

On the rationality problem for forms of moduli spaces of stable marked curves of positive genus

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Abstract. Let $M_{g,n}$ (respectively, $\overline{M}_{g,n}$) be the moduli space of smooth (respectively stable) curves of genus g with n marked points. Over the field of complex numbers, it is a classical problem in algebraic geometry to determine whether or not $M_{g,n}$ (or equivalently, $\overline{M}_{g,n}$) is a rational variety. Theorems of J. Harris, D. Mumford, D. Eisenbud and G. Farkas assert that $M_{g,n}$ is not even unirational for any $n \geq 0$ if $g \geq 22$. Moreover, P. Belorousski and A. Logan showed that $M_{g,n}$ is unirational for only finitely many pairs (g, n) with $g \geq 1$. Finding the precise range of pairs (g, n) , where $M_{g,n}$ is rational, stably rational or unirational, is a problem of ongoing interest.

In this paper we address the rationality problem for twisted forms of $\overline{M}_{g,n}$ defined over an arbitrary field F of characteristic $\neq 2$. We show that all F -forms of $\overline{M}_{g,n}$ are stably rational for $g = 1$ and $3 \leq n \leq 4$, for $g = 2$ and $2 \leq n \leq 3$, for $g = 3$ and $1 \leq n \leq 14$, $g = 4$ and $1 \leq n \leq 9$, and for $g = 5$ and $1 \leq n \leq 12$.

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1. Introduction

Let $M_{g,n}$ (respectively, $\overline{M}_{g,n}$) be the moduli space of smooth (respectively, stable) curves of genus g with n marked points. Recall that these moduli spaces are defined over the prime field ($\mathbb{Q}$ in characteristic zero and $\mathbb{F}_p$ in characteristic p). The purpose of this paper is to address the rationality problem for twisted forms of $\overline{M}_{g,n}$. Recall that a *form* of a scheme X defined over a field F is another scheme Y , also defined over F , such that X and Y become isomorphic over the separable closure F^{sep} . We will use the terms “form”, “twisted form” and “ F -form” interchangeably. Forms of $\overline{M}_{g,n}$ are of interest because they shed light on the arithmetic geometry

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of $\overline{M}_{g,n}$, and because they are coarse moduli spaces for natural moduli problems in their own right; see [9, Remark 2.4].

This paper is a sequel to [9], where two of us considered twisted forms of $\overline{M}_{0,n}$. The main results of [9] can be summarized as follows:

Theorem 1.1. *Let F be a field of characteristic $\neq 2$ and $n \geq 5$ be an integer. Then:*

(a) *All F -forms of $\overline{M}_{0,n}$ are unirational;*

(b) *If n is odd, all F -forms of $\overline{M}_{0,n}$ are rational;*

(c) *If n is even, then there exist fields E/F and E -forms of $\overline{M}_{0,n}$ that are not stably rational (or even retract rational) over E .*

In the present paper we will study the rationality problem for forms of $\overline{M}_{g,n}$ in the case where $g \geq 1$. Here the rationality problem for the usual (split) moduli space $\overline{M}_{g,n}$ (or equivalently, for $M_{g,n}$) over the field of complex numbers is already highly non-trivial. Theorems of J. Harris, D. Mumford, D. Eisenbud [6, 11] and G. Farkas [8] assert that if $g \geq 22$, then $M_{g,0}$ is not unirational (and hence, neither is $M_{g,n}$ for any $n \geq 0$). Moreover, work of P. Belorousski [2] (for $g = 1$) and A. Logan [15] (for $g \geq 2$) tells us that $M_{g,n}$ is unirational for only finitely many pairs (g, n) with $g \geq 1$. Finding the precise range of pairs (g, n) , where $M_{g,n}$ is rational, stably rational or unirational, is a problem of ongoing interest. In particular, over $\mathbb{C}$, $M_{g,n}$ is known to be rational for $1 \leq n \leq r_g$ and not unirational for $n \geq n_g$, where

g	1	2	3	4	5
r_g	10	12	14	15	12
n_g	11	-	-	16	15

see [15] and [4]. Surprisingly, we have not been able to find specific values for n_2 and n_3 in the literature, even though Logan showed that they exist; see [15, Theorem 2.4]. The main result of the present paper is as follows:

Theorem 1.2. *Let F be a field of characteristic $\neq 2$. Then every F -form of $\overline{M}_{g,n}$ is stably rational over F if*

$g = 1$ and $3 \leq n \leq 4$,

$g = 2$ and $2 \leq n \leq 3$,

$g = 3$ and $1 \leq n \leq 14$,

$g = 4$ and $1 \leq n \leq 9$,

$g = 5$ and $1 \leq n \leq 12$.

Several remarks are in order.

- (1) Stable rationality of every form of $\overline{M}_{g,n}$ is a priori much stronger than stable rationality of $\overline{M}_{g,n}$ itself. For example, $\overline{M}_{1,1} \simeq \mathbb{P}^1$ is rational, but its forms are conic curves which are not unirational in general;

- (2) Theorem 1.2 also holds for $(g, n) = (1, 2)$ (respectively, $(2, 1)$), provided $\text{char}(F) = 0$ (respectively, $\text{char}(F) \neq 2, 3$); see Remark 2.6;
- (3) By [5, Theorem 6.1(b)], every F -form of $\overline{M}_{1,n}$ is unirational for $3 \leq n \leq 9$;
- (4) The situation we encountered in Theorem 1.1(c), where some forms of $\overline{M}_{0,n}$ are stably rational and others are not, does not arise for any of the pairs (g, n) covered by Theorem 1.2. We do not know if it arises for any pair (g, n) with $g \geq 1$ and $2g + n \geq 5$.

A proof of Theorem 1.2 is outlined in Section 3 and completed in Sections 4 and 5. Our arguments rely on a theorem of B. Fantechi and A. Massarenti [7] describing the automorphism group of $\overline{M}_{g,n}$; see Section 2.5.

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2. Preliminaries

All algebraic groups in this paper will be assumed to be affine, and all algebraic varieties to be quasi-projective.

2.1. Twisting

Let G be an algebraic group defined over a field F , X be an F -variety endowed with a (left) G -action, and $P \rightarrow \text{Spec}(F)$ be a (right) G -torsor. The twisted variety ${}^P X$ is defined as ${}^P X := (P \times X)/G$, where G acts on $P \times X$ by $g: (p, x) \rightarrow (p \cdot g^{-1}, g \cdot x)$. Here $P \times X$ is, in fact, a G -torsor over $(P \times X)/G$; in particular, $P \times X \rightarrow (P \times X)/G$ is a geometric quotient. A G -equivariant morphism of F -varieties $f: X \rightarrow Y$ gives rise to a G -equivariant morphism $\text{id} \times f: P \times X \rightarrow P \times Y$ which descends to an F -morphism ${}^P f: {}^P X \rightarrow {}^P Y$. Similarly a G -equivariant rational map $f: X \dashrightarrow Y$ of F -varieties induces a rational map ${}^P f: {}^P X \dashrightarrow {}^P Y$. Some basic properties of the twisting operation are summarized in Lemma 2.1 below; see also [10, Section 2] or [5, Section 3].

Lemma 2.1. *Let G be an algebraic group defined over a field F , $f: X \rightarrow Y$ and $f': X' \rightarrow Y$ be G -equivariant morphisms of F -varieties, and $P \rightarrow \text{Spec}(F)$ be a G -torsor.*

- (a) *If f is an open (respectively, closed) immersion, then so is ${}^P f$;*
- (b) *If f is a dominant morphism (respectively, an isomorphism or a birational isomorphism), then so is ${}^P f$;*
- (c) *If f is a vector bundle of rank r , then so is ${}^P f$. In particular, ${}^P X$ is rational over ${}^P Y$;*

- (d) ${}^P(X \times_Y X')$ is isomorphic to ${}^P X \times_{{}^P Y} {}^P(X')$ over ${}^P Y$;
- (e) Moreover, if f and f' are vector bundles, then ${}^P(X \times_Y X')$ and ${}^P X \times_{{}^P Y} {}^P(X')$ are isomorphic as vector bundles over ${}^P Y$;
- (f) If f is a vector bundle of rank r , then the twisted Grassmannian bundle ${}^P \mathrm{Gr}(m, X) \simeq \mathrm{Gr}(m, {}^P X)$ is rational over ${}^P Y$ for any $1 \leq m \leq r - 1$. In particular, ${}^P \mathbb{P}(X) \simeq \mathbb{P}({}^P X)$ is rational over ${}^P Y$.

Here when we say that f is a vector bundle, we are assuming that G acts on X by vector bundle automorphisms (and similarly for f'). That is, for any $g \in G$ and $y \in Y$, g restricts to a linear map between the fibers $f^{-1}(y)$ and $f^{-1}(g(y))$.

Proof. For a proof of (a) and (b), see [5, Corollary 3.4].

(c) The first assertion is a consequence of Hilbert's Theorem 90. The second assertion follows from the first, since the vector bundle ${}^P f: {}^P X \rightarrow {}^P Y$ becomes trivial after passing to some dense Zariski open subset of ${}^P Y$.

(d) The morphism $\phi: P \times (X \times_Y X') \rightarrow (P \times X) \times_Y (P \times X')$ over Y given by $(p, x, x') \mapsto ((p, x), (p, x'))$ descends to a morphism $\bar{\phi}: {}^P(X \times_Y X') \rightarrow {}^P X \times_{{}^P Y} {}^P(X')$ over ${}^P Y$. Here the unmarked direct products are assumed to be over $\mathrm{Spec}(F)$. To show that $\bar{\phi}$ is an isomorphism, we may pass to a splitting field F'/F for P . Over F' , the G -torsor $P \rightarrow \mathrm{Spec}(F)$ becomes split, i.e., $P_{F'} \simeq G_{F'}$. Thus over F' , the morphism $P \times X \simeq G \times X \rightarrow X$ given by $(g, x) \mapsto g \cdot x$ is a G -torsor. This yields a natural isomorphism between ${}^P X$ and X . Identifying ${}^P(X')$ with X' , ${}^P Y$ with Y , and ${}^P(X \times_Y X')$ with $X \times_Y X'$ in a similar manner, we see that over F' , $\bar{\phi}$ becomes the identity map $X \times_Y X' \rightarrow X \times_Y X'$. Hence, $\bar{\phi}$ is an isomorphism over F .

(e) To check that $\bar{\phi}$ is an isomorphism of vector bundles over ${}^P Y$, we may, once again, pass to a splitting field F'/F for P . In the proof of part (d), we identified ${}^P(X')$ with X' and ${}^P(X \times_Y X')$ with $X \times_Y X'$ after passing to F' . Now we observe that these identifications are, in fact, isomorphisms of vector bundles over ${}^P Y$ (which we identified with Y). Modulo these identifications, $\bar{\phi}$ is the identity map $X \times_Y X' \rightarrow X \times_Y X'$, and part (e) follows.

(f) The second assertion follows from the first by setting $m = 1$.

Rationality of $\mathrm{Gr}(m, {}^P X)$ over ${}^P Y$ follows from the fact that any vector bundle, and in particular the vector bundle ${}^P f: {}^P X \rightarrow {}^P Y$, is locally trivial in the Zariski topology.

To show that ${}^P \mathrm{Gr}(m, X)$ is isomorphic to $\mathrm{Gr}(m, {}^P X)$ over ${}^P Y$, recall that $\mathrm{Gr}(m, X)$ is the quotient of the dense open subset $(X^m)_0$ of the n -fold fibered product $X^m = X \times_Y \cdots \times_Y X$ consisting of linearly independent m -tuples by the group GL_m (over Y). To construct ${}^P \mathrm{Gr}(m, X)$, we proceed as follows. First take the quotient of the product $P \times (X^m)_0$ by the action of GL_m . This action is trivial on the first factor, so we obtain $P \times \mathrm{Gr}(m, X)$. Now take the quotient of $P \times \mathrm{Gr}(m, X)$ by G to arrive at ${}^P \mathrm{Gr}(m, X)$.

To construct $\mathrm{Gr}(m, {}^P X)$, we also start with $P \times (X^m)_0$ and take the quotients by the same groups, but in reverse order. First we take the quotient of $P \times (X^m)_0$

by G to obtain ${}^P((X^m)_0) \simeq (({}^P X)^m)_0$ (see parts (d) and (e)); the quotient of $(({}^P X)^m)_0$ by GL_m is $\mathrm{Gr}(m, {}^P X)$. Since the actions of GL_m and G on ${}^P X \times_F (X^m)_0$ commute, we conclude that ${}^P \mathrm{Gr}(m, X)$ and $\mathrm{Gr}(m, {}^P X)$ are isomorphic over ${}^P Y$. $\square$

The F -forms of a variety X are in a natural bijective correspondence with $H^1(F, \mathrm{Aut}(X))$. Here $\mathrm{Aut}(X)$ is a functor which associates to a scheme S/F the abstract group $\mathrm{Aut}(X_S)$. In general this functor is not representable by an algebraic group defined over F . If it is, one usually says that $\mathrm{Aut}(X)$ is an algebraic group. In this case the bijective correspondence between $H^1(F, \mathrm{Aut}(X))$ (which may be viewed as a set of $\mathrm{Aut}(X)$ -torsors $P \rightarrow \mathrm{Spec}(F)$) and the set of F -forms of X (up to F -isomorphism) can be described explicitly as follows. An $\mathrm{Aut}(X)$ -torsor $P \rightarrow \mathrm{Spec}(F)$ corresponds to the twisted variety ${}^P X$, and a twisted form Y of X corresponds to the isomorphism scheme $P = \mathrm{Isom}_F(X, Y)$, which is naturally an $\mathrm{Aut}(X)$ -torsor over $\mathrm{Spec}(F)$; see [17, Section III.1.3], [19, Section 11.3].

2.2. Étale algebras

An étale algebra A/F is a commutative F -algebra of the form $F_1 \times \cdots \times F_r$, where each F_i is a finite separable field extension of F . n -dimensional étale algebras over F are F -forms of the split étale algebra $A = F \times \cdots \times F$ (n times). The automorphism group of this split algebra is the symmetric group S_n , permuting the n factors of F . Thus n -dimensional étale algebras over F are in a natural bijective correspondence with the Galois cohomology set $H^1(F, S_n)$; see, e.g., [18, Examples 2.1 and 3.2].

2.3. Weil restriction

Let A be an étale algebra over F and $X \rightarrow \mathrm{Spec}(A)$ be a variety defined over A . The Weil restriction (or Weil transfer) of X to F is, by definition, an F -variety $R_{A/F}(X)$ satisfying

$$\mathrm{Mor}_F(Y, R_{A/F}(X)) \simeq \mathrm{Mor}_A(Y_A, X), \quad (2.1)$$

where $Y_A := Y \times_{\mathrm{Spec}(F)} \mathrm{Spec}(A)$, $\mathrm{Mor}_F(Y, Z)$ denotes the set of F -morphisms $Y \rightarrow Z$, and $\simeq$ denotes an isomorphism of functors (in Y). For generalities on this notion we refer the reader to [3, Section 7.6]. For a brief summary, see [14, Section 2]. In particular, it is shown in [3, Theorem 4] that if X is quasi-projective over A , then $R_{A/F}(X)$ exists. Note that uniqueness of $R_{A/F}(X)$ follows from (2.1) by Yoneda's lemma.

The following properties of the Weil restriction will be helpful in the sequel.

Lemma 2.2. *Let A/F be an étale algebra and X be a (quasi-projective) variety defined over A .*

- (a) Let V be a free A -module of finite rank, and $X = \mathbb{A}_A(V)$ be the associated affine space. Then $R_{A/F}(X) = \mathbb{A}_F(V)$, where we view V as an F -vector space;
- (b) If X and Y are birationally isomorphic over A , then $R_{A/F}(X)$ and $R_{A/F}(Y)$ are birationally isomorphic over F ;
- (c) If X is a rational variety over A , then $R_{A/F}(X)$ is rational over F .

Proof. (a) follows directly from (2.1). For details, see [14, Lemma 1.2].

(b) Since X and Y are birationally isomorphic, there exists a variety U defined over A and open immersions $i: U \hookrightarrow X$ and $j: U \hookrightarrow Y$. After replacing U by an open subvariety, we may assume that U is quasi-projective (we may even assume that U is affine). Since Weil restriction commutes with open immersions, i and j induce open immersions of $R_{A/F}(U)$ into $R_{A/F}(X)$ and $R_{A/F}(Y)$, respectively, and part (b) follows.

(c) By our assumption, X is birationally isomorphic to $Y = \mathbb{A}^d$ over A , where d is the dimension of X . By part (b), $R_{A/F}(X)$ and $R_{A/F}(Y)$ are birationally isomorphic over F , and by part (a), $R_{A/F}(Y)$ is an affine space over F . $\square$

In the special case where X is defined over F , the Weil transfer $R_{A/F}(X_A)$ can be explicitly described as follows. The symmetric group S_n acts on the n -fold direct product X^n by permuting the factors. If $P \rightarrow \operatorname{Spec}(F)$ is a S_n -torsor, and A/F is the étale algebra of degree n representing the class of P in $H^1(F, S_n)$, then $R_{A/F}(X_A) = {}^P(X^n)$; see, e.g., [5, Proposition 3.2].

2.4. Automorphism of marked curves

We shall need the following well-known result in the sequel; see, e.g., in [13, Corollary IV.4.7] for $g = 1$ and [13, Exercise V.1.11] for $g \geq 2$.

Proposition 2.3. *Suppose $2g + n \geq 5$. Then $\operatorname{Aut}(C, p_1, \dots, p_n) = \{1\}$ for a general point $(C, p_1, \dots, p_n)$ of $\overline{M}_{g,n}$ (or equivalently, of $M_{g,n}$).*

Note that the inequality $2g + n \geq 5$ is satisfied for every pair of integers (g, n) appearing in Theorem 1.2.

2.5. Automorphisms and forms of $\overline{M}_{g,n}$

The following theorem is the starting point of our investigation.

Theorem 2.4 (A. Massarenti [16], B. Fantechi and A. Massarenti [7]). *Let F be a field of characteristic $\neq 2$. If $g, n \geq 1$, $(g, n) \neq (2, 1)$ and $2g + n \geq 5$, then the natural embedding $S_n \rightarrow \operatorname{Aut}_F(\overline{M}_{g,n})$ is an isomorphism.*

Using the bijective correspondence between F -forms of X and $\operatorname{Aut}(X)$ -torsors $P \rightarrow \operatorname{Spec}(F)$ described at the end of Section 2.1, we obtain the following.

Corollary 2.5. *For F, g, n as in Theorem 2.4, every F -form of $\overline{M}_{g,n}$ is F -isomorphic to ${}^P\overline{M}_{g,n}$ for some S_n -torsor $P \rightarrow \text{Spec}(F)$. $\square$*

Remark 2.6. Theorem 1.2 also holds in the following cases:

- (a) $g = 2$ and $n = 1$, and $\text{char}(F) = 0$;
- (b) $g = 1$ and $n = 2$ and $\text{char}(F) \neq 2$ or 3 .

In case (a), $\overline{M}_{2,1}$ has no non-trivial automorphisms by [7, Theorem 1] and hence, no non-split forms. On the other hand, the split form of $\overline{M}_{2,1}$ is known to be rational; see [4].

In case (b), the automorphism group G of $\overline{M}_{1,2}$ is non-trivial; however, it is special; see [7, Proposition 2.4]. In other words, every G -torsor over a field is split. As a consequence, $\overline{M}_{1,2}$ has no non-split forms (see [5, Remark 6.4]) and the split form of $\overline{M}_{1,2}$ is rational (see [4]).

Remark 2.7. We do not know if $\overline{M}_{g,n}$ can be replaced by $M_{g,n}$ in the statement of Theorem 2.4. If so, then $\overline{M}_{g,n}$ can also be replaced by $M_{g,n}$ in the statements of Theorems 1.2. The proof remains unchanged.

3. Proof of Theorem 1.2: the overall strategy

Let $(C, p_1, \dots, p_n)$ be a point of $M_{g,n}$.

Case I. We define a vector space V of dimension d as follows:

- If $g = 3, 4$ or 5 , then $V = H^0(C, \omega_C)^*$, where ω_C is the canonical line bundle. Here $d = g$;
- If $g = 1$ and $n = 3$ or 4 , then $V = H^0(C, \mathcal{O}_C(p_1 + \dots + p_n))^*$. Here $d = n$;
- If $g = 2$ and $n = 2$, then $V = H^0(C, \omega_C(p_1 + p_2))^*$. Here $d = 3$.

Case II.

- If $g = 2$ and $n = 3$, then in place of V we define two vector spaces, $V_1 = H^0(C, \omega_C)^*$ and $V_2 = H^0(C, \mathcal{O}_C(p_1 + p_2 + p_3))^*$. Here $\dim(V_1) = \dim(V_2) = 2$.

Remark 3.1. In Case I, for $(C, p_1, \dots, p_n)$ in a suitably defined open subset $(M_{g,n})_0$ of $M_{g,n}$, the space V is the fiber of a vector bundle $E \rightarrow (M_{g,n})_0$ obtained via push-forward from a vector bundle over the universal curve. In Case II the same is true of both V_1 and V_2 . After replacing $(M_{g,n})_0$ by a dense open subset, we may assume without loss of generality that (i) $(M_{g,n})_0$ is S_n -invariant and (ii) $\text{Aut}(C, p_1, \dots, p_n) = 1$ for any $(C, p_1, \dots, p_n) \in (M_{g,n})_0$; see Proposition 2.3.

In Case I, set

$$X = \{(C, p_1, \dots, p_n, B) \in (M_{g,n})_0 \mid B \text{ is a basis of } V, \text{ up to proportionality}\}.$$

Here two bases $B = (v_1, \dots, v_d)$ and $B' = (v'_1, \dots, v'_d)$ are called proportional if there exists a $0 \neq c \in k$ such that $v'_i = cv_i$ for every $i = 1, \dots, d$.

In Case II, the vector space V in the definition of X should be replaced by a pair of 2-dimensional vector spaces $V = (V_1, V_2)$ and the basis B by a pair $B = (B_1, B_2)$, where B_1 is a basis of V_1 and B_2 is a basis of V_2 . We identify two such pairs, $B = (B_1, B_2)$ and $B' = (B'_1, B'_2)$, if B_1 is proportional to B'_1 and B_2 is proportional to B'_2 .

In Case I, choosing a basis in V gives rise to the map $f_B: C \rightarrow \mathbb{P}^{d-1}$. Two bases, B and B' , are proportional if and only if $f_B = f_{B'}$. In Case II, we obtain two maps, $f_{B_1}, f_{B_2}: C \rightarrow \mathbb{P}^1$, which can be combined into a single morphism $f_B = f_{B_1} \times f_{B_2}: C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Once again, we identify $B = (B_1, B_2)$ and $B' = (B'_1, B'_2)$ if and only if $f_B = f_{B'}$.

In each case we will consider a diagram of the form

$$\begin{array}{ccc} & X & \\ \alpha \swarrow & & \searrow \beta \\ M_{g,n} & \xleftarrow{\text{open}} (M_{g,n})_0 & \rightarrow Y. \end{array} \quad (3.1)$$

Here α is the natural projection $(C, p_1, \dots, p_n, B) \rightarrow (C, p_1, \dots, p_n)$ which “forgets” the basis B . The second projection β “forgets” the curve C , and maps $(C, p_1, \dots, p_n, B)$ to a suitable configuration space Y for the remaining data. Specifically, we define Y and β as follows.

Case I.

- If $g = 3, 4$ or 5 , then $f_B: C \rightarrow \mathbb{P}^{g-1}$ is the canonical embedding. We define $Y = (\mathbb{P}^{g-1})^n$, and $\beta(C, p_1, \dots, p_n, B) = (f_B(p_1), \dots, f_B(p_n)) \in Y$;
- Let $g = 1$ and $n = 3$ or 4 . The constant function $1 \in H^0(C, \mathcal{O}_C(p_1 + \dots + p_n))$ cuts out a hyperplane $L \subset \mathbb{P}^{n-1}$ passing through $f_B(p_1), \dots, f_B(p_n)$. We set $Y \subset ((\mathbb{P}^{n-1})^*)^n$ to be the locally closed subvariety consisting of n -tuples $p_1, \dots, p_n$ such that $p_1, \dots, p_n$ are linearly dependent (*i.e.*, lie in a hyperplane) in $\mathbb{P}^{n-1}$ but any $n - 1$ of them are linearly independent and define $\beta(C, p_1, \dots, p_n) = (f_B(p_1), \dots, f_B(p_n))$;
- If $g = 2$ and $n = 2$, then for p_1, p_2 in general position on C , the image of f_B in $\mathbb{P}^2$ is a quartic curve C' with a node at $p = f_B(p_1) = f_B(p_2)$, and $f_B: C \rightarrow C'$ is the normalization map; see [12, Example 5.15]. Moreover, C' has two tangent lines at p , L_1 and L_2 , which correspond to p_1 and p_2 under f_B . We thus define Y as the open subvariety of $(\mathbb{P}^2)^* \times (\mathbb{P}^2)^*$ parametrizing pairs of distinct lines and $\beta(C, p_1, p_2, B) = (L_1, L_2)$.

Case II.

- Here $g = 2$ and $n = 3$, and the maps f_{B_1} and $f_{B_2}: C \rightarrow \mathbb{P}^1$ are of degree 2 and 3, respectively; see [12, Examples 5.11 and 5.13]. For p_1, p_2, p_3 in general position, $C' := f_B(C)$ is a curve of bidegree $(3, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, $f_B = f_{B_1} \times f_{B_2}$ is an isomorphism between C and C' , and $f_{B_2}(p_1) = f_{B_2}(p_2) = f_{B_2}(p_3)$ in $\mathbb{P}^1$. We set Y to be the open subvariety of $(\mathbb{P}^1)^3 \times \mathbb{P}^1$ consisting of elements of the form $((a_1, a_2, a_3), b)$, where $a_1, a_2, a_3 \in \mathbb{P}^1$ are distinct and $\beta(C, p_1, p_2, p_3, B) = ((f_{B_1}(p_1), f_{B_1}(p_2), f_{B_1}(p_3)), f_{B_2}(p_1))$.

By Corollary 2.5 it suffices to show that the twisted variety ${}^P M_{g,n}$ is stably rational over F for every S_n -torsor $P \rightarrow \text{Spec}(F)$. Here $1 \leq g \leq 5$, and (g, n) is one of the pairs appearing in Theorem 1.2. Twisting the diagram (3.1) by P and applying Lemma 2.1, we obtain the following diagram of twisted varieties.

$$\begin{array}{ccc}
 & & {}^P X \\
 & \swarrow^{P_\alpha} & \searrow^{P_\beta} \\
 {}^P M_{g,n} & \xleftarrow[\text{open}]{{}^P ((M_{g,n})_0)} & {}^P Y.
 \end{array} \tag{3.2}$$

In order to complete the proof of Theorem 1.2, we need to establish the following facts for each pair (g, n) in Theorem 1.2.

Lemma 3.2. *The rational map $\beta: X \dashrightarrow Y$ is dominant.*

Lemma 3.3.

- ${}^P Y$ is rational over F ;
- ${}^P X$ is rational over ${}^P M_{g,n}$;
- ${}^P X$ is rational over ${}^P Y$.

4. Proof of Lemma 3.2

For $g = 3, 4, 5$ we need to show that there is a canonical curve passing through n points $r_1, \dots, r_n$ in general position in $\mathbb{P}^{g-1}$.

$g = 3$. Canonical curves of genus 3 are precisely the smooth quartic curves in $\mathbb{P}^2$; see [13, Example IV.5.2.1]. Since $\dim H^0(\mathbb{P}^2, \mathcal{O}(4)) = 15$, there is a smooth quartic curve passing through n points in $\mathbb{P}^2$ in general position for any $n \leq 14$.

$g = 4$. We will use the fact that a complete intersection of a smooth quadric surface Q and a smooth cubic surface S in $\mathbb{P}^3$ is a canonical curve of genus 4; see [13, Example IV.5.2.2]. The dimensions of $H^0(\mathbb{P}^3, \mathcal{O}(2))$ and $H^0(\mathbb{P}^3, \mathcal{O}(3))$ are 10 and 20, respectively. Hence, as long as $n \leq 9$ and $r_1, \dots, r_n$ are in general position in $\mathbb{P}^3$, there exist a smooth quadric $Q \subset \mathbb{P}^3$ and smooth cubic $S \subset \mathbb{P}^3$ such

that Q and S pass through $r_1, \dots, r_n$ and intersect transversely. The intersection $Q \cap S$ is then a canonical curve of genus 4 passing through $r_1, \dots, r_n$.

$g = 5$. Let $n \leq 12$. Since $\dim H^0(\mathbb{P}^4, \mathcal{O}(2)) = 15$, for n points $r_1, \dots, r_n$ in general position in $\mathbb{P}^4$, there exist three linearly independent quadrics, Q_1, Q_2 and Q_3 such that

- (i) Q_1, Q_2 and Q_3 pass through $r_1, \dots, r_n$;
- (ii) Q_1, Q_2 and Q_3 intersect transversely;
- (iii) $C = Q_1 \cap Q_2 \cap Q_3$ is a smooth curve.

By [13, Example IV.5.5.3], C is a canonical curve of genus 5. By (i), C passes through $r_1, \dots, r_n$, as desired.

$g = 2$ and $n = 2$. The projection β is equivariant with respect to the natural GL_3 -action on X and Y . Here GL_3 acts on $(C, p_1, \dots, p_n, B) \in X$ by linear changes of the basis B , leaving C and $p_1, \dots, p_n$ invariant, and on Y via its natural action on $\mathbb{P}^2$. Since the GL_3 -action on Y is transitive, β is dominant.

$(g, n) = (2, 3), (1, 3)$ or $(1, 4)$. Let $G = \mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_3$ or GL_4 , respectively. In each case, β is G -equivariant and G acts transitively on Y , so the same argument as in the previous case shows that β is dominant. This completes the proof of Lemma 3.2. $\square$

5. Proof of Lemma 3.3

(a) Let A/F be the étale algebra associated to the S_n -torsor $P \rightarrow \mathrm{Spec}(F)$. If $g = 3, 4$ or 5 , then ${}^P Y \simeq R_{A/F}(\mathbb{P}^{g-1})$. If $g = 2$ and $n = 2$, then ${}^P Y \simeq R_{A/F}((\mathbb{P}^2)^*)$. If $g = 2$ and $n = 3$, then ${}^P Y \simeq R_{A/F}(\mathbb{P}^1) \times \mathbb{P}^1$. Here $\simeq$ stands for “birationally isomorphic over F ”. In each case ${}^P Y$ is rational over F by Lemma 2.2(c).

If $g = 1$, let $\mathcal{H} \rightarrow (\mathbb{P}^{n-1})^*$ be the tautological bundle whose fiber over the hyperplane $\{l = 0\}$ consists of the points of the affine hyperplane cut out by l in $\mathbb{A}^n$. Then Y is S_n -equivariantly birationally isomorphic to the n -fold fibered product $\mathbb{P}(\mathcal{H})^n$ over $(\mathbb{P}^{n-1})^*$. Since S_n acts trivially on $(\mathbb{P}^{n-1})^*$, and $(\mathbb{P}^{n-1})^*$ is rational over F , it suffices to show that ${}^P(\mathbb{P}(\mathcal{H})^n)$ is rational over $(\mathbb{P}^{n-1})^*$. Choosing projective coordinates $a_1, \dots, a_n$ in $\mathbb{P}^{n-1}$, we can identify the function field K of $(\mathbb{P}^{n-1})^*$ with $F(a_i/a_j \mid i, j = 1, \dots, n)$. Over K , $\mathcal{H}$ is isomorphic to the $(n-1)$ -dimensional vector subspace on K^n given by $a_1 x_1 + \dots + a_n x_n = 0$ and ${}^P(\mathbb{P}(\mathcal{H})^n)$ is isomorphic to $R_{A_K/K}(\mathbb{P}(\mathcal{H})_{A_K})$, where $A_K = A \otimes_F K$. By Lemma 2.2(c), $R_{A_K/K}(\mathbb{P}(\mathcal{H})_{A_K})$ is rational over K . This shows that ${}^P(\mathbb{P}(\mathcal{H})^n)$ is rational over $(\mathbb{P}^{n-1})^*$, as desired.

(b) Case I. Let $E \rightarrow (M_{g,n})_0$ be the vector bundle whose fiber over $(C, p_1, \dots, p_n)$ is V , as in Remark 3.1. The space of bases in V (up to equivalence) can be identified with a dense open subset of $\mathbb{P}(V^d)$. Hence, X is S_n -equivariantly birationally isomorphic to $\mathbb{P}(E^d)$ over $(M_{g,n})_0$ and consequently ${}^P X$ is birationally isomorphic to

${}^P\mathbb{P}(E^d)$. On the other hand, by Lemma 2.1(f), ${}^P\mathbb{P}(E^d)$ is rational over ${}^P(M_{g,n})_0$, and by Lemma 2.1(a), ${}^P(M_{g,n})_0$ is a dense open subset of ${}^P(M_{g,n})$. We conclude that ${}^P X \simeq {}^P\mathbb{P}(E^d)$ is rational over ${}^P M_{g,n}$.

Case II. Now suppose $(g, n) = (2, 3)$. Let E_1 and E_2 be the rank 2 vector bundles over $(M_{g,n})_0$ whose fibers over $(C, p_1, \dots, p_n)$ are V_1 and V_2 , respectively, as in Remark 3.1. The space of bases in V_i (up to equivalence) can be identified with a dense open subset of $\mathbb{P}(V_i^2)$. Hence, X is S_3 -equivariantly birationally isomorphic to a dense open subset of $\mathbb{P}(E_1^2) \times_{(M_{g,n})_0} \mathbb{P}(E_2^2)$ over $(M_{g,n})_0$. By Lemma 2.1(d), ${}^P X$ is birationally isomorphic to ${}^P\mathbb{P}(E_1^2) \times_{{}^P(M_{g,n})_0} {}^P\mathbb{P}(E_2^2)$ over ${}^P(M_{g,n})_0$. By Lemma 2.1(f), each ${}^P\mathbb{P}(E_i^2)$ is rational over ${}^P(M_{g,n})_0$ (or equivalently, over ${}^P M_{g,n}$). Hence, so is ${}^P X \simeq {}^P\mathbb{P}(E_1^2) \times_{{}^P(M_{g,n})_0} {}^P\mathbb{P}(E_2^2)$.

(c) Recall that in Case I each $(C, p_1, \dots, p_n, B) \in X$ gives rise to a map $f_B: C \rightarrow \mathbb{P}^{d-1}$. Here $d = \dim(V)$. In Case II, f_B maps C to $\mathbb{P}^1 \times \mathbb{P}^1$. Now observe that in both cases f_B (and thus B , up to equivalence) is uniquely determined by the image of $(C, p_1, \dots, p_n)$ under f_B . Indeed, in each case f_B maps C birationally onto its image. If f_B and $f_{B'}$ have the same image, then composing $f_{B'}$ with f_B^{-1} , we obtain an automorphism of $(C, p_1, \dots, p_n)$. On the other hand, for $(C, p_1, \dots, p_n) \in (M_{g,n})_0$, $\text{Aut}(C, p_1, \dots, p_n) = 1$; see Remark 3.1.

$g = 3$. Using the above observation we can S_n -equivariantly identify X with the space of tuples $(Q, r_1, \dots, r_n)$, where $Q \subset \mathbb{P}^2$ is a smooth curve of degree 4 and $r_1, \dots, r_n$ are n distinct points on Q . Let $Y_0 \subset Y = (\mathbb{P}^2)^n$ be the dense open S_n -invariant subvariety consisting of n -tuples $(r_1, \dots, r_n)$ such that (i) $(r_1, \dots, r_n)$ impose n independent conditions on quartic curves, (ii) there is a smooth quartic curve passing through $r_1, \dots, r_n$. Let $W \rightarrow Y_0 \subset (\mathbb{P}^2)^n$ be the vector bundle whose fiber over $(r_1, \dots, r_n)$ is the space of quartic polynomials in 3 variables vanishing at $r_1, \dots, r_n$. Then over Y_0 , X is S_n -equivariantly birationally isomorphic to $\mathbb{P}(W)$. By Lemma 2.1, ${}^P X \simeq {}^P\mathbb{P}(W)$ is rational over ${}^P Y$.

$g = 4$. Recall that $C \subset \mathbb{P}^3$ is a canonically embedded curve of genus 4 if and only if C is a complete intersection of an irreducible quadric surface Q and an irreducible cubic surface S in $\mathbb{P}^3$. Moreover, the quadric Q is uniquely determined by C , and the cubic polynomial s which cuts out S , is uniquely determined up to replacing s by $s' = c \cdot s + lq$, where $c \in F^*$ is a non-zero constant, q is the quadratic form cutting out Q , and l is a linear form. Conversely, any irreducible non-singular curve in $\mathbb{P}^3$, which is a complete intersection of an irreducible quadric and an irreducible cubic, is a canonically embedded curve of genus 4; see [13, Example IV.5.2.2].

Let $Y_0 \subset Y = (\mathbb{P}^3)^n$ be the open subset consisting of n -tuples of points imposing independent conditions of quadrics and cubics in $\mathbb{P}^3$. Let W be the space of $(n+1)$ -tuples $(q, r_1, \dots, r_n)$, where $(r_1, \dots, r_n) \in Y_0$ and $q \in H^0(\mathbb{P}^3, \mathcal{O}(2))$ vanishes at $r_1, \dots, r_n$. The natural projection $W \rightarrow Y_0$ given by $(q, r_1, \dots, r_n) \mapsto (r_1, \dots, r_n)$ is a vector bundle of rank $10 - n$. The fiber of the projective bundle $\mathbb{P}(W)$ over $(r_1, \dots, r_n) \in Y_0$ parametrizes quadric surfaces $Q \subset \mathbb{P}^3$ passing through $r_1, \dots, r_n$. Now let W' be the vector bundle of rank $20 - n$ over $\mathbb{P}(W)$,

whose fiber over $(Q, r_1, \dots, r_n)$ consists of cubic forms $s \in H^0(\mathbb{P}^3, \mathcal{O}(3))$ vanishing at $r_1, \dots, r_n$. Let $W'' \subset W$ be the subbundle, whose fiber over $(Q, r_1, \dots, r_n)$ consists of cubic forms $l \cdot q$, where $q \in H^0(\mathbb{P}^3, \mathcal{O}(2))$ cuts out Q and l ranges over $H^0(\mathbb{P}^3, \mathcal{O}(1))$. Now set $\overline{W} = W'/W''$. A general point $(S, Q, r_1, \dots, r_n)$ of $\mathbb{P}(\overline{W})$ gives rise to a canonical curve $Q \cap S \subset \mathbb{P}^3$ of genus 4 passing through $r_1, \dots, r_n$. Thus X is S_n -birationally isomorphic to $\mathbb{P}(\overline{W})$ over Y_0 , and we obtain the following diagram of S_n -equivariant maps

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \mathbb{P}(\overline{W}) \\ & \downarrow & \\ & \mathbb{P}(W) & \\ & \downarrow & \\ Y & \xrightarrow{\sim} & Y_0. \end{array}$$

Twisting by the S_n -torsor $P \rightarrow \text{Spec}(F)$, we obtain a diagram

$$\begin{array}{ccc} {}^P X & \xrightarrow{\sim} & {}^P \mathbb{P}(\overline{W}) \\ & \downarrow & \\ & {}^P \mathbb{P}(W) & \\ & \downarrow & \\ {}^P Y & \xrightarrow{\sim} & {}^P Y_0. \end{array}$$

By Lemma 2.1, ${}^P X$ is rational over ${}^P \mathbb{P}(W)$ and ${}^P \mathbb{P}(W)$ is rational over ${}^P Y$.

$g = 5$. Recall that a general canonical curve $C' = f_B(C)$ of genus 5 is a complete intersection of three quadric hypersurfaces Q_1, Q_2 and Q_3 in $\mathbb{P}^4$. Let $q_i \in H^0(\mathbb{P}^4, \mathcal{O}(2))$ be a defining equation for Q_i . Then the span of q_1, q_2 and q_3 is uniquely determined by the canonical curve C' , because $H^0(\mathbb{P}^4, \mathcal{I}_{C'}(2))$ is 3-dimensional. Conversely, a 3-dimensional subspace of $H^0(\mathbb{P}^4, \mathcal{O}(2))$ in general position cuts out a canonical curve of genus 5 in $\mathbb{P}^4$; see [13, Example IV.5.5.3].

Let $Y_0 \subset Y = (\mathbb{P}^4)^n$ be the open subset consisting of n -tuples of points imposing independent conditions of quadrics. Let W be the space of $(n+1)$ -tuples $(q, r_1, \dots, r_n)$, where $(r_1, \dots, r_n) \in Y_0$ and $q \in H^0(\mathbb{P}^4, \mathcal{O}(2))$ vanishes at $r_1, \dots, r_n$. The natural projection $W \rightarrow Y_0 \subset (\mathbb{P}^4)^n$ given by $(q, r_1, \dots, r_n) \mapsto (r_1, \dots, r_n)$ is a vector bundle of rank $15 - n$. By the above description, X is S_n -equivariantly birationally isomorphic to the total space of the Grassmannian bundle

$\mathrm{Gr}(3, W)$. Twisting by P , we obtain the following diagram

$$\begin{array}{ccc} {}^P X & \xrightarrow{\sim} & {}^P \mathrm{Gr}(3, W) \\ & & \downarrow \\ {}^P Y & \xleftarrow{\sim} & {}^P Y_0. \end{array}$$

By Lemma 2.1, we conclude that ${}^P X$ is rational over ${}^P Y$.

$g = 2$ and $n = 2$. Here $Y \subset (\mathbb{P}^2)^* \times (\mathbb{P}^2)^*$ parametrizes pairs (L_1, L_2) of distinct lines in $\mathbb{P}^2$. Let $W \rightarrow Y$ be the vector bundle whose fiber over (L_1, L_2) consists of quartic forms $q \in H^0(\mathbb{P}^2, \mathcal{O}(4))$ such that both $q|_{L_1}$ and $q|_{L_2}$ vanish to second order at $p = L_1 \cap L_2$. Then X is S_2 -equivariantly birationally isomorphic to $\mathbb{P}(W)$ over Y . By Lemma 2.1, we conclude that ${}^P X$ is rational over ${}^P Y$.

$g = 2$ and $n = 3$. Here X is S_3 -equivariantly birationally isomorphic to $\mathbb{P}(W)$, where W is the vector bundle over a suitable dense open subset of $Y \subset (\mathbb{P}^1)^3 \times \mathbb{P}^1$ whose fiber over $((a_1, a_2, a_3), b)$ consists of bihomogeneous polynomials $\phi \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(3, 2))$ vanishing at (a_1, b) , (a_2, b) and (a_3, b) . By Lemma 2.1, ${}^P X$ is rational over ${}^P Y$.

$g = 1$ and $n = 3$. Here $(C', r_1, r_2, r_3) = f_B(C, p_1, p_2, p_3)$ is a smooth plane cubic curve with three distinct collinear points for every $(C, p_1, p_2, p_3) \in M_{1,3}$. Conversely, every smooth cubic curve $C' \subset \mathbb{P}^2$ with three distinct collinear points $r_1, r_2, r_3 \in C'$ is of the form $f_B(C, p_1, p_2, p_3)$ for some $(C, p_1, p_2, p_3, B) \in X$, because $\mathcal{O}_{C'}(r_1 + r_2 + r_3) = \mathcal{O}_{C'}(1)$. Thus X is S_3 -equivariantly birationally isomorphic to $\mathbb{P}(W)$ over Y , where $W \rightarrow Y$ is the vector bundle whose fiber over (r_1, r_2, r_3) consists of cubic forms in $H^0(\mathbb{P}^2, \mathcal{O}(3))$ vanishing at (r_1, r_2, r_3) . Twisting by the S_3 -torsor P , we conclude that ${}^P X \simeq \mathbb{P}({}^P W)$ is rational over ${}^P Y$ by Lemma 2.1(f).

$g = 1$ and $n = 4$. For $(C, p_1, \dots, p_4)$ in general position, $(C', r_1, \dots, r_4) = f_B(C, p_1, \dots, p_4)$ is a smooth curve of genus 1 in $\mathbb{P}^3$ with four coplanar points no three of which are collinear. By [13, Exercise IV.3.6(b)], the space $H^0(\mathbb{P}^3, \mathcal{I}_{C'}(2))$ of global sections of the ideal sheaf $\mathcal{I}_{C'}(2)$ is 2-dimensional. Moreover, if q_1, q_2 is a basis of this space, then C' is a complete intersection of the quadrics Q_1 and Q_2 cut out by q_1 and q_2 . Conversely, a complete intersection of two smooth quadrics in $\mathbb{P}^3$ in general position is a smooth curve of genus 1; see [13, Exercise I.7.2]. We conclude that X is S_4 -equivariantly birationally isomorphic to the total space of the Grassmannian bundle $\mathrm{Gr}(2, W)$ over a suitably defined dense open subvariety $Y_0 \subset Y$, where $W \rightarrow Y_0$ is the vector bundle whose fiber over $(r_1, \dots, r_4) \in Y$ consists of $q \in H^0(\mathbb{P}^3, \mathcal{O}(2))$ vanishing at $(r_1, \dots, r_4)$. Twisting by a S_4 -torsor $P \rightarrow \mathrm{Spec}(F)$, we see that ${}^P X$ is birational to $\mathrm{Gr}(2, {}^P W)$ over ${}^P Y$. By Lemma 2.1(f), ${}^P X$ is rational over ${}^P Y$.

This concludes the proof of Lemma 3.3 and thus of Theorem 1.2. $\square$

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