

Contact forms with large systolic ratio in arbitrary dimensions

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Abstract. If a contact form on a $(2n + 1)$ -dimensional closed contact manifold admits closed Reeb orbits, then its systolic ratio is defined to be the quotient of $(n + 1)$ -th power of the shortest period of Reeb orbits by the contact volume. We prove that every co-oriented contact structure on any closed contact manifold admits a contact form with arbitrarily large systolic ratio. This statement generalizes the recent result of Abbondandolo *et al.* in dimension three to higher dimensions. We extend the plug construction of Abbondandolo *et al.* to any dimension, by means of generalizing the Hamiltonian disc maps studied by the authors to symplectic balls of any dimension. The plug we consider is a mapping torus and it is equipped with a special contact form, which leads to arbitrarily small contact volume, but with the smallest period of the Reeb flow bounded away from zero. Following the ideas of Abbondandolo *et al.* and using Giroux's theory of Liouville open books, we construct a special contact form on a given contact manifold whose Reeb flow leads to a circle bundle on a 'large' portion of the given contact manifold. Such a contact form can be modified on the region of periodic Reeb flow by inserting plugs so that the contact volume is sucked up while the minimal period remains the same.

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1. Introduction

One of the classical problems in Riemannian geometry is to give an upper bound on the length of the shortest non-constant closed geodesic in terms of the Riemannian area on a given closed surface. More specifically, one studies the scaling invariant functional

$$\rho(S, g) = \frac{l_{\min}(S, g)^2}{\text{area}(S, g)}, \quad (1.1)$$

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on the space of all Riemannian metrics on a given closed surface S . Here, $l_{\min}(S, g)$ denotes the length of the shortest non-constant closed g -geodesic and $\text{area}(S, g)$ denotes the area of S with respect to the metric g .

In 1949, Loewner showed that if in (1.1), $l_{\min}(S, g)$ is replaced by $\text{sys}_1(S, g)$, namely the length of a shortest non-contractible geodesic, the corresponding ratio $\rho_{\text{nc}}(\mathbb{T}^2, \cdot)$ admits an optimal bound. In 1952, Pu proved the existence of an optimal bound on $\rho_{\text{nc}}(\mathbb{RP}^2, \cdot)$. In fact, in both statements the metrics that maximize ρ_{nc} do not admit any contractible geodesic and hence they also maximize (1.1). In the early 80's, Gromov proved that

$$\rho_{\text{nc}}(S, \cdot) \leq 2$$

for any non-simply connected closed surface S , but this bound is in general non-optimal [8]. On the other hand, in the late 80's Croke gave the first upper bound on $\rho(S^2, \cdot)$ [5], which was later improved by several authors.

Actually, in [8] Gromov studied the so called *systolic ratio* in any dimension and showed that for any essential n -dimensional closed manifold M ,

$$\rho_{\text{nc}}(M, g) = \frac{\text{sys}_1(M, g)^n}{\text{vol}(M, g)}$$

admits an upper bound, which depends only on the dimension. Another natural direction for the generalization of the problem is weakening the Riemannian assumption on the metric. In fact, the ratios ρ and ρ_{nc} generalize to the Finsler setting by replacing the Riemannian area with the Holmes-Thompson area and the bounds on ρ generalize to the Finsler case [3]. For a detailed account of results about the systolic ratio in Riemannian and Finsler geometry, we refer to [1] and [2].

The systolic ratio ρ naturally generalizes to contact geometry. A *contact manifold* is a $(2n+1)$ -dimensional manifold V equipped with a maximally non-integrable hyperplane distribution ξ . That is, ξ is locally given by the kernel of a 1-form α so that $\alpha \wedge (d\alpha)^n$ is non-vanishing. In this case, one says ξ is a *contact structure* on V . If ξ is co-orientable, then there exists a global 1-form α , referred as a *contact form* on (V, ξ) , such that $\ker \alpha = \xi$. We note that if α' is another contact form on (V, ξ) then $\alpha = f\alpha'$ for some non-vanishing function f on V . A contact form α gives rise to a natural dynamical system. Namely, one defines the *Reeb vector field* R_α on V via

$$\iota_{R_\alpha} d\alpha = 0 \text{ and } \iota_{R_\alpha} \alpha = 1.$$

Then the *contact systolic ratio* on a closed contact manifold (V, ξ) is defined to be the scaling invariant functional

$$\rho(V, \alpha) := \frac{T_{\min}(V, \alpha)^{n+1}}{\text{vol}(V, \alpha)}$$

on the space of all contact forms on (V, ξ) . Here, $T_{\min}(V, \alpha)$ denotes the minimum among the periods of all orbits of the Reeb vector field R_α and

$$\text{vol}(V, \alpha) := \int_V \alpha \wedge (d\alpha)^n$$

is the *contact volume* of V associated to the contact form α .

We note that the contact systolic ratio is not merely a generalization of the notion to a dynamical system but it is strongly related to the classical question. In fact, given a smooth Finsler manifold (M, F) , the canonical Liouville 1-form pdq on the cotangent bundle T^*S , restricts to a contact form α_F on the unit cotangent bundle S_F^*M . In this case, the Reeb flow is nothing but the geodesic flow restricted to S_F^*M and up to a constant depending only on the dimension, the contact volume $\text{vol}(S_F^*M, \alpha_F)$ is the Holmes-Thompson volume of (M, F) . Hence the contact systolic ratio of (S_F^*M, α_F) recovers the classical systolic ratio of (M, F) .

But it turns out that it is not possible to bound the contact systolic ratio globally. In the case of the tight 3-sphere (S^3, ξ_{st}) , it was shown in [1] that the systolic ratio can be made arbitrarily large. Yet it was also shown that the *Zoll contact forms*, namely the contact forms for which all Reeb orbits are closed and share the same minimal period, are maximizers of the functional $\rho(S^3, \cdot)$ if the functional is restricted to a C^3 -neighbourhood of all Zoll contact forms. For any contact 3-manifold (M, ξ) , the non-existence of a global bound on $\rho(M, \cdot)$ is later proved by the same authors in [2] whereas in [4], the local bound on $\rho(S^3, \cdot)$ was generalized to all contact 3-manifolds that admit Zoll contact forms.

The aim of this paper is to prove that the contact systolic ratio is unbounded in any dimension. Here we need to point that $\rho(V, \alpha)$ makes sense only if the Reeb vector field R_α admits a closed orbit. If $\dim V = 3$, by a result of Taubes [10], we know that any contact form on V admits a closed Reeb orbit but in higher dimensions, this might not be the case. Since we aim for the non-existence of a bound on ρ , it is legitimate for us to ignore this issue.

We first prove that on any odd dimensional sphere, which is equipped with the *standard contact structure*, see Section 4, the systolic ratio is unbounded, see Theorem 4.2. The strategy of our proof is precisely the same as the proof of the corresponding 3-dimensional result in [1]. The standard contact structure on the sphere S^{2n+1} admits a contact form α_0 , so called the *standard contact form*, for which the Reeb flow leads to the Hopf fibration. In this case, the complement of a codimension two submanifold of S^{2n+1} may be identified with $S^1 \times \mathbb{B}$, where $\mathbb{B}$ is the open $2n$ -dimensional unit ball, in such a way that the Reeb flow leads to the trivial S^1 -fibration over $\mathbb{B}$, see the proof of Theorem 4.2. We note that $S^1 \times \mathbb{B}$ may be viewed as the mapping torus of the identity map on the ball. With this motivation, we recover $S^1 \times \mathbb{B}$ as the mapping torus associated to a compactly supported symplectomorphism on $\mathbb{B}$ so that it admits a contact form α that fits to α_0 near the boundary and the associated Reeb dynamics is described by the dynamics of the symplectomorphism. Namely, the closed Reeb orbits translate into the periodic points and periods of closed Reeb orbits are described by the so called *action* of the symplectomorphism. Moreover, the contact volume is, up to a certain shift, given by the *Calabi invariant* of the symplectomorphism, see Section 2. Having this dictionary at hand, we construct a suitable isotopy of compactly supported symplectomorphisms of the ball, where the action and the Calabi invariant are arranged for our purposes. As a result, we get the *plug*, that is a contact manifold $(S^1 \times \mathbb{B}, \beta)$ such that β coincides with α_0 near the boundary of $S^1 \times \mathbb{B}$ and the periods of orbits of R_β are bounded away from zero but the contact volume is arbitrarily small, see

Lemma 3.1. As an immediate corollary of the existence of the plug, we show that $\rho(S^{2n+1}, \cdot)$ is unbounded on the set of contact forms that define the standard contact structure. In order to construct the isotopy required for the plug, we extend the construction of radial Hamiltonians on the unit disc [1] to the unit ball of arbitrary dimension, which requires nothing but some minor adjustments.

In the last section, we prove the main result of this paper given as follows.

Theorem 1.1. *Let (V, ξ) be a closed connected co-oriented contact manifold and let $C > 0$ be given. Then there exists a contact form α on V such that $\ker \alpha = \xi$ and*

$$\rho(V, \alpha) \geq C.$$

Again we use the strategy in the proof of the corresponding 3-dimensional result in [2]. On a given contact manifold, we construct a contact form, for which the Reeb flow is Zoll on an invariant domain that occupies arbitrarily large portion of the total contact volume and away from this domain the periods of closed Reeb orbits are bounded away from zero, see Proposition 5.9. Then we fill this large portion with suitably re-sized plugs so that the most of the contact volume is eaten up but the minimal period is still bounded away from zero. Our construction mimics the corresponding 3-dimensional statement in [2] and relies on the results of Giroux on higher dimensional open books [6,7]. But it also requires an inductive argument, which uses the corresponding 3-dimensional statement, the Proposition 2.1 in [2], as its basis step and contains many technical aspects which do not appear in dimension three.

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2. Contact mapping tori of the symplectomorphisms of the unit ball

In this section we present the construction of contact mapping tori associated to isotopies of compactly supported symplectomorphisms of the unit ball, where the Reeb dynamics and the contact volume are described in terms of the dynamics and the classical invariants of the underlying symplectomorphisms.

The relevant notion we should recall here is a *global (hyper)surface of section* of a smooth flow ϕ^t on manifold M , which is a compact co-dimension one submanifold Σ such that

- $\partial \Sigma$ is invariant under ϕ^t ,
- all orbits that are not contained in $\partial \Sigma$ meet Σ transversally infinitely often in forward and backward time.

Given such a submanifold Σ , one defines the *first return time function*

$$\tau : \Sigma \setminus \partial \Sigma \rightarrow (0, +\infty), \quad \tau(x) := \inf\{t > 0 \mid \phi^t(x) \in \Sigma\},$$

which is a smooth function and one has the corresponding *first return map*

$$\varphi : \Sigma \setminus \partial \Sigma \rightarrow \Sigma \setminus \partial \Sigma, \quad \varphi(x) := \phi^{\tau(x)}(x),$$

which is a smooth diffeomorphism. It is clear from the above definition that away from $\partial \Sigma$, the closed orbits of ϕ^t correspond to periodic points of φ and the periods of such orbits are given by τ . The plug construction we present below is a reverse procedure in the sense that we construct a flow on a mapping torus of a symplectomorphism so that the symplectic map is the first return map and the first return time is given by the action of the symplectomorphism.

We first recall the classical notions about symplectomorphisms in a context that is sufficient for our purposes.

Let $m \geq 2$ be given and $\mathbb{B} \subset \mathbb{R}^{2m}$ be the $2m$ -dimensional *open* unit ball centered at the origin. We fix the standart symplectic form

$$\omega = \sum_{i=1}^m dx_i \wedge dy_i$$

via the coordinates $z_i = x_i + iy_i$, $(z_1, \dots, z_m) \in \mathbb{C}^m \cong \mathbb{R}^{2m}$ and we fix the primitive

$$\lambda = \frac{1}{2} \sum_{i=1}^m x_i dy_i - y_i dx_i.$$

Let $\varphi \in \text{Diff}_c(\mathbb{B}, \omega)$ be a compactly supported symplectomorphism. Then $\varphi^* \lambda - \lambda$ is a closed one form and therefore it is exact. This allows us to find a unique function, called *the action of φ* ,

$$\sigma_{\varphi, \lambda} : \mathbb{B} \rightarrow \mathbb{R}$$

that satisfies

$$d\sigma_{\varphi, \lambda} = \varphi^* \lambda - \lambda \text{ and } \sigma_{\varphi, \lambda} = 0 \text{ near } \partial \mathbb{B}. \quad (2.1)$$

As suggested by the notation, the action depends on the primitive of ω , for the details in the case of $m = 1$ see [1]. The average of the action, namely

$$\text{CAL}(\varphi) := \int_W \sigma_{\varphi, \lambda} \omega^m \quad (2.2)$$

is *the Calabi invariant of φ* , which is in turn independent of the primitive of ω .

Now we are ready to give the scheme of the plug construction and show the connection between contact geometry of the plug and the classical invariants given above.

We consider a smooth path $\{\varphi_t\}_{t \in [0, 1]} \subset \text{Diff}_c(\mathbb{B}, \omega)$ that connects the identity to the map $\varphi := \varphi^1$ and we pick $L > 0$. We define the function

$$\tau := \sigma_{\varphi, \lambda} + L : \mathbb{B} \rightarrow \mathbb{R}.$$

It is clear from the definition (2.1) that $\sigma_{\varphi,\lambda}$ vanishes near the boundary of $\mathbb{B}$ and therefore the minimum of τ on $\mathbb{B}$ exists. Now we assume that

$$\min_{\mathbb{B}} \tau > 0. \quad (2.3)$$

Then the map

$$g : \mathbb{R} \times \mathbb{B} \rightarrow \mathbb{R} \times \mathbb{B}, \quad (s, z) \mapsto (s - \tau(z), \varphi(z))$$

defines a free $\mathbb{Z}$ -action on $\mathbb{R} \times \mathbb{B}$ and leads to the quotient manifold, denoted by M . It is clear that the 1-form $ds + \lambda$ is a contact form on $\mathbb{R} \times \mathbb{B}$ and it is invariant under the action of g , namely $g^*(ds + \lambda) = ds + \lambda$. Hence $ds + \lambda$ induces a contact form η on M determined by the equation $p^*\eta = ds + \lambda$, where

$$p : \mathbb{R} \times \mathbb{B} \rightarrow M$$

is the quotient map. As g maps the graph $\{(\tau(z), z) \mid z \in \mathbb{B}\}$ of τ onto $\{0\} \times \mathbb{B}$, the subset M_0 enclosed by these two hypersurfaces is a fundamental domain. By Fubini's theorem, we have

$$\begin{aligned} \text{vol}(M, \eta) &= \text{vol}(M_0, ds + \lambda) \\ &= \int_{M_0} (ds \wedge \lambda) \wedge (ds \wedge \lambda)^m \\ &= \int_{M_0} ds \wedge \omega^m \\ &= \int_{\mathbb{B}} \tau \omega^m \\ &= \int_{\mathbb{B}} (L + \sigma) \omega^m \\ &= L\pi^m + \int_{\mathbb{B}} \sigma_{\varphi,\lambda} \omega^m \\ &= L\pi^m + \text{CAL}(\varphi). \end{aligned}$$

Now let U be a small neighborhood of the boundary of $\mathbb{B}$, on which $\tau \equiv L$ and $\varphi = \text{id}$. Then $g(s, z) = (s - L, z)$ on $\mathbb{R} \times U$ and therefore $p(\mathbb{R} \times U)$ is diffeomorphic to $\mathbb{R}/L\mathbb{Z} \times U$ and η is identified with $ds + \lambda$ on this subset. We note that with this identification, the Reeb vector field R_η associated to η reads as ∂_s and its flow is simply the translation along the coordinate s on $\mathbb{R}/L\mathbb{Z} \times U$. Moreover, it is not hard to see that $p(\{0\} \times \mathbb{B})$ is a global hypersurface of section for the flow of R_η with the first return time map τ and first return map φ . We also note that for each $s \in \mathbb{R}$, the restriction $p|_{\{s\} \times \mathbb{B}}$ pulls back $d\eta$ to ω .

After carrying out the above construction for all φ^t , we may view the contact manifold (M, η) as a smooth deformation of the contact manifold $(\mathbb{R}/L\mathbb{Z} \times \mathbb{B}, ds + \lambda)$, provided that

$$\min_{\mathbb{B}} \tau_t > 0. \quad (2.4)$$

for all $t \in [0, 1]$, where

$$\tau_t := \sigma_{\varphi^t, \lambda} + L.$$

Moreover since $\{\varphi^t\}$ is compactly supported, along this deformation, the contact forms coincide with $ds + \lambda$ on a neighbourhood of the boundary, which is identified with $\mathbb{R}/L\mathbb{Z} \times U$, where U is a neighbourhood of $\partial\mathbb{B}$.

Now we claim that by replacing the unit disc $\mathbb{D}$ with the unit ball in the proof of [1, Proposition 3.1], the manifold M may be reparametrized as $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$ in such a way that the above properties of the contact form η are inherited by a contact form β on $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$. More concretely, we have the following statement.

Lemma 2.1. *Let $\{\varphi_t\}_{t \in [0, 1]}$ be an isotopy of compactly supported symplectomorphisms on the open unit disc $\mathbb{B}$ such that $\varphi^0 = \text{id}$ and $\varphi^1 = \varphi$. Let L be a positive number such that for all $t \in [0, 1]$*

$$\tau_t := \sigma_{\varphi^t, \lambda} + L > 0$$

on $\mathbb{B}$. Then there exists a contact form β on $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$ such that the following holds:

- (a1) $\text{vol}(\mathbb{R}/L\mathbb{Z} \times \mathbb{B}; \beta) = L\pi^m + \text{CAL}(\varphi)$.
- (a2) $\beta = ds + \lambda$ on a neighbourhood of the boundary of $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$. In particular, $R_\beta = \partial_s$ near the boundary and its flow is globally defined;
- (a3) The hypersurface $\{0\} \times \mathbb{B}$ is transverse to the flow of R_β and any Reeb orbit intersects $\{0\} \times \mathbb{B}$ in future and past;
- (a4) After the identification $\{0\} \times \mathbb{B} \cong \mathbb{B}$, φ is the first return map and $\tau := \tau_1$ is the first return time associated to the hypersurface $\{0\} \times \mathbb{B}$;
- (a5) β is smoothly isotopic to $ds + \lambda$ through a path of contact forms that coincide with $ds + \lambda$ on a fixed neighbourhood of the boundary of $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$.

3. Radial Hamiltonians on the ball

The outcome of the above discussion concerning our problem is the following. If one comes up with a suitable symplectic isotopy of the ball, in the sense that the action of the isotopy satisfies (2.4), the dynamics of the time-one-map is suitably arranged and the action and the Calabi invariant are suitably bounded, then one has a contact form β on $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$ with large systolic ratio. In this section, we construct such an isotopy using radial Hamiltonians on the ball. Again, $\mathbb{B} \subset \mathbb{R}$ denotes the open unit ball and $m \geq 2$.

Lemma 3.1. *Let $L > 0$ be given. Then for any $\varepsilon > 0$, there exists a positive integer n and a compactly supported isotopy $\{\varphi_t\}_{t \in [0, 1]} \subset \text{Diff}_c(\mathbb{B}, \omega)$ such that $\varphi^0 = \text{id}$ and the following holds:*

- (b1) $\sigma_{\varphi^t, \lambda} \geq -L + L/n$ for all $t \in [0, 1]$;
- (b2) $\text{CAL}(\varphi^1) + L\pi^m \leq \varepsilon$;

- (b3) All the fixed points of φ^1 have non-negative action;
 (b4) All the periodic points of φ^1 that are not fixed points have period at least n .

Proof. As in [1], we consider an autonomous radial Hamiltonian

$$H : \mathbb{B} \rightarrow \mathbb{R}, \quad H(z) = h(r^2)$$

on the unit ball such that h is non-negative and vanishes near $r^2 = 1$. Here, $z = (z_1, \dots, z_m)$, $r^2 = \sum_{i=1}^m r_i^2$, $z_i = r_i e^{i\theta_i}$ for $i = 1, \dots, m$. As before, we take the standard primitive

$$\lambda = \frac{1}{2} \sum_{i=1}^m x_i dy_i - y_i dx_i = \frac{1}{2} \sum_{i=1}^m r_i^2 d\theta_i$$

on $\mathbb{B}$. The standard symplectic form $\omega = d\lambda$ decomposes as $\omega = \sum_{i=1}^m \omega_i$, where $\omega_i := p_i^* \omega_0$, $p_i(z) = z_i$ is the projection and ω_0 is the standard symplectic form on $\mathbb{R}^2$.

A straightforward computation shows that the Hamiltonian vector field of H reads as

$$X_H = -2h'(r^2) \sum_{i=1}^m \partial_{\theta_i}$$

and the flow of X_H is given by

$$\varphi^t(z) = e^{-2h'(r^2)ti} z.$$

It is easy to check that the action σ_{φ^t} of a compactly supported Hamiltonian isotopy φ^t associated to a time-dependent Hamiltonian H_t is given by

$$\sigma_{\varphi^t, \lambda}(z) = \int_{\{z \mapsto \varphi^s(z)\}_{s \in [0, t]}} \lambda + \int_0^t H_s(\varphi^s(z)) ds. \quad (3.1)$$

Notation. From now on, we omit the subscript for the primitive in the notation of the action whenever the standard primitive λ is meant.

Via (3.1) we compute the action of the time-one-map.

$$\begin{aligned} \sigma_{\varphi^1}(z) &= \int_{\{z \mapsto \varphi^t(z)\}} \lambda + \int_0^1 H_t(\varphi^t(z)) dt \\ &= \int_0^1 \lambda(X_H) dt + \int_0^1 H(e^{-2h'(r^2)ti} z) dt \\ &= \int_0^1 \left(\frac{1}{2} \sum_{i=1}^m r_i^2 d\theta_i \right) (-2h'(r^2) \sum_{i=1}^m \partial_{\theta_i}) dt + \int_0^1 h(|e^{-2h'(r^2)ti} z|^2) dt \\ &= h(r^2) - r^2 h'(r^2). \end{aligned}$$

Notation. We let $\Omega_{\mathbb{B}}$ and $\Omega_{S^{2m-1}}$ be the standard riemannian volume forms on the corresponding manifolds and we write $V(\cdot)$ for the volumes of these spaces (or their subsets) with respect to these volume forms.

We want to compute the Calabi invariant. To this end we fix a reparametrization of the unit ball as follows.

$$\mathbb{T} : (0, 1) \times S^{2m-1} \rightarrow \mathbb{B}, (r, z) \mapsto rz. \quad (3.2)$$

We have

$$\mathbb{T}^*(\Omega_{\mathbb{B}}) = \frac{1}{m!}(\mathbb{T}^*\omega)^m = r^{2m-1}dr \wedge \Omega_{S^{2m-1}}.$$

Hence

$$\begin{aligned} \text{CAL}(\varphi^1) &= \int_{\mathbb{B}} \sigma \omega^m \\ &= m! \int_{\mathbb{B}} \sigma \Omega_{\mathbb{B}} \\ &= m! \int_{(0,1) \times S^{2m-1}} (h(r^2) - r^2 h'(r^2)) r^{2m-1} dr \wedge \Omega_{S^{2m-1}} \\ &= m! V(S^{2m-1}) \int_0^1 (r^{2m-1} h(r^2) - r^{2m+1} h'(r^2)) dr \\ &= 2m(m+1)\pi^m \int_0^1 r^{2m-1} h(r^2) dr. \end{aligned}$$

We note that

$$\text{CAL}(\varphi^1) = (m+1) \int_{\mathbb{B}} H \omega^m. \quad (3.3)$$

Now given $\delta \in (0, 1/2)$ small enough, we pick a function $\chi_\delta : [0, \infty) \rightarrow \mathbb{R}$ supported in $[0, 1)$ satisfying:

- $\chi_\delta(s) = 1 - \delta - s$ on $[0, 1 - 2\delta]$;
- $\max\{1 - \delta - s, 0\} \leq \chi_\delta(s) \leq \max\{(1 - \delta)(1 - s)\}$;
- $\chi'_\delta(s) = -1$ for $s \in [0, 1 - 2\delta]$ and $\chi'_\delta(s) \in [-1, 0]$ for all s ;
- $0 \leq \chi_\delta(s) - s \chi'_\delta(s) \leq 1 - \delta$.

We put

$$H_+(z) = \frac{\pi}{n} \chi_\delta(r^2)$$

so that the Hamiltonian flow φ_+^t satisfies

$$\varphi_+^t(z) = e^{\frac{i2\pi t}{n}} z \text{ for } r < \sqrt{1 - 2\delta}$$

and φ_+^t is a rotation by an angle less than or equal to $2\pi/n$ on $\mathbb{B}$. We have

$$\sigma_{\varphi_+^t}(z) = \frac{t\pi}{n} \left(\chi_\delta(r^2) - r^2 \chi'_\delta(r^2) \right)$$

so that

$$0 \leq \sigma_{\varphi_+^t} \leq \frac{t\pi}{n}(1 - \delta) \leq \frac{\pi}{n} \text{ for all } t \in [0, 1]. \quad (3.4)$$

Using (3.3) and (3.4), we get

$$0 \leq \text{CAL}(\varphi_+^1) \leq \frac{\pi}{n}(m+1)!V(\mathbb{B}) = \frac{(m+1)\pi^{m+1}}{n}. \quad (3.5)$$

We identify $\mathbb{B} \setminus \{0\}$ with $(0, 1) \times S^{2m-1}$ via (3.2) and in these coordinates φ_+^t is just a rotation along the fibres of the Hopf fibration $S^{2m-1} \rightarrow \mathbb{C}P^{m-1}$. We trivialize the Hopf fibration over $\mathbb{C}^{m-1} \subset \mathbb{C}P^{m-1}$ and get a codimension-zero embedding

$$g : \mathbb{D}^* \times \mathbb{C}^{m-1} \rightarrow \mathbb{B},$$

$$g(re^{i\theta}, z) = \left(\frac{re^{i\theta}z}{\sqrt{1+||z||^2}}, \frac{re^{i\theta}}{\sqrt{1+||z||^2}} \right) \in \mathbb{C}^{m-1} \times \mathbb{C},$$

where $\mathbb{D}^*$ is the punctured unit disc.

We note that the image of g is invariant under φ_+^t . We define the sectors

$$S_k := \left\{ re^{i\theta} \in \mathbb{D}^* \mid (k-1)\frac{2\pi}{n} < \theta < k\frac{2\pi}{n} \right\}, \quad k = 1, \dots, n.$$

We note that φ_+^t is constant on the second component of the product $\mathbb{D}^* \times \mathbb{C}^{m-1}$ and on the first component, it behaves as the corresponding disc map in [1]. For each k , we define an open subset $U_k := g(S_k \times \mathbb{C}^{m-1})$ in $\mathbb{B}^{2m}$.

Let $\rho \in (0, 1)$ and $n > 0$ be given. We claim that there is a finite collection of non-overlapping closed balls in U_1 such that

- The ratio between total volume of the balls and the volume of U_1 is at least ρ ;
- The radius of each ball is less or equal to π/n .

The existence of such a collection of balls follows from the fact that the open set U_1 is a pre-compact subset of $\mathbb{B}$ with finite volume.

We have a free $\mathbb{Z}_n$ -action on $\mathbb{B} \setminus \{0\}$, given by the rotation by the angle $2\pi/n$ along the Hopf fibration, which is isometric. The corresponding action on $\mathbb{D}^* \times \mathbb{C}^{m-1}$ is just the rotation along the disc, which permutes the sectors $S_k \times \mathbb{C}^{m-1}$. Hence the $\mathbb{Z}_n$ -action permutes U_k 's. Moreover the volumes of U_k 's are identical.

Now we take the collection $(B_j := B(z_j, r_j))_{j \in J}$ given by the union of $\mathbb{Z}_n$ -orbits of the balls that we choose in U_1 . Then we have

$$V(\mathbb{B}) \geq \sum_{j \in J} V(B(z_j, r_j)) \geq \rho V(\mathbb{B})$$

and $r_j \leq \pi/n$ for all $j \in J$.

Let $\epsilon \in (0, 1/2)$ be small and $c > 0$. For each $j \in J$, we consider the Hamiltonian

$$K_j : \mathbb{B} \rightarrow \mathbb{R}, \quad K_j(z) = -c\chi_\epsilon \left(\frac{\|z - z_j\|^2}{r_j^2} \right).$$

Let ψ_j^t be the Hamiltonian flow of K_j . Since each K_j is supported in B_j , the flow ψ_j^t and the action $\sigma_{\psi_j^t}$ are supported in B_j . We compute the action via the translated Hamiltonian $\tilde{K}_j(z) = -c\chi_\epsilon \left(\frac{\|z\|^2}{r_j^2} \right)$. Let $\tilde{\psi}_j^t$ be the Hamiltonian flow of $\tilde{K}_j$. We have

$$\sigma_{\tilde{\psi}_j^t}(z) = -ct \left(\chi_\epsilon \left(\frac{\|z\|^2}{r_j^2} \right) - \frac{\|z\|^2}{r_j^2} \chi'_\epsilon \left(\frac{\|z\|^2}{r_j^2} \right) \right)$$

and on $B(z_j, r_j\sqrt{1-2\epsilon})$,

$$\sigma_{\tilde{\psi}_j^t}(z) = -ct(1 - \epsilon).$$

Moreover on B_j ,

$$0 \geq \sigma_{\tilde{\psi}_j^t} \geq -ct(1 - \epsilon) \geq -c$$

for all $t \in [0, 1]$. We compute

$$\begin{aligned} \text{CAL}(\tilde{\psi}_j^1) &= \int \sigma_{\tilde{\psi}_j^1} \omega^m \\ &\leq m! \int_{B(z_j, r_j\sqrt{1-2\epsilon})} \sigma_{\tilde{\psi}_j^1} \Omega_{\mathbb{B}} \\ &= -m!c(1 - \epsilon) V(B(z_j, r_j\sqrt{1-2\epsilon})) \\ &\leq -c\pi^m(1 - 2\epsilon)^{m+1}r_j^{2m}. \end{aligned}$$

We pick an area preserving diffeomorphism $\tau_j : \mathbb{B} \rightarrow \mathbb{B}$ such that

$$\tau_j(z) = z + z_j \quad \text{on } B(0, r_j).$$

We have

$$\tilde{\psi}_j^t = \tau_j^{-1} \circ \psi_j^t \circ \tau_j.$$

It follows from the formula (3.1) that

$$\sigma_{\psi_j^t, \lambda} \circ \tau_j = \sigma_{\tau_j^{-1} \circ \psi_j^t \circ \tau_j, \tau_j^* \lambda} = \sigma_{\tilde{\psi}_j^t, \tau_j^* \lambda}$$

and

$$\text{CAL}(\psi_j^1) = \text{CAL}(\tilde{\psi}_j^1).$$

For a detailed account on the behaviour of the action and Calabi invariant under the conjugation, we refer to [1, Lemma 2.2 and Proposition 2.4].

We put $z_j := (u_1 + i v_1, \dots, u_m + i v_m)$ and note that

$$\tau_j^* \lambda - \lambda = \frac{1}{2} \sum (u_j dy_j - v_j dx_j) \text{ on } B(0, r_j).$$

Then there is some function $\eta : \mathbb{B} \rightarrow \mathbb{R}$ such that $\tau_j^* \lambda - \lambda = d\eta$ and

$$\eta(z) = \frac{1}{2} \sum_{i=1}^m (u_i y_i - v_i x_i) = \frac{1}{2} \text{Im} \langle z, z_j \rangle_{\mathbb{C}} \text{ on } B(0, r_j).$$

Hence we get

$$\sigma_{\tilde{\psi}_j^t, \tau_j^* \lambda} = \sigma_{\tilde{\psi}_j^t, \lambda} + \eta \circ \tilde{\psi}_j^t - \eta.$$

The above equation follows essentially from the formula (3.1) and for the details, we refer to [1, Lemma 2.2].

Since $\sigma_{\tilde{\psi}_j^t}$ is supported on $B(0, r_j)$, we get

$$\Rightarrow |\sigma_{\tilde{\psi}_j^t, \tau_j^* \lambda} - \sigma_{\tilde{\psi}_j^t, \lambda}| \leq \sup_{B(0, r_j)} \eta - \inf_{B(0, r_j)} \eta \leq r_j \|z_j\| \leq r_j \leq \frac{\pi}{n}.$$

Hence we get

$$-c - \frac{\pi}{n} \leq \sigma_{\psi_j^t} \leq \frac{\pi}{n}.$$

Since the diffeomorphisms $\{\psi_j^t\}_{j \in J}$ have disjoint support, they commute pairwise.

We denote their composition by φ_-^t , which is the Hamiltonian flow of the Hamiltonian $H_- := \sum_{j \in J} K_j$. We note that the support of φ_-^t is contained in the union of B_j 's and each B_j is invariant under φ_-^t . Hence we have

$$-c - \frac{\pi}{n} \leq \sigma_{\varphi_-^t} \leq \frac{\pi}{n}, \quad (3.6)$$

and

$$\begin{aligned} \text{CAL}(\varphi_-^1) &= \sum_{j \in J} \text{CAL}(\psi_j^1) \\ &\leq \sum_{j \in J} -c \pi^m (1 - 2\epsilon)^{m+1} r_j^{2m} \\ &= -c(1 - 2\epsilon)^{m+1} \sum_{j \in J} \pi^m r_j^{2m} \\ &= -m! c (1 - 2\epsilon)^{m+1} \sum_{j \in J} V(B_j). \end{aligned}$$

Hence

$$\text{CAL}(\varphi_-^1) \leq -c(1 - 2\epsilon)^{m+1} \rho \pi^m. \quad (3.7)$$

Above in the first equation, we use the fact that for fixed a symplectic form,

$$\text{CAL} : \text{Diff}_c(\mathbb{B}, \omega) \rightarrow \mathbb{R}$$

is a group homomorphism, for details see [1], [9]. Now we assume that δ is small enough so that $\bigcup_{j \in J} B_j \subset B(0, \sqrt{1 - 2\delta})$. We consider the isotopy

$$\varphi^t = \varphi_+^t \circ \varphi_-^t,$$

which is compactly supported by the choice of δ and connects the identity map to $\varphi^1 = \varphi_+^1 \circ \varphi_-^1$. We note that the fixed points of φ^1 are the origin and the points forming a neighborhood of $\partial\mathbb{B}$, which is outside of $B(0, \sqrt{1 - 2\delta})$. As $\bigcup_{j \in J} B_j \subset B(0, \sqrt{1 - 2\delta})$, φ_+^1 permutes B_j 's so that none of the fixed points of φ_-^1 remains as a fixed point. Similarly, any periodic point of φ^1 within $B(0, \sqrt{1 - 2\delta})$ has period at least n and any other periodic point of φ^1 is simply a periodic point of φ_+^1 and has period at least n . By the formula

$$\sigma_{\varphi^t} = \sigma_{\varphi_+^t} \circ \varphi_-^t + \sigma_{\varphi_-^t},$$

which is a straightforward generalization of Lemma 2.2 (ii) in [1], we get $\sigma_{\varphi^1}(0) = \frac{\pi}{n}(1 - \delta)$ for the fixed point 0 and $\sigma_{\varphi^1}(z) \geq 0$ for any other fixed point z . Hence (b3) and (b4) are established.

Using (3.4) and (3.6), we get

$$-c - \frac{\pi}{n} \leq \sigma_{\varphi^t} \leq \frac{2\pi}{n} \quad (3.8)$$

and by (3.5) and (3.7), we get

$$\text{CAL}(\varphi^1) \leq \frac{(m+1)\pi^{m+1}}{n} - c(1 - 2\epsilon)^{m+1} \rho \pi^m. \quad (3.9)$$

We put

$$c = L - \frac{L + \pi}{n}$$

where n is large enough and get

$$-L + \frac{L}{n} \leq \sigma_{\varphi^t} \leq \frac{2\pi}{n} \quad (3.10)$$

for all $t \in [0, 1]$. This establishes (b1). From (3.9) we get

$$\text{CAL}(\varphi^1) \leq -L\pi^m(1 - 2\epsilon)^{m+1} \rho + \frac{1}{n} \left[(m+1)\pi^{m+1} + (L + \pi)(1 - 2\epsilon)^{m+1} \rho \pi^m \right].$$

Then we have

$$\begin{aligned} \text{CAL}(\varphi^1) + L\pi^m &\leq L\pi^m(1 - (1 - 2\epsilon)^{m+1}\rho) \\ &\quad + \frac{1}{n} \left[(m+1)\pi^{m+1} + (L + \pi)(1 - 2\epsilon)^{m+1}\rho\pi^m \right]. \end{aligned}$$

Now given $\varepsilon > 0$, we choose ϵ close enough to 0 and ρ close enough to 1 so that

$$(1 - (1 - 2\epsilon)^{m+1}\rho) \leq \frac{\varepsilon}{2L\pi^m}$$

and we choose n large enough so that

$$\frac{1}{n} \left[(m+1)\pi^{m+1} + (L + \pi)\pi^m \right] \leq \frac{\varepsilon}{2}.$$

With these choices we get

$$\begin{aligned} \text{CAL}(\varphi^1) + L\pi^m &\leq L\pi^m(1 - (1 - 2\epsilon)^{m+1}\rho) \\ &\quad + \frac{1}{n} \left[(m+1)\pi^{m+1} + (L + \pi)(1 - 2\epsilon)^{m+1}\rho\pi^m \right] \\ &\leq L\pi^m \frac{\varepsilon}{2L\pi^m} + \frac{1}{n} \left[(m+1)\pi^{m+1} + (L + \pi)\pi^m \right] \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This establishes (b2). □

4. The plug and the systolic ratio on spheres

In this section we first combine the lemmata in the above sections in order to define the plug of any odd dimension (compare with [2, Proposition 3.1]). Next we prove that the contact systolic ratio on any sphere with the standard contact structure is unbounded.

Proposition 4.1. *Let $\epsilon > 0$ and $L > 0$ be given. Let λ be the standard primitive of ω on the $2m$ -dimensional open unit ball $\mathbb{B}$. Then there exists a smooth contact form β on $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$ such that the following holds:*

- (p1) $\text{vol}(\mathbb{R}/L\mathbb{Z} \times \mathbb{B}, \beta) \leq \epsilon$;
- (p2) $\beta = ds + \lambda$ on a neighbourhood of the boundary of $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$. In particular, $R_\beta = \partial_s$ near the boundary and its flow is globally defined;
- (p3) All closed orbits of R_β have period at least L ;
- (p4) β is smoothly isotopic to $ds + \lambda$ through a path of contact forms that coincide with $ds + \lambda$ on a fixed neighbourhood of the boundary of $\mathbb{R}/L\mathbb{Z} \times \mathbb{B}$.

Proof. We take the isotopy given in Lemma 3.1 and apply Lemma 2.1 to get the smooth contact form β . Combining (b2) with (a1) leads to (p1). The statement (p2) and (p4) are given by (a2) and (a5) respectively.

By (a4), we know that a periodic orbit of R_β corresponds to either a fixed point of φ^1 or to a periodic point of φ^1 that is not a fixed point. In the first case, if $z \in \mathbb{B}$ is the corresponding fixed point, then the period is given by

$$\tau(z) = \sigma_{\varphi^1}(z) + L \geq L,$$

since $\sigma_{\varphi^1}(z) \geq 0$ by (b3). We note that by (b1), we have the global bound

$$\tau = \sigma_{\varphi^1} + L \geq L/n.$$

Hence in the second case, if z is the corresponding periodic point with period k , the period of the Reeb orbit is given by

$$\sum_{j=0}^{k-1} \tau(\varphi^j(z)),$$

where $\varphi^j = (\varphi^1)^j$ and by (b4), one has $k \geq n$ and therefore

$$\sum_{j=0}^{k-1} \tau(\varphi^j(z)) \geq k \frac{L}{n} \geq L. \quad \square$$

In order to provide the first application of the plug construction we recall the standard contact structure on an odd sphere. For $m \geq 2$, we consider *the standard contact form* α_0 on the unit sphere S^{2m+1} given by

$$\alpha_0 = \frac{1}{2} \sum_{i=1}^{m+1} (u_i dv_i - v_i du_i)$$

via the coordinates $z = (z_1, \dots, z_{m+1}) \in S^{2m+1} \subset \mathbb{C}^{m+1}$, $z_j = u_j + i v_j$. The contact structure $\xi_{\text{st}} := \ker \alpha_0$ is called *the standard contact structure*. The Reeb vector field R_{α_0} generates the Hopf fibration $\pi : S^{2m+1} \rightarrow \mathbb{C}P^m$ so that each closed orbit has period π . We also know that $\text{vol}(S^{2m+1}, \alpha_0) = \pi^{m+1}$, therefore

$$\rho(S^{2m+1}, \alpha_0) = \frac{(T_{\min}(\alpha_0))^{m+1}}{\text{vol}(S^{2m+1}, \alpha_0)} = \frac{\pi^{m+1}}{\pi^{m+1}} = 1.$$

Theorem 4.2. *Given any $C > 0$, there is a contact form α on S^{2m+1} such that $\ker \alpha = \xi_{\text{st}}$ and*

$$\rho(S^{2m+1}, \alpha) \geq C.$$

Proof. Using the embedding

$$\mathbb{C}^m \hookrightarrow \mathbb{C}P^m, \quad z = (z_1, \dots, z_m) \mapsto (z_1 : \dots : z_m : 1)$$

we trivialize the Hopf bundle as

$$\mathbb{R}/\pi\mathbb{Z} \times \mathbb{C}^n \rightarrow S^{2m+1}, (s, z) \mapsto \kappa(z)(e^{i2s}z, e^{i2s}) \in \mathbb{C}^{m+1}$$

where $\kappa(z) = (1 + \|z\|^2)^{-1/2}$. Via the rescaling $z \mapsto \frac{z}{\sqrt{1 - \|z\|^2}}$, we define a smooth embedding

$$f : \mathbb{R}/\pi\mathbb{Z} \times \mathbb{B} \rightarrow S^{2m+1}, f(s, z) = (e^{i2s}z, e^{i2s}\sqrt{1 - \|z\|^2}), \quad (4.1)$$

where as before, $\mathbb{B}$ is the open unit ball in $\mathbb{R}^{2m}$. On $\mathbb{B}$, we fix the primitive

$$\lambda = \frac{1}{2} \sum_{i=1}^m (x_i dy_i - y_i dx_i).$$

A straightforward computation shows that

$$f^*\alpha_0 = ds + \lambda.$$

We note that $S^{2m+1} \setminus \text{im}(f) = \pi^{-1}(\mathbb{C}P^{m-1})$ where

$$\mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m, (z_0 : \dots : z_{m-1}) \mapsto (z_0 : \dots : z_{m-1} : 0).$$

That is, f misses a subset of codimension two and hence of measure zero.

Let $C > 0$ be given. We take $\epsilon = \pi^{m+1}/C$ and $L = \pi$. By Proposition 4.1, there is a contact form β on $\mathbb{R}/\pi\mathbb{Z} \times \mathbb{B}$ satisfying (p1)-(p4). By (p4), the push forward of β by f , still denoted by β , extends to a smooth contact form on S^{2m+1} such that the minimal period is π . We also note that

$$\text{vol}(S^{2m+1}, \beta) = \text{vol}(\mathbb{R}/\pi\mathbb{Z} \times \mathbb{B}^{2m}, \beta) \leq \epsilon.$$

Hence we get

$$\rho(S^{2m+1}, \beta) \geq \frac{\pi^{m+1}}{\epsilon} = C.$$

We finally note that the contact form β on S^{m+1} is isotopic to α_0 . Hence $\ker \beta$ is isotopic to ξ_{st} . By Gray's stability theorem, there is a diffeomorphism ψ of S^{2m+1} such that $\psi_*\xi_{\text{st}} = \ker \beta$. Since systolic ratio is invariant under a diffeomorphism, $\alpha := \psi^*\beta$ is the desired contact form. $\square$

5. Large systolic ratio for arbitrary contact manifolds

In this section we prove that any co-orientable contact manifold (V, ξ) admits a contact form with arbitrarily large systolic ratio. Our statement generalizes the main theorem of [2] and our proof uses the strategy of [2] with necessary modifications.

5.1. The plug with arbitrary radius and primitive

We first note that the plug construction given above, generalizes to the balls of arbitrary radius. In fact, given $r > 0$ and $\varphi \in \text{Diff}_c(\mathbb{B}, \omega)$, we define a diffeomorphism

$$\varphi_r : r\mathbb{B} \rightarrow r\mathbb{B}, \quad \varphi_r(z) = r\varphi\left(\frac{z}{r}\right),$$

which is compactly supported and symplectic. The definitions (2.1) and (2.2) make sense for φ_r and a straightforward computation shows that

$$\sigma_{\varphi_r} = r^2 \sigma_{\varphi} \left(\frac{z}{r} \right) \quad (5.1)$$

and

$$\text{CAL}(\varphi_r) = r^{2m+2} \text{CAL}(\varphi). \quad (5.2)$$

We recover Proposition 4.1 for the ball of arbitrary radius as follows. Given $L > 0$ and $\epsilon > 0$, by Lemma 3.1, one gets an isotopy $\{\varphi^t\} \subset \text{Diff}_c(\mathbb{B}, \omega)$ such that

$$\sigma_{\varphi^t, \lambda} \geq -\frac{L}{r^2} + \frac{L}{r^2 n} \quad \forall t \in [0, 1].$$

and

$$\text{CAL}(\varphi^1) + \frac{L}{r^2} \pi^m < \frac{\epsilon}{r^{2m+2}}.$$

Combining these statements with (5.1) and (5.2), we get

$$\sigma_{\varphi_r^t, \lambda} = r^2 \sigma_{\varphi^t, \lambda} \geq -r^2 \frac{L}{r^2} + r^2 \frac{L}{r^2 n} = -L + L/n \quad (5.3)$$

and

$$\text{CAL}(\varphi_r^1) + L\pi^m r^{2m} = r^{2m+2} \text{CAL}(\varphi^1) + r^{2m+2} \frac{L}{r^2} \pi^m < r^{2m+2} \frac{\epsilon}{r^{2m+2}} = \epsilon. \quad (5.4)$$

Finally we note that repeating the construction in Lemma 2.1 with the isotopy $\{\varphi_r^t\} \subset \text{Diff}_c(r\mathbb{B}, \omega)$, we get a contact form β on $\mathbb{R}/L\mathbb{Z} \times r\mathbb{B}$ so that the property (a1) is replaced by

$$\text{vol}(\mathbb{R}/L\mathbb{Z} \times r\mathbb{B}, \beta) = L\pi^m r^{2m} + \text{CAL}(\varphi_r^1). \quad (5.5)$$

Hence combining (5.4) with (5.5) yields

$$\text{vol}(\mathbb{R}/L\mathbb{Z} \times r\mathbb{B}, \beta) \leq \epsilon.$$

We note that the statements (b3) and (b4) hold for the action $\sigma_{\varphi_r^1}$. Hence the proof of Proposition 4.1 goes through.

The final modification of the plug construction is replacing the standard primitive λ with an arbitrary primitive. After the observations made above, the proof of the following statement is a straightforward generalization of [2, Proposition 3.1].

Lemma 5.1. *Let $r, \epsilon > 0$ be given and let λ' be any primitive of the standard symplectic form ω on the open disc $r\mathbb{B}$. Then there exists a smooth contact form β' on $r\mathbb{B} \times \mathbb{R}/\mathbb{Z}$ such that the following holds:*

- (p'1) $\beta' = \lambda' + ds$ in a neighborhood of $\partial(r\mathbb{B} \times \mathbb{R}/\mathbb{Z})$;
- (p'2) β' is smoothly isotopic to the contact form $\lambda' + ds$ on $r\mathbb{B} \times \mathbb{R}/\mathbb{Z}$ through a path of contact forms which agree with $\lambda' + ds$ in a neighborhood of $\partial(r\mathbb{B} \times \mathbb{R}/\mathbb{Z})$;
- (p'3) All the closed orbits of $R_{\beta'}$ have period at least 1;
- (p'4) $\text{vol}(r\mathbb{B} \times \mathbb{R}/\mathbb{Z}, \beta') \leq \epsilon$.

Proof. From the observations made above, we know that such a contact form β exists when we use the standard primitive λ and set $L = 1$. Given an arbitrary primitive λ' , the 1-form $\lambda' - \lambda$ is closed on $r\mathbb{B}$ and hence exact. Let u be a smooth function on $r\mathbb{B}$ so that $\lambda' - \lambda = du$.

Let β_t be a smooth path of contact forms on $r\mathbb{B} \times \mathbb{R}/\mathbb{Z}$ such that $\beta_0 = \beta$, $\beta_1 = \lambda + ds$ and $\beta_t = \lambda + ds$ on $(r\mathbb{B} \setminus r'\mathbb{B}) \times \mathbb{R}/\mathbb{Z}$ for all t where $r' < r$. Let χ be a smooth function on $r\mathbb{B}$ such that $\chi = 0$ on $r'\mathbb{B}$ and $\chi = 1$ on $r\mathbb{B} \setminus r''\mathbb{B}$ where $r' < r'' < r$. We define the contact form

$$\beta' := \beta + d(\chi u).$$

In fact,

$$\beta' \wedge (d\beta')^n = \beta' \wedge (d\beta)^n = \beta \wedge (d\beta)^n + d(\chi u) \wedge (d\beta)^n = \beta \wedge (d\beta)^n$$

since on the region where $d(\chi u)$ is supported, $d\beta = d\lambda$ and ∂_s is in the kernel of both $d(\chi u)$ and $d\beta$. In particular,

$$\text{vol}(r\mathbb{B} \times \mathbb{R}/\mathbb{Z}, \beta') = \text{vol}(r\mathbb{B} \times \mathbb{R}/\mathbb{Z}, \beta) < \epsilon.$$

We also note that on the region $(r''\mathbb{B} \setminus r'\mathbb{B}) \times \mathbb{R}/\mathbb{Z}$, $R_\beta = \partial_s$ and hence

$$\beta'(R_\beta) = \beta(R_\beta) + d(\chi u)(\partial_s) = 1.$$

Therefore, $\beta'(R_\beta) = 1$ and $\iota_{R_\beta} d\beta' = 0$ on $r\mathbb{B} \times \mathbb{R}/\mathbb{Z}$, that is $R_{\beta'} = R_\beta$ and the third property is satisfied. We note that on the region $(r\mathbb{B} \setminus r''\mathbb{B}) \times \mathbb{R}/\mathbb{Z}$,

$$\beta' = \beta + du = \lambda' + ds$$

and hence first property is satisfied. We finally define

$$\tilde{\beta}^t := \beta^t + d(\chi u).$$

We note that on the region $(r''\mathbb{B} \setminus r'\mathbb{B}) \times \mathbb{R}/\mathbb{Z}$,

$$\tilde{\beta}^t = \lambda + ds \Rightarrow d\tilde{\beta}^t = d\lambda$$

and therefore $\tilde{\beta}^t$ is a smooth homotopy of contact forms so that

$$\tilde{\beta}^0 = \beta^0 + d(\chi u) = \beta + d(\chi u) = \beta'$$

and

$$\tilde{\beta}^1 = \beta^1 + d(\chi u) = \lambda + ds + d(\chi u)$$

and on the region $(r\mathbb{B} \setminus r''\mathbb{B}) \times \mathbb{R}/\mathbb{Z}$, $\tilde{\beta}^t = \lambda + ds$ for all t . Next we consider the smooth isotopy

$$\lambda + ds + tdu + (1-t)d(\chi u)$$

of contact forms that connects $\lambda + ds + d(\chi u)$ to $\lambda' + ds$, which clearly coincides with $\lambda' + ds$ on the region $(r\mathbb{B} \setminus r''\mathbb{B}) \times \mathbb{R}/\mathbb{Z}$ for all t . Hence the second property follows. $\square$

5.2. Construction of a special contact form

The next step is to construct a contact form on a given closed co-oriented contact manifold (V, ξ) such that the Reeb flow gives rise to a circle bundle structure on a domain in V , which possesses arbitrarily large portion of the total contact volume. We will later modify such a contact form via inserting plugs so that most of the contact volume is sucked up without decreasing the minimal period. We first summarize the necessary definitions and results concerning the Giroux's correspondence between the contact structures and supported open books in higher dimensions. For the details, we refer to [6] and [7].

Let F be a $2n$ -dimensional domain with boundary K and let F° denote the interior of F . A symplectic form $\omega \in \Omega^2(F^\circ)$ is called an *ideal Liouville structure*, abbreviated by ILS, on F if it admits a primitive $\lambda \in \Omega^1(F^\circ)$ such that for some/any smooth function

$$u : F \rightarrow [0, +\infty), \quad \text{where } K = u^{-1}(0) \text{ is a regular level set,} \quad (5.6)$$

the 1-form $u\lambda$ on F° extends to a smooth 1-form β on F , which is a contact form along K .

If such a 2-form ω exists, then the pair (F, ω) is called an *ideal Liouville domain*, abbreviated as ILD, and any primitive λ with the above property is called an *ideal Liouville form*, abbreviated as ILF. It turns out that given an ILD (F, ω) , the contact structure

$$\xi := \ker(\beta|_{TK})$$

depends on the 2-form ω but not on λ or u , see [7, Proposition 2]. Moreover, once λ is chosen, one can recover all possible (positive) contact forms on (K, ξ) by restricting the extension of $u\lambda$ to K as u moves among the functions with the property (5.6). Hence the pair (K, ξ) is called the *ideal contact boundary* of (F, ω) . We note that the orientation of K that is determined by the co-oriented contact structure ξ coincides with the orientation of K as the boundary of (F, ω) .

A very useful feature of an ILD is that the vicinity of its boundary admits an explicit parametrization by means of which any ILF has a very nice form.

Lemma 5.2. *Let (F, ω) be an ILD and λ be an ILF. Let u be a function satisfying (5.6) and β be the extension of $u\lambda$. Then for any contact form α_0 on (K, ξ) , there exists an embedding*

$$\iota : [0, +\infty) \times K \rightarrow F$$

such that

$$\iota^* \lambda = \frac{1}{r} \alpha_0 \text{ and } \iota(0, q) = q \text{ for all } q \in K,$$

where $r \in [0, +\infty)$. In particular,

$$\iota^* \beta = \frac{u \circ \iota}{r} \alpha_0 \text{ on } F^o$$

and for all $q \in K$

$$(\beta|_{TK})(q) = \left(\frac{\partial(\iota \circ u)}{\partial r}(0, q) \right) \alpha_0.$$

Proof. The above statement is a reformulation of [7, Proposition 3]. Here we give a proof that is essentially the same as the proof of that proposition but it is more explicit.

Let $\dim F = 2n$. As simple computation gives

$$\omega^n = (d(\beta/u))^n = u^{-n-1}(ud\beta + n\beta \wedge du) \wedge (d\beta)^{n-1}.$$

We put

$$\mu := (ud\beta + n\beta \wedge du) \wedge (d\beta)^{n-1}$$

and note that by definition, μ is a smooth positive volume form on F^o . Since $u = 0$ on K ,

$$\mu = -ndu \wedge \beta \wedge (d\beta)^{n-1} \text{ on } K$$

and it is a positive volume form since β is a positive contact form along K and u satisfies (5.6). We define the vector field X on F via

$$\iota_X \mu = n\beta \wedge (d\beta)^{n-1}.$$

Evaluating the above identity at any point on K shows that $\iota_X du = -1$ on K , that is, X is non-singular on K and points transversely outwards. On the other hand, writing $\beta = u\lambda$ on F^o yields

$$\iota_X \mu = n\beta \wedge (d\beta)^{n-1} = nu^n \lambda \wedge (d\lambda)^{n-1} = u^n \iota_Y \omega^n = u^{-1} \iota_Y \mu,$$

where Y is the Liouville vector field of λ on F^o . In particular, $Y = uX$ on F^o . We note that the 1-form β is not in general a Liouville form but here, one should think of the vector field X as the "Liouville vector field of β ". In fact,

$$L_X \beta = \rho \beta$$

for some smooth function $\rho : F \rightarrow \mathbb{R}$. To see this, we note that on F° ,

$$L_X\beta = \iota_X d\beta + d(\iota_X\beta) = \frac{1}{u}\iota_Y(du \wedge \lambda + u d\lambda) = \frac{1}{u}\iota_Y du + \lambda = \frac{1}{u}(\iota_X du + 1)\beta.$$

We claim that the function

$$\rho := \frac{1}{u}(\iota_X du + 1)$$

extends to a smooth function on F . Let β_0 be the contact form, given by the restriction of β to TK and let R_0 be the Reeb vector field on K associated to β_0 . Near K , we define a smooth vector field R via

$$R(p) := (\varphi_*^t R_0)(p); \quad \varphi^t(q) = p, \quad t \leq 0,$$

where φ^t is the flow of X . Since $R = R_0$ on K , $\beta(R) = 1$ on K and therefore on a compact neighbourhood V of K , we have $\beta(R) > 0$. On $V \setminus K$,

$$\rho = \frac{(L_X\beta)(R)}{\beta(R)}$$

and the right hand side is a smooth function on V . Hence ρ extends to a smooth function on V and $L_X\beta = \rho\beta$ on F . In particular, the flow φ^t preserves the kernel of β . Integrating the equation

$$\frac{d}{dt}(\varphi^t)^*\beta = (\varphi^t)^*(L_X\beta) = (\rho \circ \varphi^t)(\varphi^t)^*\beta,$$

we get

$$(\varphi^t)^*\beta = \mu_t\beta, \quad \mu_t = \exp\left(\int_0^t \rho \circ \varphi^s ds\right).$$

We also note that on F^0 ,

$$\beta(X) = u\lambda(X) = \lambda(Y) = 0.$$

Hence $\beta(X) = 0$ on F . Now we consider the smooth embedding

$$\Phi : (-\infty, 0] \times K \rightarrow F, \quad (t, q) \mapsto \varphi^t(q),$$

where φ^t is the flow of X . By definition, we have $\Phi^*X = \partial_t$. We put $\hat{\beta} := \Phi^*\beta$, $\hat{u} := \Phi^*u$ and $\hat{\rho} := \Phi^*\rho$. The above discussion says that $\hat{\beta}$ has no dt component. Moreover, it has the form

$$\hat{\beta}(t, q) = \exp\left(\int_0^t \hat{\rho}(s, q) ds\right) \beta_0(q),$$

where

$$\hat{\rho}(t, q) = \frac{\hat{u}_t(t, q) + 1}{\hat{u}(t, q)}.$$

Now let α_0 be a positive contact form on K . Then there is some positive function κ on K so that $\beta_0 = \kappa\alpha_0$. We put $\hat{\lambda} := \Phi^*\lambda$ and get

$$\hat{\lambda}(t, q) = \Lambda(t, q)\alpha_0(q),$$

where $\Lambda : (-\infty, 0) \times K \rightarrow (0, +\infty)$ is the smooth function given by

$$\Lambda(t, q) := \frac{1}{\hat{u}(t, q)} \kappa(q) \exp \left(\int_0^t \hat{\rho}(s, q) ds \right).$$

We note that

$$\frac{\partial \Lambda}{\partial t} = \kappa \exp \left(\int_0^t \hat{\rho}(s, \cdot) ds \right) \frac{\hat{\rho}\hat{u} - \hat{u}_t}{\hat{u}^2} = \frac{\Lambda}{\hat{u}}$$

and get

$$\Lambda(t, q) = \Lambda(-1, q) \exp \left(\int_{-1}^t \frac{1}{\hat{u}(s, q)} ds \right).$$

From $\hat{u}_t(0, q) = -1$, it follows that for all $q \in K$, $\lim_{t \rightarrow 0} \Lambda(t, q) = +\infty$. We note that since F is compact u is bounded above. In particular $1/\hat{u}$ is bounded away from zero. It follows that $\lim_{t \rightarrow -\infty} \Lambda(t, q) = 0$ for all $q \in K$. We conclude that $\Lambda(\cdot, q)$ is a diffeomorphism between $(-\infty, 0)$ and $(0, +\infty)$. We define a function

$$f : (0, +\infty) \times K \rightarrow (-\infty, 0)$$

by imposing

$$\Lambda(f(r, q), q) = \frac{1}{r}.$$

It is clear that f is also smooth along K . Moreover it extends to a smooth function on $[0, +\infty) \times K$ so that $f(0, q) = 0$ for all q . We define a smooth embedding

$$F : [0, +\infty) \times K \rightarrow (-\infty, 0] \times K, \quad F(r, q) = (f(r, q), q).$$

By definition

$$F^*\hat{\lambda} = \frac{1}{r}\alpha_0 \quad \text{and} \quad F(0, q) = (0, q)$$

for all $q \in K$. Then $\iota := \Phi \circ F$ is the desired embedding. The rest of the claim follows immediately from the equation $\iota^*\lambda = (1/r)\alpha$. $\square$

From the above lemma, it follows that given two ILF's λ_1 and λ_2 on a an ILD (F, ω) , there are two neighbourhoods U_1 and U_2 of $K := \partial F$ such that there is an exact symplectomorphism between $(U_1 \setminus K, \lambda_1)$ and $(U_2 \setminus K, \lambda_2)$, which extends to K as the identity. Using this fact and the standard Moser argument, one shows the following.

Proposition 5.3 ([7, Corollary 5]). *Let (F, ω) be an ILD and $(\lambda_t)_{t \in [0,1]}$ be a path of ILF's on F , which is smooth in the sense that there exists a smooth path of 1-forms $(\beta_t)_{t \in [0,1]}$ on F and some function u satisfying (5.6) such that $\lambda_t = \beta_t/u$ for all $t \in [0, 1]$. Then there is an isotopy $(\psi_t)_{t \in [0,1]}$ of F , relative to the boundary, such that $\psi_0 = \text{id}$ and, for every $t \in [0, 1]$ the form $\psi^* \lambda_t - \lambda_0$ is the differential of a function with compact support in F° .*

Given an ILD (F, ω) , it is not hard to see that the set of ILF's is an affine space. Hence any two IDF's can be connected by a smooth path of ILF's. Another consequence of Lemma 5.2 is the following stability result.

Proposition 5.4 ([7, Lemma 6]). *Let F be a domain and $(\omega_t)_{t \in [0,1]}$ be a smooth path of ideal Liouville structures on F , in the sense that there exists a smooth path of 1-forms $(\beta_t)_{t \in [0,1]}$ on F and some function u satisfying (5.6) such that $\omega_t = d(\beta_t/u)$. Then there exists an isotopy $(\phi_t)_{t \in [0,1]}$ of F such that $\phi_0 = \text{id}$ and $\phi_t^* \omega_t = \omega_0$ for all $t \in [0, 1]$. Moreover, we can choose this isotopy relative to $K = \partial F$ if and only if all forms ω_t induce the same boundary contact structure.*

Ideal Liouville domains are particularly useful for clarifying the existence and uniqueness of the contact structures supported by open books in higher dimensions. We first recollect some facts on open books.

An open book in a closed manifold V is a pair (K, Θ) where:

- (ob1) $K \subset V$ is a closed co-dimension two submanifold with trivial normal bundle;
- (ob2) $\Theta : V \setminus K \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is a locally trivial fibration such that K has a neighbourhood U , which admits a parametrization $(re^{ix}, q) \in \mathbb{D} \times K \cong U$ so that Θ reads as $\Theta(re^{ix}, q) = x$ on U .

The submanifold K is called the *binding* of the open book and the closures of the fibres of Θ are called the *pages*. All the pages are compact manifolds, for which the binding is the common boundary. We note that the canonical orientation of S^1 induces co-orientations on the pages and the binding. Hence if V is oriented then so are the pages and the binding. Another way of defining an open book is the following. Let $h : V \rightarrow \mathbb{C}$ be a smooth function such that:

- (df1) h vanishes transversely;
- (df2) $\Theta := h/|h| : V \setminus K \rightarrow S^1$ has no critical points, where $K := h^{-1}(0)$.

Then the pair (K, Θ) is an open book in V . Moreover, any open book in V may be recovered via a *defining function* h as above and such a defining function is unique up to multiplication by a positive function on V .

Given an open book (K, Θ) in a closed manifold V , one finds a vector field X , referred as a *spinning vector field*, on V such that:

- (m1) X lifts to a smooth vector field on the manifold with boundary obtained from V by a real oriented blow-up along K ;
- (m2) $X = 0$ on K and $(\Theta^* dx)(X) = 2\pi$ on $V \setminus K$.

Then the time-one-map of the flow of X is a diffeomorphism

$$\phi : F \rightarrow F$$

of the 0th-page $F := \Theta^{-1}(0) \cup K$, which fixes K . The isotopy class $[\phi]$ of ϕ relative the boundary is called the *monodromy* of the open book and it turns out that the open book is characterized by the pair $(F, [\phi])$. Namely, given the pair (F, ϕ) , one defines the mapping torus

$$MT(F, \phi) := ([0, 2\pi] \times F) / \sim; (2\pi, q) \sim (0, \phi(q)),$$

which is a manifold with boundary. One has the natural fibration

$$\hat{\Theta} : MT(F, \phi) \rightarrow S^1,$$

where all fibres are diffeomorphic to F and there is a natural parametrization of the fibre $\hat{\Theta}^{-1}(0)$ via the restriction of the above quotient map to $\{0\} \times F$. It turns out that if $\phi' \in [\phi]$, then there is a diffeomorphism between $MT(F, \phi)$ and $MT(F, \phi')$ that respects the fibrations over S^1 and the natural parametrizations of the 0-th pages. Now given $MT(F, \phi)$, one collapses its boundary, which is diffeomorphic to $S^1 \times K$, to K and obtains so called the *abstract open book* $OB(F, \phi)$. In fact, the closed manifold $OB(F, \phi)$ admits an open book given by the pair (K, Θ) where Θ is induced from $\hat{\Theta}$. Moreover, for $\phi' \in [\phi]$, the diffeomorphism between $MT(F, \phi)$ and $MT(F, \phi')$ descends to a diffeomorphism between corresponding abstract open books. In particular, V and $OB(F, \phi)$ may be identified together with their open book structures. We note that one may choose a vector field X that is actually smooth on V (compare with (m1)) and even 1-periodic near K . But it is not possible to obtain any given representative of the monodromy class via such a vector field, see [7, Remark 12]. In fact, in order to obtain all representatives of the monodromy class, one needs to sweep out the whole affine space of spinning vector fields.

Open books meet with the contact topology via the following definition. Let V be a closed manifold and ξ be a co-oriented contact structure on V . We say ξ is *supported* by an open book (K, Θ) on V if there is a contact form α on (V, ξ) , that is $\xi = \ker \alpha$, such that:

- α restricts to a (positive) contact form on K ;
- $d\alpha$ restricts to a (positive) symplectic form on each fibre of Θ .

It turns out that given a closed contact manifold V , the isotopy classes of co-oriented contact structures are in one-to-one correspondence of (equivalence classes of) supporting open books. This statement is a very rough summary of what is called the Gioux correspondence. We will recall certain pieces of this celebrated statement in detail.

Theorem 5.5 ([6, Theorem 10]). *Any contact structure on a closed manifold is supported by an open book with Weinstein pages.*

The above statement is the core part of the correspondence between supported open books and contact structures. In fact the existence statement for the opposite direction is relatively easy to achieve, especially in dimension three. Namely, given an open book in a 3-dimensional closed manifold, it is not hard to construct a contact form on the corresponding abstract open book, whose kernel is supported. It turns out that in higher dimensions, one needs to have an exact symplectic page and a symplectic monodromy in order to construct a contact form on an abstract open book, whose kernel is supported, see [6, Proposition 9] and [7, Proposition 17]. We will carry out such a construction in Proposition 5.9. Concerning the uniqueness features of the Giroux correspondence, we are mainly interested in one side, namely the “uniqueness” of supported contact structures. It turns out that such a statement is again more involved in higher dimensions. Philosophically, given an open book, the symplectic geometry of the pages determines the supported contact structures and in dimension three, any two symplectic structure on a page are isotopic since they are simply two area forms on a given surface. But in higher dimensions, this is not true in general.

In [7], Giroux introduced the notion of a Liouville open book, which clears out the technicalities pointed above.

A *Liouville open book*, abbreviated as LOB, in a closed manifold V is a tripple $(K, \Theta, (\omega_x)_{x \in S^1})$ where:

- (lob1) (K, Θ) is an open book on V with pages $F_x = \Theta^{-1}(x) \cup K, x \in S^1$;
- (lob2) (F_x, ω_x) is an ILD for all $x \in S^1$ and the following holds: there is a defining function $h : V \rightarrow \mathbb{C}$ for (K, Θ) and a 1-form β on V such that the restriction of $d(\beta/|h|)$ to each page is an ILF. More precisely,

$$\omega_x = d(\beta/|h|)|_{TF_x^o}$$

for all $x \in S^1$.

The 1-form β in (lob2) is called a *binding 1-form* associated to h . Note that if h' is another defining function for (K, Θ) , then $h' = \kappa h$ for some positive function κ on V and $\beta' := \kappa\beta$ is a binding 1-form associated to h' . We also note that for a fixed defining function, the set of associated binding 1-forms is an affine space.

Similar to classical open books, LOB's are characterized by the monodromy, which now has to be symplectic. Namely, one considers a *symplectically spinning vector field*, that is a vector field X satisfying (m1)-(m2) and generating the kernel of a closed 2-form on $V \setminus K$, which restricts to ω_x for all $x \in S^1$. Given such a vector field, the time-one-map of its flow, say ϕ , is a diffeomorphism of $F := F_0$, which fixes K and preserves $\omega := \omega_0$. The isotopy class $[\phi]$, among the symplectic diffeomorphisms that fixes K , is called the *symplectic monodromy* and characterizes the given LOB. For the construction of a LOB in the abstract open book $OB(F, \phi)$, where $\phi^*\omega = \omega$, we refer to [7, Proposition 17] and Proposition 5.9.

Similar to the classical open books, symplectically spinning vector fields form an affine space and all representatives of the symplectic monodromy may be obtained by sweeping out this affine space. It turns out that the obvious choice of a

symplectically spinning vector field is actually smooth and by modifying a given binding 1-form along Θ , it is possible to get a symplectically spinning vector field, whose flow is 1-periodic near the binding.

Lemma 5.6 ([7, Lemma 15]). *Let $(K, \Theta, (\omega_x)_{x \in S^1})$ be a LOB on a closed manifold V and $h : V \rightarrow \mathbb{C}$ be a defining function for (K, Θ) . Then for every binding 1-form β , the vector field X on $V \setminus K$ spanning the kernel of $d(\beta/|h|)$ and satisfying $(\Theta^*dx)(X) = 2\pi$ extends to a smooth vector field on V which is zero along K . Furthermore, β can be chosen so that X is 1-periodic near K .*

Natural sources of LOBs are contact manifolds, namely we have the following statement.

Proposition 5.7 ([7, Proposition 18]). *Let (V, ξ) be a closed contact manifold, and (K, Θ) be a supporting open book with defining function $h : V \rightarrow \mathbb{C}$. Then the contact forms α on (V, ξ) such that $d(\alpha/|h|)$ induces an ideal Liouville structure on each page form a non-empty convex cone.*

Let $(K, \Theta, (\omega_x)_{x \in S^1})$ be a LOB on a closed manifold V with a defining function h . A co-oriented contact structure ξ on V is said to be *symplectically supported* by $(K, \Theta, (\omega_x)_{x \in S^1})$ if there exists a contact form α on (V, ξ) such that α is a binding 1-form of the LOB associated to h .

By our remark following the definition of the binding 1-form, the definition of being symplectically supported is independent of the given defining function. But the crucial fact is that once a defining function is fixed, a contact binding 1-form is unique whenever it exists, see [7, Remark 20]. Hence, once a defining function h is fixed, there is a one-to-one correspondence between contact structures symplectically supported by $(K, \Theta, (\omega_x)_{x \in S^1})$ and contact binding 1-forms associated to h . Now given two contact structures ξ_0 and ξ_1 supported by $(K, \Theta, (\omega_x)_{x \in S^1})$, there exist unique contact binding 1-forms α_0 and α_1 respectively. Since the set of binding 1-forms associated to h is affine, there is a path $(\beta_t)_{t \in [0,1]}$ of binding 1-forms such that $\beta_0 = \alpha_0$ and $\beta_1 = \alpha_1$. Then by modifying β_t 's along the 1-form Θ^*dx , one gets a path of contact forms $(\beta_t^c)_{t \in [0,1]}$ and a homotopy $((\beta_t^s)_{t \in [0,1]})_{s \in [0,c]}$ between the paths $(\beta_t)_{t \in [0,1]}$ and $(\beta_t^c)_{t \in [0,1]}$ such that:

- For all $s \in [0, c]$ and $t \in [0, 1]$, β_t^s is a binding 1-form for $(K, \Theta, (\omega_x)_{x \in S^1})$ associated to h (since β_t 's stay the same along the pages through the modification);
- For all $s \in [0, c]$, β_0^s and β_1^s are contact forms (since if β_t is already a contact form then it keeps being a contact form through the modification).

In particular, whenever β_t^s is a contact form, $\ker \beta_t^s$ is symplectically supported by $(K, \Theta, (\omega_x)_{x \in S^1})$ and β_t^s is the unique contact binding 1-form associated to h . This tells us that the concatenation of the paths $(\ker \beta_0^s)_{s \in [0,c]}$, $(\ker \beta_t^c)_{t \in [0,1]}$ and $(\ker \beta_1^{c-s})_{s \in [0,c]}$ gives an isotopy between ξ_0 and ξ_1 along the contact structures that are symplectically supported by $(K, \Theta, (\omega_x)_{x \in S^1})$. In fact the following more general statement holds.

Proposition 5.8 ([7, Proposition 21]). *On a closed manifold, contact structures supported by a given Liouville open book form a non-empty and weakly contractible subset in the space of all contact structures.*

Now we are ready to construct the special contact form on given contact manifold (V, ξ) . Our construction is the generalization of Proposition 2.1 in [2] to any dimension.

Proposition 5.9. *Let (V, ξ) be a closed connected co-oriented contact manifold. Then there is an embedded compact hypersurface $F \subset V$ with the following property. Given any $\epsilon > 0$, there exists a contact form α on (V, ξ) such that F is a global hypersurface of section for the Reeb flow of α and the following holds:*

- (F1) α restricts to a contact form on $K := \partial F$, for which the closed Reeb orbits have period at least $1/2$;
- (F2) The first return time function

$$\tau : F \setminus \partial F \rightarrow (0, +\infty)$$

of the Reeb flow of α extends to a smooth function on F and the corresponding first return map

$$\varphi : F \setminus \partial F \rightarrow F \setminus \partial F$$

extends to a diffeomorphism of F ;

- (F3) There exists an open subset $U \subset F$ such that the support of φ is contained in $F \setminus U$ and $\tau \equiv 1$ on U ;
- (F4)

$$\|\tau - 1\|_\infty < \min \left\{ \epsilon, \frac{\epsilon}{\text{vol}(K, \alpha)} \right\};$$

- (F5)

$$\int_{F \setminus U} (d\alpha)^n < \epsilon,$$

where $2n + 1 = \dim V$.

Proof. We prove the statement by induction on $\dim V = 2n + 1$. For $n = 1$ the statement follows from the [2, Proposition 2.1]. In fact, all above properties follow immediately except (F4) and (F4) is a harmless modification of the property (iii). Now assume that the statement is true for $n - 1$.

Let (V, ξ) be given such that $\dim V = 2n + 1$. By Theorem 5.5, there is an open book (K, Θ) in V supported by ξ . Let $F_x := \Theta^{-1}(x)$, $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ denote the pages of the open book and let $h : V \rightarrow \mathbb{C}$ be a defining function for (K, Θ) . We want to show that the page

$$F := \Theta^{-1}(0) \cup K \tag{5.7}$$

is a hypersurface that satisfies (F1)-(F5).

Let $\epsilon > 0$ be given. By Proposition 5.7, there is a contact form α on (V, ξ) such that $(K, \Theta, d(\alpha/|h|)_{TF^o_x})$ is a LOB, which supports ξ symplectically. By Lemma 5.6, we modify the binding 1-form α and obtain a binding 1-form β , not necessarily contact, such that the associated symplectically spinning vector field X is 1-periodic near K . Hence the time-one-map of the flow of X gives us a diffeomorphism $\phi : F \rightarrow F$ such that

$$\phi^*(d\lambda) = d\lambda, \quad (5.8)$$

where $\lambda \in \Omega^1(F^o)$ is the ILF given by

$$\lambda := (\beta/|h|)|_{TF^o} = (\alpha/|h|)|_{TF^o} \quad (5.9)$$

and $\phi = \text{id}$ on some neighbourhood of K in F . Now our aim is to recover V as the abstract open book induced by the pair (F, ϕ) and to define a contact form on the abstract open book with the desired properties.

In order to construct the desired contact form, we first need a suitable contact form on $K = \partial F$. We note that $(K, \ker(\alpha|_{TK}))$ is a $(2n - 1)$ -dimensional closed connected contact manifold. In fact, since F is a Weinstein domain, see Theorem 5.5, and $\dim F \geq 4$, ∂F is connected. By the inductive hypothesis, there is a compact hypersurface $F' \subset K$ such that for a given $\epsilon' > 0$, there is contact form α' on $(K, \ker(\alpha|_{TK}))$ with the following properties.

- (F'1) α' restricts to a contact form on $K' := \partial F'$, for which the closed Reeb orbits have period at least $1/2$;
- (F'2) The first return time function

$$\tau' : F' \setminus \partial F' \rightarrow (0, +\infty)$$

of the Reeb flow of α' extends to a smooth function on F' and the corresponding first return map

$$\varphi' : F' \setminus \partial F' \rightarrow F' \setminus \partial F'$$

extends to a diffeomorphism of F' ;

- (F'3) There exists an open subset $U' \subset F'$ such that the support of φ' is contained in $F' \setminus U'$ and $\tau' \equiv 1$ on U' ;
- (F'4)

$$\|\tau' - 1\|_\infty < \min \left\{ \epsilon', \frac{\epsilon'}{\text{vol}(K', \alpha')} \right\};$$

- (F'5)

$$\int_{F' \setminus U'} (d\alpha')^{n-1} < \epsilon'.$$

For later convenience, we define the contact form

$$\alpha_0 := 2\pi\alpha' \quad (5.10)$$

on K .

Second we need a nice parametrization of F near K . By Lemma 5.2, we have an embedding

$$\iota : [0, +\infty) \times K \rightarrow F; \quad \iota^* \lambda = \alpha_0 / r, \quad r \neq 0 \quad (5.11)$$

and $\iota(0, q) = q$ for all $q \in K$.

Following the line of arguments in [2], we want to construct the desired contact form on the open book defined by the pair (F, ϕ) using a Liouville form on F^o , which has a particular behaviour near K . We cook up such a Liouville form out of λ as follows. Since $\phi = \text{id}$ near K , there is some $\rho' > 0$ such that $\rho' < 1$ and

$$\phi = \text{id} \text{ on } [0, \rho'] \times K. \quad (5.12)$$

Then there exist numbers $C > 0$, $\rho < \rho'$ and a smooth function

$$v : [0, \rho'] \rightarrow (0, +\infty)$$

such that the following holds:

- (v1) $v(r) = \frac{1}{r}$ near ρ' ;
- (v2) $v(r) = C(1 - r^2)$ for $r \leq \rho$;
- (v3) $v'(r) < 0$ for $r \in (0, \rho']$.

Finally we define the 1-form η on F via

$$\eta = \begin{cases} \lambda / C & \text{on } ([0, \rho'] \times K)^c \\ v(r)\alpha_0 / C & \text{on } [0, \rho'] \times K. \end{cases} \quad (5.13)$$

By (v1), η is a smooth 1-form and we note that on $(0, \rho') \times K$,

$$Cd\eta = v'dr \wedge \alpha_0 + v d\alpha_0 \Rightarrow C^n(d\eta)^n = (n-1)v'v^{n-1}dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}.$$

Hence by (v3), $d\eta$ is symplectic on $(0, \rho') \times K$ and therefore on $F \setminus K$.

Now we are ready to construct the open book associated to the pair (F, ϕ) and the required contact form. We first consider the mapping torus

$$MT(F, \phi) := ([0, 2\pi] \times F) / \sim; \quad (2\pi, q) \sim (0, \phi(q)).$$

Via (5.11) and (5.12), we have a neighborhood W of $\partial MT(F, \phi)$ in $MT(F, \phi)$ with the coordinates

$$(x, q, r) \in W := \mathbb{R}/2\pi\mathbb{Z} \times K \times [0, \rho] \quad (5.14)$$

and the open book is given by

$$OB(F, \phi) := (MT(F, \phi) \sqcup K \times \rho\mathbb{D}) / \sim; \quad (x, q, r) \sim (q, r e^{ix}). \quad (5.15)$$

Then the set

$$W' := K \times \rho \mathbb{D}$$

is a compact neighbourhood of $K = \{0\} \times K$ in $OB(F, \phi)$, containing $W \setminus \partial MT(F, \phi)$.

On $[0, 2\pi] \times F$, we define a family of 1-forms

$$\tilde{\alpha}_s = dx + s((1 - \beta(x))\eta + \beta(x)\phi^*\eta), \quad (5.16)$$

where s is a positive real parameter and $\beta : [0, 2\pi] \rightarrow [0, 1]$ is a smooth function such that $\beta(0) = 0$, $\beta(2\pi) = 1$ and $\text{supp}(\beta') \subset (0, 2\pi)$. By the choice of β , $\tilde{\alpha}_s$ descends to a family of 1-forms on $MT(F, \phi) \setminus \partial MT(F, \phi)$ and equivalently on $OB(F, \phi) \setminus K$. We note that on W , (5.16) reads as

$$\tilde{\alpha}_s = dx + s(1 - r^2)\alpha_0. \quad (5.17)$$

We fix $\delta > 0$ and, as in the proof of [2, Proposition 2.1], there exists $s_1 > 0$ depending on ρ and δ so that the following holds. For any $s \in (0, s_1)$, there is a curve

$$\gamma : [0, \rho] \rightarrow \mathbb{C}, \quad \gamma(r) = f(r) + i g(r)$$

with $f, g \geq 0$ satisfying:

(fg1) $\gamma(r) = 1 + is(1 - r^2)$ on $[r_1, \rho]$, for some $r_1 \in (0, \rho)$;

(fg2) $g' < 0$ on $(0, \rho]$;

(fg3) there exists some $r_0 \in (0, r_1)$ such that

$$\gamma(r) = r^2 + i(1 + \delta - r^2)$$

for $r \in [0, r_0]$ and

$$g(r_0) - g(\rho) = g(r_0) - s(1 - \rho^2) \leq 2\delta;$$

(fg4) $\frac{g'f - gf'}{g^2 + f^2} < 0$ on $(0, \rho]$;

(fg5) $\frac{g''f' - f''g'}{(g')^2 + (f')^2} \leq 0$ on $[0, \rho]$.

Now we define a family of smooth 1-forms on $OB(F, \phi)$ by

$$\alpha_s = \begin{cases} \frac{\tilde{\alpha}_s}{2\pi(1 + \delta)} & \text{on } OB(F, \phi) \setminus W' \\ \frac{f(r)dx + g(r)\alpha_0}{2\pi(1 + \delta)} & \text{on } W' \end{cases} \quad (5.18)$$

By (fg3) one has

$$\alpha_s = \frac{r^2 dx + (1 + \delta - r^2)\alpha_0}{2\pi(1 + \delta)},$$

near $K \subset W'$ so that α_s is a smooth 1-form on W' for all $s \in (0, s_1)$ and by (5.10),

$$\alpha_s = \frac{1}{2\pi} \alpha_0 = \alpha' \text{ on } K. \quad (5.19)$$

We note that for $\epsilon' < 1/2$, any closed Reeb orbit of α' that passes through $F' \setminus K'$ has period at least $1/2$ due to (F'4). Moreover, by (F'1), any closed Reeb orbit of α' that is contained in the contact submanifold $K' \subset K$ has period at least $1/2$. Hence for all $s \in (0, s_1)$ the condition (F1) is satisfied for α_s if $\epsilon' < 1/2$.

Lemma 5.10. *There exists $s_2 \in (0, s_1)$, depending on $\delta, \rho, \phi, \eta, \beta$ such that α_s is a contact form on $OB(F, \phi)$ for all $s \in (0, s_2)$.*

Proof. We first check the statement on W' . We compute

$$\begin{aligned} \alpha_s \wedge (d\alpha_s)^n &= \frac{1}{(2\pi(1+\delta))^{n+1}} [(f dx + g\alpha_0) \wedge (f' dr \wedge dx + g' dr \wedge \alpha_0 + g d\alpha_0)^n] \\ &= \frac{(n-1)g^{n-1}}{(2\pi(1+\delta))^{n+1}} [(f dx + g\alpha_0) \wedge (f' dr \wedge dx \wedge (d\alpha_0)^{n-1} \\ &\quad + g' dr \wedge d\alpha_0 \wedge (d\alpha_0)^{n-1})] \\ &= \frac{(n-1)g^{n-1}(fg' - gf')}{(2\pi(1+\delta))^{n+1}} (dx \wedge dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}) \\ &= \frac{(n-1)g^{n-1}(f'g - fg')}{r(2\pi(1+\delta))^{n+1}} ((r dr \wedge dx) \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}) \end{aligned}$$

Now for $r \neq 0, g > 0$ and by (fg4) we get

$$\frac{(n-1)g^{n-1}(f'g - fg')}{r(2\pi(1+\delta))^{n+1}} > 0.$$

For r close to 0, by (fg3), we get

$$\begin{aligned} \frac{(n-1)g^{n-1}(f'g - fg')}{r(2\pi(1+\delta))^{n+1}} &= \frac{(n-1)g^{n-1}(2r(1+\delta-r^2) - r^2(-2r))}{r(2\pi(1+\delta))^{n+1}} \\ &= \frac{(n-1)g^{n-1}2r(1+\delta)}{r(2\pi(1+\delta))^{n+1}} \\ &= \frac{2(n-1)(1+\delta-r^2)^{n-1}}{(2\pi)^{n+1}(1+\delta)^n}. \end{aligned}$$

We note that the limit of this expression is positive as r tends to 0. Hence α_s is a positive contact form for all $s \in (0, s_1)$.

Next we consider $OB(F, \phi) \setminus W'$. We have

$$\begin{aligned} 2\pi(1+\delta) d\alpha_s &= s(-\beta' dx \wedge \eta + (1-\beta)d\eta + \beta' dx \wedge \phi^* \eta + \beta \phi^* d\eta) \\ &= s(\beta' dx \wedge (h^* \eta - \eta) + (1-\beta)d\eta + \beta d\eta) \\ &= s(\beta' dx \wedge \eta_\Delta + d\eta) \end{aligned}$$

where $\eta_\Delta := \phi^*\eta - \eta$. The second equation follows from the fact that ϕ is a symplectomorphism. We get

$$(2\pi(1+\delta))^n (d\alpha_s)^n = s^n ((n-1)\beta' dx \wedge \eta_\Delta \wedge (d\eta)^{n-1} + (d\eta)^n).$$

Hence up to a positive constant we have

$$\begin{aligned} \frac{\alpha_s \wedge (d\alpha_s)^n}{s^n} &= [dx + s(\eta + \beta\eta_\Delta)] \wedge [(n-1)\beta' dx \wedge \eta_\Delta \wedge (d\eta)^{n-1} + (d\eta)^n] \\ &= dx \wedge (d\eta)^n + s(n-1)\beta'\eta \wedge dx \wedge \eta_\Delta \wedge (d\eta)^{n-1}. \end{aligned}$$

Now since $dx \wedge (d\eta)^n$ is a volume form, we can choose $s_2 \in (0, s_1)$ small enough so that α_s is a contact form on $OB(F, \phi) \setminus W'$ for all $s \in (0, s_2)$. $\square$

After identifying V with the abstract open book $OB(F, \phi)$, for each $s \in (0, s_2)$, we have two contact forms on $OB(F, \phi)$, namely the one induced by α on (V, ξ) , still denoted by α and α_s defined above. We note that the statement of the proposition is invariant under a diffeomorphism, hence it is enough to show that $F \subset OB(F, \phi)$ has the following property: given $\epsilon > 0$, there is a contact form α_s such that the statements (F1)-(F5) hold for α_s and $\ker \alpha_s$ isotopic to $\ker \alpha$.

In order to relate $\ker \alpha_s$ to $\ker \alpha$, we first want to show that the obvious open book structure on $OB(F, \phi)$ is a Liouville open book with the binding form α_s . Let

$$\hat{\Theta} : OB(F, \phi) \setminus K \rightarrow S^1$$

be the fibration induced by the projection $MT(F, \phi) \rightarrow S^1$. We pick a suitable defining function $\hat{h}$ as follows. We define a smooth function

$$\hat{u} : F \rightarrow [0, \infty)$$

such that for some suitably chosen $b > 0$:

- (û1) $\hat{u}(r, q) = \frac{r(1+\delta-r^2)}{2\pi(1+\delta)}$ for $(r, q) \in [0, \rho] \times K$ and r small enough;
- (û2) $\hat{u} > 0$ on $(0, \rho'] \times K$ and $\hat{u} \equiv b$ on $([0, \rho'] \times K)^c$;
- (û3) On $[0, \rho'] \times K$, $\hat{u}$ depends only on r and $\hat{u}_r \geq 0$.

We note that imposing (û1) for small enough r guarantees that $\hat{u} > 0$ for $r \neq 0$ and $\hat{u}_r \geq 0$. Hence (û1) is compatible with (û2) and (û3). Since $\hat{u}$ is constant on $\text{supp}(\phi)$, the S^1 -invariant extension of $\hat{u}$ is a well-defined smooth function on $MT(F, \phi)$. We define $\hat{h}$ on $MT(F, \phi)$ via $|\hat{h}| := \hat{u}$ and $\hat{h}/|\hat{h}| := \hat{\Theta}$. Note that $\hat{h}$ descends to a smooth function on $OB(F, \phi)$ since $r\hat{\Theta}$ and $1 + \delta - r^2$ are smooth functions on a neighbourhood of $K \times \{0\}$ in $K \times \rho\mathbb{D}$. Moreover $\hat{h}$ is a defining function for the open book $(K, \hat{\Theta})$ on $OB(F, \phi)$ by its very definition and the fact that $\hat{u}_r(0, q) > 0$ for all $q \in K$, compare with (df1).

Lemma 5.11. *For any $s \in (0, s_2)$, $d(\alpha_s/|\hat{h}|)$ induces an ideal Liouville structure on each fibre of $\hat{\Theta}$.*

Proof. We put

$$\lambda_x^s := (\alpha_s/|\hat{h}|)_{|TF_x^o}, \quad (5.20)$$

where $F_x = \hat{\Theta}^{-1}(x)$. We study λ_x^s on pieces of F_x separately.

- On $\{x\} \times (0, \rho] \times K$: we have

$$\lambda_x^s = \frac{g(r)\alpha_0}{2\pi(1+\delta)\hat{u}}. \quad (5.21)$$

Hence, up to positive constants, we get

$$\begin{aligned} d\lambda_x^s &= \frac{g'\hat{u} - g\hat{u}_r}{\hat{u}^2} dr \wedge \alpha_0 + \frac{g}{\hat{u}} d\alpha_0 \\ \Rightarrow (d\lambda_x^s)^n &= (n-1)g^{n-1} \frac{g'\hat{u} - g\hat{u}_r}{\hat{u}^{n+1}} dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}. \end{aligned}$$

We note that due to the parametrization (5.11), $dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}$ is a negative volume form. Combining (fg2), (û2), (û3) and the fact that $g > 0$, we get $g'\hat{u} - g\hat{u}_r < 0$. Hence $d\lambda_x^s$ is a positive symplectic form for all s .

- On $\{x\} \times (\rho, \rho') \times K$: we note that $\phi = \text{id}$ on this set. Hence, up to a positive constant, we have

$$\lambda_x^s = s \frac{\eta}{\hat{u}} \quad (5.22)$$

By (û3), we have

$$\begin{aligned} s^{-1}d\lambda_x^s &= -\frac{1}{\hat{u}^2}\hat{u}_r dr \wedge \eta + \frac{1}{\hat{u}} d\eta \\ \Rightarrow s^{-n}(d\lambda_x^s)^n &= -\frac{(n-1)}{\hat{u}^{n+1}}\hat{u}_r dr \wedge \eta \wedge (d\eta)^{n-1} + \frac{1}{\hat{u}^n} (d\eta)^n. \end{aligned}$$

We also note that

$$\begin{aligned} \eta &= \frac{v}{C}\alpha_0 \Rightarrow C(d\eta) = v' dr \wedge \alpha_0 + v d\alpha_0 \\ \Rightarrow C^{n-1}(d\eta)^{n-1} &= (n-2)v^{n-2}v' dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-2} + v^{n-1} (d\alpha_0)^{n-1} \\ \Rightarrow C^n(d\eta)^n &= (n-1)v^{n-1}v' dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1} \\ \Rightarrow C^n \eta \wedge (d\eta)^{n-1} &= v^n \alpha_0 \wedge (d\alpha_0)^{n-1}. \end{aligned}$$

Combining all these computations, we get

$$s^{-n} C^n (d\lambda_x^s)^n = \left[-\frac{(n-1)}{\hat{u}^{n+1}} \hat{u}_r v^n + \frac{1}{\hat{u}^n} (n-1) v^{n-1} v' \right] dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}.$$

By (v3), $v' < 0$ and by (û2), $\hat{u}_r \geq 0$. Hence λ_x^s is a positive symplectic form.

- On $\{x\} \times ((0, \rho') \times K)^c$: we have $\hat{u} \equiv b$ by (û2). Hence

$$\lambda_x^s = s \left[(1 - \beta(x)) \frac{\eta}{b} + \beta(x) \frac{\phi^* \eta}{b} \right] \Rightarrow d\lambda_x^s = s \frac{d\eta}{b}. \quad (5.23)$$

Since $d\lambda_x^s$ coincides with $d\lambda$ up to a positive constant, it is symplectic for all s . $\square$

Now we are in the following situation. On $OB(F, \phi)$, we have the Liouville open book

$$(K, \hat{\Theta}, d(\alpha/|h|)|_{TF_x^0}), \quad (5.24)$$

which symplectically supports the contact structure $\ker \alpha$ and for any $s \in (0, s_2)$ we have the second Liouville open book

$$(K, \hat{\Theta}, d(\alpha_s/|\hat{h}|)|_{TF_x^0}), \quad (5.25)$$

which symplectically supports the contact structure $\ker \alpha_s$. We want to show that there exists a diffeomorphism

$$\Phi : OB(F, \phi) \rightarrow OB(F, \phi) \quad (5.26)$$

such that $\Phi \circ \hat{\Theta} = \hat{\Theta} \circ \Phi$ and the restriction of Φ to each fibre is symplectic, that is, for all $x \in S^1$,

$$\Phi^* d(\alpha_s/|\hat{h}|)|_{TF_x^0} = d(\alpha/|h|)|_{TF_x^0}.$$

If such a diffeomorphism exists, then $\ker(\Phi_* \alpha_s)$ and $\ker \alpha$ are two contact structures on $OB(F, \phi)$, which symplectically support the Liouville open book (5.24). Hence they are isotopic by Proposition 5.8. Then it is enough to establish the dynamical properties given by (F1)-(F5) for F and α_s . Now our task is to construct the diffeomorphism Φ .

Lemma 5.12. *Let $s \in (0, s_2)$ be fixed. Then there exists a diffeomorphism*

$$\Phi : OB(F, \phi) \rightarrow OB(F, \phi) \quad (5.27)$$

such that $\Phi \circ \hat{\Theta} = \hat{\Theta} \circ \Phi$ and the restriction of Φ to each fibre is symplectic, that is, for all $x \in S^1$,

$$\Phi^* d(\alpha_s/|\hat{h}|)|_{TF_x^0} = d(\alpha/|h|)|_{TF_x^0}.$$

Proof. We first take a closer look at the setting. By the equations (5.21), (5.22) and (5.23), the ideal Liouville structures $(d(\alpha_s/|\hat{h}|)|_{TF_x^0})_{x \in S^1}$ may be identified with the ideal Liouville structure on the 0-th page, namely

$$\omega^s := d(\alpha_s/|\hat{h}|)|_{TF^0}. \quad (5.28)$$

Similarly, the symplectic form

$$\omega := d\lambda = d(\alpha/|h|)|_{TF^0} \quad (5.29)$$

gives the ideal Liouville structure on each page $\{x\} \times F \cong F$, after identifying V with $OB(F, \phi)$ via the flow of the symplectically spinning vector field X .

We first show that

$$\omega_t := (1-t)\omega + t\omega^s$$

is symplectic on F^o for all $t \in [0, 1]$. In fact, we claim that

$$\lambda_t = (1-t)\lambda + t\lambda^s, \quad t \in [0, 1] \quad (5.30)$$

is a Liouville form on F^o for all t , where λ is the primitive of ω given by (5.9) and λ^s is the primitive of ω^s given by (5.20). Again we compute $d\lambda_t$ on separate pieces of F^o .

- On $(0, \rho] \times K$: we have

$$\begin{aligned} \lambda_t &= (1-t)\frac{\alpha_0}{r} + t\frac{g(r)\alpha_0}{2\pi(1+\delta)\hat{u}(r)} = \left[(1-t)\frac{1}{r} + t\frac{g}{2\pi(1+\delta)\hat{u}} \right] \alpha_0 =: \kappa(r)\alpha_0 \\ &\Rightarrow d(\lambda_t) = \kappa' dr \wedge \alpha_0 + \kappa d\alpha_0 \\ &\Rightarrow (d\lambda_t)^n = (n-1)\kappa^{n-1}\kappa' dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}. \end{aligned}$$

Due to the parametrization (5.11), $dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}$ is a negative volume form. Hence $d\lambda_t$ is a positive symplectic form if and only if $\kappa' < 0$. We have

$$\Rightarrow \kappa' = -(1-t)\frac{1}{r^2} + \frac{t}{2\pi(1+\delta)\hat{u}^2}(g'\hat{u} - g\hat{u}_r) < 0,$$

since $g'\hat{u} - g\hat{u}_r < 0$, see the proof of Lemma 5.11.

- On $(\rho, \rho'] \times K$: (5.30) reads as

$$\begin{aligned} \lambda_t &= (1-t)\frac{1}{r}\alpha_0 + t\frac{sv}{C2\pi(1+\delta)\hat{u}}\alpha_0 \\ &= \left[(1-t)\frac{1}{r} + t\frac{sv}{C2\pi(1+\delta)\hat{u}} \right] \alpha_0 =: \kappa(r)\alpha_0. \end{aligned}$$

We have

$$\kappa' = -(1-t)\frac{1}{r^2} + \frac{st}{C2\pi(1+\delta)\hat{u}^2}(\hat{u}v' - v\hat{u}_r).$$

By (v3) and (û3), $\hat{u}v' - v\hat{u}_r < 0$ and therefore $\kappa' < 0$.

- On $((0, \rho'] \times K)^c$: by (5.23), we have

$$\lambda_t = (1-t)\lambda + ts\frac{\eta}{b} = \left[(1-t) + \frac{ts}{bC} \right] \lambda.$$

Since $d\lambda$ is positive symplectic and

$$(1-t) + \frac{ts}{bC} > 0,$$

$d\lambda_t$ is a positive symplectic form for all s and t .

Hence $\omega_t = d\lambda_t$ is symplectic on F^o for all $t \in [0, 1]$. Moreover by (fg3) and (û1), we have

$$\lambda^s = \frac{\alpha_s}{|\hat{h}|} = \frac{g(r)}{2\pi(1+\delta)\hat{u}}\alpha_0 = \frac{1+\delta-r^2}{2\pi(1+\delta)} \cdot \frac{2\pi(1+\delta)}{r(1+\delta-r^2)}\alpha_0 = \frac{1}{r}\alpha_0 = \lambda$$

on some neighbourhood of K in F . Hence after reparametrizing $t \in [0, 1]$ as $x \in [0, 2\pi]$, we get a smooth path of ideal Liouville structures $(\omega_x)_{x \in [0, 2\pi]}$ such that $\omega_0 = \omega$, $\omega_{2\pi} = \omega^s$ and the difference $\lambda - \lambda^s$ has compact support in F^0 . We apply the standard Moser argument and obtain an isotopy $(\psi_x)_{x \in [0, 2\pi]}$ of F such that:

- (i1) $\psi_0 = \text{id}$;
- (i2) $\psi_x = \text{id}$ near K for all $x \in [0, 2\pi]$;
- (i3) $\psi_x^*\omega_x = \omega_0 = \omega$ for all $x \in [0, 2\pi]$.

We remark that such an isotopy exists by Lemma 5.4 in the case where ω and ω^s do not necessarily coincide near K but just define the same boundary contact structure. In this case the isotopy in general just fixes K . But in our case it is better to apply Moser argument directly in order to get a isotopy, which is identity near K .

Now we define $\Phi : [0, 2\pi] \times F \rightarrow [0, 2\pi] \times F$,

$$\Phi(x, p) := (x, \psi_{2\pi} \circ \psi_x^{-1} \circ \phi^{-1} \circ \psi_x(p)). \quad (5.31)$$

We note that

$$\Phi(2\pi, p) = (2\pi, \psi_{2\pi} \circ \psi_{2\pi}^{-1} \circ \phi^{-1} \circ \psi_{2\pi}(p)) = (2\pi, \phi^{-1} \circ \psi_{2\pi}(p)),$$

and by (i1),

$$\begin{aligned} \Phi(0, \phi(p)) &= (0, \psi_{2\pi} \circ \psi_0^{-1} \circ \phi^{-1} \circ \psi_0(\phi(p))) \\ &= (0, \psi_{2\pi}(p)) = (0, \phi(\phi^{-1} \circ \psi_{2\pi}(p))). \end{aligned}$$

Hence Φ descends to a diffeomorphism on $MT(F, \phi)$. By (i2) and the fact that $\phi = \text{id}$ near K , $\Phi = \text{id}$ near $\partial MT(F, \phi)$. Hence Φ descends to a diffeomorphism

on $OB(F, \phi)$. By definition Φ commutes with $\hat{\Theta}$. It is clear that $\phi^* \omega_x = \omega_x$ for all $x \in [0, 2\pi]$. Hence by (i3), we get

$$\begin{aligned}
 \Phi^*(\alpha_s/|\hat{h}|)|_{TF_x^0} &= \Phi^* \omega^s \\
 &= (\psi_{2\pi} \circ \psi_x^{-1} \circ \phi^{-1} \circ \psi_x)^* \omega^s \\
 &= \psi_x^*(\phi^{-1})^*(\psi_x^{-1})^* \psi_{2\pi}^* \omega_{2\pi} \\
 &= \psi_x^*(\phi^{-1})^*(\psi_x^{-1})^* \omega_0 \\
 &= \psi_x^*(\phi^{-1})^* \omega_x \\
 &= \psi_x^* \omega_x \\
 &= \omega_0 = \omega = d(\alpha/|h|)|_{TF_x^0}. \quad \square
 \end{aligned}$$

Now our task is to establish that for a given $\epsilon > 0$ the statements (F1)-(F5) hold for some α_s . To this end we first study the Reeb vector field R_s of α_s on $OB(F, \phi) \setminus W'$. We know that $d\eta$ is well-defined on $OB(\phi) \setminus W'$ and restricts to a symplectic form on each fibre F_x^0 . We define the vector field Y on $OB(\phi) \setminus W'$ so that it is tangent to F_x for each x and satisfies

$$\iota_Y d\eta = -\beta' \eta_\Delta$$

along F_x for each x . Since $\text{supp}(\phi) \subset ([0, \rho] \times K)^c$, Y is compactly supported in $OB(\phi) \setminus W'$. We note that

$$\begin{aligned}
 2\pi(1 + \delta)\iota_{(\partial_x + Y)} d\alpha_s &= s(\iota_{(\partial_x + Y)} \beta' dx \wedge \eta_\Delta + \iota_{(\partial_x + Y)} d\eta) \\
 &= s(\beta' dx (\partial_x + Y) \eta_\Delta - \eta_\Delta (\partial_x + Y) \beta' dx + \iota_Y d\eta) \\
 &= s(\beta' \eta_\Delta + d\eta(Y, Y) \beta' dx - \beta' \eta_\Delta) \\
 &= 0.
 \end{aligned}$$

Hence on $OB(\phi) \setminus W'$, the Reeb vector field of α_s reads as

$$R_s = \frac{\partial_x + Y}{\alpha_s(\partial_x + Y)}. \quad (5.32)$$

We note that since Y is supported in $OB(\phi) \setminus W'$ and $OB(\phi) \setminus W'$ is invariant under the flow of ∂_x , $OB(\phi) \setminus W'$ is invariant under the flow of R_s and hence W' is invariant as well. Moreover the ∂_x component of R_s never vanishes and since Y is tangent F_x , R_s is transverse to $F_x \setminus W'$ for all x .

On W' , by (5.18) we have

$$R_s = 2\pi(1 + \delta) \frac{f' R_0 - g' \partial_x}{f' g - f g'}, \quad (5.33)$$

where R_0 is the Reeb vector field of α_0 on K . We note that since $g' < 0$ on $(0, \rho]$, ∂_x component of R_s does not vanish hence R_s is transverse to F_x for all x . For r close to 0, by (fg3) we get

$$R_s = 2\pi(R_0 + \partial_x) \quad (5.34)$$

and as ∂_x vanishes on K ,

$$R_s = 2\pi R_0 \quad (5.35)$$

on K . Since R_s leaves K invariant and have a non-vanishing ∂_x component, $F = F_0$ is a global hypersurface of section for the flow of R_s and we have well-defined first return time function and first return map, namely

$$\tau : F \setminus K \rightarrow (0, \infty), \quad \varphi : F \setminus K \rightarrow F \setminus K.$$

We compute the flow of R_s on $W' \setminus K$ via solving the system

$$\dot{r} = 0, \quad \dot{x} = \frac{-g'2\pi(1+\delta)}{f'g - fg'}, \quad \dot{q} = \frac{f'2\pi(1+\delta)}{f'g - fg'} R_0(q)$$

in the coordinates given by (5.14). The components of the flow reads as

$$r(t) = r, \quad x(t) = x - \frac{g'2\pi(1+\delta)}{f'g - fg'} t, \quad q(t) = \phi_{R_0}^{\frac{f'2\pi(1+\delta)}{f'g - fg'} t}(q)$$

where $\phi_{R_0}^s$ is the flow of R_0 on K . Hence we have

$$\tau(r, q) = \frac{-f'g + fg'}{g'(1+\delta)}, \quad \varphi(r, q) = (r, \phi_{R_0}^{-2\pi \frac{f'}{g'}}(q)). \quad (5.36)$$

Hence for $r \leq r_0$, by (fg3), we have

$$\tau = 1, \quad \varphi(r, q) = (r, \phi_{R_0}^{2\pi}(q)). \quad (5.37)$$

Hence τ and φ extend smoothly to K , that is, the statement (F2) is satisfied. We note that

$$2\pi R_0 = R'$$

where R' is the Reeb vector field of α' on K . Now we invoke our assumptions on the contact form α' on K . By (F'3), we have a circle bundle over the open subset $U' \subset F'$, which is given by the closed Reeb orbits of α' . Let V' be the total space of this circle bundle. Then V' is an open subset of K and $\phi_{R'}^1(q) = q$ for all $q \in V'$. Hence we have $\phi_{R_0}^{2\pi}(q) = q$ for all $q \in V'$. We define the open subset

$$U := (0, r_0) \times V' \subset F. \quad (5.38)$$

Then (5.37) leads to

$$\tau|_U = 1, \quad \varphi|_U = \text{id}, \quad (5.39)$$

which establishes (F3).

Now we want to show that suitable choices of ϵ' , δ and $s > 0$ lead to (F4) and (F5).

We first observe the following.

Lemma 5.13. *There exists $s_3 \in (0, s_2)$ such that for all $s \in (0, s_3)$,*

$$\|\tau - 1\|_\infty < 2\delta.$$

Proof. By (5.36), τ depends only on r on $(0, \rho] \times K$ and by (fg5), we have

$$\frac{\partial \tau}{\partial r} = \frac{g(f'g'' - f''g')}{(g')^2(1 + \delta)} \geq 0.$$

For any s , by (fg1) we have

$$\tau = \frac{1}{1 + \delta}$$

near ρ . Hence for all s

$$\frac{1}{1 + \delta} \leq \tau \leq 1$$

on $[0, \rho] \times K$. Hence

$$\sup_{[0, \rho] \times K} |\tau - 1| \leq \frac{1}{1 + \delta} - 1 \leq \delta$$

for any $s \in (0, s_2)$.

We note that on $OB(F, \phi) \setminus W'$, we get

$$dx(R_s) = \frac{1}{\alpha_s(\partial_x + Y)}$$

using (5.32) and the fact that Y is tangent to the fibres of $OB(F, \phi)$. We note that as $s \rightarrow 0$

$$\alpha_s(\partial_x + Y) = \frac{1}{2\pi(1 + \delta)} \tilde{\alpha}_s(\partial_x + Y) = \frac{1}{2\pi(1 + \delta)} \left(1 + s((1 - \beta)\eta(Y) + \beta(\phi^*\eta)(Y)) \right)$$

converges to $\frac{1}{2\pi(1 + \delta)}$ uniformly on $OB(F, \phi) \setminus W'$ since Y is compactly supported. Hence τ converges to $\frac{1}{(1 + \delta)}$ uniformly as $s \rightarrow 0$. Combining this with the above estimates proves the lemma. $\square$

Now we want to estimate the $d\alpha_s$ -volume of the complement of U in F .

Lemma 5.14. *There exists $s_4 \in (0, s_3)$ such that for all $s \in (0, s_4)$,*

$$\int_{([0, \rho] \times K)^c} (d\alpha_s)^n < 2\delta.$$

Proof. We recall from the proof of Lemma 5.10 that on $([0, \rho] \times K)^c$,

$$2\pi(1 + \delta)(d\alpha_s)|_{TF} = s(\beta' dx \wedge \eta_\Delta + d\eta)|_{TF} = s d\eta.$$

Hence the integral

$$\int_{([0, \rho] \times K)^c} (d\alpha_s)^n \searrow 0 \text{ as } s \rightarrow 0. \quad \square$$

Next we consider the region $[r_0, \rho] \times K$. We have

$$\begin{aligned} & 2\pi(1+\delta)(d\alpha_s)|_{TF} \\ &= (f' dr \wedge dx + (g' dr \wedge \alpha_0 + g d\alpha_0))|_{TF} = g' dr \wedge \alpha_0 + g d\alpha_0 \\ &\Rightarrow (2\pi(1+\delta))(d\alpha_s)|_{TF}^n = (n-1)g' g^{n-1} dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}. \end{aligned}$$

We note that $dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1}$ is a negative volume form due to the parametrization (5.11). Hence

$$\begin{aligned} \int_{([r_0, \rho] \times K)} (d\alpha_s)^n &= \frac{n-1}{(2\pi)^n(1+\delta)^n} \int_{([r_0, \rho] \times K)} g' g^{n-1} dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1} \\ &= \frac{n-1}{(2\pi)^n(1+\delta)^n} \text{vol}(K, \alpha_0) \int_{r_0}^{\rho} (-g' g^{n-1}) dr \\ &\leq \frac{n-1}{(2\pi)^n(1+\delta)^n} \text{vol}(K, \alpha_0) (1+\delta)^{n-1} \int_{r_0}^{\rho} (-g') dr \\ &= \frac{n-1}{(2\pi)^n(1+\delta)} \text{vol}(K, \alpha_0) (g(r_0) - g(\rho)) \\ &\leq \frac{n-1}{(2\pi)^n} \text{vol}(K, \alpha_0) 2\delta \\ &= (n-1) \text{vol}(K, \alpha') 2\delta. \end{aligned}$$

The remaining piece of F is the subset $(0, r_0) \times (K \setminus V')$. We note that

$$\begin{aligned} & \int_{(0, r_0) \times (K \setminus V')} (d\alpha_s)^n \\ &= \frac{1}{(2\pi)^n(1+\delta)^n} \int_{(0, r_0) \times (K \setminus V')} \left(2r dr \wedge dx - 2r dr \wedge \alpha_0 + (1+\delta-r^2) d\alpha_0 \right)^n \\ &= \frac{(n-1)}{(2\pi)^n(1+\delta)^n} \int_{(0, r_0) \times (K \setminus V')} -2r(1+\delta-r^2)^{n-1} dr \wedge \alpha_0 \wedge (d\alpha_0)^{n-1} \\ &= \frac{(n-1)}{(2\pi)^n(1+\delta)^n} \text{vol}(K \setminus V', \alpha_0) \int_0^{r_0} 2r(1+\delta-r^2)^{n-1} dr \\ &= \frac{(n-1)}{n(2\pi)^n(1+\delta)^n} \text{vol}(K \setminus V', \alpha_0) (1+\delta-r^2)^n \Big|_{r_0}^0 \\ &\leq \frac{(n-1)}{n(2\pi)^n(1+\delta)^n} \text{vol}(K \setminus V', \alpha_0) (1+\delta)^n \leq \frac{\text{vol}(K \setminus V', \alpha_0)}{(2\pi)^n} \\ &= \text{vol}(K \setminus V', \alpha'). \end{aligned}$$

In order to estimate the above volume, we use (F'4), (F'5) and get

$$\text{vol}(K \setminus V', \alpha') = \int_{F' \setminus U'} \tau' (d\alpha')^{n-1} \leq (1 + \epsilon')\epsilon'$$

and hence

$$\int_{(0, r_0) \times (K \setminus V')} (d\alpha_s)^n \leq 2\epsilon' \quad (5.40)$$

if $\epsilon' < 1/2$.

Now given $\epsilon > 0$, we first choose $\epsilon' > 0$ such that

$$\epsilon' < \min\{\epsilon/6, 1/2\}. \quad (5.41)$$

Then we get the contact form α' on K satisfying (F'1)-(F'5) with respect to ϵ' . Then using the definition (5.10) and the parametrization (5.11), we construct α_s , where we pick $\delta > 0$ satisfying

$$\delta < \min\left\{\frac{\epsilon}{6(n-1)\text{vol}(K, \alpha')}, \frac{\epsilon}{6}\right\}. \quad (5.42)$$

We note that once $\epsilon' > 0$ is fixed, the 1-form α' on K is fixed.

By Lemma 5.14, there is some $s_4 \in (0, s_3)$ such that for all $s \in (0, s_4)$,

$$\begin{aligned} \int_{F \setminus U} (d\alpha_s)^n &\leq \int_{([0, \rho) \times K)^c} (d\alpha_s)^n + \int_{([r_0, \rho) \times K)} (d\alpha_s)^n + \int_{(0, r_0) \times (K \setminus V')} (d\alpha_s)^n \\ &\leq 2\delta + (n-1)\text{vol}(K, \alpha')2\delta + 2\epsilon' \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

where the second line follows from the above estimates and the last line follows from (5.41) and (5.42). This establishes (F5). We note that the estimate (5.42) implies

$$2\delta < \min\left\{\epsilon, \frac{\epsilon}{\text{vol}(K, \alpha_s)}\right\}.$$

Hence from Lemma 5.13, it follows that for any $s \in (0, s_4)$,

$$\|\tau - 1\|_\infty < \min\left\{\epsilon, \frac{\epsilon}{\text{vol}(K, \alpha_s)}\right\}.$$

Hence the statement (F4) holds. $\square$

Now we are ready to prove our main theorem.

Theorem 5.15. *Let (V, ξ) be a closed connected co-oriented contact manifold and let $C > 0$ be given. Then there exists a contact form α on V such that $\ker \alpha = \xi$ and*

$$\rho(V, \alpha) \geq C.$$

Proof. Let $F \subset V$ be the hypersurface with $\partial F = K$ given in Proposition 5.9 and let $\epsilon > 0$ be given. Then there is a contact form α on (V, ξ) such that the statements (F1)-(F5) hold. Let $\dim V = 2m + 1$.

We put

$$c := \int_F (d\alpha)^m = \int_K \alpha \wedge (d\alpha)^{m-1}, \quad a := \int_U (d\alpha)^m$$

so that by (F5),

$$c - a < \epsilon.$$

Let $\epsilon' > 0$ be given. We claim that there is a finite collection of smooth embeddings

$$\psi_j : r_j \mathbb{B} \rightarrow U, \quad \psi_j^* \omega = d\alpha; \quad j = 1, \dots, k,$$

where $\mathbb{B}$ is the open unit ball of dimension $2m$ and ω is the standard symplectic form, such that the images of these embeddings are mutually disjoint and

$$\sum_j \text{vol}(r_j \mathbb{B}, \omega) = (1 - \epsilon')a.$$

We note that the open subset U is precompact in F . So we can take a finite open cover of U consisting of Darboux coordinate charts. After making this finite collection mutually disjoint, it is enough to fill each corresponding open subset of $\mathbb{R}^{2m}$ with balls of arbitrary center and radius so that the total volume is $(1 - \epsilon')a$.

Using (F2) and (F3), we define the embeddings

$$\Psi_j : \mathbb{R}/\mathbb{Z} \times r_j \mathbb{B} \rightarrow V$$

such that $\Psi_j(0, \cdot) = \psi_j$ and $\Psi_j^* R_\alpha = \partial_s$ where $s \in \mathbb{R}/\mathbb{Z}$. We claim that

$$\alpha_j := \Psi_j^* \alpha = \lambda_j + ds,$$

where λ_j is a primitive of ω on $r_j \mathbb{B}$. By construction, ∂_s is the Reeb vector field of α_j . Hence α_j is invariant under the translation along the s coordinate and has the form $\alpha_j = \lambda_j + u ds$ where λ_j is a 1-form on $r_j \mathbb{B}$ and u is a function on $r_j \mathbb{B}$. Since $\alpha_j(\partial_s) = 1$ one has $u \equiv 1$ and again by construction

$$\omega = \psi_j^* d\alpha = \Psi_j^* d\alpha|_{r_j \mathbb{B} \times \{0\}} = d\alpha_j|_{r_j \mathbb{B} \times \{0\}} = d\lambda_j.$$

We denote the image of Ψ_j by W_j . We have

$$\begin{aligned} \sum_j \text{vol}(W_j, \alpha) &= \sum_j \text{vol}(\mathbb{R}/\mathbb{Z} \times r_j \mathbb{B}, \alpha_j) \\ &= \sum_j \text{vol}(\mathbb{R}/\mathbb{Z} \times r_j \mathbb{B}, \lambda_j + ds) \\ &= \sum_j \text{vol}(r_j \mathbb{B}, \omega) \\ &= (1 - \epsilon')a. \end{aligned}$$

Using (F4), we get

$$\begin{aligned}\tau &\leq 1 + \frac{\epsilon}{c} \\ \Rightarrow \operatorname{vol}(V, \alpha) &= \int_F \tau (d\alpha)^m \leq \left(1 + \frac{\epsilon}{c}\right) c = c + \epsilon.\end{aligned}$$

Using the fact that W_j 's are disjoint, we get

$$\begin{aligned}\operatorname{vol}(V \setminus \bigcup_j W_j, \alpha) &= \operatorname{vol}(V, \alpha) - \sum_j \operatorname{vol}(W_j, \alpha) \\ &\leq (c + \epsilon) - (1 - \epsilon')a \\ &= (c - a) + \epsilon + \epsilon'a \\ &\leq \epsilon + \epsilon + \epsilon'a \\ &= 2\epsilon + \epsilon'a.\end{aligned}$$

Now for each j , we apply Lemma 5.1, with $r = r_j$, $\lambda' = \lambda_j$ and get the contact form β_j on $r_j\mathbb{B} \times \mathbb{R}/\mathbb{Z}$ satisfying (p'1)-(p'3) and

$$\operatorname{vol}(r_j\mathbb{B} \times \mathbb{R}/\mathbb{Z}, \beta_j) < \frac{\epsilon'}{k}.$$

We define $\alpha_j = \Psi_j^* \beta_j$ on W_j . By construction, α_j 's fit α near the boundary of W_j and define a contact form $\hat{\alpha}$ on V . Since $\hat{\alpha}$ is isotopic to α , $\ker \hat{\alpha}$ is diffeomorphic to $\ker \tilde{\alpha}$. We have the estimate

$$\begin{aligned}\operatorname{vol}(V, \hat{\alpha}) &= \operatorname{vol}(V \setminus \bigcup_j W_j, \hat{\alpha}) + \sum_j \operatorname{vol}(W_j, \hat{\alpha}) \\ &= \operatorname{vol}(V \setminus \bigcup_j W_j, \alpha) + \sum_j \operatorname{vol}(W_j, \beta_j) \\ &\leq 2\epsilon + \epsilon'a + \epsilon' \\ &= 2\epsilon + \epsilon'(a + 1).\end{aligned}$$

We note that the open subsets W_j are invariant under the Reeb flow of $\hat{\alpha}$. The closed orbits that are contained in W_j 's have period at least 1 due to the construction of contact forms $\{\beta_j\}$. Due to (F1), closed orbits of $\hat{\alpha}$, which are contained in K have period at least $1/2$. Taking $\epsilon < 1/2$, the remaining closed orbits have period at least $1/2$ due to (F4). Hence we get

$$T_{\min}(\hat{\alpha}) \geq 1/2$$

and therefore

$$\rho(V, \hat{\alpha}) = \frac{(T_{\min}(\hat{\alpha}))^{m+1}}{\operatorname{vol}(V, \hat{\alpha})} \geq \frac{(1/2)^{m+1}}{2\epsilon + \epsilon'(a + 1)}.$$

Now given $C > 0$, taking ϵ and ϵ' small enough yields

$$\rho(V, \hat{\alpha}) \geq C.$$

We note that once ϵ is chosen, a is fixed and one may choose ϵ' for the given a . $\square$

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