $k_1^{(n)} + k_2^{(n)} + \cdots + k_{d_n}^{(n)}$ increases by 1. The choice of a nice and slow Følner sequence is equivalent to fixing a “sequence of directions” (i_n) , in which every natural number appears infinitely many times, and

¹³ Recall that a direct sum is the subset of the Cartesian product (with addition acting coordinate-wise) consisting of points with at most finitely many nonzero coordinates.

letting $k_i^{(n)} = |\{j \in \{1, \dots, n\} : i_j = i\}|$. There are several fairly natural options for choosing the sequence (i_n) , for example:

$$1; 1, 2; 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5; \dots \text{ the staircase type,} \quad (6.3)$$

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, \dots \text{ the Toeplitz type.} \quad (6.4)$$

Now we can translate all this to the multiplicative representation (\mathbb{N}, \times) . In (\mathbb{N}, \times) we have the natural partial order given by $m \preceq M \iff m|M$. The set \mathbb{N} equipped with this order is a *directed set* (i.e., every two elements have a common upper bound; in this case a common multiple). A sequence of natural numbers (L_n) *multiplicatively tends to infinity*, if for any $m \in \mathbb{N}$ there exists n_0 such that $m \preceq L_n$ for all $n \geq n_0$. If, in addition, the sequence L_n strictly increases with respect to the multiplicative order, we will say that (L_n) *multiplicatively increases to infinity*. For an anchored rectangular box $F \subset \mathbb{G}$ (see (6.1)) the number $L = p_1^{k_1} p_2^{k_2} \cdots p_d^{k_d}$ will be called the *leading parameter* of F . Interpreting F as a subset of (\mathbb{N}, \times) , notice that $F = \{m : m \preceq L\}$ (i.e., F is the set of all divisors of L). With this terminology, a sequence of anchored rectangular boxes in (\mathbb{N}, \times) is a Følner sequence (resp. nice Følner sequence) if and only if the sequence (L_n) of their leading parameters multiplicatively tends (resp. multiplicatively increases) to infinity. A nice Følner sequence (F_n) in (\mathbb{N}, \times) is nice and slow if and only if $\frac{L_{n+1}}{L_n}$ is a prime for every n . Notice that even if (F_n) is nice and slow, the cardinalities $|F_n|$ grow relatively fast. Indeed, from time to time the dimension d_{n+1} of F_{n+1} has to increase, i.e., a new direction has to be included, and then the cardinality doubles: $|F_{n+1}| = 2|F_n|$. Otherwise the cardinality is multiplied by a factor smaller than 2, but in any case a rectangular box of dimension $d_n - 1$ is added. In particular, $|F_{n+1}| - |F_n| > 1$ for $n > 1$.

Obviously, there are many other Følner sequences in \mathbb{G} . The rectangular boxes need not be anchored at zero, and moreover, they can be replaced by other shapes. For instance, it is possible to create a Følner sequence with $|F_n| = n$, but it is not going to be rectangular (however, it may have a nice and slow Følner subsequence). We skip further details. While there is no preferred “canonical” choice for a Følner sequence in \mathbb{G} , it will be convenient for our purposes to focus on anchored rectangular, and, in particular, on nice Følner sequences (mainly due to advantageous arithmetic properties of their multiplicative interpretation).

6.2. Multiplicative Champernowne set

The construction of a Champernowne set in (\mathbb{N}, \times) can be made significantly more transparent than in the general case discussed in Section 5. This is due to the fact that the semigroup (\mathbb{N}, \times) admits a system of (congruent, deterministic, Følner, syndetic) *monotilings*, i.e., tilings with only one shape. In fact, any rectangular box tiles the semigroup, while a congruent system of tilings is obtained from a specific Følner sequence, which we will call *doubling*. This will enable us to create “condensed” packages which contain every brick exactly once (like in the classical Champernowne construction). For every k , the k th chain still has to contain more

than one repetition of every package of order k (this we would have to do even in the two-dimensional semigroup $(\mathbb{N}^2, +)$), but we will use the least possible number of repetitions to fill a rectangular box the size of the next order package. In this manner we will obtain a “compensious” Champernowne set, which will turn out to be normal at least with respect to the same doubling Følner sequence which is used in its construction. Later we will present a slight modification of the same construction, which produces a “net-normal” set, *i.e.*, normal with respect to any nice Følner sequence, at the cost of repeating each package of order k an infinite number of times.

We begin by formally introducing the notion of a doubling Følner sequence. Again, we will interpret (\mathbb{N}, \times) as the additive semigroup \mathbb{G} .

Definition 6.1. A nice Følner sequence (F_n) is called *doubling* if F_{n+1} is a disjoint union $F_n \cup (v_n + F_n)$ for some $v_n \in \mathbb{G}$.

Note that since F_{n+1} is an anchored rectangular box, v_n must be equal to one of the vectors spanning F_n , *i.e.*, if

$$F_n = \{0, 1, \dots, k_1^{(n)}\} \times \{0, 1, \dots, k_2^{(n)}\} \times \dots \times \{0, 1, \dots, k_{d_n}^{(n)}\} \times \{0\} \times \{0\} \times \dots,$$

then v_n is of the form $(0, 0, \dots, 0, k_i^{(n)} + 1, 0, 0, \dots)$, where $k_i^{(n)} + 1$ occurs as the i th term, $i = 1, 2, \dots, d_n$, or $v_n = (0, 0, \dots, 0, 1, 0, 0, \dots)$, where 1 occurs at a position larger than d_n .

Any doubling Følner sequence can be obtained by the following procedure. As before, in the construction of a nice and slow Følner sequence, we fix a sequence of directions $(i_n)_{n \geq 1}$ in which each natural i is repeated infinitely many times. We begin with the “zero Følner set” $F_0 = \{0\}$. Once the Følner set F_n is determined, the next one, F_{n+1} , instead of growing by a unit in the direction i_{n+1} , is *doubled* in that direction. The cardinality of F_n will hence be equal to 2^n . For example, if (i_n) is the staircase sequence $1; 1, 2; 1, 2, 3; \dots$, the first six Følner sets are

$$\begin{aligned} F_0 &= \{0\} \\ F_1 &= \{0, 1\} \\ F_2 &= \{0, 1, 2, 3\} \\ F_3 &= \{0, 1, 2, 3\} \times \{0, 1\} \\ F_4 &= \{0, 1, 2, 3, 4, 5, 6, 7\} \times \{0, 1\} \\ F_5 &= \{0, 1, 2, 3, 4, 5, 6, 7\} \times \{0, 1, 2, 3\} \\ F_6 &= \{0, 1, 2, 3, 4, 5, 6, 7\} \times \{0, 1, 2, 3\} \times \{0, 1\}. \end{aligned}$$

(for convenience we skip the infinite product of singletons $\{0\}$ that should follow to the right in the formula for each of the above sets).

6.2.1. The construction

We shall now construct an (F_n) -normal element $x \in \{0, 1\}^{\mathbb{N}}$ for a doubling Følner sequence. We describe how we build the bricks and packages, and how we place

them in x (we will use the language of chains rather than that of mixed tilings). The bricks of order k will simply be blocks over the Følner set F_k (of cardinality 2^k), *i.e.*, the bricks will belong to $\{0, 1\}^{F_k}$. We accept as bricks of order k all blocks over F_k . Thus there will be 2^{2^k} different bricks, which, when concatenated together (each used exactly once), produce a package of cardinality 2^{2^k+k} . Since the sizes in all directions of all our objects (Følner sets, bricks, packages, etc.) are powers of 2, we can arrange the package so that it is a block over F_{2^k+k} (the index $2^k + k$ plays the role of $r(k)$ from the general construction). We define the 0th chain as the concatenation of 4 copies of the package of order zero arranged to fill a block over F_3 (see Figure 6.1 below). For $k \geq 1$ we assume inductively that the $(k-1)$ st chain is a block over the same set as the package of order k (for $k = 1$ this holds). This assumption guarantees that the $(k-1)$ st chain is saturated with respect to the tiling number r_k , so it can be concatenated together with packages of order k without gaps or overlaps. Now we can build the k th chain. It consists of

- (i) the $(k-1)$ st chain occupying the “lower left” corner, *i.e.*, containing the origin, and
- (ii) $2^{2^k+1} - 1$ shifted copies of the k th order package,

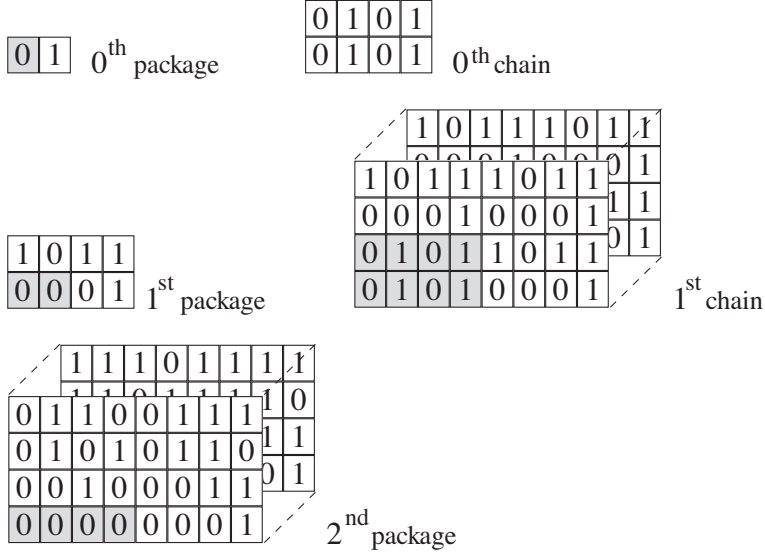
so that the chain has cardinality $2^{2^{k+1}+k+1}$. The chain can be arranged to be a block over $F_{2^{k+1}+k+1}$, *i.e.*, over the same set as the package of order $(k+1)$, as required in the induction.

Figure 6.1 corresponds to the (mentioned above) “staircase type” doubling Følner sequence. It shows the package of order 0 with the initial “zero” brick of order 0 shaded, and next to it the 0th chain, which is a concatenation of four such packages. In the next line we show the package of order 1 with the initial “zero” brick of order 1 shaded, and next to it the 1st chain which is a concatenation of the preceding chain (shaded) and seven identical packages of order 1. The last picture shows the package of order 2 with the initial “zero” brick of order 2 shaded. The 2nd chain is too large to be shown. It is a concatenation of the 1st chain and 31 copies of the package of order 2, and it is a block over F_{11} (which is four-dimensional).

The chains converge to an element $x \in \{0, 1\}^{\mathbb{G}}$. The set $\{g \in \mathbb{G} : c(g) = 1\}$ and the real number with binary expansion x will be called the *multiplicative Champernowne set* and *multiplicative Champernowne number*, respectively.

6.2.2. (F_n) -normality of x

Recall that given a nonempty finite set K and $\varepsilon > 0$, for some large k_0 , packages of orders $k \geq k_0$ are multiplicatively (K, ε) -normal. So, to prove (F_n) -normality of x we need to check that, as n increases, we have $\frac{a_n}{|F_n|} \rightarrow 1$, where a_n is the cardinality of the portion F'_n of F_n such that $x|_{F'_n}$ is a concatenation of packages of orders $k \geq k_0$. First, we will check this for indices n of the special form $n = r(k) = 2^k + k$. For such an n , F_n is filled with the k th chain, consisting of the $(k-1)$ st chain and many (precisely, $2^{2^k+1} - 1$) packages of order k (having the same size as the $(k-1)$ st chain), so these packages “dominate” in $x|_{F_n}$ (precisely, $\frac{a_n}{|F_n|} \geq 1 - \frac{1}{2^{2^k+1}}$). If a large

**Figure 6.1.**

n is not of this form, then, for some $k \geq k_0$, we have $2^k + k < n < 2^{k+1} + k + 1$, and the block $x|_{F_n}$ is a concatenation of the k th chain and some number of packages of order $k + 1$, so $\frac{a_n}{|F_n|} > \frac{a_{2^k+k}}{|F_{2^k+k}|}$. This completes the proof of (F_n) -normality of x .

Remark 6.2. We can also deduce multiplicative normality of x with respect to the nice and slow Følner sequence of which (F_k) is a subsequence (to obtain such a nice and slow Følner sequence, instead of doubling a direction we increase it by 1 several times). We omit the details. On the other hand, x is definitely not normal for some other nice Følner sequences. For instance, if the Følner sets increase in the first direction much faster than in other directions (elongated shapes) then the symbol 0 will prevail. We skip the details again.

6.3. Net-normal sets

The notion of an anchored rectangular box or Følner sequence is meaningful not only in \mathbb{G} , but also in \mathbb{N}^d ($d \in \mathbb{N}$) with addition. The elements of such a sequence are d -dimensional anchored rectangular boxes given by

$$F = \{0, 1, \dots, k_1\} \times \{0, 1, \dots, k_2\} \times \dots \times \{0, 1, \dots, k_d\} \quad (6.5)$$

(cf. (6.1)). Denoting by G either \mathbb{G} or \mathbb{N}^d for some $d \in \mathbb{N}$, let \mathcal{F}_G stand for the family of all anchored rectangular boxes in G . In either case, this family, ordered by inclusion¹⁴, is a directed set: any two such boxes are contained in a third one.

¹⁴ When working with (\mathbb{N}, \times) rather than with \mathbb{G} , the above order on \mathcal{F}_G coincides with the multiplicative order \preceq (see Section 6.1) applied to the respective leading parameters.

So any function with domain \mathcal{F}_G is a *net*. With slight abuse of terminology, the directed set \mathcal{F}_G will be called the *Følner net* (formally, this term should refer to the identity function on \mathcal{F}_G).

Now we introduce the notion of net-normality.

Definition 6.3. Let G be either \mathbb{G} or \mathbb{N}^d for some $d \in \mathbb{N}$. A set $A \subset G$ (as well as its indicator function $1_A \in \{0, 1\}^G$) is *net-normal* if for any finite set $K \subset G$ and every block $B \in \{0, 1\}^K$, the net of averages (indexed by $F \in \mathcal{F}_G$)

$$\frac{1}{|F|} \mathbf{N}(B, x, F)$$

converges¹⁵ to $2^{-|K|}$ (comp. with (1.7)). If $G = \mathbb{G}$ is interpreted as the multiplicative semigroup (\mathbb{N}, \times) , a net-normal set $A \subset \mathbb{N}$ (and its indicator function $1_A \in \{0, 1\}^{\mathbb{N}}$) will be called *multiplicatively net-normal*.

Proposition 6.4. *A set $A \subset \mathbb{N}$ is multiplicatively net-normal if and only if it is (F_n) -normal with respect to every anchored rectangular Følner sequence (F_n) in (\mathbb{N}, \times) . Also, A is multiplicatively net-normal if and only if it is (F_n) -normal with respect to every nice (i.e., anchored rectangular and increasing by inclusion) Følner sequence (F_n) in (\mathbb{N}, \times) .*

Proof. Every anchored rectangular Følner sequence is a *subnet* of the Følner net, hence net-normality implies normality with respect to any anchored rectangular Følner sequence. The fact that normality with respect to every nice Følner sequence implies net-normality follows from the trivial observation that the failure of convergence of any countable net can be detected along some increasing subsequence of that net (in our case, an increasing subsequence of the Følner net is a nice Følner sequence). \square

It is natural to inquire about the existence of net-normal sets and their typicality, in \mathbb{N}^d and in \mathbb{G} . Curiously enough, it turns out that the answers are different for \mathbb{N}^d and \mathbb{G} . This difference is captured by the following two theorems.

Theorem 6.5. *For any $d \geq 1$, almost every (with respect to the Bernoulli measure λ) element of $\{0, 1\}^{\mathbb{N}^d}$ is net-normal.*

Proof. It is not hard to check that the proof of Theorem 4.2 works also for countable Følner nets. It now suffices to notice that in \mathbb{N}^d the sum $\sum_{F \in \mathcal{F}_{\mathbb{N}^d}} e^{-|F|}$ is finite. \square

The above argument fails for \mathbb{G} , because the sum $\sum_{F \in \mathcal{F}_{\mathbb{G}}} e^{-|F|}$ diverges (for example, there are infinitely many anchored rectangular boxes of cardinality 2). In fact, we have the following theorem.

¹⁵ A net $\iota \mapsto a_\iota$ of real numbers, indexed by a directed set (I, \geq) , converges to a limit a if for every $\varepsilon > 0$ there exists ι_0 such that $|a_\iota - a| < \varepsilon$ for every $\iota \geq \iota_0$ in I .

Theorem 6.6. *The collection of all net-normal elements in $\{0, 1\}^{\mathbb{G}}$ has measure zero for the Bernoulli measure λ .*

Proof. For λ -almost every $x \in \{0, 1\}^{\mathbb{G}}$ we will construct a nice (in fact doubling) Følner sequence $(F_n(x))_{n \in \mathbb{N}}$ for which x is not $(F_n(x))$ -normal. By Proposition 6.4 this will imply that any such x is not net-normal. For even n the definition of $F_n(x)$ will depend on x and will apply to a subset of full measure of the set of points x for which $F_{n-1}(x)$ was defined. For odd n , $F_n(x)$ will be defined for all points x for which $F_{n-1}(x)$ is defined (at step 1 this will be the whole space $\{0, 1\}^{\mathbb{G}}$). For even n the rectangular box $F_n(x)$ will grow (relatively to $F_{n-1}(x)$) in a “random” (i.e., depending on x) direction. However, for $F_n(x)$ to be a Følner sequence, the rectangles must grow in every direction infinitely many times. This property will be guaranteed by judicious (deterministic) choice of the directions at odd steps of the construction.

We start by defining $F_1(x)$ (for every $x \in X_1 = \{0, 1\}^{\mathbb{G}}$) as the “zero rectangle”:

$$F_1(x) = \{0\} \times \{0\} \times \{0\} \times \cdots.$$

Next, for each x and every $k \geq 1$ we consider the rectangle which is “doubled” in the k th direction:

$$F_1(x, k) = \{0\} \times \{0\} \times \{0\} \times \cdots \times \{0\} \times \{0, 1\} \times \{0\} \times \cdots,$$

where $\{0, 1\}$ appears at the k th position in the product. Note that the sets $F_1(x, k) \setminus F_1(x)$ (which at this step of construction are singletons) are disjoint for different k ’s, and hence the functions $x \mapsto x_{g_k}$ where $g_k \in F_1(x, k) \setminus F_1(x)$ form an i.i.d. sequence of random variables. Thus, there exists a full measure set $X_2 \subset \{0, 1\}^{\mathbb{G}}$, such that for every $x \in X_2$ there exists k such that $x_{g_k} = 0$. We let $k_1(x)$ be the smallest such k and we define $F_2(x)$ as $F_1(x, k_1(x))$. In this manner, for almost every x , we have guaranteed at least half of the symbols $x_g, g \in F_2(x)$, to be zeros. From now on we consider only the points $x \in X_2$.

Next, we produce $F_3(x)$ by doubling $F_2(x)$ in the direction provided (for example) by the staircase sequence (6.3). Since the first term of the staircase sequence is 1, we simply double the first coordinate:

$$F_3(x) = \begin{cases} \{0, 1\} \times \{0\} \times \{0\} \times \cdots \times \{0\} \times \{0, 1\} \times \{0\} \times \cdots & \text{if } k_1(x) > 1 \\ \{0, 1, 2, 3\} \times \{0\} \times \{0\} \times \cdots & \text{if } k_1(x) = 1. \end{cases}$$

This time for any x some two “random” symbols x_g with $g \in F_3(x) \setminus F_2(x)$ appear in the block $x|_{F_3(x)}$. We let $X_3 = X_2$. At the fourth step, for each x and every $k > k_1(x)$,¹⁶ we consider the rectangle which is “doubled” in the k th direction:

$$F_3(x, k) = \begin{cases} \{0, 1\} \times \{0\} \times \cdots \times \{0\} \times \{0, 1\} \times \{0\} \times \cdots \times \{0\} \times \{0, 1\} \times \{0\} \times \cdots & \text{if } k_1(x) > 1 \\ \{0, 1, 2, 3\} \times \{0\} \times \{0\} \times \cdots \times \{0\} \times \{0, 1\} \times \{0\} \times \cdots & \text{if } k_1(x) = 1, \end{cases}$$

¹⁶ The requirement $k > k_1(x)$ is inessential. We put it only to reduce the variety of possible formulas for $F_3(x, k)$.

where the last appearance of $\{0, 1\}$ takes place at the position k in the product. As before, there exists a set of full measure, $X_4 \subset X_3$, such that for every $x \in X_4$ there exists k for which all symbols x_g with $g \in F_3(x, k) \setminus F_3(x)$ are zeros (again, it is essential that the sets $F_3(x, k) \setminus F_3(x)$ are disjoint for different k 's). We let $k_2(x) > k_1(x)$ be the smallest such k and define $F_4(x) = F_3(x, k_2(x))$. In this way, we have guaranteed at least the fraction $\frac{1}{2} + \frac{1}{8}$ of zeros in the block $x|_{F_4(x)}$.

Continuing in this way, at the odd steps we will double the rectangles in the directions provided by the staircase sequence, and at the even steps (restricting to a full measure set) we will double the rectangles so that all the symbols x_g with $g \in F_n(x) \setminus F_{n-1}(x)$ (which constitutes half of $F_n(x)$) will be zeros.

It is clear that eventually, for λ -almost every x (more precisely for $x \in \bigcap_n X_n$), we will obtain a doubling Følner sequence $(F_n(x))$, such that the lower $(F_n(x))$ -density of zeros in x is at least $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{2}{3}$. Thus x is not $(F_n(x))$ -normal. \square

We remark that the set of net-normal elements of $\{0, 1\}^G$ is an intersection of the sets of (F_n) -normal elements for a family of Følner sequences (F_n) , hence, by Proposition 4.7, it is of the first category. Further, in view of the Theorem 6.6, in the case $G = \mathbb{G}$, it is not only topologically, but also measure-theoretically small. Nevertheless, we will prove in Theorem 6.8 that this set is nonempty, and moreover, as follows from Remark 6.9 below, it is even uncountable.

We begin with a preparatory lemma.

Lemma 6.7. *Fix some nonempty finite set $K \subset \mathbb{N}$ and $\varepsilon > 0$. Let $Z(K, \varepsilon)$ denote the union of all anchored rectangular boxes which are not multiplicatively (K, ε) -invariant. Then $Z(K, \varepsilon)$ has density zero with respect to any (not necessarily rectangular) Følner sequence (F_n) in \mathbb{G} .*

Proof. For sake of convenience we will write $[0, n]$ instead of $\{0, 1, 2, \dots, n\}$. Intuitively, a rectangular box is not multiplicatively (K, ε) -invariant if it is “narrow” in some direction, and narrow sets have density zero. More precisely, let $\bar{K} = [0, k_1] \times [0, k_2] \times \cdots \times [0, k_q]$ be the smallest rectangular box containing K . If a rectangular box $B = [0, b_1] \times [0, b_2] \times \cdots \times [0, b_r]$ is not (K, ε) -invariant, then it is not (\bar{K}, ε) -invariant, i.e., there is an index $i \in \{1, 2, \dots, q\}$ such that $b_i < \alpha_i$, where $\alpha_i = \frac{2k_i}{\varepsilon}$. Thus, the union $Z(K, \varepsilon)$ of all such rectangles B is contained in the finite union $\bigcup_{i=1}^q X_i$, where X_i is the set of all vectors in \mathbb{G} whose i th coordinate is smaller than α_i . It is clear that each set X_i has density zero with respect to any Følner sequence (F_n) , because its size in one of the directions is bounded. The proof is complete since density zero is preserved under finite unions. \square

Theorem 6.8. *There exists an effectively defined net-normal set $A \subset \mathbb{G}$.*

Remark 6.9. Given one net-normal set A we can easily produce a Cantor set of net-normal elements of $\{0, 1\}^G$, by altering the indicator function 1_A in all possible ways along some infinite subset of \mathbb{G} which has density zero for all nice Følner sequences (an example of such a subset is provided by any finitely-generated sub-semigroup; see also Lemma 6.7 above).

Proof of Theorem 6.8. The construction is a modification of the construction of an (F_n) -normal element for a doubling Følner sequence (F_n) (see Subsection 6.2.1). The bricks and packages will be the same (they depend on the choice of the sequence (F_n)). The mixed tiling will be different: this time, for each $k \geq 1$ it will contain infinitely many tiles of $\mathcal{T}_{r(k)}$. We can now describe the modification of the mixed tiling Θ (or, equivalently, of the chains) appearing in the construction 6.2.1. We continue to use the notation $r(k) = 2^k + k$ and keep denoting by $\mathcal{T}_{r(k)}$ the tiling by shifted copies of F_{2^k+k} . Step 0 is unchanged: the 0th chain is the concatenation of 4 packages of order zero arranged to fill a block over $F_3 = F_{r(1)}$. In the language of tilings, this defines Θ on $F_{r(1)}$ (as a partition into 4 rectangles), which clearly is a $\mathcal{T}_{r(1)}$ -saturated set. For $k \geq 1$ assume that at the steps $1, \dots, k-1$ we have defined Θ on a $\mathcal{T}_{r(k)}$ -saturated set. Now, at the step k , we consider the set $Z(F_k, \frac{1}{k})$, and its saturation \mathbf{Z}_k with respect to the tiling $\mathcal{T}_{r(k+1)}$. Part of \mathbf{Z}_k has been tiled in preceding steps (by tiles of orders $\mathcal{T}_{r(i)}$ with $i < k$), and this part is $\mathcal{T}_{r(k)}$ -saturated. We now tile the remaining part of \mathbf{Z}_k by the tiles of $\mathcal{T}_{r(k)}$. Due to the congruency of the system of tilings (\mathcal{T}_k) , in this manner we tile exactly the set \mathbf{Z}_k (which is $\mathcal{T}_{r(k+1)}$ -saturated), so that the inductive assumption is fulfilled for $k+1$. Notice that the sets \mathbf{Z}_k eventually fill up the whole group, thus the mixed tiling Θ is well defined on \mathbb{G} and it determines an element $x = 1_A \in \{0, 1\}^{\mathbb{G}}$.

By Proposition 6.4, it remains to verify multiplicative normality of x with respect to any nice Følner sequence (H_n) . As in (6.2.2), we need to show that for each k_0 we have $\frac{a_n}{|H_n|} \rightarrow 1$, where a_n is the cardinality of the portion H'_n of H_n such that $x|_{H'_n}$ is a concatenation of packages of orders $k \geq k_0$, equivalently, the portion of H_n tiled by the tiles belonging to \mathcal{T}_{r_k} with $k \geq k_0$. In other words, we need to show the convergence $\frac{b_n}{|H_n|} \rightarrow 0$, where b_n is the cardinality of the portion $H_n \setminus H'_n$ of H_n tiled by the tiles belonging to \mathcal{T}_{r_k} with $k < k_0$. This convergence follows directly from three facts:

- tiles belonging to \mathcal{T}_{r_k} with $k < k_0$ appear only in \mathbf{Z}_{k_0} ;
- by Lemma 6.7, $Z(F_{k_0}, \frac{1}{k_0})$ has (H_n) -density zero;
- for any Følner sequence (H_n) in any countably infinite group G , if a set has (H_n) -density zero, then so does its saturation with respect to any proper (i.e., having finitely many shapes) tiling of the group, in particular, \mathbf{Z}_{k_0} has (H_n) -density zero. \square

Remark 6.10. Notice that for any countably infinite amenable (semi)group G it is impossible to find an element $x \in \{0, 1\}^G$ which is normal with respect to all Følner sequences. For instance, if A is (F_n) -normal for some Følner sequence (F_n) in $(\mathbb{N}, +)$, then its complement A^c contains arbitrarily long intervals, which constitute a Følner sequence disjoint from A . A similar argument applies to any amenable semigroup. The existence of a multiplicatively net-normal subset of (\mathbb{N}, \times) shows that the restriction to anchored rectangular boxes is a well balanced level of generality.

7. Combinatorial and Diophantine properties of additively and multiplicatively normal sets in \mathbb{N}

In this section we will be focusing on the combinatorial and Diophantine richness of normal sets in $(\mathbb{N}, +)$ and (\mathbb{N}, \times) . Before starting the discussion we review some terminology.

- (i) We say that a set $S \subset \mathbb{N}$ is *additively (multiplicatively) large* if there exists a Følner sequence (F_n) in $(\mathbb{N}, +)$ (resp. (\mathbb{N}, \times)) for which $\bar{d}_{(F_n)}(S) > 0$.
- (ii) A set S in a semigroup G is called *thick* if it contains a right translate of every finite set. The family of thick sets in G is denoted by $\mathcal{T}(G)$. Note that $S \in \mathcal{T}(\mathbb{N}, +)$ if and only if S contains arbitrarily long intervals, and that $S \in \mathcal{T}(\mathbb{N}, \times)$ if and only if S contains arbitrarily large sets of the form $a_n\{1, 2, \dots, n\} = \{a_n, 2a_n, \dots, na_n\}$.
- (iii) We say that $S \subset \mathbb{N}$ is *additively normal* if it is (F_n) -normal for some Følner sequence (F_n) in $(\mathbb{N}, +)$. If $(F_n) = (\{1, 2, \dots, n\})$, we will call S a *classical normal set*. Similarly, a set S is called *multiplicatively normal* if it is (F_n) -normal for some Følner sequence (F_n) in (\mathbb{N}, \times) (there is no classical notion in this case).¹⁷
- (iv) Let $(n_i)_{i=1}^\infty$ be a sequence of (not necessarily distinct) positive integers. The set

$$FS(n_i)_{i=1}^\infty = \{n_{i_1} + n_{i_2} + \dots + n_{i_k} : i_1 < i_2 < \dots < i_k, k \in \mathbb{N}\}$$

is called an *additive IP-set*. Likewise, the set

$$FP(n_i)_{i=1}^\infty = \{n_{i_1}n_{i_2} \dots n_{i_k} : i_1 < i_2 < \dots < i_k, k \in \mathbb{N}\}$$

is called a *multiplicative IP-set*.

7.1. Multiplicative versus additive density and normality – some basic observations

Recall that (F_n) -normality (additive or multiplicative) of an element $x \in \{0, 1\}^\mathbb{N}$ is defined as the property that for any finite set $K \subset \mathbb{N}$ and every block $B \in \{0, 1\}^K$ the (additive or multiplicative) shifts of B occur in x with (F_n) -density $2^{-|K|}$. We emphasize that the additive and multiplicative shifts of a block are quite different. For example, if w is a word over $\{1, 2, \dots, k\}$, its additive shift occurs at a position

¹⁷ We remark that in contrast to the set of classical normal numbers (which is of first category, see Corollary 4.8), the set of additively normal numbers is residual. Indeed, given a nonempty finite set $K \subset \mathbb{N}$ and $\varepsilon > 0$, it is easy to see that the set $S(K, \varepsilon)$ of all 0-1 sequences $x \in \{0, 1\}^\mathbb{N}$, such that there exists an interval $I \subset \mathbb{N}$ for which $x|_I$ is (K, ε) -normal, is open and dense. The countable intersection $\bigcap_{K, \varepsilon} S(K, \varepsilon)$ over all nonempty finite sets K and all rational $\varepsilon \in (0, 1)$ is residual and consists of additively normal sequences. Residuality of the set of additively normal numbers in $[0, 1]$ now follows by a proof similar to that of Corollary 4.8.

An analogous argument establishes the residuality of the set of multiplicatively normal numbers.

n of some x if $x|_{\{n+1, n+2, \dots, n+k\}} = w$, while its multiplicative shift occurs at n if $x|_{\{n, 2n, 3n, \dots, kn\}} = w$.

Theorem 4.2 implies that for any Følner sequence (F_n) in $(\mathbb{N}, +)$ such that $|F_n|$ increases, λ -almost every $x \in \{0, 1\}^{\mathbb{N}}$ is (F_n) -normal and, similarly, for any Følner sequence (K_n) in (\mathbb{N}, \times) such that $|K_n|$ increases, λ -almost every x is (K_n) -normal. So λ -almost every x is both additively (F_n) -normal and multiplicatively (K_n) -normal. On the other hand, the two notions of normality are “in general position”: additively normal sets can be multiplicatively trivial (have multiplicative density 0 or 1), and vice-versa, multiplicatively normal sets can have additive density 0 or 1. More precisely, the following holds.

Theorem 7.1. *For any Følner sequence $(K_n)_{n \in \mathbb{N}}$ in (\mathbb{N}, \times) there exists a set $A \subset \mathbb{N}$ with (K_n) -density 1 and having universal additive density 0 (here universal means that the additive density can be computed with respect to an arbitrary Følner sequence in $(\mathbb{N}, +)$).*

Proof. First observe that for any m the set $m\mathbb{N}$ (being a multiplicative shift of a set of (K_n) -density 1) has (K_n) -density 1. Let n_m be such that for each $n > n_m$ the fraction of multiples of $m!$ in K_n is larger than $1 - \frac{1}{m}$. The set A is defined as the union $K_1 \cup K_2 \cup \dots \cup K_{n_2}$ to which we add all multiples of $2!$ contained in the union $K_{n_2+1} \cup K_{n_2+2} \cup \dots \cup K_{n_3}$, all multiples of $3!$ contained in the union $K_{n_3+1} \cup K_{n_3+2} \cup \dots \cup K_{n_4}$, etc. It is obvious that the (K_n) -density of A equals 1. On the other hand, since A has gaps which tend to infinity in length, its additive density is equal to 0 (for any additive Følner sequence). \square

Note that the complementary set A^c has (K_n) -density 0 and universal additive density 1. Further, given a Følner sequence (F_n) in $(\mathbb{N}, +)$, let $B \subset \mathbb{N}$ be a set which is both multiplicatively (K_n) -normal and additively (F_n) -normal. Then $A \cap B$ is multiplicatively (K_n) -normal, while it has universal additive density zero, and on the other hand, $A^c \cap B$ is additively (F_n) -normal and has multiplicative (K_n) -density 0. These examples justify our claim above that the notions of multiplicative and additive normality are in “general position”.

Remark 7.2. A statement symmetric to Theorem 7.1, in which one fixes a Følner sequence (F_n) in $(\mathbb{N}, +)$ (for instance the classical one) and looks for a set of (F_n) -density 1 and universal multiplicative density 0, does not hold. As a matter of fact, any set of upper density 1 with respect to the classical Følner sequence in $(\mathbb{N}, +)$ has density 1 with respect to some Følner sequence in (\mathbb{N}, \times) . Indeed, in the proof of [9, Theorem 6.3] it is shown that if A has classical upper density 1, so does $A/n \cap A$ for every n (see Definition 7.15 below). By an obvious iteration, we get that $A \cap A/2 \cap A/3 \cap \dots \cap A/n$ is nonempty, which implies that A contains arbitrarily large sets of the form $a_n\{1, 2, \dots, n\}$, i.e., A is multiplicatively thick (see (ii) above). This, in turn, implies that for some Følner sequence (K_n) in (\mathbb{N}, \times) one has $d_{(K_n)}(A) = 1$.

On the other hand, for every $\varepsilon > 0$ there are sets $A \subset (\mathbb{N}, +)$ with $d(A) \geq 1 - \varepsilon$ such that A has universal multiplicative density zero (take for example all numbers not divisible by some large n).

7.2. Elementary combinatorial properties of additively and multiplicatively normal sets

The above Theorem 7.1 and Remark 7.2 hint that, in general, the combinatorial properties of additively and multiplicatively normal sets are distinct. We will see below that this is indeed the case.

In this subsection we will focus on properties of additively/multiplicatively normal sets which follow from the fact that these sets are additively/multiplicatively thick. Since we are interested in properties of normal sets, in the statements of our theorems we will make the ostensibly stronger assumption that the sets in question are normal rather than just thick. Note that, since every thick set obviously contains an (F_n) -normal set for some Følner sequence (F_n) , in all theorems in this subsection the normality and thickness assumptions are in fact equivalent.

For example, it is not hard to see that every thick, in particular every normal, set contains an IP-set (this applies to both additive and multiplicative setups). Now, IP-sets can be defined as solutions of (an infinite) system of certain equations, and in our quest for patterns in normal sets, it is natural to inquire which Diophantine equations and systems thereof are always solvable in normal sets. The following two theorems shed some light on this question.

Theorem 7.3. *If S is a multiplicatively normal set then any homogeneous system of finitely many polynomial equations (with several variables) which is solvable in \mathbb{N} is solvable in S .*

Proof. Since $S \in \mathcal{T}(\mathbb{N}, \times)$ (i.e., is multiplicatively thick), S contains arbitrarily large sets of the form $a_n\{1, 2, \dots, n\}$. If a given homogeneous system is solvable in \mathbb{N} then it is solvable in $\{1, 2, \dots, n\}$ for some n and hence, due to homogeneity, also in $a_n\{1, 2, \dots, n\}$. \square

Remark 7.4. Note that it follows from Theorem 7.3 that any multiplicatively normal set contains, for any $m \in \mathbb{N}$, “finite-sums sets” of the form

$$FS(n_i)_{i=1}^m = \{n_{i_1} + n_{i_2} + \dots + n_{i_k} : i_1 < i_2 < \dots < i_k \leq m, k \in \{1, 2, \dots, m\}\}.$$

Indeed, these sets can be described as solutions of finite homogeneous systems of linear equations¹⁸. On the other hand, we will now show that, in general, multiplicatively normal sets need not contain additive IP-sets $FS(n_i)_{i=1}^\infty$ or shifts thereof. Take any Følner sequence (F_n) in (\mathbb{N}, \times) and let A be the set of (F_n) -density 1 constructed in the proof of Theorem 7.1 (A has universal additive density zero). For

¹⁸ The set $FS(n_i)_{i=1}^m$ is the solution of the following system of equations (with variables n_T):

$$n_T = \sum_{i \in T} n_i,$$

where T ranges over all nonempty finite subsets of the set $\{1, 2, \dots, m\}$ (this applies also to $m = \infty$).

each n , this set contains only finitely many numbers not divisible by n . On the other hand, it is well known (and also easy to see) that every additive IP set contains, for arbitrarily large n , infinitely many numbers divisible by n as well as infinitely many numbers not divisible by n . Thus any (shifted or not) additive IP-set contains infinitely many numbers not divisible by n , and hence cannot be contained in A .

Remark 7.5. We remark that additively normal sets (even the classical ones) need not contain multiplicative IP-sets. In fact, they do not need to contain triples of the form $\{a, b, ab\}$ (see [25]).

Theorem 7.6. *Let $A\vec{x} = 0$ be a partition-regular (see Introduction) system of finitely many linear equations with n variables. Then, for any additively normal set S one can find a solution $\vec{x} = (x_1, x_2, \dots, x_n)$ with all entries in S .*

Proof. The proof is short but uses same facts from Ramsey theory, topological dynamics and topological algebra in the Stone–Čech compactification $\beta\mathbb{N}$ viewed as a semitopological semigroup obtained by an extension of the operation in $(\mathbb{N}, +)$. Since this theorem forms only a rather small fragment of a big picture, in order to save space, we will be using some terms and results without giving all the needed details (but remedying this by providing pertinent references).

First, note that our additively normal set is thick and hence is a member of a minimal idempotent in $(\beta\mathbb{N}, +)$. Further, any member of a minimal idempotent in $(\beta\mathbb{N}, +)$ is a central set (see, for example, Definition 5.8 and Lemma 5.10 in [8]). Now it only remains to invoke the theorem due to Furstenberg which states that any central set contains solutions to any partition-regular system $A\vec{x} = 0$ ([29, Theorem 8.22]).¹⁹ \square

One can actually show that a system of linear equations is partition-regular if and only if it is solvable in any additively normal set (equivalently, in any thick set). For sake of simplicity we prove this equivalence in the case of one equation with three variables. Note that any such equation (which has at least one solution) can be written as $ia + jb = kc$ with $i, j, k \in \mathbb{N}$ and a, b, c as unknowns.

Theorem 7.7. *Let i, j, k be three natural coefficients. The following conditions are equivalent:*

- (1) $k \in \{i, j, i + j\}$;
- (2) the equation $ia + jb = kc$ is partition-regular;
- (3) the equation $ia + jb = kc$ is solvable in any thick set;
- (4) the equation $ia + jb = kc$ is solvable in any additively normal set.

Proof. Equivalence of (1) and (2) is well known. As a matter of fact, a necessary and sufficient condition for partition-regularity of an equation $i_1a_1 + i_2a_2 + \dots + i_na_n = 0$ is that some subset of coefficients sums up to zero, see for example [30].

¹⁹ See [26, Theorem 4.1] for a more general result of this kind, obtained by a different method.

Conditions (3) and (4) are equivalent since every additively normal set is thick, while every thick set contains the union $\bigcup_n F_n$ of some Følner sequence and—within this union—an additively normal set. Solvability of partition-regular linear equations in thick (and hence additively normal) sets is our Theorem 7.6. It remains to consider coefficients for which (1) does not hold and construct a thick set $A \subset \mathbb{N}$, which contains no solutions, *i.e.*, is such that $(iA + jA) \cap kA = \emptyset$.

Since $k \notin \{i, j, i + j\}$ there exist a rational number $\delta > 0$ such that

$$k[1, 1 + \delta] \cap (i[1, 1 + 2\delta] \cup j[1, 1 + 2\delta] \cup (i + j)[1, 1 + \delta]) = \emptyset.$$

Let

$$A = \bigcup_{n=1}^{\infty} I_n, \text{ where } I_n = r_n[1, 1 + \delta],$$

where the numbers $r_n \in \mathbb{N}$ are such that $r_n\delta \in \mathbb{N}$ and grow geometrically with a large ratio. Obviously, A is a thick set. Choose any $a, b \in A$. If a, b belong to the same interval I_n , then $ia + jb \in (i + j)r_n[1, 1 + \delta]$. If a, b belong to two different intervals, say I_m, I_n with $m < n$, then, since r_m is much smaller than r_n , $ia + jb$ is either in $ir_n[1, 1 + 2\delta]$ or in $jr_n[1, 1 + 2\delta]$. In any case, $ia + jb \notin kI_n$ and, due to the fast growth of r_n , $ia + jb \notin kI_l$ for any other l . So, $(iA + jA) \cap kA = \emptyset$, as needed. \square

Remark 7.8. We will show later (see Corollary 7.19) that any *classical* normal set A has the stronger property that any equation $ia + jb = kc$ with $i, j, k \in \mathbb{N}$ is solvable in A .

We conclude this subsection with a simple observation that additively normal sets always contain at least some modest amount of multiplicative structure.

Theorem 7.9. *Any additively normal set A contains “consecutive product sets” of the form $\{y_1, y_1y_2, \dots, y_1y_2 \cdots y_k\}$ with arbitrarily large k and $y_n \geq 2$.*

Proof. The result follows from the (almost obvious) fact that any thick set contains arbitrarily large product sets. \square

7.3. Covering property of translates of normal sets

The special case (for $(\mathbb{Z}, +)$) of the following result is implicit in [10]. We give a short proof for arbitrary countably infinite amenable groups (and cancellative semi-groups).

Lemma 7.10. *Let G be a countably infinite amenable group in which we fix arbitrarily a Følner sequence (F_n) . If A is an (F_n) -normal set and $B \subset G$ is infinite, then the set BA has (F_n) -density 1.*

Proof. Observe that if $K \subset G$ is nonempty finite then KA has (F_n) -density precisely $1 - 2^{-|K|}$. Indeed, $g \notin KA$ is equivalent to $K^{-1}g \cap A = \emptyset$, i.e., the indicator function $1_A|_{K^{-1}g} = 0$. By normality of A , the last equality holds for elements g whose (F_n) -density is $2^{-|K|}$. If B is infinite, the lower (F_n) -density of BA is larger than $1 - 2^{-k}$ for any k , so BA has (F_n) -density 1. \square

Corollary 7.11. *By Theorem 2.12, Lemma 7.10 holds in countably infinite amenable cancellative semigroups, in particular in $(\mathbb{N}, +)$ and (\mathbb{N}, \times) . Moreover, we also have that $B^{-1}A$ defined as the set of such $g \in G$ that $bg \in A$ for some $b \in B$, has (F_n) -density 1.*

Remark 7.12. The following useful observation generalizes [10, Theorem 2]: If $C \subset G$ has positive upper (F_n) -density then $bA \cap C$ has (F_n) -density zero for at most finitely many $b \in \mathbb{N}$. Indeed, if k is such that $\overline{d}_{(F_n)}(C) > 2^{-k}$, then for any $K \subset \mathbb{N}$ with $|K| = k$ one has $\overline{d}_{(F_n)}(KA \cap C) > 0$. If there were k different elements $b_1, \dots, b_k \in \mathbb{N}$ satisfying $d_{(F_n)}(b_i A \cap C) = 0$, then the set $K = \{b_1, \dots, b_k\}$ would violate the last inequality.

Example 7.13. The following example, in the classical setup of $(\mathbb{N}, +)$, shows that for an additively (in particular, classical) normal set A the complement of $B + A$ need not be finite, even if $B = A$. Start the construction by choosing a finite word w_1 with good normality properties. Let \tilde{w} denote the block obtained from w by switching zeros and ones and writing the symbols in reverse order. The concatenated word $v_1 = w_1 \tilde{w}_1$ also has good normality properties and is antisymmetric, i.e., it satisfies $v_1(k) = 1 - v_1(n_1 - k)$, for all $0 \leq k < n_1$, where n_1 is the length of v_1 . Note that if A is any set whose indicator function 1_A starts with v_1 then $A + A$ misses n_1 . Let w_2 be a word much longer than w_1 and with much better normality properties. Define v_2 as the concatenation $v_1 w_2 \tilde{w}_2 v_1$. This word starts with v_1 , is antisymmetric and has nearly as good normality properties as w_2 . If 1_A starts with v_2 then $A + A$ misses both n_1 and $n_2 = |v_2|$. Continuing in this fashion we will end up with an infinite set A which is normal and such that $A + A$ misses infinitely many integers.

Remark 7.14. On the other hand, thickness alone easily implies that $A - A = \mathbb{N}$ (where $A - A$ is understood as the set of positive differences of elements from A).

7.4. Divisibility properties of classical normal sets. First applications

Until the end of Section 7 we will be dealing with classical normal sets in $(\mathbb{N}, +)$ (and also, briefly, with net-normal sets in (\mathbb{N}, \times)). As we will see, they exhibit especially rich combinatorial structure (not shared by general additively or multiplicatively normal sets). In this subsection we focus on general linear equations with three variables in classical normal sets.

Definition 7.15. Given $A \subset \mathbb{N}$, and $n \in \mathbb{N}$, denote by A/n the set $\{m : nm \in A\}$ (formally, this is $\frac{1}{n}(A \cap n\mathbb{N})$ or, invoking the multiplicative shift, $1_{A/n} = \rho_n(1_A)$).

Lemma 7.16. *If A is a classical normal set, so is A/n for any $n \in \mathbb{N}$.*

Remark 7.17.

- (i) Lemma 7.16 says that $(x_k) \in \{0, 1\}^{\mathbb{N}}$ is classical normal if and only if, for every $n \in \mathbb{N}$, the sequence (x_{nk}) is classical normal. This result was proved in D. Wall's thesis [46]. We provide a different, ergodic proof.
- (ii) A nontrivial fact which is implicitly used in the proof is the *divisibility property* of the classical Følner sequence $F_n = \{1, 2, \dots, n\}$ in $(\mathbb{N}, +)$: for any $k \in \mathbb{N}$, (F_n/k) is essentially the same Følner sequence. For example, for $k = 3$, (F_n/k) is $(\emptyset, \emptyset, F_1, F_1, F_1, F_2, F_2, F_2, F_3, F_3, F_3, \dots)$.

Proof of Lemma 7.16. Suppose A/n is not normal, i.e., some word w having length k does not occur in the indicator function $1_{A/n}$ of A/n with the correct frequency 2^{-k} . This means that the “scattered” block \hat{w} (in which the entries of w appear along the arithmetic progression $\{n, 2n, \dots, kn\}$) occurs in 1_A starting at coordinates $m \in n\mathbb{N}$ with, say, upper density different from $2^k \frac{1}{n}$. Let y be the periodic sequence $y(i) = 1 \iff n|i$. Now consider the pair $(1_A, y)$ (it is convenient to imagine this pair as a two-row sequence with 1_A written above y). This pair is a sequence over four symbols in $\{0, 1\}^2$. This means that the scattered block (\hat{w}, \hat{v}) where \hat{v} denotes the block of just 1's at each bottom position $\{n, 2n, \dots, kn\}$, occurs in $(1_A, y)$ with the upper density different from $2^k \frac{1}{n}$. There is a subsequence n_i such that the upper density is achieved along intervals $\{1, \dots, n_i\}$, and moreover, the corresponding sequence of normalized counting measures supported by the sets

$$\{(1_A, y), \sigma((1_A, y)), \sigma^2((1_A, y)), \dots, \sigma^{n_i}((1_A, y))\}$$

(where $\sigma((x, y)) = (\sigma(x), \sigma(y))$) converges in the weak-star topology to a shift-invariant measure μ on $(\{0, 1\}^2)^{\mathbb{N}}$. Since 1_A is normal (i.e., generic for the Bernoulli measure λ) and y is periodic (hence generic for the unique invariant probability measure ξ on the periodic orbit of y), the marginal measures of μ are λ and ξ . Now, $\mu([\hat{w}] \times [\hat{v}]) \neq 2^k \frac{1}{n} = \mu([\hat{w}])\xi([\hat{v}])$, which means that $\mu \neq \lambda \times \xi$. This contradicts disjointness of Bernoulli measures from periodic measures (which is a particular case of disjointness between K-systems and entropy zero systems, see [28]). \square

We can now derive another fact in the classical case.

Theorem 7.18. *If A is a classical normal set and $n, m \in \mathbb{N}$ are coprime, then both $nA + mA$ and $nA - mA$ (restricted to \mathbb{N}) have density 1.*

Proof. The theorem follows from the fact that, relatively, in every residue class mod n , (i.e., in the set $n\mathbb{N} + i$ for each $i = 0, 1, \dots, n-1$), the set $nA \pm mA$ has density 1. Indeed, since m, n are coprime, the set $(mA \mp i)/n$ is infinite (this follows already from the thickness of A). Then, by Lemma 7.10, the set $A \pm (mA \mp i)/n$, which we can write as $(nA \pm mA - i)/n$, has density 1. Hence $nA \pm mA$ has density 1 in the residue class of i . \square

Corollary 7.19. *For any $i, j, k \in \mathbb{N}$ there are $a, b, c \in A$ solving the equation $ia + jb = kc$.²⁰*

Proof. Restricting to $n\mathbb{N}$, where $n = \text{LCD}(i, j, k)$, we can assume that some two coefficients, for example i and j , are coprime (the other cases can be treated similarly). By Theorem 7.18, $iA + jA$ has relative density 1 in $n\mathbb{N}$, while kA has positive relative density in $n\mathbb{N}$, so $(iA + jA) \cap kA \neq \emptyset$. \square

Actually, one has a more general fact. In the theorem below we use the following terminology: a set $B \subset \mathbb{N}$ is called *divisible* if it contains multiples of every natural number n (note that then B/n is infinite for each n), and it is called *substantially divisible* if B/n has positive upper density for every n .

Theorem 7.20. *Let A, B, C be subsets of \mathbb{N} and assume that A is classical normal, B is divisible, and C is substantially divisible. Fix any $i, j, k \in \mathbb{N}$. Then the equation $ia + jb = kc$ is solvable with $a \in A, b \in B, c \in C$.*

Remark 7.21. The assumptions are satisfied when A, B, C are classical normal sets (the special case $A = B = C$ was treated in Corollary 7.19).

Proof of Theorem 7.20. Note that the set kC/i can be interpreted in two ways: as $k \cdot C/i$ or as $(kC)/i$ with the latter set being possibly larger than the former. Nevertheless, both sets have positive upper density. Also, regardless of the interpretation, the set jB/i is infinite. By Lemma 7.10, $(A + jB/i) \cap kC/i$ has positive upper density (the same as kC/i). Multiplying by i we obtain that $(iA + jB) \cap kC$ has positive upper density, in particular is nonempty. So, there exist (many) desired solutions. \square

7.5. Solvability of certain equations in net-normal sets

Motivated by the preceding subsection, let us now turn to the multiplicative semigroup (\mathbb{N}, \times) and multiplicative normality. The analogue of the equation $ia + jb = kc$ reads $a^i b^j = c^k$. To see this analogy even better, let us view (\mathbb{N}, \times) again as the direct sum \mathbb{G} which is an additive semigroup. Now the multiplicative equation $a^i b^j = c^k$ takes on the familiar additive form $ia + jb = kc$. The problem we immediately encounter in this “infinite-dimensional” semigroup is that if A is multiplicatively normal (even net-normal) then the set A/n (multiplicatively this is the set $\{m : m^n \in A\}$) need not be multiplicatively normal. In fact, it can even be empty, because the set $n\mathbb{G}$ (multiplicatively this is the set of n th powers) has universal multiplicative density zero. Below we provide an easy example of failure for the multiplicative equation $a^2 b^2 = c^3$, regardless of the Følner sequence in (\mathbb{N}, \times) .

²⁰ For a more general result of this type, presented in a different language and with a different proof see [26, Theorem 1.3.2].

Example 7.22. Let (F_n) be a Følner sequence in (\mathbb{N}, \times) and let A be an (F_n) -normal set (alternatively, it can be net-normal). By removing from A all squares (note that the set of squares is a set of universal multiplicative density 0), we can assume that A contains no squares. Then the elements c^3 with $c \in A$ are not squares either. Thus A contains no solutions of $a^2b^2 = c^3$.

This is why we will restrict our attention only to the case with $i = j = 1$, i.e., consider only equations of the form $ab = c^k$.

Theorem 7.23. *Let A and B be net-normal sets and let C contain infinitely many pairs (n, rn) , where r is fixed, while n tends to infinity with respect to the multiplicative order \preceq (i.e., for any $k \in \mathbb{N}$, large enough n is a multiple of k ; this holds, for instance, if C is (F_n) -normal with respect to some fixed Følner sequence in (\mathbb{N}, \times)). Then for any natural k there exist $a \in A, b \in B$ and $c \in C$ such that $ab = c^k$.*

Corollary 7.24. *If A is net-normal then for any natural k the equation $ab = c^k$ is solvable in A .*

Proof of Theorem 7.23. We continue to switch freely between the sets A, B, C and their indicator functions denoted $1_A, 1_B, 1_C$. We view (\mathbb{N}, \times) again as the additive semigroup \mathbb{G} . Thus our task becomes to find solutions of the equation $a + b = kc$ with $a \in A, b \in B, c \in C$. From now on, adjectives “small”, “nearly”, “close”, etc will refer to quantities (error terms, distances) that are estimated above by functions of ε tending to zero as $\varepsilon \rightarrow 0$. Fix a small $\varepsilon > 0$. By the assumption, we can find in C two elements n and $n + r$, where ($r \in \mathbb{G}$ is fixed a priori), with n multiplicatively so large that any anchored rectangle F with a leading parameter multiplicatively larger than or equal to $3n$ is (kr, ε) -invariant and has the property that both A and B have in F a proportion nearly $\frac{1}{2}$ (it is here that we are using net-normality of A and B).

Now suppose that there are no triples $a \in A, b \in B, c \in C$ satisfying $a + b = kc$. This implies that within the rectangle F with the leading parameter kn , A is disjoint from $kn - B$. Since the proportion of both sets in F is nearly $\frac{1}{2}$, these two sets are in fact nearly complementary within F , i.e., we can write $1_A(m) = 1 \iff 1_B(kn - m) = 0$ and this will be true except for a small percentage of m 's in F . The same holds with $n + r$ replacing n within the rectangle F' with the leading parameter $k(n + r)$, in particular, also in F (because by (kr, ε) -invariance, F is negligibly smaller than F'). This implies that the configuration of symbols in 1_A (and also in 1_B) within F is nearly invariant under the shift by kr , i.e., in most places $m \in F$ the symbols at m and $m + kr$ are the same. This contradicts net-normality of 1_A (and likewise of 1_B): if F is large enough then the proportion of pairs of identical symbols at positions m and $m + kr$ with $m \in F$ should be close to $\frac{1}{2}$, not to 1. \square

Example 7.25. Using an idea similar to that utilized in the proof of Theorem 7.7, we will show that assuming multiplicative normality with respect to just one nice Følner sequence may be insufficient for the solvability of the equation $ab = c^3$. We

continue to use the additive notation of \mathbb{G} . Let L_n be a multiplicatively increasing to infinity sequence of natural numbers. We assume that $5L_n \preccurlyeq L_{n+1}$ (recall that multiplicatively this means $L_n^5 | L_{n+1}$). Let F_n be the rectangle with the leading parameter $3L_n$ and let B_n be F_n with the rectangle with the leading parameter $2L_n$ removed. Let $B = \bigcup_n B_n$. Note that as soon as L_n is high-dimensional, say of a large dimension d (multiplicatively, this means that L_n is a product of [powers of] d different primes) then B_n constitutes the large fraction $1 - (\frac{2}{3})^d$ of F_n . It is now obvious that B has (F_n) -density 1. Consider the sum $a + b$ of two elements of B . Let n be the maximal index such that B_n contains either a or b . Then $a + b$ belongs to the rectangle with the leading parameter $6L_n$ with the rectangle with the leading parameter $2L_n$ removed (call this difference C_n). It is easy to see (it suffices to consider the one-dimensional case) that the union $\bigcup_n C_n$ is disjoint from $3B$. We have shown that $a + b = 3c$ has no solutions in B . Since B has (F_n) -density 1, it now suffices to intersect it with any (F_n) -normal set to get an (F_n) -normal set without the considered solutions.

It is now natural to ask: are all multiplicative equations $ab = c^k$ solvable in classical normal sets? Here the answer is known to be negative. In [25], A. Fish constructed normal sets of the form $A = \{n : f(n) = -1\}$, where f is a multiplicative function (so-called random Liouville function) $f : \mathbb{N} \rightarrow \{-1, 1\}$. In such sets there are clearly no solutions of the equations $ab = c^k$ for any odd k .²¹

7.6. Pairs $\{a + b, ab\}$ in classical additively normal sets

In this subsection we establish yet another nontrivial property of classical normal sets.

Theorem 7.26. *Let A be a classical normal set. For given $a \in \mathbb{N}$ define*

$$S_a = \{b : a + b \in A, ab \in A\}.$$

Then for every $a \in \mathbb{N}$ either S_a or S_{a^2} has positive upper density. In particular, A contains pairs $\{a + b, ab\}$ with arbitrarily large a and b .

Remark 7.27. The property stipulated in Theorem 7.26 does not necessarily hold for general additively normal sets. Indeed, one can construct an additively thick set which does not contain pairs $\{a + b, ab\}$ [9, Theorem 6.2]. Clearly, such a thick set contains an additively normal set with no pairs $\{a + b, ab\}$.

Proof of Theorem 7.26. It follows from the definition of the set A/n that $b \in S_a \iff b \in A/a \cap (A - a)$. Fix some $a \geq 2$ and suppose that both S_a and S_{a^2} have density zero. This can be written as

$$A \cap (A/a + a) \approx \emptyset \text{ and } A \cap (A/a^2 + a^2) \approx \emptyset,$$

²¹ On the other hand, the equation $ab = c^2$ is solvable in any classical normal set. This follows from the fact that classical normal sets contain geometric progressions of length 3, see Theorems 7.33 or 7.35 below.

where \approx means equality up to a set of density zero. Since every set in the above intersections has density $\frac{1}{2}$, we get $A/a + a \approx \mathbb{N} \setminus A$ and $A/a^2 + a^2 \approx \mathbb{N} \setminus A$, and in particular

$$A/a + a \approx A/a^2 + a^2.$$

Multiplying both sides by a we obtain

$$(A \cap a\mathbb{N}) + a^2 \approx (A/a \cap a\mathbb{N}) + a^3.$$

Since A/a and $A - a$ are nearly disjoint (the intersection has zero density), we also have that $(A/a \cap a\mathbb{N}) + a^3$ is nearly disjoint from $(A \cap a\mathbb{N}) + a^3 - a$. Plugging this into the last displayed formula we conclude that $(A \cap a\mathbb{N})$ is nearly disjoint from $(A \cap a\mathbb{N}) + a^3 - a^2 - a$. Dividing both sets by a , we get that A/a is nearly disjoint from $A/a + a^2 - a - 1$. Since A/a has density $\frac{1}{2}$, we have proved that the indicator function of A/a has the property that for n 's of density 1 its values at n and at $n + r$ (where $r = a^2 - a - 1$) are different. This contradicts Lemma 7.16 (normality of A/a), as in normal sets the density of such ns should be $\frac{1}{2}$. \square

7.7. Multiplicative configurations in classical normal sets

In this subsection we show that every classical normal set contains (up to scaling) all configurations which are known to be present in multiplicatively large sets. The following theorem is the main technical result allowing us to prove this fact.

Theorem 7.28. *Let $A \subset \mathbb{N}$ be a classical normal set. Then, for any Følner sequence (K_n) in (\mathbb{N}, \times) there exists a set E of (K_n) -density $\frac{1}{2}$ such that for any nonempty finite subset $\{n_1, n_2, \dots, n_k\} \subset E$ the intersection $A/n_1 \cap A/n_2 \cap \dots \cap A/n_k$ has positive upper density in $(\mathbb{N}, +)$.*

The key role in the proof of Theorem 7.28 will be played by the following theorem (cf. [5, Theorem 4.19] and [4, Theorem 2.1]).

Theorem 7.29. *Let (F_n) be a Følner sequence in $(\mathbb{N}, +)$, let $a \in (0, 1)$, and let $\mathcal{F} = \{A_1, A_2, \dots\}$ be a countable family of subsets in \mathbb{N} such that $d_{(F_n)}(A) \geq a$ for all $A \in \mathcal{F}$. Then there exists an invariant mean L on the space $B_{\mathbb{C}}(\mathbb{N})$ of bounded complex-valued functions such that*

- (i) $L(1_A) = d_{(F_n)}(A)$ for every $A \in \mathcal{F}$,
- (ii) for any $k \in \mathbb{N}$ and any $n_1, n_2, \dots, n_k \in \mathbb{N}$,

$$\overline{d}_{(F_n)}(A_{n_1} \cap A_{n_2} \cap \dots \cap A_{n_k}) \geq L(1_{A_{n_1}} \cdot 1_{A_{n_2}} \cdot \dots \cdot 1_{A_{n_k}}),$$

- (iii) *there exists a compact metric space X , a regular measure μ on $\mathbb{B}(X)$ (the Borel σ -algebra of X), and sets $\tilde{A}_n \in \mathbb{B}(X)$, $n \in \mathbb{N}$, such that for any $n_1, n_2, \dots, n_k \in \mathbb{N}$ one has*

$$L(1_{A_{n_1}} \cdot 1_{A_{n_2}} \cdot \dots \cdot 1_{A_{n_k}}) = \mu(\tilde{A}_{n_1} \cap \tilde{A}_{n_2} \cap \dots \cap \tilde{A}_{n_k}).$$

Proof. In the proof, when convenient, we will view L as a finitely additive measure on the family $\mathcal{P}(\mathbb{N})$ of all subsets of \mathbb{N} . Let \mathbb{S} be the (countable) family of all finite intersections of the form $A_{n_1} \cap A_{n_2} \cap \cdots \cap A_{n_k}$, where $A_{n_j} \in \mathcal{F}$, $j = 1, \dots, k$. By using the diagonal procedure we arrive at a subsequence (F_{n_i}) of our Følner sequence (F_n) , such that for any $S \in \mathbb{S}$ the limit

$$L(S) = \lim_{i \rightarrow \infty} \frac{|S \cap F_{n_i}|}{|F_{n_i}|} = \lim_{i \rightarrow \infty} \frac{1}{|F_{n_i}|} \sum_{m \in F_{n_i}} 1_S(m)$$

exists. Notice that $L(A) = d_{(F_n)}(A)$ for any $A \in \mathcal{F}$, and that for any $n_1, n_2, \dots, n_k \in \mathbb{N}$ we have

$$\begin{aligned} \bar{d}_{(F_n)} \left(\bigcap_{j=1}^k A_{n_j} \right) &= \limsup_{n \rightarrow \infty} \frac{\left| \left(\bigcap_{j=1}^k A_{n_j} \right) \cap F_n \right|}{|F_n|} \\ &\geq \lim_{i \rightarrow \infty} \frac{\left| \left(\bigcap_{j=1}^k A_{n_j} \right) \cap F_{n_i} \right|}{|F_{n_i}|} = L \left(\bigcap_{j=1}^k A_{n_j} \right). \end{aligned}$$

Extending by linearity, we get a linear functional L on a subspace $V \subset B_{\mathbb{R}}(\mathbb{N})$. By invoking the Hahn-Banach Theorem²², we can extend L from V to $B_{\mathbb{R}}(\mathbb{N})$. This L naturally extends to a functional on the space $B_{\mathbb{C}}(\mathbb{N})$, which satisfies conditions (i) and (ii).

We move now to proving (iii). Let \mathcal{A} be the uniformly closed and closed under conjugation algebra of functions on \mathbb{N} , which is generated by indicator functions 1_A of sets $A \in \mathcal{F}$. Then \mathcal{A} is a separable C^* -subalgebra of $\ell^\infty(\mathbb{N}, \|\cdot\|_\infty)$, and, by the Gelfand Representation Theorem, $\mathcal{A} \cong C(X)$, where X is a compact metric space. The restriction $L_{\mathcal{A}}$ of the mean L , which we constructed above, induces a positive linear functional \tilde{L} on $C(X)$, which by the Riesz Representation Theorem is given by a Borel measure μ .

Note that the isomorphism $\mathcal{A} \cong C(X)$ sends indicator functions of subsets of \mathbb{N} to indicator functions of subsets of X (because the isomorphism provided by the Gelfand transform preserves algebraic operations, and the indicator functions are the only ones which satisfy the equation $f^2 = f$). Let \tilde{A}_j be the subsets of X which correspond to sets $A_j \in \mathcal{F}$ (note that since $1_{\tilde{A}_j} \in C(X)$ for each j , the sets \tilde{A}_j are measurable). Clearly, we have

$$L_{\mathcal{A}}(A_{n_1} \cap A_{n_2} \cap \cdots \cap A_{n_k}) = \mu(\tilde{A}_{n_1} \cap \tilde{A}_{n_2} \cap \cdots \cap \tilde{A}_{n_k})$$

for any n_1, n_2, \dots, n_k . This completes the proof. \square

²² We remark that for our applications we need only a “restricted” version of Theorem 7.29 which deals with functional $L_{\mathcal{A}}$ on \mathcal{A} and does not need appealing to the Hahn-Banach Theorem.

The last result which is needed for the proof of Theorem 7.28, is the following theorem.

Theorem 7.30 (see Lemma 5.10 in [7]). *Let (K_n) be a Følner sequence in (\mathbb{N}, \times) , let (X, \mathcal{B}, μ) be a probability space, and let A_j , $j \in \mathbb{N}$, be measurable sets in X , satisfying $\mu(A_j) \geq a$ for some $a > 0$. Then there exists a set $E \in \mathbb{N}$ with $\bar{d}_{(K_n)}(E) \geq a$, such that for any nonempty finite set $F \subset E$, one has $\mu(\bigcap_{j \in F} A_j) > 0$.*

Proof of Theorem 7.28. The result in question follows from Theorem 7.29 applied to the Følner sequence $F_n = \{1, 2, \dots, n\}$ in $(\mathbb{N}, +)$ and $\mathcal{F} = \{A/n : n \in \mathbb{N}\}$ (and then we apply Theorem 7.30). Note that by Lemma 7.16, each A/n is a classical normal set and hence has density $a = \frac{1}{2}$. \square

It was shown in [6] that multiplicatively large sets in \mathbb{N} have very rich combinatorial structure (which is quite a bit richer than that of additively large sets). For example, any multiplicatively large set contains not only arbitrarily long geometric and arithmetic progressions, but also all kinds of more complex structures which involve both the addition and multiplication operations. Theorem 7.28 allows us to conclude that classical normal sets in \mathbb{N} are, in a way, as combinatorially rich as multiplicatively large sets. For example, we can combine it with the following theorems.

Theorem 7.31 (Theorem 3.10 in [6]). *Let $\mathcal{S}^a, \mathcal{S}^m$ be two families of finite subsets of \mathbb{N} with the following properties:*

- (i) *Any additively large set in \mathbb{N} contains a configuration of the form $a + F$, where $F \in \mathcal{S}^a$;*
- (ii) *Any multiplicatively large set in \mathbb{N} contains a configuration of the form bF , where $F \in \mathcal{S}^m$.*

Then any multiplicatively large set E contains a configuration of the form $bF_2(a + F_1)$, where $F_1 \in \mathcal{S}^a$ and $F_2 \in \mathcal{S}^m$.

Theorem 7.32 (Theorem 3.11 in [6]). *Let $E \subset \mathbb{N}$ be a multiplicatively large set. Let $S_1, S_2 \subset \mathbb{N}$ be two infinite sets and let $IP^a(S_1)$ and $IP^m(S_2)$ be the additive and multiplicative IP sets generated by S_1 and S_2 , respectively. Then for any $n \in \mathbb{N}$, there exist $a, b \in \mathbb{N}$, $d \in IP^a(S_1)$, and $q \in IP^m(S_2)$ such that*

$$\{bq^j(a + id), 0 \leq i, j \leq n\} \subset E.$$

Then we get the following result.

Theorem 7.33. *Let \mathcal{S}^a and \mathcal{S}^m be two families of finite sets in \mathbb{N} which have the following properties:*

- (i) *any additively large set in \mathbb{N} contains a configuration of the form $a + F_1$ for some $F_1 \in \mathcal{S}^a$ and $a \in \mathbb{N}$,²³*

²³ An example of \mathcal{S}^a is the family $\{\{r, 2r, \dots, nr\} : r \in \mathbb{N}\}$ with any fixed n (this follows from the classical Szemerédi Theorem).

- (ii) any multiplicatively large set in \mathbb{N} contains a configuration of the form bF_2 for some $F_2 \in \mathcal{S}^m$ and $b \in \mathbb{N}$.²⁴

Then any classical normal set $A \subset \mathbb{N}$ contains a configuration $bF_2(a + F_1)$ with $F_1 \in \mathcal{S}^a$, $F_2 \in \mathcal{S}^m$ and $a, b \in \mathbb{N}$.

In particular, any classical normal set A contains, for any $n \in \mathbb{N}$, configurations of the form $\{q^j(a + id) : 0 \leq i, j \leq n\}$ with some $q > 1$, $a, d \in \mathbb{N}$.

Proof. Theorem 7.31 tells us that any multiplicatively large set E contains a configuration of the form $bF_2(a + F_1)$ with $a, b \in \mathbb{N}$, $F_1 \in \mathcal{S}^a$, $F_2 \in \mathcal{S}^m$. For a classical normal set $A \subset \mathbb{N}$, we can take as E the set given by Theorem 7.28. Thus, E contains a set $\{n_1, n_2, \dots, n_k\}$ of the above form $bF_2(a + F_1)$. Then the intersection $A/n_1 \cap A/n_2 \cap \dots \cap A/n_k$ has positive additive upper density. In particular, this intersection contains some natural number c , and then $cbF_2(a + F_1) \subset A$.

To get the last statement of the theorem, one has to use Theorem 7.32 which guarantees the existence of configurations of the form $\{bq^j(a + id) : 0 \leq i, j \leq n\}$ with some $q > 1$, $a, b, d \in \mathbb{N}$. Observe that we may write $bq^j(a + id)$ as $q^j(a' + id')$. \square

Similarly, we can invoke another theorem.

Theorem 7.34 (Theorem 3.15 in [6]). *Let $E \subset \mathbb{N}$ be a multiplicatively large set. For any $k \in \mathbb{N}$ there exist $a, b, d \in \mathbb{N}$ such that $\{b(a + id)^j, 0 \leq i, j \leq k\} \subset E$.*

Then one gets the following result.

Theorem 7.35. *Any classical normal set contains, for any $n \in \mathbb{N}$, sets of the form $\{b(a + id)^j; 0 \leq i, j \leq n\}$ with some $a, b, d \in \mathbb{N}$.*

Remark 7.36. While Theorems 7.33 and 7.35 guarantee that any classical normal set contains arbitrarily long *finite* geometric progressions, it need not contain infinite geometric progressions. Indeed, it is not hard to construct a set of density zero which contains, for any $b, q \in \mathbb{N}$, a number of the form bq^j for some j . Removing this set from a classical normal set results in a desired example.

Remark 7.37. Note that Theorem 7.28, and thus Theorems 7.33 and 7.35, are not valid for general additively normal sets in $(\mathbb{N}, +)$. For example, one can show (see [3, Theorem 3.5]) that there exist additively thick sets which do not contain geometric progressions of length 3, $\{c, cr, cr^2\}$, where $r \in \mathbb{Q} \setminus \{1\}$ (cf. Remark 7.27).

Many of the results of Section 7 are valid in a wider setup, where one replaces normal sets with more general sets having strong enough randomness properties. See for example [26], where configurations in so-called weakly mixing sets are studied.

²⁴ An example of \mathcal{S}^m is the family $\{q, q^2, \dots, q^n\} : q \geq 1\}$ with any fixed n (see [6, Theorem 3.11]).

8. (F_n) -normal Liouville numbers

Let us recall that an irrational number x is called a *Liouville number* if for every natural k there exists a rational number $\frac{p}{q}$ such that $|x - \frac{p}{q}| < \frac{1}{q^k}$. Clearly, “for every k ” can be equivalently replaced by “for arbitrarily large k ” (if $\frac{p}{q}$ is good for k , it is also good for all $k' < k$). It is well known that the set \mathcal{L} of Liouville numbers is residual (dense G_δ) but its Lebesgue measure equals zero. This can be expressed concisely by saying that this set is T-large and M-small. On the other hand, Theorem 4.2 and Corollary 4.8 imply that the set $\mathcal{N}((F_n))$ of (F_n) -normal numbers is M-large and T-small (this applies to both additive and multiplicative normality; we recall that Theorem 4.2 requires a mild assumption on (F_n) which is satisfied e.g. when the sequence $|F_n|$, $n = 1, 2, \dots$ is strictly increasing). Thus, it is a priori not clear whether the sets \mathcal{L} and $\mathcal{N}((F_n))$ have a nonempty intersection. In this section we will show that if (F_n) is any Følner sequence in $(\mathbb{N}, +)$ or any nice Følner sequence in (\mathbb{N}, \times) (see Section 6.1) then $\mathcal{L} \cap \mathcal{N}((F_n))$ is not only nonempty but in fact uncountable (contains a Cantor set). For results dealing with Liouville numbers in the context of classical normality see e.g. [14].

Theorem 8.1. *For every Følner sequence (F_n) in $(\mathbb{N}, +)$ there exists an (F_n) -normal Liouville number.*

The proof will be preceded by some generalities about Følner sequences in $(\mathbb{N}, +)$. Recall that two Følner sequences (F_n) and (F'_n) in an amenable semigroup G are called equivalent if $\frac{|F_n \Delta F'_n|}{|F_n|} \rightarrow 0$, and that if $(F_n), (F'_n)$ are equivalent Følner sequences then the notions of (F_n) -normality and (F'_n) -normality coincide.

Lemma 8.2. *Let (F_n) be an arbitrary Følner sequence in $(\mathbb{N}, +)$. There exists a sequence of natural numbers (ℓ_n) tending to infinity and a Følner sequence (F'_n) equivalent to (F_n) such that each set F'_n is a disjoint union of intervals, each of length at least ℓ_n .*

Proof. Fix a sequence $(\varepsilon_\ell)_{\ell \geq 1}$ decreasing to zero. For each ℓ there exists n_ℓ such that for every $n \geq n_\ell$, the set F_n is $(K_\ell, \frac{\varepsilon_\ell}{2\ell})$ -invariant, where K_ℓ stands for $\{1, 2, \dots, \ell\}$. Then, by Lemma 2.4, the K_ℓ -core of F_n , which we denote by F_{n, K_ℓ} , is an ε_ℓ -modification of F_n . For each n we define ℓ_n as the unique ℓ satisfying the inequalities $n_\ell \leq n < n_{\ell+1}$. We set $F'_n = F_n$ for $n < n_1$, and for $n \geq n_1$, $F'_n = F_{n, K_{\ell_n}} + K_{\ell_n}$. Now, for each n , F'_n is an ε_{ℓ_n} -modification of F_n (hence (F'_n) is a Følner sequence equivalent to (F_n)), and it is a union of (not necessarily disjoint) intervals of length ℓ_n . The “connected components”²⁵ of F'_n are disjoint intervals of lengths at least ℓ_n , as required. \square

We will establish now some technical facts about (a subclass of) Liouville numbers which will be utilized in the proof of Theorem 8.1.

²⁵ By a connected component of a set $F \subset \mathbb{N}$ we mean an interval $I = \{a, a+1, \dots, b\} \subset F$ such that $a-1 \notin F$ (this includes the case $a-1=0$) and $b+1 \notin F$.

Definition 8.3. We will call a binary sequence $w \in \{0, 1\}^{\mathbb{N}}$ *repetitive* if it is the limit of a sequence of words w_k ($k \geq 1$) defined inductively, as follows:

- (1) $w_1 = u_1$ is an arbitrary nonempty 0-1 word,
- (2) for $k > 1$, $w_k = w_{k-1}w_{k-1} \dots w_{k-1}u_k$, where w_{k-1} is repeated at least $k - 1$ times, and u_k is an arbitrary nonempty 0-1 word.

Proposition 8.4. Any not eventually periodic repetitive sequence w is the binary expansion of a Liouville number x .

Proof. Given k , consider the rational number $\frac{p}{q}$ represented by the periodic sequence $w_k w_k w_k \dots$. Then $q < 2^{|w_k|}$, hence $\frac{1}{q^k} > 2^{-k|w_k|}$. The difference $|x - \frac{p}{q}|$ is a number whose first nonzero binary digit appears at a position larger than $k|w_k|$, which means that $|x - \frac{p}{q}| \leq 2^{-k|w_k|}$, hence $|x - \frac{p}{q}| < \frac{1}{q^k}$. Since w is not eventually periodic, x is irrational, and thus it is a Liouville number. \square

Notice that since in Definition 8.3 the word u_k is completely arbitrary, in particular it may have the form $v_k v_k \dots v_k$ (where the word v_k and the number of repetitions are also arbitrary). Using this observation, we can isolate a special class of repetitive sequences.

Definition 8.5. Let (v_k) be a sequence of nonempty binary words. A repetitive sequence w is said to be *balanced* with respect to (v_k) if, for each $k \geq 1$, $u_k = v_k v_k \dots v_k$, where the number of repetitions is such that the following two conditions hold:

$$\delta_k = \frac{\max\{|v_k|, |v_{k+1}|, |v_{k+2}|, |w_{k-1}|\}}{|w_k|} \rightarrow 0, \quad (8.1)$$

$$1 - \gamma_k = \frac{|u_k|}{|w_k|} \rightarrow 1. \quad (8.2)$$

It is easy to see that given any sequence of nonempty words (v_k) , one can construct a repetitive sequence w which is balanced with respect to (v_k) . One just needs to apply large enough number of repetitions of v_k in u_k (depending on the lengths $|v_k|$, $|v_{k+1}|$ and $|v_{k+2}|$).

Lemma 8.6. Let (v_k) be a sequence of nonempty binary words and let w be a repetitive sequence balanced with respect to (v_k) . For each $k \geq 2$ define $\varepsilon_k = 2(\delta_k + \gamma_k)$ (see Definition 8.5). If W is a subword of w_{k+2} with $|W| \geq |w_k|$ then, for some $r, s, t \geq 0$ satisfying $\frac{r|v_k| + s|v_{k+1}| + t|v_{k+2}|}{|W|} \geq 1 - \varepsilon_k$, W contains $r + s + t$ nonoverlapping subwords of which r are copies of v_k , s are copies of v_{k+1} and t are copies of v_{k+2} .

Proof. Note that w_{k+2} has the following structure: $(w_{k+1})^a (v_{k+2})^{a'}$, and likewise $w_{k+1} = (w_k)^b (v_{k+1})^{b'}$, $w_k = (w_{k-1})^c (v_k)^{c'}$ ($a \geq k + 1, b \geq k, c \geq k - 1, a', b', c' \geq 1$). By successive substitution (two times), we obtain that w_{k+2} is

a concatenation of (shifted) copies of v_k, v_{k+1}, v_{k+2} and w_{k-1} . So is any subword W of w_{k+2} , except that the copies covering the ends of W may extend beyond W in which case the concatenation representing W includes (at most two) *end words* V_1, V_2 which are subwords of either v_k, v_{k+1}, v_{k+2} or w_{k-1} .

To finish the proof we need to show that,

$$\frac{p|w_{k-1}| + |V_1| + |V_2|}{|W|} < \varepsilon_k,$$

where p is the number of copies of w_{k-1} in the concatenation representing W . The fraction $\frac{p|w_{k-1}|}{|W|}$ is largest precisely when $W = (w_{k-1})^c (v_k)^{c'} (w_{k-1})^c$ and then we have

$$\frac{p|w_{k-1}|}{|W|} = \frac{2c|w_{k-1}|}{|w_k| + c|w_{k-1}|} < 2\frac{c|w_{k-1}|}{|w_k|} = 2\gamma_k.$$

The joint length of the end words not larger than $2 \max\{|v_k|, |v_{k+1}|, |v_{k+2}|, |w_{k-1}|\}$, so $\frac{|V_1|+|V_2|}{|W|} < 2\delta_k$. We have shown that the joint length of the (nonoverlapping) copies of v_k, v_{k+1} and v_{k+2} which are subwords of W is least $(1 - 2\delta_k - 2\gamma_{k-1})|W| = (1 - \varepsilon_k)|W|$ and this is precisely what we needed to show. \square

Proof of Theorem 8.1. Fix a Følner sequence (F_n) in $(\mathbb{N}, +)$. In view of Lemma 8.2 we can assume without loss of generality that if ℓ_n denotes the length of the shortest connected component of F_n then the sequence (ℓ_n) tends to infinity. For each natural j we define t_j as the largest element of the set

$$\bigcup_{\{n: \ell_n < j\}} F_n,$$

i.e., t_j is such that if F_n has at least one connected component shorter than j then $F_n \subset \{1, 2, \dots, t_j\}$.

Let $v \in \{0, 1\}^{\mathbb{N}}$ be a classical normal sequence and let $v_k = v|_{\{1, 2, \dots, k\}}$. Note that the words (v_k) are *asymptotically normal* in the following sense: for any nonempty finite $K \subset \mathbb{N}$ and any $\varepsilon > 0$, if k is sufficiently large then v_k is (K, ε) -normal.

Let w be a repetitive sequence which is balanced with respect to (v_k) . By choosing the numbers of repetitions of v_{k+2} in u_{k+2} (see Definition 8.5) sufficiently large, we can arrange that $|w_{k+2}| \geq t_{|w_{k+1}|}$, for each k . For each n let k_n be the unique integer satisfying the inequalities $|w_{k_n}| \leq \ell_n < |w_{k_n+1}|$. Notice that since the numbers ℓ_n tend to infinity with n , so do the numbers k_n . By the definition of the numbers t_j and since $\ell_n < |w_{k_n+1}|$, we have $F_n \subset \{1, 2, \dots, t_{|w_{k_n+1}|}\} \subset \{1, 2, \dots, |w_{k_n+2}|\}$. Thus, for any connected component I of F_n , the word $W = w|_I$ is a subword of length at least $|w_{k_n}|$ of w_{k_n+2} . Now, Lemma 8.6 implies that at least the fraction $1 - \varepsilon_{k_n}$ of $w|_I$ is a constituted by nonoverlapping copies of the words v_{k_n}, v_{k_n+1} and v_{k_n+2} . Since $\varepsilon_{k_n} \rightarrow 0$, it is now obvious that the blocks

$w|_{F_n}$ ²⁶ are asymptotically normal as n grows to infinity, *i.e.*, that w is (F_n) -normal. In particular, the number x (whose binary expansion is w) is irrational²⁷, hence it is an (F_n) -normal Liouville number. \square

Remark 8.7. If in the above construction we vary the classical normal element v (used to define the words v_k), while keeping the numbers of repetitions of v_k in u_k unchanged, we obtain a continuous and injective map $v \mapsto w$ sending classical normal sequences to (F_n) -normal repetitive sequences. Moreover, since every (F_n) -normal number is irrational, also the map $w \mapsto x$ (where x is the number whose binary expansion is w) is injective and continuous. Thus, for every compact set C consisting of classical normal sequences, the restriction to C of the composition $v \mapsto w \mapsto x$ is a homeomorphism of C onto its image. Since the set of classical normal sequences contains a Cantor set, so does the set of (F_n) -normal Liouville numbers.

We now turn to constructing Liouville numbers which are (multiplicatively) normal with respect to nice Følner sequences. As we shall see, repetitive sequences are naturally well fitted for this kind of normality. Recall (see Section 6.1) that for $m, M \in \mathbb{N}$ we write $m \preccurlyeq M$ when $m|M$. If $m \preccurlyeq M$ and $M \neq m$, we will write $m < M$. Recall also that a nice Følner sequence (F_n) in (\mathbb{N}, \times) corresponds to a multiplicatively increasing sequence (L_n) of the leading parameters, *i.e.*, natural numbers such that, for each n , $L_n < L_{n+1}$ and $F_n = \{m : m \preccurlyeq L_n\}$.

Lemma 8.8. *Given $k \geq 1$ and $\varepsilon > 0$, there exists an $m_{k,\varepsilon}$ such that for any m and M satisfying $m_{k,\varepsilon} \preccurlyeq m \preccurlyeq M$, the interval $\{m+1, \dots, (k+1)m\}$ contains at most a fraction ε of all divisors of M , *i.e.*,*

$$\frac{|\{i : i \preccurlyeq M, m+1 \leq i \leq (k+1)m\}|}{|\{i : i \preccurlyeq M\}|} \leq \varepsilon.$$

Proof. Let p be the smallest prime number strictly larger than k . Let $r \in \mathbb{N}$ be such that $\frac{1}{r} \leq \varepsilon$, and put $m_{k,\varepsilon} = p^r$. Let m be any multiple of $m_{k,\varepsilon}$ and let M be any multiple of m . The set of all divisors of M (which can be visualized as the anchored rectangular box with the leading parameter M , see Section 6.1) splits into disjoint union of one-dimensional sets of the form $aI = \{a, ap, ap^2, \dots, ap^s\}$, where a is not a multiple of p , and p^s is the largest power of p dividing M . Clearly, $s \geq r$. Since $p \geq k+1$, at most one element from any set aI may fall in $\{m+1, \dots, (k+1)m\}$. Thus at most the fraction $\frac{1}{s} \leq \frac{1}{r} \leq \varepsilon$ of all divisors of M may fall in $\{m+1, \dots, (k+1)m\}$. \square

Theorem 8.9. *For any nice Følner sequence (F_n) in (\mathbb{N}, \times) there exists an (F_n) -normal Liouville number.*

²⁶ We use the term “block” because F_n need not be an interval.

²⁷ Rational numbers are neither additively nor multiplicatively normal because their additive as well as multiplicative orbits are finite.

Proof. The proof relies on choosing an arbitrary (F_n) -normal 0-1-sequence \tilde{w} and modifying it on a set of (F_n) -density zero. Clearly, then the modified sequence w maintains (F_n) -normality. On the other hand, we will make the sequence w repetitive. Since the number x whose binary expansion is w is multiplicatively normal is not rational, Proposition 8.4 will imply that x is the desired (F_n) -normal Liouville number.

Given $k \geq 1$, Lemma 8.8 applied for k and $\varepsilon = 2^{-k}$ provides a number $m_{k,2^{-k}}$. Let n_k be the smallest index n such that $m_{k,2^{-k}} \in F_n$ (i.e., $m_{k,2^{-k}} \preccurlyeq L_n$) and let $m_k = \text{LCM}(m_{k,2^{-k}}, L_{n_k-1})$. In this manner, we have assured that $L_{n_k-1} \preccurlyeq m_k \preccurlyeq L_{n_k}$. Since m_k is a multiple of $m_{k,2^{-k}}$, the following holds:

$$m_k \preccurlyeq M \text{ implies } \frac{|\{i : i \preccurlyeq M, m_k + 1 \leq i \leq (k+1)m_k\}|}{|\{i : i \preccurlyeq M\}|} \leq 2^{-k}. \quad (8.3)$$

Further, it is obvious that m_k can be replaced by $m_{k'}$ with any $k' \geq k$ ($m_{k'}$ has the above property with k' thus also with k). Hence, passing if necessary to a subsequence, we can assume that $m_{k+1} > (k+1)m_k$ for each k . Although the property $L_{n_k-1} \preccurlyeq m_k \preccurlyeq L_{n_k}$ may be lost, we still have for any natural indices k and n , either $m_k \preccurlyeq L_n$ or $L_n \preccurlyeq m_k$.

Now we are in a position to define w . We let $u_1 = w_1 = \tilde{w}|_{\{1, \dots, m_1\}}$. Next, we define $w_2 = w_1 u_1 u_2$, where $u_2 = \tilde{w}|_{\{2m_1+1, \dots, m_2\}}$. Notice that the coordinates on which w_2 disagrees with $\tilde{w}|_{\{1, \dots, m_2\}}$ (if any) are contained in the interval $\{m_1 + 1, \dots, 2m_1\}$. Then we define $w_3 = w_2 u_2 u_3$, where $u_3 = \tilde{w}|_{\{3m_2+1, m_3\}}$. Similarly, the coordinates where w_3 disagrees with $\tilde{w}|_{\{1, m_3-1\}}$ (if any) are contained in the union $\{m_1 + 1, \dots, 2m_1\} \cup \{m_2 + 1, \dots, 3m_2\}$. Continuing in this way we will define a sequence of words w_k converging to a sequence w which agrees with \tilde{w} on the complement of the set

$$\bigcup_{k \geq 1} \{m_k + 1, \dots, (k+1)m_k\}. \quad (8.4)$$

According to Definition 8.3, w is a repetitive sequence. It remains to show that the (F_n) -density of the union (8.4) is zero. Given an $n \in \mathbb{N}$, we divide the indices k into three classes (some of them possibly empty): $k \in \mathbb{S}_n$ if $km_k \leq |F_n|^{\frac{1}{3}}$, $k \in \mathbb{L}_n$ if $m_k \geq L_n$ and $\mathbb{M}_n = \mathbb{N} \setminus (\mathbb{S}_n \cup \mathbb{L}_n)$.

- For $k \in \mathbb{S}_n$ we have

$$\frac{|\{m_k + 1, \dots, (k+1)m_k\} \cap F_n|}{|F_n|} = \frac{km_k}{|F_n|} \leq |F_n|^{-\frac{2}{3}}.$$

Because $|\mathbb{S}_n| \leq |F_n|^{\frac{1}{3}}$, we have

$$\frac{1}{|F_n|} \left| \bigcup_{k \in \mathbb{S}_n} \{m_k + 1, \dots, (k+1)m_k\} \cap F_n \right| \leq |F_n|^{-\frac{1}{3}}.$$

- For $k \in \mathbb{L}_n$, F_n is disjoint from $\{m_k + 1, \dots, (k + 1)m_k\}$, hence

$$\frac{1}{|F_n|} \left| \bigcup_{k \in \mathbb{L}_n} \{m_k + 1, \dots, (k + 1)m_k\} \cap F_n \right| = 0.$$

- For $k \in \mathbb{M}_n$, we have $L_n > m_k$, in particular $L_n \not\leq m_k$ and thus $m_k \preceq L_n$. By (8.3), we have

$$\frac{1}{|F_n|} |\{m_k + 1, \dots, (k + 1)m_k\} \cap F_n| \leq 2^{-k}.$$

Putting the above three cases together, we get

$$\frac{1}{|F_n|} \left| \bigcup_{k \in \mathbb{N}} \{m_k + 1, \dots, (k + 1)m_k\} \cap F_n \right| \leq |F_n|^{-\frac{1}{3}} + \sum_{k \in \mathbb{M}_n} 2^{-k}.$$

Since $|F_n| \rightarrow \infty$, the right hand side tends to zero with n (note that every k eventually falls in \mathbb{S}_n). \square

Remark 8.10. Denote by \mathbb{D} the complement in \mathbb{N} of the union (8.4). Then \mathbb{D} has (F_n) -density 1 in (\mathbb{N}, \times) , and $w|_{\mathbb{D}} = \tilde{w}|_{\mathbb{D}}$, where \tilde{w} is the (F_n) -normal sequence chosen at the beginning of the proof of Theorem 8.9. The construction of w uses only the subwords of \tilde{w} appearing in $\tilde{w}|_{\mathbb{D}}$, hence the mapping $\tilde{w}|_{\mathbb{D}} \mapsto w$ is injective (and obviously it is also continuous). It is easy to see that there exists a Cantor set consisting of (F_n) -normal elements \tilde{w} on which the map $\tilde{w} \mapsto \tilde{w}|_{\mathbb{D}}$ is injective. On this Cantor set, the map $\tilde{w} \mapsto w$ is injective and continuous. Arguing as in Remark 8.7, we get that the map $w \mapsto x$ is injective and continuous on this Cantor set. This implies that the set of (multiplicatively) (F_n) -normal Liouville numbers contains a Cantor set.

Remark 8.11. The technique employed in the proof of Theorem 8.9 can be utilized to obtain Liouville numbers with other properties. Let (F_n) be a nice Følner sequence in (\mathbb{N}, \times) and let P be any property satisfied by a nonempty set of numbers and preserved under zero (F_n) -density modifications of the binary expansions (for example, the property of being generic for some multiplicatively invariant, not necessarily Bernoulli, measure). Then there exist Liouville numbers with property P .

We conclude this section (and the paper) with an open problem.

Question 8.12. Do there exist net-normal Liouville numbers?

We remark that our technique does not allow us to produce such numbers. Indeed, for any fixed sequence of intervals of the form $\{m_k + 1, \dots, km_k\}$, the union (8.4) has upper density at least $\frac{1}{2}$ for a suitable nice Følner sequence. To prove this, it suffices to indicate for any leading parameter L a multiple pL such that half of divisors of pL belong to one of the intervals $\{m_k + 1, \dots, km_k\}$. To this end, choose $k \geq 2L$ and a prime number p in $\{m_k + 1, \dots, 2m_k\}$ (such p exists by Bertrand's postulate). Then at least half of the divisors of pL have the form pl , where $l \preceq L$ (in particular $l \leq L$) and then $m_k < p \leq pl \leq 2m_k L \leq km_k$.

Appendix

In this appendix we briefly discuss the original proof in [12] of the fact that the set of normal numbers in $[0, 1]$ has full Lebesgue measure, and the controversies it generated. The proof has two parts. In the first part Borel defines a number $x \in [0, 1]$ to be *simply normal in base b* if the frequency of every digit $0, 1, \dots, b-1$ in the expansion of x equals $\frac{1}{b}$. He then shows that the set of numbers in $[0, 1]$ which are simply normal in base b is of full Lebesgue measure. One can view this result as a special case of the Strong Law of Large Numbers (SLLN). The proof is based on what is now known as the Borel-Cantelli Lemma. We remark that in this part it is inessential that \mathbb{Z} is a group. What matters is that the functions $X_i = \lfloor b^i x \rfloor \bmod b$ (which express the digits in the base b expansion of x) form a countable family of independent identically distributed random variables and that the averaging sets F_n (in this case $\{1, 2, \dots, n\}$) strictly increase in cardinality. The Følner property and the inclusions $F_n \subset F_{n+1}$ are not used. In the second part of the proof, Borel defines a number x to be *completely normal* if for every $k, m \geq 1$ numbers $b^m x$ (considered modulo 1) are simply normal in base b^k . As a countable intersection of sets of full measure, the set of completely normal numbers also has full measure. Then Borel writes (for $b = 10$):

“La propriété caractéristique d’un nombre normal est la suivante: *un groupement quelconque de p chiffres consécutifs étant considéré, si l’on désigne par c_n le nombre de fois que se rencontre ce groupement dans les n premiers chiffres décimaux, on a:*

$$\lim_{n \rightarrow \infty} \frac{c_n}{n} = \frac{1}{10^p} . \quad (*)$$

(The characteristic property of a normal number is the following: for any grouping of p consecutive digits being considered, denoting by c_n the number of times this grouping occurs in the first n decimal digits, one has (*).)

This “characteristic property” is exactly normality in terms of our Definition 1.1 (adapted to base 10). Borel does not prove equivalence between his definition of “complete normality” and the “propriété caractéristique”. Perhaps Borel intentionally skipped the proof (considering it fairly obvious), but this omission triggered a long-lasting controversy (and confusion). In particular, Champernowne [16], Koksma [34], Copeland and Erdős [18], Hardy and Wright [33] explicitly or implicitly used the unproved equivalence. To illustrate how far from obvious this equivalence was at that time, let us quote what Donald D. Wall claimed in his dissertation [46] (written in 1949 under the supervision of Derrick H. Lehmer):

“Actually, there seems to be little reason to believe that the classes are identical.”

In fact, Wall believed to be close to finding a counterexample:

“Certain aspects of the problem are discussed in some detail here, and the main result is a new method of constructing some class II numbers – a method which seems to give hope of finding a class II number which is not in class III.”

Eventually the equivalence was established by I. Niven and H. S. Zuckerman in 1951 [40] (see also [15]). Today this equivalence is no longer controversial, once we understand that normality (in the sense of Borel's "characteristic property") of a sequence x implies normality of x restricted to any infinite arithmetic progression²⁸.

Borel was also criticized for other gaps in his proof. One such criticism appears in the 1910 book of Georg Faber [24] on page 400. It seems that Faber finds it unclear that the Lebesgue measure on $[0, 1]$ corresponds to the distribution of the i.i.d. process $\{X_i\}_{i \geq 1}$, where the X_i 's are the random variables defined above.

"Sodann hat Herr Borel kürzlich nach Aufstellung geeigneter Definitionen über Wahrscheinlichkeit bei einer abzählbaren Menge von Dingen bewiesen, dass die Wahrscheinlichkeit dafür, dass ein Punkt der obigen Menge angehört, gleich Null ist. Die Vergleichung des obigen Satzes mit dem Borelschen Resultat legt die Frage nahe:

Ist die Wahrscheinlichkeit – nach der Borelschen Festsetzung die eventuell zur Beantwortung dieser Fragen zu erweitern wäre –, dass eine Zahl einer bestimmten vorgelegten Menge vom Masse Null angehört, immer gleich Null? Und umgekehrt: Ist eine Menge immer vom Masse Null, wenn die Wahrscheinlichkeit, dass ein Punkt ihr angehört, gleich Null ist?"

(Next, shortly after establishing appropriate definitions about probability associated with a countable set, Mr. Borel has proved that the probability for a point to be an element of the above set equals zero. The comparison of the above theorem with Borel's result suggests the following question:

Is the probability – which, according to Borel's definition, possibly has to be extended in order to answer these questions – that a number belongs to a certain given set of mass zero, always zero? And conversely: Does a set always have mass zero, if the probability that a point belongs to it is zero?^{a)})

^{a)} We thank Christoph Kawan for helping us with the translation.

Because of Faber's somewhat antiquated style and terminology, we are not exactly sure what is bothering him, but from today's perspective, the equivalence between the above two meanings of a null set leaves no doubts. It is worth mentioning that Faber provides his own, different proof of SLLN. A reference to Faber's proof is made in the following passage in the survey [22] by Joseph L. Doob, where he indicates that Borel has actually proved only convergence in measure rather than almost everywhere:

"Classical elementary probability calculations imply that this sequence of averages converges in measure to $1/2$, but a stronger mathematical version of the law of large numbers was the fact deduced by Borel—in an unmendably faulty proof—that this sequence of averages converges to $1/2$ for (Lebesgue measure) almost every value of x . A correct proof was given a year later by Faber, and

²⁸ Ironically, this implication was first proved by Wall in his dissertation, but apparently he has not realized that it solves the "equivalence problem". In modern times the implication follows immediately from the fact that K-systems (in particular Bernoulli systems) and systems with entropy zero (in particular periodic) are disjoint in the sense of Furstenberg.

much simpler proofs have been given since. [Fréchet remarked tactfully: <<Borel's proof is excessively short. It omits several intermediate arguments and assumes certain results without proof.>>]"

So, what is actually wrong with Borel's proof of the SLLN? A careful examination of Borel's proof reveals the following:

- (1) On pages 250–252, it is proved, under the (implicit) assumption that a sequence of sets A_n is independent, that if the sequence of probabilities $\mathbb{P}(A_n)$ is summable then the upper limit $\bigcap_{m \geq 1} \bigcup_{n \geq m} A_n$ has measure zero (which is a special case of what is today called the Borel–Cantelli Lemma).
- (2) On page 259, in the proof of the fact that simply normal numbers form a set of full measure (in other words, in the proof of the SLLN for 0-1 valued random variables), the Borel–Cantelli Lemma is applied to sets A_n which are not independent.
- (3) The proof of the full version of the Borel–Cantelli Lemma is missing. Without it, Borel's proof indeed establishes (as pointed out by Doob) only the version of the Law of Large Numbers which involves the convergence in measure.

So, formally speaking, Borel's proof does contain a gap. But does that mean that the proof is “unmendably faulty”? We are inclined to accept Fréchet's assessment, that the proof was just excessively short.

We conclude with a comment concerning the possibility of adapting Borel's method to more general amenable groups.

The key property of \mathbb{Z} which is behind the equivalence between the two Borel's definitions of normality is that \mathbb{Z} admits, for each k , a *monotiling* (tiling with one shape) with the shape being the interval $\{0, 1, \dots, k-1\}$. It is plausible that for monotileable groups²⁹ Borel's definition of normality and his proof that (F_n) -normal elements form a set of full measure λ can be adapted with not too much effort to a large class of Følner sequences. But it seems impossible to extend Borel's definition of normality to elements of $\{0, 1\}^G$, where G is any infinitely countable amenable group or semigroup. Although the notion of *simple* normality can be naturally defined in this case, and moreover, by essentially the same proof as in the case of $\{0, 1\}^{\mathbb{Z}}$, one can show that almost every element $x \in \{0, 1\}^G$ is simply normal, it is not clear what is the analog of the operation of changing the base from b to b^k . One would need to find a large finite set S which tiles the group (*i.e.*, is a shape of a monotiling \mathcal{T}) and then treat the blocks $B = x|_T$ (where $T = Sc$, $c \in C_S$, are the tiles of \mathcal{T}) as new symbols (from the alphabet $\{0, 1, \dots, b-1\}^S$) associated to the centers c of the tiles. It is not known which groups (except residually finite) admit monotilings with arbitrarily large shapes. In fact, it is an open problem whether all countable amenable groups are monotileable. This is the reason why in the proof of Theorem 4.2 we must use tiling with many shapes which complicates the proof of this theorem.

²⁹ A group G is *monotileable* if it admits monotilings with arbitrarily large shapes, by which we mean that any finite set $K \in G$ is eventually a subset of the shape of some monotiling.

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