

Traces of Newton-Sobolev, Hajlasz-Sobolev, and BV functions on metric spaces

PANU LAHTI, XINING LI AND ZHUANG WANG

Abstract. We study the boundary traces of Newton-Sobolev, Hajlasz-Sobolev, and BV (bounded variation) functions. Assuming less regularity of the domain than is usually done in the literature, we show that all of these function classes achieve the same “boundary values”, which in particular implies that the trace spaces coincide provided that they exist. Many of our results seem to be new even in Euclidean spaces but we work in a more general complete metric space equipped with a doubling measure and supporting a Poincaré inequality.

Mathematics Subject Classification (2020): 46E35 (primary); 26B30, 30L99 (secondary).

1. Introduction

Boundary traces for various function classes, especially functions of bounded variation (BV functions), have been studied in recent years in the setting of metric measure spaces (X, d, μ) . In [31], the authors studied the boundary traces, or traces for short, of BV functions in suitably regular domains. Typically, the boundary trace Tu of a function u in a domain Ω is defined by the condition

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |u - Tu(x)| d\mu = 0 \quad (1.1)$$

for a.e. $x \in \partial\Omega$ with respect to the codimension-1 Hausdorff measure $\mathcal{H}$. In [33] (see also references therein for previous works in Euclidean spaces) the authors considered the corresponding extension problem, that is, the problem of finding a function whose trace is a prescribed L^1 -function on the boundary. They showed

X. L. is supported by NNSF of China (No. 11701582). Z. W. is supported by the Academy of Finland via Centre of Excellence in Analysis and Dynamics Research (No. 307333) and partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund.

Received December 23, 2019; accepted May 19, 2020.

Published online September 2021.

that in sufficiently regular domains, the trace operator of BV functions is surjective, and that in fact the extension can always be taken to be a Newton-Sobolev function. This implies that the trace space of both $BV(\Omega)$ and $N^{1,1}(\Omega)$ is $L^1(\partial\Omega)$. This trace and extension problem is motivated by Dirichlet problems for functions of least gradient, in which one minimizes the total variation among BV functions with prescribed boundary data, see [5, 13, 24, 34, 38].

In the current paper, we consider boundary traces from a different viewpoint. Unlike in the existing literature, we assume very little regularity of the domain, meaning that traces need not always exist. We are nonetheless able to show in various cases that for a given function, it is possible to find a more regular function that “achieves the same boundary values”. In particular, if the original function has a boundary trace, then the more regular function has the same trace. This sheds further light on the extension problem. To prove our results, we apply some existing approximation results for BV and Newton-Sobolev functions, and develop some new ones.

We will always assume that (X, d, μ) is a complete metric space equipped with a doubling measure μ and supporting a $(1, 1)$ -Poincaré inequality. Let $\Omega \subset X$ be a nonempty open set. For BV functions we prove the following three theorems. The exponent s is sometimes called the homogeneous dimension of the space. $N^{1,1}(\Omega)$ is a generalization of the Sobolev class $W^{1,1}(\Omega)$ to metric spaces; see Section 2 for definitions.

Theorem 1.1. *Let $u \in BV(\Omega)$. Then there exists $v \in N^{1,1}(\Omega) \cap \text{Lip}_{\text{loc}}(\Omega)$ such that*

$$\int_{B(x,r) \cap \Omega} |v - u|^{s/(s-1)} d\mu \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

uniformly for all $x \in \partial\Omega$.

In particular, *whenever* there exists a BV extension of a given function defined on the boundary, it is possible to also find a Newton-Sobolev extension. If we give up the requirement that v is locally Lipschitz, we can replace $s/(s-1)$ by an arbitrarily large exponent.

Theorem 1.2. *Let $u \in BV(\Omega)$ and let $1 \leq q < \infty$. Then there exists $v \in N^{1,1}(\Omega)$ such that*

$$\int_{B(x,r) \cap \Omega} |v - u|^q d\mu \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

uniformly for all $x \in \partial\Omega$.

If we also allow v to have a small (approximate) *jump set* S_v , then we can include the case $q = \infty$. The class of *special functions of bounded variation*, denoted by $SBV(\Omega)$, is defined as those BV functions whose variation measure only has an absolutely continuous part (like Sobolev functions) and a jump part. The class was introduced by De Giorgi and Ambrosio [2] as a natural class in which to solve various variational problems, e.g., the minimization of Mumford–Shah functional.

Theorem 1.3. *Let $u \in \text{BV}(\Omega)$ and let $\varepsilon > 0$. Denote $\Omega(r) := \{x \in \Omega : \text{dist}(x, X \setminus \Omega) > r\}$ for $r > 0$. Then there exists $v \in \text{SBV}(\Omega)$ such that $\mathcal{H}(S_v) < \varepsilon$ and*

$$\|v - u\|_{L^\infty(\Omega \setminus \Omega(r))} \rightarrow 0 \quad \text{as } r \rightarrow 0^+.$$

See also [10] for some previous results on traces of BV functions whose variation measure only has an absolutely continuous part, jump part, or *Cantor part*.

Note that $v \in \text{SBV}(\Omega)$ belongs to $N^{1,1}(\Omega)$ if and only if $\mathcal{H}(S_v) = 0$ (see [23, Theorem 4.1], (2.12), and [18, Theorem 4.6]). Thus we could equivalently require

- $v \in \text{SBV}(\Omega) \cap \text{Lip}_{\text{loc}}(\Omega)$ (in particular, $S_v = \emptyset$) in Theorem 1.1,
- $v \in \text{SBV}(\Omega)$ with $\mathcal{H}(S_v) = 0$ in Theorem 1.2, and
- $v \in \text{SBV}(\Omega)$ with $\mathcal{H}(S_v) < \varepsilon$ in Theorem 1.3,

illustrating how we get better boundary approximation by relaxing the regularity requirements on v .

From Theorem 1.1 (or Theorem 1.2), we obtain the following corollary.

Corollary 1.4. *The trace spaces of $\text{BV}(\Omega)$ and $N^{1,1}(\Omega)$ are the same.*

The definitions of trace and trace space are given in Definition 2.2 and Definition 2.3. Here and throughout this paper, for two function spaces $\mathbb{X}(\Omega)$ and $\mathbb{Y}(\Omega)$, that the trace spaces of $\mathbb{X}(\Omega)$ and $\mathbb{Y}(\Omega)$ are the same means that if the function space $\mathbb{Z}(\partial\Omega)$ is the trace space of $\mathbb{X}(\Omega)$, then it is also the trace space of $\mathbb{Y}(\Omega)$, and vice versa.

Corollary 1.4 is stronger than we expected; it says that we can obtain the existence of the trace and the trace space of $\text{BV}(\Omega)$ by only knowing the existence of the trace and the trace space of $N^{1,1}(\Omega)$, which is nontrivial, since $N^{1,1}(\Omega)$ is a strict subset of $\text{BV}(\Omega)$.

The so-called Hajlasz-Sobolev space $M^{1,p}(\Omega)$, $p \geq 1$, introduced in [14], is a subspace of $N^{1,p}(\Omega)$. For $p > q \geq 1$ and Ω supporting a $(1, q)$ -Poincaré inequality and a doubling measure, we have $N^{1,p}(\Omega) = M^{1,p}(\Omega)$ with equivalent norms, see [15], and hence the traces of $M^{1,p}(\Omega)$ and $N^{1,p}(\Omega)$ will be the same. But for $p = 1$, even under these strong assumptions, $M^{1,1}(\Omega)$ is only a strict subspace of $N^{1,1}(\Omega)$ and it seems that trace results for $M^{1,1}$ are lacking in the literature. One can also define a local version $M_{c_H}^{1,1}(\Omega)$, see Section 2 and Remark 4.5 for more information. For these classes, we prove the following results.

Theorem 1.5. *Suppose Ω satisfies the measure density condition (2.4). Then there exists $0 < c_H < 1$ such that for any $u \in N^{1,1}(\Omega)$, there is $v \in M_{c_H}^{1,1}(\Omega) \cap \text{Lip}_{\text{loc}}(\Omega)$ satisfying $\|v\|_{M_{c_H}^{1,1}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}$ and*

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |v - u| d\mu = 0$$

for $\mathcal{H}$ -a.e. $x \in \partial\Omega$, where $\mathcal{H}$ is the codimension-1 Hausdorff measure.

If additionally Ω is a uniform domain, then v can be chosen in $M^{1,1}(\Omega) \cap \text{Lip}_{\text{loc}}(\Omega)$.

With the exception of this theorem, our results are not written in terms of the codimension-1 Hausdorff measure $\mathcal{H}$ (defined in (2.2) and (2.3)) which is used in most existing literature. In Theorems 1.1–1.3, the results hold for every point on the boundary. On the other hand, the space or domain may be endowed with a measure μ for which the codimension-1 Hausdorff measure is not σ -finite on the boundary of the domain (see Example 5.6). More precisely, in Example 5.6 we define a weighted measure on the Euclidean half-space $\mathbb{R}_+^2$ whose codimension-1 Hausdorff measure is infinity for any open interval of $\partial\mathbb{R}_+^2 = \mathbb{R}$. But on $\mathbb{R}_+^2$, it is natural to study instead the trace with respect to the 1-dimensional Lebesgue measure on $\mathbb{R}$, which we do in Example 5.7. Another motivation for us is that in certain Dirichlet problems one needs to consider the trace with respect to a measure different from $\mathcal{H}$, see [24, Definition 4.1].

More generally, instead of only studying the codimension-1 Hausdorff measure, we may study any arbitrary boundary measure $\tilde{\mathcal{H}}$ on $\partial\Omega$. In order to study such problems, we first replace the codimension-1 Hausdorff measure $\mathcal{H}$ with $\tilde{\mathcal{H}}$ in the previous definition of trace to give the definition of trace with respect to $\tilde{\mathcal{H}}$, see Definition 2.4. Then we prove the following result.

Theorem 1.6. *Suppose Ω satisfies the measure doubling condition (2.5). Let $\tilde{\mathcal{H}}$ be any Radon measure on $\partial\Omega$. There exists a constant $0 < c_H < 1$ with the following property: for a given $u \in N^{1,1}(\Omega)$, if there exists a function Tu such that*

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |u - Tu(x)| d\mu = 0$$

for $\tilde{\mathcal{H}}$ -a.e. $x \in \partial\Omega$, then there exists a function $v \in M_{c_H}^{1,1}(\Omega) \cap \text{Lip}_{\text{loc}}(\Omega)$ such that $\|v\|_{M_{c_H}^{1,1}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}$ and

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |v - Tu(x)| d\mu = 0$$

for $\tilde{\mathcal{H}}$ -a.e. $x \in \partial\Omega$.

If additionally Ω is a uniform domain, then the above v can be chosen in $M^{1,1}(\Omega) \cap \text{Lip}_{\text{loc}}(\Omega)$.

The constant c_H in Theorem 1.5 and Theorem 1.6 can be chosen as $1/50\lambda$ (see the proof of Theorem 4.1), where λ is the dilation constant in the $(1, 1)$ -Poincaré inequality; see Section 2 for more information about Poincaré inequalities.

Since μ is a doubling measure on X , it follows from [9, Theorem 4.4] that Ω satisfies the measure doubling condition (2.5) if Ω is a uniform domain. Hence similarly to Corollary 1.4, from Theorem 1.6 we obtain the following corollary.

Corollary 1.7. *Let $\Omega \subset X$ be a uniform domain. Then for any given boundary measure $\tilde{\mathcal{H}}$, the trace spaces of $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$ with respect to any boundary measure $\tilde{\mathcal{H}}$ on $\partial\Omega$ are the same.*

The paper is organized as follows. In Section 2, we give the necessary preliminaries. In Section 3, we study the traces of $N^{1,1}$ and BV and give the proofs of Theorems 1.1–1.3 and Corollary 1.4. In Section 4, we study the traces of $N^{1,1}$ and $M^{1,1}$ and give the proofs of Theorem 1.5, Theorem 1.6, and Corollary 1.7. Finally, in Section 5, apart from giving several examples that we refer to in Section 3 and Section 4, we also discuss some trace results and examples obtained as applications of Corollary 1.4 and Corollary 1.7.

ACKNOWLEDGEMENTS. The authors would like to thank Pekka Koskela, Nageswari Shanmugalingam, Lukáš Malý, and Anders Björn for reading the manuscript and giving comments that helped improve the paper.

2. Preliminaries

In this section we introduce the notation, definitions, and assumptions used in the paper.

Throughout this paper, (X, d, μ) is a complete metric space that is equipped with a metric d and a Borel regular outer measure μ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x, r) := \{y \in X : d(y, x) < r\}$. By iterating the doubling condition, for every $0 < r \leq R$ and $y \in B(x, R)$, we have

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq 4^{-s} \left(\frac{r}{R}\right)^s, \quad (2.1)$$

for any $s \geq \log_2 C_d$. See [15, Lemma 4.7] or [6] for a proof of this. We fix such an $s > 1$ and call it the *homogeneous dimension*.

The letters c, C (sometimes with a subscript) will denote positive constants that usually depend only on the space and may change at different occurrences; if C depends on $a, b, \dots$, we write $C = C(a, b, \dots)$. The notation $A \approx B$ means that there is a constant C such that $1/C \cdot A \leq B \leq C \cdot A$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a constant C such that $A \leq C \cdot B$ ($A \geq C \cdot B$).

All functions defined on X or its subsets will take values in $[-\infty, \infty]$. A complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact. For an open set $\Omega \subset X$, a function is in the class $L^1_{\text{loc}}(\Omega)$ if and only if it is in $L^1(\Omega')$ for every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω . Other local spaces of functions are defined similarly.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension-1 is defined as

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{j \in I} \frac{\mu(B(x_j, r_j))}{r_j} : A \subset \bigcup_{j \in I} B(x_j, r_j), r_j \leq R, I \subset \mathbb{N} \right\}. \quad (2.2)$$

The codimension-1 Hausdorff measure of $A \subset X$ is then defined as

$$\mathcal{H}(A) := \lim_{R \rightarrow 0^+} \mathcal{H}_R(A). \quad (2.3)$$

Given an open set $\Omega \subset X$, we can regard it as a metric space in its own right, equipped with the metric induced by X and the measure $\mu|_\Omega$ which is the restriction of μ to subsets of Ω . This restricted measure $\mu|_\Omega$ is a Radon measure, see [22, Lemma 3.3.11].

We say that an open set Ω satisfies a *measure density condition* if there is a constant $c_m > 0$ such that

$$\mu(B(x, r) \cap \Omega) \geq c_m \mu(B(x, r)) \quad (2.4)$$

for every $x \in \overline{\Omega}$ and every $r \in (0, \text{diam}(\Omega))$. We say that Ω satisfies a *measure doubling condition* if the measure $\mu|_\Omega$ is a doubling measure, i.e., there is a constant $c_d > 0$ such that

$$0 < \mu(B(x, 2r) \cap \Omega) \leq c_d \mu(B(x, r) \cap \Omega) < \infty \quad (2.5)$$

for every $x \in \overline{\Omega}$ and every $r > 0$. Notice that if Ω satisfies the measure density condition, then it satisfies the measure doubling condition.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into X . A non-negative Borel function g on X is an upper gradient of a function u on X if for all nonconstant curves γ , we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds, \quad (2.6)$$

where x and y are the end points of γ and the curve integral is defined by using an arc-length parametrization, see [21, Section 2] where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite.

We say that a family of curves Γ is of zero 1-modulus if there is a non-negative Borel function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho \, ds$ is infinite. A property is said to hold for 1-almost every curve if it fails only for a curve family with zero 1-modulus. If g is a non-negative μ -measurable function on X and (2.6) holds for 1-almost every curve, we say that g is a 1-weak upper gradient of u . By only considering curves γ in $A \subset X$, we can talk about a function g being a (1-weak) upper gradient of u in A .

Given a μ -measurable set $H \subset X$, we let

$$\|u\|_{N^{1,1}(H)} := \|u\|_{L^1(H)} + \inf \|g\|_{L^1(H)},$$

where the infimum is taken over all 1-weak upper gradients g of u in H . The substitute for the Sobolev space $W^{1,1}$ in the metric setting is the Newton-Sobolev space

$$N^{1,1}(H) := \{u : \|u\|_{N^{1,1}(H)} < \infty\},$$

which was first introduced in [37]. It is known that for any $u \in N^{1,1}_{\text{loc}}(H)$ there exists a minimal 1-weak upper gradient of u in H , always denoted by g_u , satisfying $g_u \leq g$ μ -a.e. in H , for any 1-weak upper gradient $g \in L^1_{\text{loc}}(H)$ of u in H , see [6, Theorem 2.25].

Next we present the basic theory of functions of bounded variation on metric spaces. This was first developed in [1, 35]; see also the monographs [3, 11–13, 44] for the classical theory in Euclidean spaces. We will always denote by Ω an open subset of X . Given a function $u \in L^1_{\text{loc}}(\Omega)$, we define the total variation of u in Ω by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i \rightarrow \infty} \int_{\Omega} g_{u_i} d\mu : u_i \in N^{1,1}_{\text{loc}}(\Omega), u_i \rightarrow u \text{ in } L^1_{\text{loc}}(\Omega) \right\}, \quad (2.7)$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i in Ω . (In [35], local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in BV(\Omega)$, if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf \{ \|Du\|(W) : A \subset W, W \subset X \text{ is open} \}.$$

Proposition 2.1 ([35, Theorem 3.4]). *If $u \in L^1_{\text{loc}}(\Omega)$, then $\|Du\|(\cdot)$ is a Borel measure on Ω .*

For any $u, v \in L^1_{\text{loc}}(\Omega)$, it is straightforward to show that

$$\|D(u+v)\|(\Omega) \leq \|Du\|(\Omega) + \|Dv\|(\Omega). \quad (2.8)$$

The BV norm is defined by

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

We will assume throughout the paper that X supports a $(1, 1)$ -Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient g of u , we have

$$\int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq C_P r \int_{B(x,\lambda r)} g d\mu,$$

where

$$u_{B(x,r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

Recall the exponent $s > 1$ from (2.1). The $(1, 1)$ -Poincaré inequality implies the so-called Sobolev-Poincaré inequality, see, e.g., [6, Theorem 4.21], and by applying the latter to approximating locally Lipschitz functions in the definition of the total variation, we get the following Sobolev-Poincaré inequality for BV functions. For every ball $B(x, r)$ and every $u \in L^1_{\text{loc}}(X)$, we have

$$\left(\int_{B(x,r)} |u - u_{B(x,r)}|^{s/(s-1)} \, d\mu \right)^{(s-1)/s} \leq C_{SP} \frac{\|Du\|(B(x, 2\lambda r))}{\mu(B(x, 2\lambda r))}, \quad (2.9)$$

where $C_{SP} = C_{SP}(C_d, C_P, \lambda) \geq 1$ is a constant.

For an open set $\Omega \subset X$ and a μ -measurable set $E \subset X$ with $\|D\chi_E\|(\Omega) < \infty$, we know that for any Borel set $A \subset \Omega$,

$$\|D\chi_E\|(A) = \int_{\partial^* E \cap A} \theta_E \, d\mathcal{H}, \quad (2.10)$$

where $\theta_E: X \rightarrow [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [4, Theorem 4.6]. The following coarea formula is given in [35, Proposition 4.2]: if $\Omega \subset X$ is an open set and $u \in L^1_{\text{loc}}(\Omega)$, then

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt. \quad (2.11)$$

The lower and upper approximate limits of a function u on Ω are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \rightarrow 0} \frac{\mu(\{u < t\} \cap B(x, r))}{\mu(B(x, r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \rightarrow 0} \frac{\mu(\{u > t\} \cap B(x, r))}{\mu(B(x, r))} = 0 \right\}.$$

Then the jump set S_u is defined as the set of points $x \in \Omega$ for which $u^\wedge(x) < u^\vee(x)$. It is straightforward to check that $u^\wedge$ and $u^\vee$ are Borel functions.

By [4, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set $\Omega \subset X$ and $u \in \text{BV}(\Omega)$, we have for any Borel set $A \subset \Omega$

$$\begin{aligned} \|Du\|(A) &= \|Du\|^a(A) + \|Du\|^s(A) \\ &= \|Du\|^a(A) + \|Du\|^c(A) + \|Du\|^j(A) \\ &= \int_A a \, d\mu + \|Du\|^c(A) + \int_{A \cap S_u} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u > t\}}(x) \, dt \, d\mathcal{H}(x), \end{aligned} \quad (2.12)$$

where $a \in L^1(\Omega)$ is the density of the absolutely continuous part $\|Du\|^a(A)$ of $\|Du\|(A)$ and the functions $\theta_{\{u>t\}} \in [\alpha, C_d]$ are as in (2.10).

Next, we introduce the Hajlasz-Sobolev space. Let $0 < p < \infty$. Given a μ -measurable set $K \subset X$, we define $M^{1,p}(K)$ to be the set of all functions $u \in L^p(K)$ for which there exists $0 \leq g \in L^p(K)$ and a set $A \subset K$ of measure zero such that for all $x, y \in K \setminus A$ we have the estimate

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)). \quad (2.13)$$

The corresponding norm (when $p \geq 1$) is obtained by setting

$$\|u\|_{M^{1,p}(K)} = \|u\|_{L^p(K)} + \inf \|g\|_{L^p(K)},$$

where the infimum is taken over all admissible functions g in (2.13). We refer to [14, 15] for more properties of the Hajlasz-Sobolev space $M^{1,p}$. The space $M_{c_H}^{1,p}(K)$ is defined exactly in the same manner as the space $M^{1,p}(K)$ except for one difference: in the definition of $M_{c_H}^{1,p}(K)$, the condition (2.13) is assumed to hold only for points $x, y \in K \setminus A$ that satisfy the condition

$$d(x, y) \leq c_H \cdot \min\{d(x, X \setminus K), d(y, X \setminus K)\}, \quad (2.14)$$

where $0 < c_H < 1$ is a constant.

We give the following definitions for the boundary trace, or trace for short, of a function defined on an open set Ω .

Definition 2.2. Let $\Omega \subset X$ be an open set and let u be a μ -measurable function on Ω . A number $Tu(x)$ is the trace of u at $x \in \partial\Omega$ if we have

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |u - Tu(x)| d\mu = 0. \quad (2.15)$$

We say that u has a trace Tu in $\partial\Omega$ if $Tu(x)$ exists for $\mathcal{H}$ -almost every $x \in \partial\Omega$.

Moreover, we give the following definitions for the trace space of a function space defined on an open set Ω .

Definition 2.3. Let Ω be an open set and let $\mathbb{X}(\Omega)$ be a function space on Ω . A function space $\mathbb{Y}(\partial\Omega, \mathcal{H})$ on $\partial\Omega$ is the trace space of $\mathbb{X}(\Omega)$ if the trace operator $u \mapsto Tu$ defined in Definition 2.2 is a bounded linear surjective operator from $\mathbb{X}(\Omega)$ to $\mathbb{Y}(\partial\Omega, \mathcal{H})$.

Definition 2.4. Let Ω be an open set and $\tilde{\mathcal{H}}$ be a measure on $\partial\Omega$. Let $\mathbb{X}(\Omega)$ be a function space on Ω . A function space $\mathbb{Y}(\partial\Omega, \tilde{\mathcal{H}})$ on $\partial\Omega$ is the trace space of $\mathbb{X}(\Omega)$ with respect to $\tilde{\mathcal{H}}$, if the trace operator $u \mapsto Tu$ defined in Definition 2.2 by replacing $\mathcal{H}$ by $\tilde{\mathcal{H}}$ is a bounded linear surjective operator from $\mathbb{X}(\Omega)$ to $\mathbb{Y}(\partial\Omega, \tilde{\mathcal{H}})$.

3. Traces of $N^{1,1}(\Omega)$ and $BV(\Omega)$

In this section, let $\Omega \subset X$ be an arbitrary nonempty open set. Recall the definition of the number $s > 1$ from (2.1).

Lemma 3.1. *Let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then there exists a sequence $(u_i) \subset \text{Lip}_{\text{loc}}(\Omega)$ such that $u_i \rightarrow u$ in $L^{s/(s-1)}_{\text{loc}}(\Omega)$ and*

$$\|Du\|(\Omega) = \lim_{i \rightarrow \infty} \int_{\Omega} g_{u_i} d\mu.$$

Proof. By the Sobolev-Poincaré inequality (2.9), we have $u \in L^{s/(s-1)}_{\text{loc}}(\Omega)$. Take open sets $\Omega_1 \subseteq \Omega_2 \subseteq \dots \subseteq \Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Now $u \in L^{s/(s-1)}(\Omega_j)$ for each $j \in \mathbb{N}$. Define the truncations

$$u_M := \min\{M, \max\{-M, u\}\}, \quad M > 0.$$

For each $j \in \mathbb{N}$ we find a number $M_j > 0$ such that $\|u_{M_j} - u\|_{L^{s/(s-1)}(\Omega_j)} < 1/j$. From the definition of the total variation, take a sequence $(v_i) \subset \text{Lip}_{\text{loc}}(\Omega)$ such that $v_i \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$ and

$$\|Du\|(\Omega) = \lim_{i \rightarrow \infty} \int_{\Omega} g_{v_i} d\mu.$$

Then also $(v_i)_{M_j} \rightarrow u_{M_j}$ in $L^{s/(s-1)}(\Omega_j)$ for all $j \in \mathbb{N}$. Thus we can pick indices $i(j) \geq j$ such that $\|(v_{i(j)})_{M_j} - u_{M_j}\|_{L^{s/(s-1)}(\Omega_j)} < 1/j$ for each $j \in \mathbb{N}$. Defining $u_j := (v_{i(j)})_{M_j}$, we now have

$$\|u_j - u\|_{L^{s/(s-1)}(\Omega_j)} < 2/j \quad \text{for all } j \in \mathbb{N}$$

and so $u_j \rightarrow u$ in $L^{s/(s-1)}_{\text{loc}}(\Omega)$. Moreover, since truncation does not increase energy,

$$\limsup_{j \rightarrow \infty} \int_{\Omega} g_{u_j} d\mu \leq \|Du\|(\Omega).$$

But by lower semicontinuity, also $\|Du\|(\Omega) \leq \liminf_{j \rightarrow \infty} \int_{\Omega} g_{u_j} d\mu$. $\square$

We have the following standard fact; for a proof see, e.g., [18, Proposition 3.8].

Lemma 3.2. *Let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$ and let $(u_i) \subset N^{1,1}_{\text{loc}}(\Omega)$ with $u_i \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$ and*

$$\|Du\|(\Omega) = \lim_{i \rightarrow \infty} \int_{\Omega} g_{u_i} d\mu.$$

Then we also have the weak convergence $g_{u_i} d\mu \xrightarrow{*} d\|Du\|$.*

Lemma 3.3. *Let $\Omega_1 \subseteq \Omega_2 \subseteq \dots \subseteq \bigcup_{j=1}^{\infty} \Omega_j = \Omega$ be open sets, let $\Omega_0 := \emptyset$, and let $\eta_j \in \text{Lip}_c(\Omega_j)$ be such that $0 \leq \eta_j \leq 1$ on X and $\eta_j = 1$ in Ω_{j-1} for each $j \in \mathbb{N}$, with $\eta_1 \equiv 0$. Let $1 \leq q < \infty$. Moreover, let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$, and for each $j \in \mathbb{N}$ let $(u_{j,i}) \subset N^{1,1}(\Omega_j)$ such that $u_{j,i} - u \rightarrow 0$ in $L^q(\Omega_j)$ and*

$$\lim_{i \rightarrow \infty} \int_{\Omega_j} g_{u_{j,i}} d\mu = \|Du\|(\Omega_j),$$

where each $g_{u_{j,i}}$ is the minimal 1-weak upper gradient of $u_{j,i}$ in Ω_j . Finally, let $\delta_j > 0$ for each $j \in \mathbb{N}$, and let $\varepsilon > 0$. Then for each $j \in \mathbb{N}$ we find an index $i(j)$ such that letting $u_j := u_{j,i(j)}$ and

$$v := \sum_{j=2}^{\infty} (\eta_j - \eta_{j-1}) u_j,$$

we have

$$\max\{\|v - u\|_{L^1(\Omega_j \setminus \Omega_{j-1})}, \|v - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})}\} < \delta_j \quad \text{for all } j \in \mathbb{N},$$

and $\int_{\Omega} g_v d\mu < \|Du\|(\Omega) + \varepsilon$.

Note that neither u nor the functions $u_{j,i}$ need to be in $L^q(\Omega_j)$, only in $L^1(\Omega_j)$, but still we can have $u_{j,i} - u \rightarrow 0$ in $L^q(\Omega_j)$ for each $j \in \mathbb{N}$. We can also see that in $\Omega_j \setminus \Omega_{j-1}$, the function v can be written as the finite sum (let $\eta_0 \equiv 0$)

$$\sum_{i=2}^{\infty} (\eta_i - \eta_{i-1}) u_i = (\eta_j - \eta_{j-1}) u_j + (\eta_{j+1} - \eta_j) u_{j+1} = \eta_j u_j + (1 - \eta_j) u_{j+1}. \quad (3.1)$$

Proof. By Lemma 3.2, for each $j \in \mathbb{N}$ we have $g_{u_{j,i}} d\mu \xrightarrow{*} d\|Du\|$ as $i \rightarrow \infty$ in Ω_j . For each $j \in \mathbb{N}$, let $L_j > 0$ denote a Lipschitz constant of η_j ; we can take this to be an increasing sequence. Set $\delta_0 := 1$, $L_0 := 1$. Letting $u_j := u_{j,i(j)}$ for suitable indices $i(j) \in \mathbb{N}$, we get

$$\max\{\|u_j - u\|_{L^1(\Omega_j)}, \|u_j - u\|_{L^q(\Omega_j)}\} < \min\{\delta_{j-1}, \delta_j, 2^{-j-1} \varepsilon / L_j\} / 2 \quad (3.2)$$

for all $j \in \mathbb{N}$, and

$$\int_{\Omega_j} (\eta_j - \eta_{j-1}) g_{u_j} d\mu < \int_{\Omega_j} (\eta_j - \eta_{j-1}) d\|Du\| + 2^{-j} \varepsilon \quad (3.3)$$

for all $j = 2, 3, \dots$. We get for all $j \in \mathbb{N}$

$$\begin{aligned} \|v - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} &= \left\| \sum_{i=2}^{\infty} (\eta_i - \eta_{i-1}) u_i - u \right\|_{L^q(\Omega_j \setminus \Omega_{j-1})} \\ &\stackrel{(3.1)}{=} \|\eta_j u_j + (1 - \eta_j) u_{j+1} - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} \\ &= \|\eta_j u_j + (1 - \eta_j) u_{j+1} - \eta_j u - (1 - \eta_j) u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} \\ &\leq \|u_j - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} + \|u_{j+1} - u\|_{L^q(\Omega_j \setminus \Omega_{j-1})} < \delta_j \end{aligned}$$

by (3.2) as desired, and similarly for the L^1 -norm. Let $v_2 := u_2$ in Ω_2 , and recursively $v_{i+1} := \eta_i v_i + (1 - \eta_i)u_{i+1}$ in Ω_{i+1} . We see that $v = \lim_{i \rightarrow \infty} v_i$ (at every point in Ω). By the proof of the Leibniz rule in [6, Lemma 2.18], the minimal 1-weak upper gradient of v_3 in Ω_3 satisfies

$$g_{v_3} \leq g_{\eta_2}|u_2 - u_3| + \eta_2 g_{u_2} + (1 - \eta_2)g_{u_3}.$$

Inductively, we get for $i = 3, 4, \dots$

$$g_{v_i} \leq \sum_{j=2}^{i-1} g_{\eta_j}|u_j - u_{j+1}| + \sum_{j=2}^{i-1} (\eta_j - \eta_{j-1})g_{u_j} + (1 - \eta_{i-1})g_{u_i} \quad \text{in } \Omega_i;$$

to prove this, assume that it holds for the index i . Then we have by applying a Leibniz rule as above, and noting that g_{η_i} can be non-zero only in $\Omega_i \setminus \Omega_{i-1}$ (see [6, Corollary 2.21]), where $v_i = u_i$,

$$\begin{aligned} g_{v_{i+1}} &\leq g_{\eta_i}|v_i - u_{i+1}| + \eta_i g_{v_i} + (1 - \eta_i)g_{u_{i+1}} \\ &= g_{\eta_i}|u_i - u_{i+1}| + \eta_i g_{v_i} + (1 - \eta_i)g_{u_{i+1}} \\ &\stackrel{\text{Induction}}{\leq} g_{\eta_i}|u_i - u_{i+1}| + \sum_{j=2}^{i-1} g_{\eta_j}|u_j - u_{j+1}| \\ &\quad + \sum_{j=2}^{i-1} (\eta_j - \eta_{j-1})g_{u_j} + (\eta_i - \eta_{i-1})g_{u_i} + (1 - \eta_i)g_{u_{i+1}} \\ &= \sum_{j=2}^i g_{\eta_j}|u_j - u_{j+1}| + \sum_{j=2}^i (\eta_j - \eta_{j-1})g_{u_j} + (1 - \eta_i)g_{u_{i+1}} \quad \text{in } \Omega_{i+1}. \end{aligned}$$

This completes the induction. In each $\Omega_j \setminus \Omega_{j-1}$, by (3.1) we have

$$v = \eta_j u_j + (1 - \eta_j)u_{j+1} = \eta_j v_j + (1 - \eta_j)u_{j+1} = v_{j+1},$$

and so in fact $v = v_{j+1}$ in Ω_j , for each $j \in \mathbb{N}$. Thus the minimal 1-weak upper gradient of v in Ω_i satisfies

$$g_v = g_{v_{i+1}} \leq \sum_{j=2}^{\infty} g_{\eta_j}|u_j - u_{j+1}| + \sum_{j=2}^{\infty} (\eta_j - \eta_{j-1})g_{u_j}.$$

Thus

$$\begin{aligned} \int_{\Omega_i} g_v d\mu &\leq \sum_{j=2}^{\infty} \int_{\Omega_j} g_{\eta_j}|u_j - u_{j+1}| d\mu + \sum_{j=2}^{\infty} \int_{\Omega_j} (\eta_j - \eta_{j-1})g_{u_j} d\mu \\ &\leq \sum_{j=2}^{\infty} L_j \|u_j - u_{j+1}\|_{L^1(\Omega_j \setminus \Omega_{j-1})} \\ &\quad + \sum_{j=2}^{\infty} \left(\int_{\Omega_j} (\eta_j - \eta_{j-1}) d\|Du\| + 2^{-j}\varepsilon \right) \quad \text{by (3.3)} \\ &\leq \varepsilon/2 + \|Du\|(\Omega) + \varepsilon/2 \quad \text{by (3.2), (3.3)} \\ &= \|Du\|(\Omega) + \varepsilon. \end{aligned}$$

Note that g_v does not depend on i , see [6, Lemma 2.23], and so it is well defined on Ω . Since g_v is the minimal 1-weak upper gradient of v in each Ω_i , it is clearly also (the minimal) 1-weak upper gradient of v in Ω . Then by Lebesgue's monotone convergence theorem,

$$\int_{\Omega} g_v d\mu \leq \|Du\|(\Omega) + \varepsilon. \quad \square$$

Theorem 1.1 of the introduction follows from the following theorem.

Theorem 3.4. *Let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$ and let $\varepsilon > 0$. Then there exists $v \in N^{1,1}_{\text{loc}}(\Omega) \cap \text{Lip}_{\text{loc}}(\Omega)$ such that $\|v - u\|_{L^1(\Omega)} < \varepsilon$, $\|v - u\|_{L^{s/(s-1)}(\Omega)} < \varepsilon$, $\int_{\Omega} g_v d\mu < \|Du\|(\Omega) + \varepsilon$, and*

$$\int_{B(x,r) \cap \Omega} |v - u|^{s/(s-1)} d\mu \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

uniformly for all $x \in \partial\Omega$.

Note that if $u \in BV(\Omega)$ as in the formulation of Theorem 1.1, then $v \in L^1(\Omega)$ and so $v \in N^{1,1}(\Omega)$.

Proof. Fix $x_0 \in X$. Define $\Omega_0 := \emptyset$ and pick numbers $d_j \in (2^{-j}, 2^{-j+1})$, $j \in \mathbb{N}$, such that the sets

$$\Omega_j := \{x \in \Omega : d(x, X \setminus \Omega) > d_j\} \cap B(x_0, d_j^{-1})$$

satisfy $\|Du\|(\partial\Omega_j) = 0$. For each $j \in \mathbb{N}$, take $\eta_j \in \text{Lip}_c(\Omega_j)$ such that $0 \leq \eta_j \leq 1$ on X and $\eta_j = 1$ in Ω_{j-1} , and $\eta_1 \equiv 0$. Note that for a fixed $r > 0$, the function

$$x \mapsto \mu(B(x, r) \cap \Omega), \quad x \in \partial\Omega,$$

is lower semicontinuous and strictly positive. Since $\partial\Omega \cap \overline{B}(x_0, d_j^{-1})$ is compact for every $j \in \mathbb{N}$, the numbers

$$\beta_j := \inf\{\mu(B(x, 2^{-j}) \cap \Omega) : x \in \partial\Omega \cap \overline{B}(x_0, d_{j+2}^{-1})\}, \quad j \in \mathbb{N},$$

are strictly positive. Set

$$\delta_j := 2^{-j} \min\left\{\varepsilon, \beta_j^{s/(s-1)}\right\}.$$

By Lemma 3.1 we find functions $(u_i) \subset \text{Lip}_{\text{loc}}(\Omega)$ such that $u_i \rightarrow u$ in $L^{s/(s-1)}_{\text{loc}}(\Omega)$ and

$$\lim_{i \rightarrow \infty} \int_{\Omega} g_{u_i} d\mu = \|Du\|(\Omega).$$

Then also $u_i \rightarrow u$ in $L^{s/(s-1)}(\Omega_j)$ for every $j \in \mathbb{N}$, and by Lemma 3.2 and the fact that $\|Du\|(\partial\Omega_j) = 0$ we get

$$\lim_{i \rightarrow \infty} \int_{\Omega_j} g_{u_i} d\mu = \|Du\|(\Omega_j).$$

Then apply Lemma 3.3 to obtain a function $v \in \text{Lip}_{\text{loc}}(\Omega)$. By the lemma, we have $\int_{\Omega} g_v d\mu < \|Du\|(\Omega) + \varepsilon$ as desired, and from the condition

$$\max\{\|v - u\|_{L^1(\Omega_j \setminus \Omega_{j-1})}, \|v - u\|_{L^{s/(s-1)}(\Omega_j \setminus \Omega_{j-1})}\} < \delta_j \leq 2^{-j}\varepsilon \quad \text{for all } j \in \mathbb{N}$$

we easily get $\|v - u\|_{L^1(\Omega)} < \varepsilon$ and $\|v - u\|_{L^{s/(s-1)}(\Omega)} < \varepsilon$. In particular, $v \in N_{\text{loc}}^{1,1}(\Omega)$ as desired.

Fix $x \in \partial\Omega$. Choose the smallest $l \in \mathbb{N}$ such that $x \in B(x_0, d_{l+2}^{-1})$. Note that then $B(x, 1) \cap B(x_0, d_{l-1}^{-1}) = \emptyset$ (if $l \geq 2$) and so for any $k \in \mathbb{N}$,

$$B(x, 2^{-k+1}) \cap \Omega = B(x, 2^{-k+1}) \cap \left(\bigcup_{j=\max\{k,l\}}^{\infty} (\Omega_j \setminus \Omega_{j-1}) \right).$$

Now

$$\begin{aligned} & \frac{1}{\mu(B(x, 2^{-k}) \cap \Omega)} \int_{B(x, 2^{-k+1}) \cap \Omega} |v - u|^{s/(s-1)} d\mu \\ &= \frac{1}{\mu(B(x, 2^{-k}) \cap \Omega)} \sum_{j=\max\{k,l\}}^{\infty} \int_{B(x, 2^{-k+1}) \cap \Omega_j \setminus \Omega_{j-1}} |v - u|^{s/(s-1)} d\mu \\ &\leq \frac{1}{\mu(B(x, 2^{-k}) \cap \Omega)} \sum_{j=\max\{k,l\}}^{\infty} \int_{\Omega_j \setminus \Omega_{j-1}} |v - u|^{s/(s-1)} d\mu \\ &\leq \frac{1}{\mu(B(x, 2^{-k}) \cap \Omega)} \sum_{j=\max\{k,l\}}^{\infty} \delta_j^{(s-1)/s} \\ &\leq \sum_{j=\max\{k,l\}}^{\infty} \frac{2^{-j} \beta_j}{\mu(B(x, 2^{-j}) \cap \Omega)} \\ &\leq \sum_{j=\max\{k,l\}}^{\infty} 2^{-j} \leq 2^{-k+1}. \end{aligned}$$

Now it clearly follows that

$$\int_{B(x,r) \cap \Omega} |v - u|^{s/(s-1)} d\mu \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

uniformly for all $x \in \partial\Omega$. □

We have the following approximation result for BV functions in the L^q -norm.

Theorem 3.5. *Let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$ and let $1 \leq q < \infty$. Then there exists a sequence $(u_i) \subset N^{1,1}_{\text{loc}}(\Omega)$ such that $u_i - u \rightarrow 0$ in $L^1(\Omega) \cap L^q(\Omega)$ and*

$$\int_{\Omega} g_{u_i} d\mu \rightarrow \|Du\|(\Omega).$$

Proof. For each $k = 0, 1, \dots$ define the truncation of u at levels k and $k + 1$

$$u_k := \min\{1, (u - k)_+\}.$$

Then $u_k \in L^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ for each $k = 0, 1, \dots$ and $u_+ = \sum_{k=0}^{\infty} u_k$. Also note that by the coarea formula (2.11),

$$\|Du_k\|(\Omega) = \int_{-\infty}^{\infty} P(\{u_k > t\}, \Omega) dt = \int_k^{k+1} P(\{u > t\}, \Omega) dt.$$

For each $k = 0, 1, \dots$, from the definition of the total variation we get a sequence $(v_{k,i}) \subset N^{1,1}_{\text{loc}}(\Omega)$ with $v_{k,i} \rightarrow u_k$ in $L^1_{\text{loc}}(\Omega)$ and

$$\int_{\Omega} g_{v_{k,i}} d\mu \rightarrow \|Du_k\|(\Omega) \quad \text{as } i \rightarrow \infty.$$

In the proof of Theorem 3.4 we saw that in fact we can get $v_{k,i} - u_k \rightarrow 0$ in $L^1(\Omega)$. Since $0 \leq u_k \leq 1$, by truncation we can assume that also $0 \leq v_{k,i} \leq 1$. Then also $v_{k,i} - u_k \rightarrow 0$ in $L^q(\Omega)$. Let $\varepsilon > 0$. For a suitable choice of indices $i = i(k)$, for $v_k := v_{k,i(k)}$ we have $\|v_k - u_k\|_{L^1(\Omega)} < 2^{-k-2}\varepsilon$, $\|v_k - u_k\|_{L^q(\Omega)} < 2^{-k-2}\varepsilon$, and

$$\int_{\Omega} g_{v_k} d\mu < \|Du_k\|(\Omega) + 2^{-k-1}\varepsilon = \int_k^{k+1} P(\{u > t\}, \Omega) dt + 2^{-k-1}\varepsilon.$$

Then for $v := \sum_{k=0}^{\infty} v_k$ we have $\|v - u_+\|_{L^1(\Omega)} < \varepsilon/2$ and $\|v - u_+\|_{L^q(\Omega)} < \varepsilon/2$. Moreover, using, e.g., [6, Lemma 1.52] we get $g_v \leq \sum_{k=0}^{\infty} g_{v_k}$ and then

$$\begin{aligned} \int_{\Omega} g_v d\mu &\leq \sum_{k=0}^{\infty} \int_{\Omega} g_{v_k} d\mu \leq \sum_{k=0}^{\infty} \left(\int_k^{k+1} P(\{u > t\}, \Omega) dt + 2^{-k-1}\varepsilon \right) \\ &= \int_0^{\infty} P(\{u > t\}, \Omega) dt + \varepsilon/2 \\ &= \|Du_+\|(\Omega) + \varepsilon/2 \end{aligned}$$

again by the coarea formula. Similarly we find a function $w \in N^{1,1}_{\text{loc}}(\Omega)$ with $\|w - u_-\|_{L^1(\Omega)} < \varepsilon/2$, $\|w - u_-\|_{L^q(\Omega)} < \varepsilon/2$, and $\int_{\Omega} g_w d\mu < \|Du_-\|(\Omega) + \varepsilon/2$. Then for $h := v - w$ we have $\|h - u\|_{L^1(\Omega)} < \varepsilon$, $\|h - u\|_{L^q(\Omega)} < \varepsilon$, and

$$\int_{\Omega} g_h d\mu < \|Du_+\|(\Omega) + \varepsilon/2 + \|Du_-\|(\Omega) + \varepsilon/2 = \|Du\|(\Omega) + \varepsilon$$

using the coarea formula once more. In this way we get the desired sequence. $\square$

Theorem 1.2 of the introduction follows from the following theorem. In Example 5.1 we will show that here we cannot take u to be continuous or even locally bounded in Ω .

Theorem 3.6. *Let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$, let $1 \leq q < \infty$, and let $\varepsilon > 0$. Then there exists $v \in N^{1,1}_{\text{loc}}(\Omega)$ such that $\|v - u\|_{L^1(\Omega)} < \varepsilon$, $\|v - u\|_{L^q(\Omega)} < \varepsilon$, $\int_{\Omega} g_v d\mu < \|Du\|(\Omega) + \varepsilon$, and*

$$\int_{B(x,r) \cap \Omega} |v - u|^q d\mu \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

uniformly for all $x \in \partial\Omega$.

Proof. The proof is essentially the same as for Theorem 3.4; the difference is that here we apply Theorem 3.5 to find sequences $(u_{j,i})_i \subset N^{1,1}(\Omega_j)$, $j \in \mathbb{N}$, such that $\|u_{j,i} - u\|_{L^q(\Omega_j)} \rightarrow 0$ and $\lim_{i \rightarrow \infty} \int_{\Omega_j} g_{u_{j,i}} d\mu = \|Du\|(\Omega_j)$ as $i \rightarrow \infty$. $\square$

We say that $w \in \text{SBV}(\Omega)$ if $w \in \text{BV}(\Omega)$ and $\|Dw\|^c(\Omega) = 0$ (recall the decomposition (2.12)). Recall also that the jump set S_u is the set of points $x \in \Omega$ for which $u^\wedge(x) < u^\vee(x)$. Denote $\Omega(r) := \{x \in \Omega : \text{dist}(x, X \setminus \Omega) > r\}$. We have the following approximation result for BV functions by SBV functions.

Theorem 3.7. *Let $u \in \text{BV}(\Omega)$ and let $\varepsilon > 0$. Then there exists $w \in \text{SBV}(\Omega)$ such that $\|w - u\|_{L^1(\Omega)} < \varepsilon$, $\|w - u\|_{L^\infty(\Omega)} < \varepsilon$, $\|Dw\|(\Omega) < \|Du\|(\Omega) + \varepsilon$, $\mathcal{H}(S_w \setminus S_u) = 0$, and*

$$\lim_{r \rightarrow 0^+} \|w - u\|_{L^\infty(\Omega \setminus \Omega(r))} = 0.$$

Proof. This is given in [29, Corollary 5.15]; for the above limit see [29, equations (3.8), (3.11)]. $\square$

The following approximation result for BV functions by means of functions with a jump set of finite Hausdorff measure is given as part of [30, Theorem 5.3].

Theorem 3.8. *Let $u \in \text{BV}(\Omega)$ and let $\varepsilon, \delta > 0$. Then we find $w \in \text{BV}(\Omega)$ such that $\|w - u\|_{L^1(\Omega)} < \varepsilon$,*

$$\|D(w - u)\|(\Omega) \leq 2\|Du\|(\{0 < u^\vee - u^\wedge < \delta\}) + \varepsilon,$$

$$\|w - u\|_{L^\infty(\Omega)} \leq 10\delta, \text{ and } \mathcal{H}(S_w \setminus \{u^\vee - u^\wedge \geq \delta\}) = 0.$$

We apply this theorem first to obtain the following proposition.

Proposition 3.9. *Let $u \in \text{BV}(\Omega)$ and let $\varepsilon > 0$. Then we find $v \in \text{BV}(\Omega)$ such that $\|v - u\|_{\text{BV}(\Omega)} < \varepsilon$, $\|v - u\|_{L^\infty(\Omega)} < \varepsilon$, $\mathcal{H}(S_v) < \infty$, and*

$$\lim_{r \rightarrow 0^+} \|v - u\|_{L^\infty(\Omega \setminus \Omega(r))} = 0.$$

Proof. Take numbers $\delta_j \searrow 0$, $0 < \delta_j < \varepsilon/20$, such that

$$\sum_{j=2}^{\infty} \|Du\|(\{0 < u^\vee - u^\wedge < \delta_j\}) < \frac{\varepsilon}{4}. \quad (3.4)$$

Note that by the decomposition (2.12), $\mathcal{H}(\{u^\vee - u^\wedge > t\}) < \infty$ for all $t > 0$. Thus we can take a strictly decreasing sequence of $r_j \searrow 0$ so that the sets $\Omega_j := \Omega(r_j)$ satisfy (let $\Omega_0 := \emptyset$)

$$\mathcal{H}((\Omega_j \setminus \Omega_{j-2}) \cap \{u^\vee - u^\wedge \geq \delta_j\}) \leq \mathcal{H}((\Omega \setminus \Omega_{j-2}) \cap \{u^\vee - u^\wedge \geq \delta_j\}) < 2^{-j}\varepsilon$$

for all $j = 3, 4, \dots$. Then

$$\sum_{j=2}^{\infty} \mathcal{H}((\Omega_j \setminus \Omega_{j-2}) \cap \{u^\vee - u^\wedge \geq \delta_j\}) < \mathcal{H}(\{u^\vee - u^\wedge \geq \delta_2\}) + \varepsilon. \quad (3.5)$$

Also choose functions $\eta_j \in \text{Lip}(X)$ supported in Ω_j , $j \in \mathbb{N}$, such that $0 \leq \eta_j \leq 1$ on X and $\eta_j = 1$ in Ω_{j-1} , with $\eta_1 \equiv 0$. For each $j \in \mathbb{N}$, apply Theorem 3.8 to find a function $v_j \in BV(\Omega)$ satisfying

$$\max\{\|g_{\eta_j} + g_{\eta_{j-1}}\|_{L^\infty(\Omega)}, 1\} \cdot \|v_j - u\|_{L^1(\Omega)} < 2^{-j-1}\varepsilon \quad (3.6)$$

as well as

$$\|D(v_j - u)\|(\Omega) \leq 2\|Du\|(\{0 < u^\vee - u^\wedge < \delta_j\}) + 2^{-j-1}\varepsilon, \quad (3.7)$$

$\|v_j - u\|_{L^\infty(\Omega)} \leq 10\delta_j$, and $\mathcal{H}(S_{v_j} \setminus \{u^\vee - u^\wedge \geq \delta_j\}) = 0$. Let

$$v := \sum_{j=2}^{\infty} (\eta_j - \eta_{j-1})v_j. \quad (3.8)$$

Then

$$\begin{aligned} \|v - u\|_{L^1(\Omega)} &= \left\| \sum_{j=2}^{\infty} (\eta_j - \eta_{j-1})(v_j - u) \right\|_{L^1(\Omega)} \\ &\leq \sum_{j=2}^{\infty} \|v_j - u\|_{L^1(\Omega)} \\ &\leq \sum_{j=2}^{\infty} 2^{-j-1}\varepsilon = \varepsilon/4. \end{aligned}$$

Since $\|v_j - u\|_{L^\infty(\Omega)} \leq 10\delta_j < \varepsilon/2$, also $\|v - u\|_{L^\infty(\Omega)} < \varepsilon$. It is also easy to check that $\lim_{r \rightarrow 0^+} \|v - u\|_{L^\infty(\Omega \setminus \Omega(r))} = 0$.

Clearly $\sum_{j=2}^k (\eta_j - \eta_{j-1})(v_j - u) \rightarrow v - u$ in $L^1_{\text{loc}}(\Omega)$ as $k \rightarrow \infty$. Thus by lower semicontinuity and a Leibniz rule (see [19, Lemma 3.2]),

$$\begin{aligned}
 \|D(v - u)\|(\Omega) &\leq \liminf_{k \rightarrow \infty} \left\| D \sum_{j=2}^k (\eta_j - \eta_{j-1})(v_j - u) \right\|(\Omega) \\
 &\leq \sum_{j=2}^{\infty} \|D((\eta_j - \eta_{j-1})(v_j - u))\|(\Omega) \quad \text{by (2.8)} \\
 &\leq \sum_{j=2}^{\infty} \left(\|D(v_j - u)\|(\Omega) + \int_{\Omega} (g_{\eta_j} + g_{\eta_{j-1}})|v_j - u| d\mu \right) \\
 &< \sum_{j=2}^{\infty} \left(2\|Du\|(\{0 < u^{\vee} - u^{\wedge} < \delta_j\}) + 2^{-j-1}\varepsilon \right) + \sum_{j=2}^{\infty} 2^{-j-1}\varepsilon \quad \text{by (3.7), (3.6)} \\
 &< \varepsilon/2 + \varepsilon/4 + \varepsilon/4 \quad \text{by (3.4)} \\
 &= \varepsilon.
 \end{aligned}$$

Finally we want to show that $\mathcal{H}(S_v) < \infty$. Note that (3.8) is a locally finite sum. If $x \in S_{(\eta_j - \eta_{j-1})v_j}$, then we get $x \in S_{v_j}$, and so $S_v \subset \bigcup_{j=2}^{\infty} (S_{v_j} \cap (\Omega_j \setminus \Omega_{j-2}))$. By the fact that $\mathcal{H}(S_{v_j} \setminus \{u^{\vee} - u^{\wedge} \geq \delta_j\}) = 0$ for all $j \in \mathbb{N}$ and by (3.5), we find that

$$\begin{aligned}
 \mathcal{H}(S_v) &\leq \sum_{j=2}^{\infty} \mathcal{H}(S_{v_j} \cap (\Omega_j \setminus \Omega_{j-2})) \leq \sum_{j=2}^{\infty} \mathcal{H}(\{u^{\vee} - u^{\wedge} \geq \delta_j\} \cap (\Omega_j \setminus \Omega_{j-2})) \\
 &< \mathcal{H}(\{u^{\vee} - u^{\wedge} \geq \delta_2\}) + \varepsilon < \infty,
 \end{aligned}$$

as desired. $\square$

Now we can prove Theorem 1.3 of the introduction. In Example 5.2 we will show that here we cannot have $\mathcal{H}(S_v) = 0$.

Proof of Theorem 1.3. First apply Proposition 3.9 to find $\widehat{w} \in \text{BV}(\Omega)$ such that $\|\widehat{w} - u\|_{\text{BV}(\Omega)} < \varepsilon/4$, $\mathcal{H}(S_{\widehat{w}}) < \infty$, and

$$\lim_{r \rightarrow 0^+} \|\widehat{w} - u\|_{L^\infty(\Omega \setminus \Omega(r))} = 0.$$

Then apply Theorem 3.7 to find $w \in \text{SBV}(\Omega)$ such that $\|w - \widehat{w}\|_{L^1(\Omega)} < \varepsilon/4$, $\|Dw\|(\Omega) < \|D\widehat{w}\|(\Omega) + \varepsilon/4$, $\mathcal{H}(S_w \setminus S_{\widehat{w}}) = 0$, and

$$\lim_{r \rightarrow 0^+} \|w - \widehat{w}\|_{L^\infty(\Omega \setminus \Omega(r))} = 0.$$

In total, we have $w \in \text{SBV}(\Omega)$ such that $\|w - u\|_{L^1(\Omega)} < \varepsilon/2$, $\|Dw\|(\Omega) < \|Du\|(\Omega) + \varepsilon/2$, $\mathcal{H}(S_w) < \infty$, and

$$\lim_{r \rightarrow 0^+} \|w - u\|_{L^\infty(\Omega \setminus \Omega(r))} = 0.$$

Take $\Omega' \Subset \Omega$ such that $\|Dw\|(\Omega \setminus \Omega') < \varepsilon/2$ and $\mathcal{H}(S_w \setminus \Omega') < \varepsilon$, and take a function $\eta \in \text{Lip}_c(\Omega)$ with $0 \leq \eta \leq 1$ on X and $\eta = 1$ in Ω' . From the definition of the total variation, take a sequence $(w_i) \subset \text{Lip}_{\text{loc}}(\Omega)$ such that $w_i \rightarrow w$ in $L^1_{\text{loc}}(\Omega)$ and $\lim_{i \rightarrow \infty} \|Dw_i\|(\Omega) = \|Dw\|(\Omega)$. Define for each $i \in \mathbb{N}$

$$v_i := \eta w_i + (1 - \eta)w.$$

Then clearly $\lim_{i \rightarrow \infty} \|v_i - w\|_{L^1(\Omega)} = 0$ and by a Leibniz rule (see [19, Lemma 3.2]) and since g_η is bounded,

$$\begin{aligned} \|Dv_i\|(\Omega) &\leq \int_{\Omega} |w_i - w| g_\eta d\mu + \|Dw_i\|(\Omega) + \|Dw\|(\Omega \setminus \Omega') \\ &\rightarrow \|Dw\|(\Omega) + \|Dw\|(\Omega \setminus \Omega') < \|Du\|(\Omega) + \varepsilon. \end{aligned}$$

Thus if we choose $v := v_i$ for suitably large $i \in \mathbb{N}$, we have $\|v - u\|_{L^1(\Omega)} < \varepsilon$ and $\|Dv\|(\Omega) < \|Du\|(\Omega) + \varepsilon$, and so in particular $v \in \text{BV}(\Omega)$. It is then easy to check that in fact $v \in \text{SBV}(\Omega)$. Since $S_{w_i} = \emptyset$ for all $i \in \mathbb{N}$, we have $S_{v_i} \subset S_w \setminus \Omega'$ for all $i \in \mathbb{N}$, and since $\mathcal{H}(S_w \setminus \Omega') < \varepsilon$, in fact $\mathcal{H}(S_v) < \varepsilon$. Finally,

$$\lim_{r \rightarrow 0^+} \|v - u\|_{L^\infty(\Omega \setminus \Omega(r))} = \lim_{r \rightarrow 0^+} \|w - u\|_{L^\infty(\Omega \setminus \Omega(r))} = 0$$

as required. $\square$

To complete this section, we give the proof of Corollary 1.4 using Theorem 3.4 (or Theorem 3.6).

Proof of Corollary 1.4. Assume that $\mathbb{Z}(\partial\Omega, \mathcal{H})$ is the trace space of $\text{BV}(\Omega)$, i.e., the trace operator $u \mapsto Tu$ in Definition 2.2 is a bounded linear surjective operator from $\text{BV}(\Omega)$ to $\mathbb{Z}(\partial\Omega, \mathcal{H})$. From the definition of the total variation (2.7) we immediately get $N^{1,1}(\Omega) \subset \text{BV}(\Omega)$ with $\|\cdot\|_{\text{BV}(\Omega)} \leq \|\cdot\|_{N^{1,1}(\Omega)}$. Thus the trace operator $u \mapsto Tu$ is still a bounded linear operator from $N^{1,1}(\Omega)$ to $\mathbb{Z}(\partial\Omega, \mathcal{H})$. Hence it remains to show the surjectivity. For any $f \in \mathbb{Z}(\partial\Omega, \mathcal{H})$, we know that there is a function $u \in \text{BV}(\Omega)$ such that $Tu = f$. It follows from Theorem 3.4 (or Theorem 3.6) that there is a function $v \in N^{1,1}(\Omega)$ such that $Tv = Tu = f$, since

$$\begin{aligned} \lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |v - f(x)| d\mu &\leq \lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |u - v| + |u - f(x)| d\mu \\ &\leq \lim_{r \rightarrow 0^+} \left(\int_{B(x,r) \cap \Omega} |u - v|^{s/s-1} d\mu \right)^{(s-1)/s} \\ &\quad + \lim_{r \rightarrow 0^+} \int_{B(x,r) \cap \Omega} |u - f(x)| d\mu \\ &= 0, \quad \text{for } \mathcal{H}\text{-a.e. } x \in \partial\Omega. \end{aligned} \tag{3.9}$$

This gives the surjectivity, as desired.

Conversely, assume that $\mathbb{Z}(\partial\Omega, \mathcal{H})$ is the trace space of $N^{1,1}(\Omega)$, i.e., the trace operator $u \mapsto Tu$ in Definition 2.2 is a bounded linear surjective operator from $N^{1,1}(\Omega)$ to $\mathbb{Z}(\partial\Omega, \mathcal{H})$. Then for any $h \in \text{BV}(\Omega)$, without loss of generality, we may assume that $\|h\|_{\text{BV}(\Omega)} > 0$. By Theorem 3.4, choosing $\varepsilon = \|h\|_{\text{BV}(\Omega)}/2$, there is a function $v \in N^{1,1}(\Omega)$ with $\|v\|_{N^{1,1}(\Omega)} \leq 2\|h\|_{\text{BV}(\Omega)}$ and

$$\int_{B(x,r) \cap \Omega} |v - h|^{s/(s-1)} d\mu \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

uniformly for all $x \in \partial\Omega$. Then we have that $Th = Tv$ by a similar argument to (3.9), and that

$$\|Th\|_{\mathbb{Z}(\partial\Omega, \mathcal{H})} = \|Tv\|_{\mathbb{Z}(\partial\Omega, \mathcal{H})} \lesssim \|v\|_{N^{1,1}(\Omega)} \leq 2\|h\|_{\text{BV}(\Omega)}.$$

Hence the trace Th exists for any $h \in \text{BV}(\Omega)$ and the trace operator $h \rightarrow Th$ is linear and bounded from $\text{BV}(\Omega)$ to $\mathbb{Z}(\partial\Omega, \mathcal{H})$. Moreover, the surjectivity of the trace operator follows immediately from $N^{1,1}(\Omega) \subset \text{BV}(\Omega)$. Thus $\mathbb{Z}(\partial\Omega, \mathcal{H})$ is also the trace space of $\text{BV}(\Omega)$. $\square$

Remark 3.10. The trace spaces of $\text{BV}(\Omega)$ and $N^{1,1}(\Omega)$ are also the same with respect to any given boundary measure $\tilde{\mathcal{H}}$ under Definition 2.4.

4. Traces of $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$

In this section let $\Omega \subset X$ be an arbitrary nonempty open set with nonempty complement.

We will work with Whitney coverings of open sets. For a ball $B = B(x, r)$ and a number $a > 0$, we use the notation $aB := B(x, ar)$. We can choose a Whitney covering $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ of Ω such that:

1. For each $j \in \mathbb{N}$,

$$r_j = \text{dist}(x_j, X \setminus \Omega)/100\lambda;$$

2. For each $k \in \mathbb{N}$, the ball $20\lambda B_k$ meets at most $C_0 = C_0(C_d)$ balls $20\lambda B_j$ (that is, a bounded overlap property holds);
3. If $20\lambda B_k$ meets $20\lambda B_j$, then $r_j \leq 2r_k$;

see, e.g., [22, Proposition 4.1.15] and its proof. Given such a covering of Ω , we find a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to the covering, that is, for each $j \in \mathbb{N}$ the function ϕ_j is c/r_j -Lipschitz, $c = c(C_d)$, with $\text{spt}(\phi_j) \subset 2B_j$ and $0 \leq \phi_j \leq 1$, such

that $\sum_j \phi_j = 1$ on Ω ; see, e.g., [22, page 103]. We define a *discrete convolution* u_W of $u \in L^1_{\text{loc}}(\Omega)$ with respect to the Whitney covering by

$$u_W := \sum_{j=1}^{\infty} u_{B_j} \phi_j.$$

In general, $u_W \in \text{Lip}_{\text{loc}}(\Omega) \subset L^1_{\text{loc}}(\Omega)$.

Theorem 4.1. *For any function $u \in N^{1,1}(\Omega)$, there exists a constant $0 < c_H = c_H(\lambda) < 1$ such that the discrete convolution u_W of u with respect to the Whitney covering $\{B_j = B(x_j, r_j)\}_{j=1}^{\infty}$ is in $M^{1,1}_{c_H}(\Omega)$ with the norm estimate*

$$\|u_W\|_{M^{1,1}_{c_H}(\Omega)} \lesssim \|u\|_{N^{1,1}(\Omega)}.$$

Proof. First we consider the L^1 -norm of w_W . By the bounded overlap property of the Whitney covering $\{B_j = B(x_j, r_j)\}_{j=1}^{\infty}$, it follows from the facts $\text{spt}(\phi_j) \subset 2B_j$ and $0 \leq \phi_j \leq 1$ that

$$\|u_W\|_{L^1(\Omega)} \leq \sum_{j=1}^{\infty} \mu(2B_j) \int_{B_j} |u| d\mu \leq C_d \sum_{j=1}^{\infty} \int_{B_j} |u| d\mu \lesssim \|u\|_{L^1(\Omega)}.$$

Next, for the minimal 1-weak upper gradient g_u of u , we will give an admissible function g that satisfies (2.13) when the pair of points x, y satisfy (2.14) with $c_H = 1/50\lambda$. We claim that the admissible function g can be defined as follows: for any point $x \in \Omega$, we define

$$g(x) := C \sum_{j=1}^{\infty} \chi_{B_j}(x) \int_{60\lambda B_j} g_u d\mu \quad (4.1)$$

with $C = C(C_d, C_P, \lambda)$. Indeed, for any pair of points $x, y \in \Omega$ satisfying (2.14), without loss of generality, we may assume that $\text{dist}(x, X \setminus \Omega) \leq \text{dist}(y, X \setminus \Omega)$ and $x \in B_j, y \in B_i$ for some $i, j \in \mathbb{N}$. Recalling the properties of the Whitney covering, we have that

$$\text{dist}(x, X \setminus \Omega) \leq \text{dist}(x_j, X \setminus \Omega) + r_j = (100\lambda + 1)r_j.$$

Hence we have

$$d(y, x_j) \leq d(x, y) + r_j \leq \frac{1}{50\lambda} \text{dist}(x, X \setminus \Omega) + r_j < 4r_j,$$

which means $y \in 4B_j$. Hence $20\lambda B_i \cap 20\lambda B_j \neq \emptyset$, and so $r_i \leq 2r_j$. Hence $B_i \subset 10B_j$. Moreover, if $2B_k \cap B_i \neq \emptyset$, then $r_k \leq 2r_i$ and so $B_k \subset 6B_i \subset 20B_j$. Recall that the function ϕ_k is c/r_k -Lipschitz for any $k \in \mathbb{N}$ and that $\sum_k \phi_k = 1$

on Ω . Then by the bounded overlap property of the Whitney covering and the Poincaré inequality for u and g_u , we have that

$$\begin{aligned}
 |u_W(x) - u_W(y)| &= \left| \sum_{k=1}^{\infty} u_{B_k} \phi_k(x) - \sum_{k=1}^{\infty} u_{B_k} \phi_k(y) \right| \\
 &= \left| \sum_{k=1}^{\infty} (u_{B_k} - u_{B_j}) \phi_k(x) - \sum_{k=1}^{\infty} (u_{B_k} - u_{B_j}) \phi_k(y) \right| \\
 &\leq \sum_{k=1}^{\infty} |u_{B_k} - u_{B_j}| |\phi_k(x) - \phi_k(y)| \\
 &\leq d(x, y) \sum_{\{k: 2B_k \cap (B_j \cup B_i) \neq \emptyset\}} \frac{c}{r_k} |u_{B_k} - u_{B_j}| \quad (4.2) \\
 &\lesssim d(x, y) \frac{c}{r_j} \int_{20B_j} |u - u_{20B_j}| d\mu \\
 &\leq Cd(x, y) \int_{20\lambda B_j} g_u d\mu,
 \end{aligned}$$

where C is a constant depending on λ, c, C_d, C_P and C_0 only, and thus in fact only on C_d, C_P, λ . Thus, the function g defined in (4.1) is an admissible function for u_W .

At last, we show the L^1 -norm estimate for g . It follows from the bounded overlap property of the Whitney covering that

$$\begin{aligned}
 \int_{\Omega} g(x) d\mu(x) &\leq \sum_{j=1}^{\infty} \int_{B_j} g(x) d\mu(x) \lesssim \sum_{j=1}^{\infty} \mu(B_j) \int_{20\lambda B_j} g_u d\mu \\
 &\lesssim \sum_{j=1}^{\infty} \int_{20\lambda B_j} g_u(x) d\mu(x) \lesssim \int_{\Omega} g_u(x) d\mu(x) = \|g_u\|_{L^1(\Omega)}. \quad \square
 \end{aligned}$$

Recall the homogeneous dimension $s > 1$ from (2.1).

Theorem 4.2 ([15, Theorem 9.2]). *Let $\sigma > 1$ and let $B = B(x, r)$ be a ball in X . If $u \in M^{1,p}(\sigma B, d, \mu)$ and g is an admissible function in (2.13), where $p \geq s/(s+1)$, then*

$$\int_B |u - u_B| d\mu \leq Cr \left(\int_{\sigma B} g^p d\mu \right)^{1/p}, \quad (4.3)$$

with C depending on C_d, p , and σ only.

Next we will consider the relationship between $M_{CH}^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$. The next theorem shows that when $\Omega \subset X$ is a uniform domain, $M_{CH}^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$ are the same. The case $X = \mathbb{R}^n$, i.e. the Euclidean case was proved in [25, Theorem 19]. Before stating the theorem, we first give the definition of uniform domain.

Definition 4.3. A domain $\Omega \subset X$ is called *uniform* if there is a constant $c_U \in (0, 1]$ such that every pair of distinct points $x, y \in \Omega$ can be connected by a curve $\gamma: [0, \ell_\gamma] \rightarrow \Omega$ parametrized by arc-length such that $\gamma(0) = x$, $\gamma(\ell_\gamma) = y$, $\ell_\gamma \leq c_U^{-1}d(x, y)$, and

$$\text{dist}(\gamma(t), X \setminus \Omega) \geq c_U \min\{t, \ell_\gamma - t\} \quad \text{for all } t \in [0, \ell_\gamma]. \quad (4.4)$$

Theorem 4.4. Assume $\Omega \subset X$ is a uniform domain. Then for any $0 < c_H < 1$, we have $M_{c_H}^{1,1}(\Omega) = M^{1,1}(\Omega)$ with equivalent norms.

Proof. Choose arbitrary $x, y \in \Omega$. By modifying the standard covering argument in uniform domains (see [16, 17, 25] for details), from the uniformity condition we deduce easily that there is a chain of balls B_k resembling a cigar that joins the points x and y . More precisely, there are balls $B_k := B(z_k, r_k)$ with $k \in \mathbb{Z}$ and $z_k \in \Omega$ such that for each k one has for some $c' = c'(\lambda, c_H, c_U)$

$$15\lambda/c_H B_k \subset \Omega \quad \text{and} \quad r_k \geq \frac{1}{c'} \min\{d(z_k, x), d(z_k, y)\},$$

with also $B_k \cap B_{k+1} \neq \emptyset$, and $r_k/2 \leq r_{k+1} \leq 2r_k$. In addition, $\lim_{k \rightarrow +\infty} d(x, B_k) = 0 = \lim_{k \rightarrow -\infty} d(y, B_k)$. Finally, we may assume that $\sum_{k \in \mathbb{Z}} r_k \leq C'd(x, y)$.

Let $u \in M_{c_H}^{1,1}(\Omega)$ with admissible function $g \in L^1(\Omega)$. We can zero extend g outside Ω . Since $15\lambda/c_H B_k \subset \Omega$ and $c_H < 1$, then for any $x_0, y_0 \in 5\lambda B_k$, we have

$$d(x_0, y_0) \leq 10\lambda r_k \leq c_H(15\lambda/c_H - 5\lambda)r_k \leq c_H \min\{\text{dist}(x_0, X \setminus \Omega), \text{dist}(y_0, X \setminus \Omega)\}.$$

Hence, for any $x_0, y_0 \in 5\lambda B_k$, the condition (2.14) is satisfied. Thus, $u \in M^{1,1}(5\lambda B_k)$ for any $k \in \mathbb{Z}$. It follows from the Poincaré inequality in Theorem 4.2 on the ball $5B_k$ with $\sigma = \lambda$ that

$$\begin{aligned} |u_{B_k} - u_{B_{k+1}}| &\lesssim \int_{5B_k} |u - u_{5B_k}| \lesssim r_k \left(\int_{5\lambda B_k} g^{s/(s+1)} d\mu \right)^{(s+1)/s} \\ &\lesssim r_k \left(\int_{(5\lambda+2c')B_k} g^{s/(s+1)} d\mu \right)^{(s+1)/s} \\ &\lesssim r_k \left(\left(\mathcal{M}g^{s/(s+1)}(x) \right)^{(s+1)/s} + \left(\mathcal{M}g^{s/(s+1)}(y) \right)^{(s+1)/s} \right), \end{aligned}$$

where s is the associated homogeneous dimension. Here the last inequality follows from the fact that either x or y is contained in $2c'B_k \subset (5\lambda + 2c')B_k$.

If x, y are Lebesgue points of u , we have $|u(x) - u(y)| \leq \sum_{k \in \mathbb{Z}} |u_{B_k} - u_{B_{k+1}}|$. By summing over k , it follows that

$$|u(x) - u(y)| \leq d(x, y)(\tilde{g}(x) + \tilde{g}(y)),$$

where $\tilde{g}(x) = 2C(\mathcal{M}g^{s/(s+1)}(x))^{(s+1)/s}$. The conclusion follows from the Hardy-Littlewood maximal inequality. $\square$

Remark 4.5. From the proof of Theorem 4.4, we know that if X is a geodesic space, i.e., for any $x, y \in X$, there exists a curve γ in X such that $\ell_\gamma = d(x, y)$, then $M_{c_1}^{1,1}(\Omega) = M_{c_2}^{1,1}(\Omega)$ with equivalent norms for any two constants $0 < c_1, c_2 < 1$. This fact coincides with the case $\Omega \subset \mathbb{R}^n$, where $\mathbb{R}^n$ is a geodesic space. When $\Omega \subset \mathbb{R}^n$, for any $0 < c_H < 1$, we obtain $M_{c_H}^{1,1}(\Omega) = M_{ball}^{1,1}(\Omega)$. Here we refer to [25, 43] for more details about the space $M_{ball}^{1,1}(\Omega)$.

To “achieve” the boundary values, we need the following proposition.

Proposition 4.6 ([31, Proposition 6.5]). *Let $u \in \text{BV}(\Omega)$. Then the discrete convolution u_W of u satisfies*

$$\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap \Omega} |u_W - u| d\mu = 0$$

for $\mathcal{H}$ -a.e. $x \in \partial\Omega$.

The above proposition considers the measure $\mathcal{H}$ on $\partial\Omega$, that is, the codimension-1 Hausdorff measure. But this may not be the measure we really want to study. For example, a classical problem is to study the trace spaces of weighted Sobolev spaces on Euclidean spaces. For the half plane $\Omega = \mathbb{R}_+^2 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and the measure $d\mu(x) = w_\lambda(x) dm_2(x)$ with m_2 the 2-dimensional Lebesgue measure and $w_\lambda(x) := \log^\lambda(\max\{e, e/|x_2|\})$, $\lambda > 0$, the codimension-1 Hausdorff measure on $\partial\mathbb{R}_+^2 = \mathbb{R}$ is not even σ -finite and hence is not the 1-dimensional Lebesgue measure that we usually study, see Example 5.6. Thus, it is reasonable to consider the equivalence of the traces of $N^{1,1}(\Omega)$ and $M^{1,1}(\Omega)$ under any general boundary measure $\tilde{\mathcal{H}}$ on $\partial\Omega$. Thus, we introduce the following lemma.

Lemma 4.7. *Assume Ω satisfies a measure doubling condition (2.5), i.e., $\mu|_\Omega$ is doubling. Let $u \in L_{\text{loc}}^1(\Omega)$ and $z \in \partial\Omega$. Assume that there is $a \in \mathbb{R}$ such that*

$$\lim_{r \rightarrow 0^+} \int_{B(z, r) \cap \Omega} |u - a| d\mu = 0.$$

Then the discrete convolution u_W of u satisfies

$$\lim_{r \rightarrow 0^+} \int_{B(z, r) \cap \Omega} |u_W - a| d\mu = 0.$$

Proof. In the Whitney covering $\{B_k\}_{k=1}^\infty$, recall that for any $B_k = B(x_k, r_k)$ we have $r_k = \text{dist}(x_k, X \setminus \Omega)/100\lambda$. If $2B_k \cap B(z, r) \neq \emptyset$, then

$$2r_k + r \geq d(x_k, z) \geq \text{dist}(x_k, X \setminus \Omega) = 100\lambda r_k,$$

which implies

$$\bigcup_{\{k \in \mathbb{N} : 2B_k \cap B(z, r) \neq \emptyset\}} B_k \subset B(z, 2r).$$

Then we have

$$\begin{aligned}
 \int_{B(z,r) \cap \Omega} |u_W - a| d\mu &= \int_{B(z,r) \cap \Omega} \left| \sum_{k=1}^{\infty} (\phi_k u_{B_k} - \phi_k a) \right| d\mu \\
 &\leq \int_{B(z,r) \cap \Omega} \sum_{k=1}^{\infty} |\phi_k| |u_{B_k} - a| d\mu \\
 &\leq \int_{B(z,r) \cap \Omega} \sum_{k=1}^{\infty} \chi_{2B_k} |u_{B_k} - a| d\mu \\
 &\leq \int_{B(z,r) \cap \Omega} \sum_{k=1}^{\infty} \chi_{2B_k} \int_{B_k} |u - a| d\mu d\mu \\
 &\leq C_d \sum_{\{k \in \mathbb{N}: 2B_k \cap B(z,r) \neq \emptyset\}} \int_{B_k} |u - a| d\mu \\
 &\lesssim \int_{B(z,2r) \cap \Omega} |u - a| d\mu
 \end{aligned}$$

by the bounded overlap property. Thus, the doubling property of $\mu|_{\Omega}$ gives the estimate

$$\int_{B(z,r) \cap \Omega} |u_W - a| d\mu \lesssim \int_{B(z,2r) \cap \Omega} |u - a| d\mu.$$

The result follows by passing to the limit. $\square$

Proof of Theorem 1.5, Theorem 1.6, and Corollary 1.7. Theorem 1.5 follows immediately by combining Theorem 4.1, Theorem 4.4 and Proposition 4.6, while Theorem 1.6 follows immediately by combining Theorem 4.1, Theorem 4.4 and Lemma 4.7.

For Corollary 1.7, by adapting the proof of Corollary 1.4, we obtain the result using Theorem 1.6. Note that $M^{1,1}(\Omega) \subset N^{1,1}(\Omega)$ with $\|\cdot\|_{N^{1,1}(\Omega)} \lesssim \|\cdot\|_{M^{1,1}(\Omega)}$, see [15, Theorem 8.6]. $\square$

5. Examples and applications

The following example shows that in Theorem 1.2 we cannot take a function $v \in \text{Lip}_{\text{loc}}(\Omega)$, or even $v \in L_{\text{loc}}^{\infty}(\Omega)$.

Example 5.1. Let $X = \mathbb{R}^2$ (unweighted) and let $\Omega := B(0, 1)$. We find a sequence $\{x_k\}$ that is dense in $B(0, 1)$. Take

$$u_k(x) := |x - x_k|^{-1+1/k}, \quad k \in \mathbb{N}.$$

Then $\|u_k\|_{L^1(\Omega)} < \infty$ and the minimal 1-weak upper gradient satisfies (see [6, Proposition A.3])

$$g_{u_k}(x) = |\nabla u_k(x)| = (-1 + 1/k)|x - x_k|^{-2+1/k}$$

and so

$$\int_{B(0,1)} g_{u_k} dx \lesssim \int_{B(0,1)} |x - x_k|^{-2+1/k} dx \leq \int_{B(0,2)} |x|^{-2+1/k} dx < \infty.$$

Let

$$u(x) := \sum_k 2^{-k} \frac{u_k}{\|u_k\|_{N^{1,1}(B(0,1))}}.$$

Then using, e.g., [6, Lemma 1.52] we see that u has a 1-weak upper gradient

$$\sum_k 2^{-k} \frac{g_{u_k}}{\|u_k\|_{N^{1,1}(B(0,1))}},$$

which implies $u \in N^{1,1}(B(0,1))$. We know that the homogeneous dimension s of $\mathbb{R}^2$ is 2, and then $\frac{s}{s-1} = 2$. On the other hand, we can see that for any $q > 2$, we have for all sufficiently large $k \in \mathbb{N}$

$$\int_{B(x_k, r) \cap B(0,1)} |u_k|^q dx = \infty \quad \text{for all } r > 0,$$

and then for all balls $B \cap B(0,1) \neq \emptyset$ we have $\int_{B \cap B(0,1)} |u|^q dx = \infty$. Given $v \in \text{Lip}_{\text{loc}}(B(0,1))$, we know that $v \in L^q_{\text{loc}}(B(0,1))$. Therefore we have $\|v - u\|_{L^q(B \cap B(0,1))} = \infty$ for all balls $B \cap B(0,1) \neq \emptyset$, which contradicts the desired conclusion in Theorem 1.2.

The following example shows that in Theorem 1.3 we cannot take a function v with $\mathcal{H}(S_v) = 0$.

Example 5.2. Let $X = \mathbb{R}^2$ (unweighted) and let $\Omega := (-1, 1) \times (0, 1)$. Define $u \in \text{BV}(\Omega)$ by

$$u(x_1, x_2) := \begin{cases} 0 & \text{when } x_1 < 0 \\ 1 & \text{when } x_1 \geq 0. \end{cases}$$

Let $v \in \text{BV}(\Omega)$ with $\mathcal{H}(S_v) = 0$. Since $\mathcal{H}(\{0\} \times (0, 1)) > 0$, it is now easy to check that $\|v - u\|_{L^\infty(\Omega \setminus \Omega(r))} \geq 1/2$ for all $r > 0$.

A direct consequence of Corollary 1.4 and Corollary 1.7 is that under a proper setting, the trace spaces of the BV space, Newton-Sobolev space, and Hajlasz-Sobolev space are the same. Hence we can obtain many trace results for the BV and Hajlasz-Sobolev space directly from trace results for the Newton-Sobolev space obtained in the literature. In particular, from [32, Theorem 1.1] we are able to obtain the following result.

Theorem 5.3. *Let $\Omega \subset X$ be a bounded uniform domain satisfying the measure doubling condition (2.5). Assume also that $(\Omega, d, \mu|_{\Omega})$ admits a $(1, 1)$ -Poincaré inequality. Let $\partial\Omega$ be endowed with an Ahlfors codimension θ -regular measure ν for some $0 < \theta < 1$. Then the trace spaces of $N^{1,1}(\Omega, \mu)$, $BV(\Omega, \mu)$ and $M^{1,1}(\Omega, \mu)$ are the same, namely the Besov space $B_{1,1}^{1-\theta}(\partial\Omega, \nu)$.*

We say that $\partial\Omega$ is endowed with an Ahlfors codimension θ -regular measure ν if there is a σ -finite Borel measure ν on $\partial\Omega$ and a constant $c_{\theta} > 0$ such that

$$c_{\theta}^{-1} \frac{\mu(B(x, r) \cap \Omega)}{r^{\theta}} \leq \nu(B(x, r) \cap \partial\Omega) \leq c_{\theta} \frac{\mu(B(x, r) \cap \Omega)}{r^{\theta}} \quad (5.1)$$

for all $x \in \partial\Omega$ and $0 < r < 2 \operatorname{diam} \Omega$. The Besov space $B_{1,1}^{1-\theta}(\partial\Omega, \nu)$ consists of L^1 -functions of finite Besov norm that is given by

$$\|u\|_{B_{1,1}^{1-\theta}(\partial\Omega, \nu)} = \|u\|_{L^1(\partial\Omega, \nu)} + \int_0^{\infty} \int_{\partial\Omega} \int_{B(y,t)} \frac{|u(x) - u(y)|}{t^{1-\theta}} d\nu(x) d\nu(y) \frac{dt}{t}.$$

The above theorem seems to be new even for BV and $M^{1,1}$ functions in the (weighted) Euclidean setting. As an illustration, we give an example in weighted Euclidean spaces.

Example 5.4. Let $\Omega = \mathbb{D} \subset \mathbb{R}^2$ be the unit disk with $\partial\Omega = \mathbb{S}^1$ the unit circle. Take the measure $d\mu(x) = \operatorname{dist}(x, \mathbb{S}^1)^{-\alpha} dm_2(x)$ with $0 < \alpha < 1$ and m_2 two-dimensional Lebesgue measure. Then by a direct computation, $\operatorname{dist}(x, \mathbb{S}^1)^{-\alpha}$ with $0 < \alpha < 1$ is an A_1 -weight and hence μ supports a $(1, 1)$ -Poincaré inequality, see [20, Chapter 15]. Moreover, it is easy to check that the 1-dimensional Hausdorff measure $\mathcal{H}^1$ on $\mathbb{S}^1$ is an Ahlfors codimension $(1 - \alpha)$ -regular measure, i.e., $\mathcal{H}^1$ on $\mathbb{S}^1$ satisfies (5.1) with $\theta = 1 - \alpha$. Hence we obtain from Theorem 5.3 that the trace spaces of $N^{1,1}(\mathbb{D}, \mu)$, $BV(\mathbb{D}, \mu)$, and $M^{1,1}(\mathbb{D}, \mu)$ are $B_{1,1}^{\alpha}(\mathbb{S}^1, \mathcal{H}^1)$. It is also known from the classical trace results of weighted Sobolev spaces that the trace space of $N^{1,1}(\mathbb{D}, \mu)$ is the classical Besov space $B_{1,1}^{\alpha}(\mathbb{S}^1, \mathcal{H}^1)$. Here we refer to [36, 40, 41] for the trace results for weighted Sobolev spaces on Euclidean spaces and refer to the seminal monographs by Triebel [39] for more information on Besov spaces.

On the other hand, using our theory it is also possible to obtain new trace results for Hajlasz-Sobolev or Newton-Sobolev functions from the known trace results for BV functions. In particular, from [33, Corollary 1.4] we are able to obtain the following trace results.

Theorem 5.5. *Let $\Omega \subset X$ be a bounded uniform domain that satisfies the measure density condition (2.4) and admits a $(1, 1)$ -Poincaré inequality. Assume also that the codimension 1 Hausdorff measure $\mathcal{H}$ is Ahlfors codimension-1-regular. Then we have that the trace spaces of $BV(\Omega, \mu)$, $N^{1,1}(\Omega, \mu)$ and $M^{1,1}(\Omega, \mu)$ are the same, namely the space $L^1(\partial\Omega, \mathcal{H})$.*

When $\Omega = \mathbb{D}$, $\partial\Omega = \mathbb{S}^1$, $\mu = m_2$ the 2-dimension Lebesgue measure and $\mathcal{H} \approx \mathcal{H}^1$ the 1-dimension Hausdorff measure, the above theorem coincides with the classical results that the trace spaces of $BV(\mathbb{D})$ and $N^{1,1}(\mathbb{D})$ are both $L^1(\mathbb{S}^1)$. Moreover, the above theorem gives that $L^1(\mathbb{S}^1)$ is also the trace space of $M^{1,1}(\mathbb{D})$, which seems to be new even in this case.

The above Theorem 5.3 and Theorem 5.5 both require that the boundaries are endowed with some codimension Ahlfors regular measure. In the following, we will give an example where the measure on the boundary do not satisfy any codimension Ahlfors regularity.

Example 5.6. Let $\Omega = \mathbb{R}_+^2 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and take the measure $d\mu(x) = w_\lambda(x) dm_2(x)$ with m_2 the 2-dimensional Lebesgue measure and $w_\lambda(x) = \log^\lambda(\max\{e, e/|x_2|\})$, $\lambda > 0$. For any $x \in \mathbb{R} = \partial\Omega$ and $0 < r < e^{-2\lambda}$, let $Q(x, r)$ denote the cube parallel to the coordinate axes with center x and sidelength r . Then we have the estimate

$$\begin{aligned} \mu(Q(x, r)) &= 2 \int_0^r \int_0^{r/2} \log^\lambda(e/|x_2|) dx_2 dx_1 \\ &= 2r \int_0^{r/2} \log^\lambda(e/t) dt \approx r^2 \log^\lambda(e/r). \end{aligned} \quad (5.2)$$

Here the last equality holds since we have

$$(t \log^\lambda(e/t))' = \log^\lambda(e/t) \left(1 - \frac{\lambda}{\log(e/t)}\right) \approx \log^\lambda(e/t), \quad \text{for } 0 < t \leq r < e^{-2\lambda}.$$

By using the estimate (5.2), it follows from the definition of the codimension-1 Hausdorff measure (2.3) that for any nonempty interval $[a, b]$ in $\mathbb{R} = \partial\mathbb{R}_+^2$, we have that

$$\mathcal{H}([a, b]) = \lim_{R \rightarrow 0^+} \mathcal{H}_R([a, b]) \approx \lim_{R \rightarrow 0^+} |a - b| \log^\lambda(e/R) = \infty.$$

Hence the codimension-1 Hausdorff measure $\mathcal{H}$ on $\mathbb{R}$ is not even σ -finite and is not the 1-dimensional Lebesgue measure that we usually study.

Moreover, the weight w_λ defined above is a Muckenhoupt A_1 -weight, since it is easy to check from estimate (5.2) that

$$\frac{\mu(B(z, r))}{r^2} \lesssim \inf_{x \in B(z, r)} w_\lambda(x), \quad \text{for any } z \in \mathbb{R}_+^2 \text{ and } r > 0.$$

We refer to [8] and [20, Chapter 15] for definitions, properties and examples of Muckenhoupt class weights.

Example 5.7. Let Ω, μ be as in the above example. Then it is easy to check from estimate (5.2) that the 1-dimensional Lebesgue measure on $\mathbb{R}$ does not satisfy the condition (5.1) for any θ . We denote by $\mathcal{Q}$ the collection of dyadic semi-open

intervals in $\mathbb{R}$, i.e. the intervals of the form $I := 2^{-k}((0, 1] + m)$, where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$. Write $\ell(I)$ for the edge length of $I \in \mathcal{Q}$, i.e. 2^{-k} in the preceding representation, and $\mathcal{Q}_k$ for the cubes $Q \in \mathcal{Q}$ such that $\ell(Q) = 2^{-k}$. For any $I \in \mathcal{Q}_{2^j}$, denote by $\tilde{I}$ the interval in $\mathcal{Q}_{2^{j-1}}$ containing the interval I . By applying the methods used in [41] and [28, Theorem 1.3], we are able to use the dyadic norm similar with the ones used in [27] and [28] to characterize the trace space of $N^{1,1}(\mathbb{R}_+^2, \mu)$, which is the Besov-type space $\mathcal{B}_1^\lambda(\mathbb{R})$. The Besov-type space $\mathcal{B}_1^\lambda(\mathbb{R})$ consists of functions in $L^1(\mathbb{R})$ of finite dyadic norm that is given by

$$\|u\|_{\mathcal{B}_1^\lambda(\mathbb{R})} = \|u\|_{L^1(\mathbb{R})} + \sum_{j=1}^{+\infty} 2^{-\lambda j} \sum_{I \in \mathcal{Q}_{2^j}} 2^{-2^j} |u_I - u_{\tilde{I}}|.$$

We omit the detailed proof here. Since $\mathbb{R}_+^2$ is uniform domain and satisfies the measure doubling condition (2.5), hence we obtain that the trace spaces of $BV(\mathbb{R}_+^2, \mu)$, $N^{1,1}(\mathbb{R}_+^2, \mu)$ and $M^{1,1}(\mathbb{R}_+^2, \mu)$ are the same, the Besov-type space $\mathcal{B}_1^\lambda(\mathbb{R})$.

Example 5.8. The recent papers [7, 28, 42] studied trace results on regular trees. We refer to [7, Section 2] or [28, Section 2.1] for the definition of regular trees. It is easy to check that a regular tree is uniform and that it supports $(1, 1)$ -Poincaré inequality by modifying the proof in [7, Theorem 4.2] under the setting in [7, 28]. Even the definition of trace in [7, 28, 42] looks different from the one we used here, but [26] shows that they are equivalent. Hence the trace results of $N^{1,1}$ in [7, 28] can be immediately applied to BV and $M^{1,1}$. We omit the detail here and leave it to the interested reader.

References

- [1] L. AMBROSIO, *Fine properties of sets of finite perimeter in doubling metric measure spaces. Calculus of variations, nonsmooth analysis and related topics*, Set-Valued Anal. **10** (2002), 111–128.
- [2] L. AMBROSIO and E. DE GIORGI, *New functionals in the calculus of variations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **82** (1988), 199–210.
- [3] L. AMBROSIO, N. FUSCO and D. PALLARA, “Functions of Bounded Variation and Free Discontinuity Problems”, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
- [4] L. AMBROSIO, M. MIRANDA JR. and D. PALLARA, *Special functions of bounded variation in doubling metric measure spaces*, In: “Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi”, Quad. Mat., Vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, 1–45.
- [5] G. ANZELLOTTI and M. GIAQUINTA, *Funzioni BV e tracce*, Rend. Semin. Mat. Univ. Padova **60** (1978), 1–21.
- [6] A. BJÖRN and J. BJÖRN, “Nonlinear Potential Theory on Metric Spaces”, EMS Tracts in Mathematics, Vol. 17, European Mathematical Society (EMS), Zürich, 2011.
- [7] A. BJÖRN, J. BJÖRN, J. GILL and N. SHANMUGALINGAM, *Geometric analysis on Cantor sets and trees*, J. Reine Angew. Math. **725** (2017), 63–114.

- [8] J. BJÖRN, *Poincaré inequalities for powers and products of admissible weights*, Ann. Acad. Sci. Fenn. Math. **26** (2001), 175–188.
- [9] J. BJÖRN and N. SHANMUGALINGAM, *Poincaré inequalities, uniform domains and extension properties for Newton-Sobolev functions in metric spaces*, J. Math. Anal. Appl. **332** (2007), 190–208.
- [10] G. DAL MASO and R. TOADER, *Decomposition results for functions with bounded variation*, Boll. Unione Mat. Ital. (9) **1** (2008), 497–505.
- [11] L. C. EVANS and R. F. GARIEPY, “Measure Theory and Fine Properties of Functions”, Studies in Advanced Mathematics series, CRC Press, Boca Raton, 1992.
- [12] H. FEDERER, “Geometric Measure Theory”, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.
- [13] E. GIUSTI, “Minimal Surfaces and Functions of Bounded Variation”, Monographs in Mathematics, Vol. 80, Birkhäuser Verlag, Basel, 1984.
- [14] P. HAJLASZ, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), 403–415.
- [15] P. HAJLASZ, *Sobolev spaces on metric-measure spaces*, In: “Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces” (Paris, 2002), Contemp. Math., Vol. 338, Amer. Math. Soc., Providence, RI, 2003, 173–218.
- [16] P. HAJLASZ and P. KOSKELA, *Isoperimetric inequalities and imbedding theorems in irregular domains*, J. London Math. Soc. (2) **58** (1998), 425–450.
- [17] P. HAJLASZ and O. MARTIO, *Traces of Sobolev functions on fractal type sets and characterization of extension domains*, J. Funct. Anal. **143** (1997), 221–246.
- [18] H. HAKKARAINEN, J. KINNUNEN, P. LAHTI and P. LEHTELÄ, *Relaxation and integral representation for functionals of linear growth on metric measure spaces*, Anal. Geom. Metr. Spaces **4** (2016), 288–313.
- [19] H. HAKKARAINEN, R. KORTE, P. LAHTI and N. SHANMUGALINGAM, *Stability and continuity of functions of least gradient*, Anal. Geom. Metr. Spaces **3** (2015), 123–139.
- [20] J. HEINONEN, T. KILPELÄINEN and O. MARTIO, “Nonlinear Potential Theory of Degenerate Elliptic Equations”, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993.
- [21] J. HEINONEN and P. KOSKELA, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.
- [22] J. HEINONEN, P. KOSKELA, N. SHANMUGALINGAM and J. TYSON, “Sobolev Spaces on Metric Measure Spaces. An Approach Based on Upper Gradients”, New Mathematical Monographs, Vol. 27, Cambridge University Press, Cambridge, 2015.
- [23] J. KINNUNEN, R. KORTE, N. SHANMUGALINGAM and H. TUOMINEN, *Lebesgue points and capacities via the boxing inequality in metric spaces*, Indiana Univ. Math. J. **57** (2008), 401–430.
- [24] R. KORTE, P. LAHTI, X. LI and N. SHANMUGALINGAM, *Notions of Dirichlet problem for functions of least gradient in metric measure spaces*, Rev. Mat. Iberoam. **35** (2019), 1603–1648.
- [25] P. KOSKELA and E. SAKSMAN, *Pointwise characterizations of Hardy-Sobolev functions*, Math. Res. Lett. **15** (2008), 727–744.
- [26] P. KOSKELA, K. N. NGUYEN and Z. WANG, *Trace operators on regular trees*, Anal. Geom. Metr. Spaces **8** (2020), 396–409.
- [27] P. KOSKELA, T. SOTO and Z. WANG, *Traces of weighted function spaces: dyadic norms and Whitney extensions*, Sci. China Math. **60** (2017), 1981–2010.
- [28] P. KOSKELA and Z. WANG, *Dyadic norm Besov-type spaces as trace spaces on regular trees*, Potential Anal. **53** (2020), 1317–1346.
- [29] P. LAHTI, *Approximation of BV by SBV functions in metric spaces*, J. Funct. Anal. **279** (2020), 108763, 33 pp.
- [30] P. LAHTI, *Discrete convolutions of BV functions in quasiopen sets in metric spaces*, Calc. Var. Partial Differential Equations **59** (2020), Paper No. 27, 23 pp.

- [31] P. LAHTI and N. SHANMUGALINGAM, *Trace theorems for functions of bounded variation in metric spaces*, J. Funct. Anal. **274** (2018), 2754–2791.
- [32] L. MALÝ, *Trace and extension theorems for Sobolev-type functions in metric spaces*, arXiv:1704.06344.
- [33] L. MALÝ, N. SHANMUGALINGAM and M. SNIPES, *Trace and extension theorems for functions of bounded variation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18** (2018), 313–341.
- [34] M. MIRANDA SR., *Comportamento delle successioni convergenti di frontiere minimali*, Rend. Sem. Mat. Univ. Padova **38** (1967), 238–257.
- [35] M. MIRANDA, JR., *Functions of bounded variation on “good” metric spaces*, J. Math. Pures Appl. (9) **82** (2003), 975–1004.
- [36] P. MIRONESCU and E. RUSS, *Traces of weighted Sobolev spaces. Old and new*, Nonlinear Anal. **119** (2015), 354–381.
- [37] N. SHANMUGALINGAM, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoam. **16** (2) (2000), 243–279.
- [38] P. STERNBERG, G. WILLIAMS and W.P. ZIEMER, *Existence, uniqueness, and regularity for functions of least gradient*, J. Reine Angew. Math. **430** (1992), 35–60.
- [39] H. TRIEBEL, “Theory of Function Spaces”, Monographs in Mathematics, Vol. 78, Birkhäuser Verlag, Basel, 1983.
- [40] A. I. TYULENEV, *Description of traces of functions in the Sobolev space with a Muckenhoupt weight*, Proc. Steklov Inst. Math. **284** (2014), 280–295.
- [41] A. I. TYULENEV, *Traces of weighted Sobolev spaces with Muckenhoupt weight. The case $p = 1$* , Nonlinear Anal. **128** (2015), 248–272.
- [42] Z. WANG, *Trace spaces of Orlicz-Sobolev spaces on regular trees*, J. Math. Anal. Appl. **494** (2021), 124646, 21 pp.
- [43] Y. ZHOU, *Hajlasz-Sobolev imbedding and extension*, J. Math. Anal. Appl. **382** (2011), 577–593.
- [44] W. P. ZIEMER, “Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation”, Graduate Texts in Mathematics, Vol. 120, Springer-Verlag, New York, 1989.

Institut für Mathematik
 Universität Augsburg
 Universitätsstr. 14
 86159 Augsburg, Germany
 and
 Academy of Mathematics and Systems Science
 Chinese Academy of Sciences
 Beijing 100190, PR China
 panulahti@amss.ac.cn

School of Mathematics(Zhuhai)
 Sun Yat-Sen University
 519082, Zhuhai, China
 lixingning3@mail.sysu.edu.cn

LCSM(MOE)
 School of Mathematics and Statistics
 Hunan Normal University
 410081, Changsha, China.
 zhuang.z.wang@foxmail.com