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Hénon equation involving nearly critical Sobolev exponent in a general domain

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Abstract. In this paper we are concerned with the following Hénon problem

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{2^*-1-\varepsilon} \ \ u > 0 & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega, \end{cases}$$

where $N \geq 4$, $2^* = \frac{2N}{N-2}$, $\alpha > 0$, ε is a small positive parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$ and $0 \in \Omega$. Most of the previous works for the Hénon problems were investigated in special domains, such as balls and annuli. In this paper we will study the case when Ω is a more general domain, which does not satisfy symmetry any more. We first investigate the necessary condition on the location of the blow-up point for the peak solution to the above the Hénon problem. Then, we prove that, as $\varepsilon \to 0$, the above problem has a positive solution with multiple bubbles under a suitable condition on the geometry of Ω.

Mathematics Subject Classification (2010): 35J60 (primary); 35J65 (secondary).

1. Introduction

The Hénon problem

$$\begin{cases} -\Delta u = |x|^{\alpha} |u|^{p-1} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

was originally raised originally by M. Hénon in [12] to study the rotating stellar structures. It turns out that there is a rich structure of mathematical phenomena related to the solutions of the Hénon equation, and that there has been some intensive study mainly on the Dirichlet boundary value problems. In [16], by a compactness

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lemma for radial function, Ni proved that problem (1.1) possesses a positive radial solution if $\Omega = B_1(0)$ is the unit ball of $\mathbb{R}^N$ and $p \in (1, p_\alpha)$, where $p_\alpha = \frac{N+2+2\alpha}{N-2}$. Using the Pohozaev identity [20], one sees that problem (1.1) has no solution in any domain that is star-shaped with respect to the origin for $p \geq p_\alpha$. Hence, p_α is a critical exponent for the Hénon problem. Because of the weight being increasing, the classical symmetry results of Gidas-Ni-Nirenberg do not apply and the symmetric property of positive solutions becomes an interesting topic. After a numerical simulation in [9], which illustrates a symmetry breaking of a least-energy solution, Smets-Su-Willem [25] proved the symmetry breaking phenomenon for ground-state solutions when $N \geq 2$, $p \in (1, 2^* - 1)$, and α is large enough. Moreover, the authors of [1–3] found the asymptotic profiles of the ground-state solutions to depend upon some geometric property of the boundary when p is a Sobolev subcritical exponent ($1 < p < \frac{N+2}{N-2}$).

On the one hand, when $p = p_\alpha - \varepsilon$ with small $\varepsilon > 0$, the authors in [10,11] showed that there exists a solution concentrating at the origin provided $0 < \alpha \leq 1$ and α is not an even integer. In [13], the asymptotic behavior of the radial solutions was analyzed and it was shown that the solution tends to the fundamental solution of the Laplacian operator as $\varepsilon \to 0^+$.

On the other hand, it is worth pointing out that many researchers focused on the critical case $p = 2^* - 1$. In [24], Serra proved that problem (1.1) has a non-radial solution provided α is large enough. More recently, Wei and Yan [27] showed that there are infinitely many non-radial positive solutions for problem (1.1) with $\alpha > 0$ if Ω is $B_1(0)$. We refer readers to [7,14] and references therein for more multiplicity results. For $p = 2^* - 1 - \varepsilon$, Cao and Peng [4] showed that the ground-state solution is non-radial and blows up near the boundary of $B_1(0)$ as $\varepsilon \to 0$. Recently, Liu and Peng [15] studied the existence of a positive peak solution to problem (1.1) when $\Omega = B_1(0)$ and $p = 2^* - 1 + \varepsilon$ with $\varepsilon > 0$ small enough. More results for p near the critical exponent are contained in [5,8,17,23,26] and the references therein.

If Ω has no spherical symmetry, no compactness result is available, which leads to some difficulties to prove the existence of a solution for $p \in (1, p_\alpha)$. Naturally, one would like to know whether the Hénon problem has peak solutions for $p < p_\alpha$ in a general domain. To the best of our knowledge, it seems that there are a few results for this question. The aim of this paper is to obtain multi-peak solutions for the following Hénon problem with nearly critical growth in a general domain

$$\begin{cases} -\Delta u = |x|^\alpha u^{2^*-1-\varepsilon}, \quad u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\alpha > 0$, $2^* = \frac{2N}{N-2}$, $N \geq 4$, ε is a parameter, and $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain containing the origin.

It is well known that

$$-\Delta u = u^{2^*-1} \text{ in } \mathbb{R}^N$$

has a unique positive solution, up to translation and scaling, that is,

$$U_{x_0,\mu}(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}\mu^{\frac{N-2}{2}}}{(1+\mu^2|x-x_0|^2)^{\frac{N-2}{2}}}, \qquad \mu > 0,\ x_0 \in \mathbb{R}^N.$$

Assume that $x_0 \neq 0$, then we denote $W_{x_0,\mu} = |x_0|^{-\frac{\alpha}{2^*-2}}U_{x_0,\mu}$. Let $PW_{x_0,\mu}$ be the projection of $W_{x_0,\mu}$ into $H_0^1(\Omega)$, that is,

$$\begin{cases} -\Delta PW_{x_0,\mu} = |x_0|^\alpha W_{x_0,\mu}^{2^*-1} & \text{in } \Omega \\ PW_{x_0,\mu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Next, we will recall the definition of a single peak solution and state our main results.

Definition 1.1. We say u_ε is a single peak solution of (1.2), if u_ε is a solution of (1.2) with a form

$$u_\varepsilon = PW_{x_\varepsilon,\mu_\varepsilon} + \omega_\varepsilon, \tag{1.4}$$

where $\langle \omega_\varepsilon, \partial_{\mu_\varepsilon} PW_{x_\varepsilon,\mu_\varepsilon}\rangle_{H_0^1(\Omega)} = \langle \omega_\varepsilon, \partial_{x_{\varepsilon,j}} PW_{x_\varepsilon,\mu_\varepsilon}\rangle_{H_0^1(\Omega)} = 0$ for $j = 1, \cdots, N$, $\|\omega_\varepsilon\|_{H_0^1(\Omega)} \to 0$, and $\mu_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

One of our main results is in the following:

Theorem 1.2. *Assume that $0 \in \Omega$ and Ω is a smooth bounded domain in $\mathbb{R}^N$. If u_ε is a single peak solution of (1.2), then u_ε satisfies the following conditions: as $\varepsilon \to 0$,*

(1) $d_\varepsilon \varepsilon^{-1} \to \gamma_1$;
(2) $\mu_\varepsilon \varepsilon^{\frac{N-1}{N-2}} \to \gamma_2$;
(3) $x_\varepsilon \to x_0 \in \partial\Omega$ *and x_0 is a critical point of the function $|x|$ restricted on $\partial\Omega$,*

where γ_1 and γ_2 are constants, $d_\varepsilon = \operatorname{dist}(x_\varepsilon, \partial\Omega)$.

Remark 1.3. If $x_0 \in \partial\Omega$ is a critical point of the function $|x|$ restricted on $\partial\Omega$, then $x_0 = |x_0|\nu$, where ν is a unit outer normal of $\partial\Omega$ at x_0. In fact, if $\partial\Omega$ is represented by the equation $F(x) = 0$, then we have $\frac{x_0}{|x_0|} = \lambda\nabla F(x_0)$ for some constant λ, which in turn implies $x_0 = \lambda|x_0|\nabla F(x_0) = \lambda_1|x_0|\nu$. From $|x_0| = |\lambda_1||x_0|$ and $0 \in \Omega$, we conclude $\lambda_1 = 1$.

The necessary conditions imply that the peak of solutions is located near a special point x_0 on the boundary, that is, x_0 is a critical point of the distance from the origin restricted on the boundary of Ω. However, when Ω is a unit ball, the peak can locate near any point on the boundary due to the rotational symmetry of the problem (see [18,19]). In [22], Rey showed that the single peak solution concentrates on a critical point of Robin function $\varphi(x) = H(x,x)$ provided $\alpha = 0$, where $H(x,x)$ is the regular part of Green's function. So we see that the weight $|x|^\alpha$ in the Hénon problem has a stronger effect on the location of the blow-up point. Theorem 1.2 will inspire us to find a suitable condition on the domain, so we can construct peak solutions for (1.2). In the following, the existence of multi-peak solutions for (1.2) will be proved in a general domain via the Lyapunov-Schmit reduction method.

Before stating our existence results, we add some assumptions for Ω:

(A) The function $|x|$ restricted on $\partial\Omega$ has k different critical points $\xi_{i,0} \in \partial\Omega$, which are non-degenerate in the sense that at $\xi_{i,0}$, $\det(D_{\tau_h\tau_j}|x|)_{(N-1)\times(N-1)} \neq 0$, where $\{\tau_1, \cdots, \tau_{N-1}\}$ forms a base for the tangential plane of $\partial\Omega$ at $\xi_{i,0}$.

Theorem 1.4. *Assume that Ω is a smooth bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$ and satisfies* (A). *Then, there exists $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, problem* (1.2) *has a solution*

$$u_\varepsilon = \sum_{i=1}^{k} PW_{\mu_{i,\varepsilon},\xi_{i,\varepsilon}} + o_\varepsilon(1).$$

Moreover, as $\varepsilon \to 0$,

$$\|o_\varepsilon(1)\|_{H_0^1(\Omega)} \to 0,\ \xi_{i,\varepsilon} \to \xi_{i,0},\ d_{i,\varepsilon}\varepsilon \to \gamma_i,\ \mu_{i,\varepsilon}\varepsilon^{\frac{N-1}{N-2}} \to \tau_i,$$

where $d_{i,\varepsilon} = \mathrm{dist}(\xi_{i,\varepsilon}, \partial\Omega)$, γ_i and τ_i are constants, $i = 1, 2, \cdots, k$.

This paper is organized as follows. In Section 2, we will give the proof for Theorem 1.2. The existence result of Theorem 1.4 will be proved in Section 3. In order to state the proof of existence clearly, we list some needed estimates in Appedix A.

2. Necessary results for single peak solutions

At first, we will introduce the Green's function $G(\cdot, y)$ of $-\Delta$ in Ω with Dirichlet boundary condition, that is, for any $y \in \Omega$,

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega \\ G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

and its regular part is $H(\cdot, y)$, which is the solution of the equation

$$\begin{cases} -\Delta H(\cdot, y) = 0 & \text{in } \Omega \\ H(\cdot, y) = \frac{1}{(N-2)\omega_{N-1}|\cdot - y|^{N-2}} & \text{on } \partial\Omega, \end{cases}$$

where ω_{N-1} is the area of the unit sphere $\mathbb{S}^{N-1}$. By the comparison principle, the following estimations hold

$$\begin{aligned} |G(x, y)| &\le \frac{1}{(N-2)\omega_{N-1}|x-y|^{N-2}} \quad \text{and} \\ |H(x, y)| &\le \frac{1}{(N-2)\omega_{N-1}|x-y|^{N-2}}. \end{aligned} \tag{2.1}$$

To estimate ω_ε, we firstly prove the following lemmas.

Lemma 2.1. *The blow-up point* $x_\varepsilon \to x_0 \neq 0$ *as* $\varepsilon \to 0$.

Proof. Assume that $x_0 = 0$ and u_ε is a solution of (1.2) with $\mu_\varepsilon^{\frac{N-2}{2}} = \max_{x\in\Omega} u_\varepsilon(x) = u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Set $v_\varepsilon(z) = \mu_\varepsilon^{-\frac{N-2}{2}} u_\varepsilon(\frac{z}{\mu_\varepsilon} + x_\varepsilon)$. Then $v_\varepsilon(z)$ satisfies

$$\begin{cases} -\Delta v_\varepsilon(z) = \mu_\varepsilon^{-\frac{(N-2)\varepsilon}{2}} |\frac{z}{\mu_\varepsilon} + x_\varepsilon|^\alpha v_\varepsilon^{2^*-1-\varepsilon} & \text{in } \Omega_\varepsilon =: \{z : \frac{z}{\mu_\varepsilon} + x_\varepsilon \in \Omega\} \\ v_\varepsilon(z) = 0 & \text{on } \partial\Omega_\varepsilon \\ v_\varepsilon(0) = \max_{z\in\Omega_\varepsilon} v_\varepsilon(z) = 1. \end{cases}$$

As $\varepsilon \to 0$, by the elliptic regularity, we have $v_\varepsilon \to v$ in C^2 and v satisfies

$$-\Delta v = 0, \text{ in } \mathbb{R}^N.$$

Since v is bounded and $v \to 0$ as $|z| \to +\infty$, the harmonic function $v \equiv 0$, which contradicts with $v(0) = 1$. □

By blow-up analysis, we have $\mu_\varepsilon d_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Next, let us consider the relation between μ_ε and ε.

Lemma 2.2. *It holds* $(\mu_\varepsilon)^\varepsilon \to 1$, *as* $\varepsilon \to 0$. *That is,* $\varepsilon \ln \mu_\varepsilon \to 0$, *as* $\varepsilon \to 0$.

Proof. By (1.4), we have $\int_\Omega |\nabla u_\varepsilon|^2 dx = |x_0|^{-\frac{2\alpha}{2^*-2}} \int_{\mathbb{R}^N} U_{0,1}^{2^*} + o(1)$, which gives $\int_\Omega |x|^\alpha u_\varepsilon^{2^*-\varepsilon} = |x_0|^{-\frac{2\alpha}{2^*-2}} \int_{\mathbb{R}^N} U_{0,1}^{2^*} + o(1)$.

If $\varepsilon \ln \mu_\varepsilon \to c_0 > 0$, then

$$\begin{aligned} \int_\Omega |x|^\alpha u_\varepsilon^{2^*-\varepsilon}\, dx &= \int_\Omega |x|^\alpha W_{x_\varepsilon,\mu_s}^{2^*-\varepsilon} + o(1) = |x_0|^{-\frac{2\alpha}{2^*-2}} \int_{B_{d_\varepsilon}(x_\varepsilon)} U_{x_\varepsilon,\mu_\varepsilon}^{2^*-\varepsilon} + o(1) \\ &= |x_0|^{-\frac{2\alpha}{2^*-2}} \int_{\mathbb{R}^N} U_{0,1}^{2^*} dx - \frac{N-2}{2} \varepsilon \ln \mu_\varepsilon \int_{\mathbb{R}^N} U_{0,1}^{2^*} dx + o(1) \\ &\leq |x_0|^{-\frac{2\alpha}{2^*-2}} \int_{\mathbb{R}^N} U_{0,1}^{2^*} dx - \delta, \end{aligned}$$

which implies a contradiction. Thus, $\varepsilon \ln \mu_\varepsilon \to 0$, as $\varepsilon \to 0$. □

Since u_ε is a solution of (1.2) with the form (1.4), ω_ε satisfies

$$L_\varepsilon(\omega_\varepsilon) := -\Delta\omega_\varepsilon - (2^* - 1 - \varepsilon)|x|^\alpha PW_{x_\varepsilon,\mu_\varepsilon}^{2^*-2-\varepsilon} \omega_\varepsilon = f + R(\omega_\varepsilon), \tag{2.2}$$

where

$$f = |x|^\alpha PW_{x_\varepsilon,\mu_\varepsilon}^{2^*-1-\varepsilon} - |x_\varepsilon|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}$$

and

$$R(\omega_\varepsilon) = |x|^\alpha (PW_{x_\varepsilon,\mu_\varepsilon} + \omega_\varepsilon)^{2^*-1-\varepsilon} - |x|^\alpha PW_{x_\varepsilon,\mu_\varepsilon}^{2^*-1-\varepsilon} - (2^*-1-\varepsilon)|x|^\alpha PW_{x_\varepsilon,\mu_\varepsilon}^{2^*-2-\varepsilon} \omega_\varepsilon.$$

Similar to [19], we have:

Lemma 2.3. *It holds that, for $N \geq 4$,*

$$\|\omega_\varepsilon\|_{H_0^1(\Omega)} = O(\|f\|_{H^{-1}(\Omega)}) = O(\varepsilon \ln \mu_\varepsilon) + O(\mu_\varepsilon^{-1}) + V(\varepsilon, N), \tag{2.3}$$

where

$$V(\varepsilon, N) = \begin{cases} O\Big(\frac{1}{(\mu_\varepsilon d_\varepsilon)^{N-2}}\Big) & N = 4, 5 \\ O\left(\frac{\Big(\ln(\mu_\varepsilon d_\varepsilon)\Big)^{\frac{2}{3}}}{(\mu_\varepsilon d_\varepsilon)^4}\right) & N = 6 \\ O\Big(\frac{1}{(\mu_\varepsilon d_\varepsilon)^{\frac{N+2}{2}}}\Big), & N > 6. \end{cases}$$

Proof. For simplicity, we denote $\|\cdot\|_{H_0^1(\Omega)}$ by $\|\cdot\|$. The estimate of $\|f\|_{H^{-1}(\Omega)}$ is obtained from the following, for any $\omega \in H_0^1(\Omega)$,

$$\begin{aligned}
\langle f, \omega\rangle &= \int_\Omega \Big(|x|^\alpha P W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1-\varepsilon} - |x_\varepsilon|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}\Big)\omega\, dx \\
&= \int_{B_{d_\varepsilon}(x_\varepsilon)} \Big(|x|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1-\varepsilon} - |x_\varepsilon|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}\Big)\omega\, dx \\
&\quad + O\Big(|\psi_{x_\varepsilon,\mu_\varepsilon}|_{L^\infty} \int_{B_{d_\varepsilon}(x_\varepsilon)} W^{2^*-2}\omega\, dx\Big) + O\big((\mu_\varepsilon d_\varepsilon)^{-\frac{N+2}{2}}\big)\|\omega\| \\
&= \int_{B_{d_\varepsilon}(x_\varepsilon)} \Bigg[|x|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}\Big(1 - \frac{N-2}{2}\varepsilon \ln \mu_\varepsilon + O(\varepsilon \ln \mu_\varepsilon)^2\Big) \\
&\qquad \cdot \Big(1+ \frac{N-2}{2}\varepsilon \ln(1 + \mu_\varepsilon^2|x - x_\varepsilon|^2) + O(\varepsilon^2)\Big) - |x_\varepsilon|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}\Bigg]\omega\, dx \\
&\quad + O\left((\mu_\varepsilon d_\varepsilon)^{2-N}\left[\int_{B_{\mu_\varepsilon d_\varepsilon}(0)} (1 + |z|^2)^{-\frac{4N}{N+2}}\, dz\right]^{\frac{N+2}{2N}}\right)\|\omega\| \\
&\quad + O\big((\mu_\varepsilon d_\varepsilon)^{-\frac{N+2}{2}}\big)\|\omega\| \\
&= \int_{B_{d_\varepsilon}(x_\varepsilon)} \big(|x|^\alpha - |x_\varepsilon|^\alpha\big) W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}\omega\, dx - \frac{N-2}{2}\varepsilon \ln \mu_\varepsilon \int_{B_{d_\varepsilon}(x_\varepsilon)} |x|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}\omega \\
&\quad + \frac{N-2}{2}\varepsilon \int_{B_{d_\varepsilon}(x_\varepsilon)} |x|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1} \ln(1 + \mu_\varepsilon^2|x - x_\varepsilon|^2)\omega\, dx \\
&\quad + O((\varepsilon \ln \mu_\varepsilon)^2)\|\omega\| + O\left((\mu_\varepsilon d_\varepsilon)^{2-N}\left(\int_{B_{\mu_\varepsilon d_\varepsilon}(0)} (1+|z|^2)^{-\frac{4N}{N+2}}\, dz\right)^{\frac{N+2}{2N}}\right)\|\omega\| \\
&\quad + O\big((\mu_\varepsilon d_\varepsilon)^{-\frac{N+2}{2}}\big)\|\omega\| \\
&= O(\varepsilon \ln \mu_\varepsilon)\|w\| + O(\mu_\varepsilon^{-1})\|w\| + V(\varepsilon, N)\|w\|.
\end{aligned}$$

Thus, the estimate of $\|\omega_\varepsilon\|$ is obtained by

$$\|f\|_{H^{-1}(\Omega)} = O(\varepsilon \ln \mu_\varepsilon) + O(\mu_\varepsilon^{-1}) + V(\varepsilon, N).$$

□

Next, let us give the estimate of u_ε away from x_ε.

Proposition 2.4. *Suppose that* $x \in \Omega \setminus B_{R\mu_\varepsilon^{-1}}(x_\varepsilon)$ *for* $R > 0$ *is any fixed large constant. Then*

$$u_\varepsilon(x) = A|x_\varepsilon|^{-\frac{\alpha}{2^*-2}} \frac{G(x, x_\varepsilon)}{\mu_\varepsilon^{\frac{N-2}{2}}} \left(1 + O\left(\frac{\ln(\mu_\varepsilon \tau)}{\mu_\varepsilon^2 \tau^2}\right)\right) + \omega_\varepsilon \quad \textit{in } \Omega \setminus B_{R\mu_\varepsilon^{-1}}(x_\varepsilon),$$

and

$$\nabla u_\varepsilon(x) = A|x_\varepsilon|^{-\frac{\alpha}{2^*-2}} \frac{\nabla_x G(x, x_\varepsilon)}{\mu_\varepsilon^{\frac{N-2}{2}}} \left(1 + O\left(\frac{\ln(\mu_\varepsilon \tau)}{\mu_\varepsilon^2 \tau^2}\right)\right) + \nabla\omega_\varepsilon \quad \textit{in } \Omega \setminus B_{R\mu_\varepsilon^{-1}}(x_\varepsilon).$$

where $A = \int_{\mathbb{R}^N} U_{0,1}^{2^*-1}\, dz$, *and* $\tau = \frac{1}{2}|x - x_\varepsilon|$.

Proof. By (1.3), $PW_{x_\varepsilon,\mu_\varepsilon}$ is expressed by

$$\begin{aligned} PW_{x_\varepsilon,\mu_\varepsilon} &= \int_\Omega G(x, y)|x_\varepsilon|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy \\ &= |x_\varepsilon|^\alpha \int_{B_\tau(x_\varepsilon)} G(x, y) W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy + O\left(\int_{\Omega \setminus B_\tau(x_\varepsilon)} G(x, y) U_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy\right). \end{aligned}$$

If $y \in \Omega \setminus (B_\tau(x_\varepsilon) \cup B_\tau(x))$, then $G(x, y) \le \frac{C}{|x-y|^{N-2}} \le C\tau^{2-N}$, which gives

$$\begin{aligned} \int_{\Omega \setminus (B_\tau(x_\varepsilon) \cup B_\tau(x))} G(x, y) U_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy &\le C\tau^{2-N} \int_{\Omega \setminus B_\tau(x_\varepsilon)} U_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy \\ &\le C\mu_\varepsilon^{-\frac{N-2}{2}} \tau^{2-N} \int_{\mathbb{R}^N \setminus B_{\mu_\varepsilon \tau}(0)} |z|^{-N-2}\, dz \\ &\le C\mu_\varepsilon^{-\frac{N+2}{2}} \tau^{-N}. \end{aligned}$$

If $y \in B_\tau(x)$, then $|y - x_\varepsilon| \ge |x - x_\varepsilon| - |y - x| \ge \tau$. Thus

$$\begin{aligned} \int_{B_\tau(x)} G(x, y) U_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy &\le C\mu_\varepsilon^{-\frac{N+2}{2}} \tau^{-N-2} \int_{B_\tau(x)} |x - y|^{2-N}\, dy \\ &\le C\mu_\varepsilon^{-\frac{N+2}{2}} \tau^{-N}. \end{aligned}$$

Therefore,

$$O\left(\int_{\Omega\setminus B_\tau(x_\varepsilon)} G(x,y)U_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy\right) = O\Big(\mu_\varepsilon^{-\frac{N+2}{2}}\tau^{-N}\Big).$$

On the other hand,

$$\begin{aligned}
&|x_\varepsilon|^\alpha \int_{B_\tau(x_\varepsilon)} G(x,y)W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}(y)dy\\
&= |x_\varepsilon|^{-\frac{\alpha}{2^*-2}} \int_{B_\tau(x_\varepsilon)} \Big(G(x,x_\varepsilon) + O(|D^2G||y-x_\varepsilon|^2)\Big)U_{x_\varepsilon,\mu_\varepsilon}^{2^*-1}dy\\
&= |x_\varepsilon|^{-\frac{\alpha}{2^*-2}} G(x,x_\varepsilon)\mu_\varepsilon^{-\frac{N-2}{2}} \int_{\mathbb{R}^N} U_{0,1}^{2^*-1}\,dz + O\left(\frac{\ln(\mu_\varepsilon\tau)}{\mu_\varepsilon^{\frac{N+2}{2}}\tau^N}\right).
\end{aligned}$$

Thus, the first one is proved, and the other is similar. □

To obtain our main results, we will apply the following Pohozaev identities, which are inspired by [21]:

$$\begin{aligned}
&-\int_{\partial B_\sigma(x_\varepsilon)} \frac{\partial u_\varepsilon}{\partial \nu}\big\langle x-x_\varepsilon, \nabla u_\varepsilon\big\rangle\\
&+\frac{1}{2}\int_{\partial B_\sigma(x_\varepsilon)} |\nabla u_\varepsilon|^2\big\langle x-x_\varepsilon,\nu\big\rangle + \frac{2-N}{2}\int_{\partial B_\sigma(x_\varepsilon)} \frac{\partial u_\varepsilon}{\partial\nu}u_\varepsilon\\
&= \frac{1}{2^*-\varepsilon}\int_{\partial B_\sigma(x_\varepsilon)} |x|^\alpha u_\varepsilon^{2^*-\varepsilon}\big\langle x-x_\varepsilon,\nu\big\rangle + \left(\frac{N-2}{2} - \frac{N}{2^*-\varepsilon}\right)\int_{B_\sigma(x_\varepsilon)} |x|^\alpha u_\varepsilon^{2^*-\varepsilon}\,dx\\
&-\frac{1}{2^*-\varepsilon}\int_{B_\sigma(x_\varepsilon)} u_\varepsilon^{2^*-\varepsilon}\big\langle x-x_\varepsilon, \nabla(|x|^\alpha)\big\rangle\,dx
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_{\partial B_\sigma(x_\varepsilon)} |\nabla u_\varepsilon|^2\nu_i\,dS - \int_{\partial B_\sigma(x_\varepsilon)} \langle\nabla u_\varepsilon,\nu\rangle\frac{\partial u_\varepsilon}{\partial x_i}dS\\
&= \frac{1}{2^*-\varepsilon}\int_{\partial B_\sigma(x_\varepsilon)} |x|^\alpha u_\varepsilon^{2^*-\varepsilon}\nu_i\,dS - \frac{1}{2^*-\varepsilon}\int_{B_\sigma(x_\varepsilon)} \frac{\partial(|x|^\alpha)}{\partial x_i}u_\varepsilon^{2^*-\varepsilon}dx,
\end{aligned} \tag{2.5}$$

where ν is the outward unit normal of $\partial B_\sigma(x_\varepsilon)$.

Proof of Theorem 1.2. Firstly, we have:

$$d_\varepsilon^{\frac{2}{N}}\int_{B_{2d_\varepsilon/3}(x_\varepsilon)\setminus B_{d_\varepsilon/2}(x_\varepsilon)} (|\nabla\omega_\varepsilon|^2 + \omega_\varepsilon^2) + \int_{d_\varepsilon/2}^{2d_\varepsilon/3}\left(\int_{\partial B_t(x_\varepsilon)} |\omega_\varepsilon|^{2^*}\right)^{2/2^*} \le Cd_\varepsilon^{\frac{2}{N}}\|\omega_\varepsilon\|^2.$$

So we take a $\sigma_\varepsilon \in (\frac{d_\varepsilon}{2}, \frac{2d_\varepsilon}{3})$, such that

$$\frac{d_\varepsilon}{6}\left(d_\varepsilon^{\frac{2}{N}}\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}(|\nabla\omega_\varepsilon|^2+\omega_\varepsilon^2)+\left(\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}|\omega_\varepsilon|^{2^*}\right)^{2/2^*}\right)\le Cd_\varepsilon^{\frac{2}{N}}\|\omega_\varepsilon\|^2,$$

which gives

$$\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}(|\nabla\omega_\varepsilon|^2+\omega_\varepsilon^2)\le Cd_\varepsilon^{-1}\|\omega_\varepsilon\|^2 \text{ and } \left(\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}|\omega_\varepsilon|^{2^*}\right)^{2/2^*}\le Cd_\varepsilon^{\frac{2}{N}-1}\|\omega_\varepsilon\|^2.$$

Let $\sigma = \sigma_\varepsilon$ in the Pohozaev identities (2.4) and (2.5). By Proposition 2.4 and Lemma 2.3, it holds

$$\begin{aligned}\text{LHS of (2.4)} = A^2|x_\varepsilon|^{-\frac{2\alpha}{2^*-2}}\mu_\varepsilon^{2-N}\bigg\{&-\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}\frac{\partial G}{\partial\nu}\langle x-x_\varepsilon,\nabla G\rangle\\ &+\frac{1}{2}\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}|\nabla G|^2\langle x-x_\varepsilon,\nu\rangle\quad+\frac{2-N}{2}\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}\frac{\partial G}{\partial\nu}G\bigg\}\\ &+O\Big((\mu_\varepsilon d_\varepsilon)^{-\frac{N-2}{2}}\big(\varepsilon\ln\mu_\varepsilon+\mu_\varepsilon^{-1}+V(\varepsilon,N)\big)\Big).\end{aligned}$$

It is easy to calculate

$$\left|\frac{1}{2^*-\varepsilon}\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}|x|^\alpha u_\varepsilon^{2^*-\varepsilon}\langle x-x_\varepsilon,\nu\rangle\right| = O((\mu_\varepsilon^N d_\varepsilon^N)^{-1}).$$

For the second term in the right-hand side, since

$$\frac{N}{2^*-\varepsilon}=\frac{N}{2^*}+\frac{N}{(2^*)^2}\varepsilon+O(\varepsilon^2),$$

we have

$$\begin{aligned}&\left(\frac{N-2}{2}-\frac{N}{2^*-\varepsilon}\right)\int_{B_{\sigma_\varepsilon}(x_\varepsilon)}|x|^\alpha u_\varepsilon^{2^*-\varepsilon}\,dx\\ &=\left(-\frac{N}{(2^*)^2}\varepsilon+O(\varepsilon^2)\right)\int_{B_{\sigma_\varepsilon}(x_\varepsilon)}|x|^\alpha W_{x_\varepsilon,\mu_\varepsilon}^{2^*-\varepsilon}(x)\,dx\\ &\quad+O\left(\varepsilon\int_{B_{\sigma_\varepsilon}(x_\varepsilon)}W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1-\varepsilon}(x)\big(|\psi_{x_\varepsilon,\mu_\varepsilon}|+|\omega_\varepsilon|\big)\right)\\ &=\left(-\frac{N}{(2^*)^2}\varepsilon+O(\varepsilon^2)\right)|x_\varepsilon|^{-\frac{2^*\alpha}{2^*-2}}\mu_\varepsilon^{-\frac{(N-2)\varepsilon}{2}}\int_{B_{\mu_\varepsilon\sigma_\varepsilon}(0)}\left|\frac{z}{\mu_\varepsilon}+x_\varepsilon\right|^\alpha U_{0,1}^{2^*-\varepsilon}(z)dz\\ &\quad+O\Big(\frac{\varepsilon}{(\mu_\varepsilon d_\varepsilon)^{N-2}}\Big)+O\Big(\varepsilon\|\omega_\varepsilon\|\Big)=-\frac{N}{(2^*)^2}\varepsilon|x_\varepsilon|^{-\frac{2\alpha}{2^*-2}}\int_{\mathbb{R}^N}U_{0,1}^{2^*}(z)\,dz\\ &\quad+O\big(\varepsilon^2\ln\mu_\varepsilon\big)+O\left(\frac{\varepsilon}{\mu_\varepsilon^2}\right)+O\left(\frac{\varepsilon}{(\mu_\varepsilon d_\varepsilon)^{N-2}}\right)+O\Big(\varepsilon\|\omega_\varepsilon\|\Big).\end{aligned}$$

The last term of equation (2.4) can be estimated as

$$\begin{aligned}
&-\frac{1}{2^*-\varepsilon}\int_{B_{\sigma_\varepsilon}(x_\varepsilon)} u_\varepsilon^{2^*-\varepsilon}\big\langle x-x_\varepsilon, \nabla(|x|^\alpha)\big\rangle\,dx\\
&= -\frac{1}{2^*-\varepsilon}\int_{B_{\sigma_\varepsilon}(x_\varepsilon)} W_{x_\varepsilon,\mu_\varepsilon}^{2^*-\varepsilon}\big\langle x-x_\varepsilon, \nabla(|x|^\alpha)\big\rangle dx\\
&\quad + O\left(\int_{B_{\sigma_\varepsilon}(x_\varepsilon)} W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1-\varepsilon}\big(|\psi_{x_\varepsilon,\mu_\varepsilon}|+|\omega_\varepsilon|\big)\big\langle x-x_\varepsilon, \nabla(|x|^\alpha)\big\rangle\,dx\right)\\
&= -\frac{1}{2^*-\varepsilon}|x_\varepsilon|^{-\frac{(2^*-\varepsilon)\alpha}{2^*-2}}\mu_\varepsilon^{-\frac{(N-2)\varepsilon}{2}}\int_{B_{\mu_\varepsilon\sigma_\varepsilon}(0)} U_{0,1}^{2^*-\varepsilon}\Big\langle \frac{z}{\mu_\varepsilon}, \nabla\Big(|\frac{z}{\mu_\varepsilon}+x_\varepsilon|^\alpha\Big)\Big\rangle\,dx\\
&\quad + O\big(\mu_\varepsilon^{1-N}d_\varepsilon^{2-N}\big)+O\Big(\mu_\varepsilon^{-1}\|\omega_\varepsilon\|\Big)\\
&= O\big(\mu_\varepsilon^{-2}\big)+O\big(\mu_\varepsilon^{1-N}d_\varepsilon^{2-N}\big)+O\Big(\mu_\varepsilon^{-1}\|\omega_\varepsilon\|\Big).
\end{aligned}$$

By [6, Proposition 6.2.4], we know that

$$\begin{aligned}
&-\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)} \frac{\partial G(x,x_\varepsilon)}{\partial\nu}\big\langle x-x_\varepsilon, \nabla G(x,x_\varepsilon)\big\rangle\\
&+\frac{1}{2}\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)} |\nabla G(x,x_\varepsilon)|^2\big\langle x-x_\varepsilon, \nu\big\rangle\\
&+\frac{2-N}{2}\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)} \frac{\partial G(x,x_\varepsilon)}{\partial\nu}G(x,x_\varepsilon)\\
&= -\frac{N-2}{2}H(x_\varepsilon,x_\varepsilon).
\end{aligned}$$

Combining with the above estimates, we have

$$\begin{aligned}
&-\frac{N-2}{2}A^2|x_\varepsilon|^{-\frac{2\alpha}{2^*-2}}\mu_\varepsilon^{2-N}H(x_\varepsilon,x_\varepsilon)\\
&= -\frac{N}{(2^*)^2}\varepsilon|x_\varepsilon|^{-\frac{2\alpha}{2^*-2}}\int_{\mathbb{R}^N} U_{0,1}^{2^*}(z)\,dz + O\big(\varepsilon^2\ln\mu_\varepsilon\big)\\
&\quad + O\left(\frac{\varepsilon}{(\mu_\varepsilon d_\varepsilon)^{N-2}}\right)+O\Big(\varepsilon\|\omega_\varepsilon\|\Big)+O\big(\mu_\varepsilon^{-2}\big) \qquad (2.6)\\
&\quad + O\big(\mu_\varepsilon^{1-N}d_\varepsilon^{2-N}\big)\\
&\quad + O\Big(\mu_\varepsilon^{-1}\|\omega_\varepsilon\|\Big)+O\Big((\mu_\varepsilon d_\varepsilon)^{-\frac{N-2}{2}}\big(\varepsilon\ln\mu_\varepsilon+\mu_\varepsilon^{-1}+V(\varepsilon,N)\big)\Big).
\end{aligned}$$

On the other hand, we estimate each term in the Pohozaev identity (2.5) through Proposition 2.4. Similarly,

$$\begin{aligned}
&\text{LHS of (2.5)}\\
&= A^2|x_\varepsilon|^{-\frac{2\alpha}{2^*-2}}\mu_\varepsilon^{2-N}\\
&\quad\times\left\{\frac{1}{2}\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}|\nabla G(x,x_\varepsilon)|^2\nu_i\,dS-\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}\frac{\partial G(x,x_\varepsilon)}{\partial\nu}\frac{\partial G(x,x_\varepsilon)}{\partial x_i}\,dS\right\}\\
&\quad+d_\varepsilon^{-1}O\Big((\mu_\varepsilon d_\varepsilon)^{-\frac{N-2}{2}}\big(\varepsilon\ln\mu_\varepsilon+\mu_\varepsilon^{-1}+V(\varepsilon,N)\big)\Big).
\end{aligned}$$

It is easy to estimate

$$\left|\int_{\partial B_{\sigma_\varepsilon}(x_\varepsilon)}|x|^\alpha u_\varepsilon^{2^*-\varepsilon}\nu_i\,dS\right|=O\left(\frac{1}{\mu_\varepsilon^N d_\varepsilon^{N+1}}\right).$$

Since $\mu_\varepsilon d\to+\infty$, we can evaluate the second term in the right-hand side of (2.5):

$$\begin{aligned}
&-\frac{1}{2^*-\varepsilon}\int_{B_{\sigma_\varepsilon}(x_\varepsilon)}\frac{\partial(|x|^\alpha)}{\partial x_i}u_\varepsilon^{2^*-\varepsilon}\,dx\\
&=-\frac{1}{2^*-\varepsilon}\int_{B_{\sigma_\varepsilon}(x_\varepsilon)}\frac{\partial(|x|^\alpha)}{\partial x_i}W_{x_\varepsilon,\mu_\varepsilon}^{2^*-\varepsilon}\,dx\\
&\quad+O\left(\int_{B_{\sigma_\varepsilon}(x_\varepsilon)}\frac{\partial(|x|^\alpha)}{\partial x_i}W_{x_\varepsilon,\mu_\varepsilon}^{2^*-1-\varepsilon}\big(|\psi_{x_\varepsilon,\mu_\varepsilon}|+|\omega_\varepsilon|\big)\,dx\right)\\
&=-\frac{1}{2^*}\big(1+O(\varepsilon)\big)|x_\varepsilon|^{-\frac{2^*\alpha}{2^*-2}}\frac{\partial(|x|^\alpha)}{\partial x_i}\Big|_{x=x_\varepsilon}\int_{\mathbb{R}^N}U_{0,1}^{2^*}\,dz\,(1+O(\varepsilon\ln\mu_\varepsilon))\\
&\quad+O\Big(\frac{\partial(|x|^\alpha)}{\partial x_i}\Big|_{x=x_\varepsilon}\big((\mu_\varepsilon d_\varepsilon)^{2-N}+\|\omega_\varepsilon\|\big)\Big)\\
&=-\frac{1}{2^*}|x_\varepsilon|^{-\frac{2^*\alpha}{2^*-2}}\frac{\partial(|x|^\alpha)}{\partial x_i}\Big|_{x=x_\varepsilon}\int_{\mathbb{R}^N}U_{0,1}^{2^*}\,dz\Big(1+O(\varepsilon\ln\mu_\varepsilon)\\
&\qquad\qquad+O\big((\mu_\varepsilon d_\varepsilon)^{2-N}\big)+O\big(\|\omega_\varepsilon\|\big)\Big).
\end{aligned}$$

By [6, Proposition 6.2.3], we have:

$$\begin{aligned}
&A^2|x_\varepsilon|^{-\frac{2\alpha}{2^*-2}}\mu_\varepsilon^{2-N}\frac{\partial H(x,x_\varepsilon)}{\partial x_i}\Big|_{x=x_\varepsilon}\\
&=-\frac{1}{2^*}|x_\varepsilon|^{-\frac{2^*\alpha}{2^*-2}}\frac{\partial(|x|^\alpha)}{\partial x_i}\Big|_{x=x_\varepsilon}\int_{\mathbb{R}^N}U_{0,1}^{2^*}\,dz\Big(1+O(\varepsilon\ln\mu_\varepsilon)\\
&\qquad\qquad+O\big((\mu_\varepsilon d_\varepsilon)^{2-N}\big)+O\big(\|\omega_\varepsilon\|\big)\Big)\\
&\quad+d_\varepsilon^{-1}O\Big((\mu_\varepsilon d_\varepsilon)^{-\frac{N-2}{2}}\big(\varepsilon\ln\mu_\varepsilon+\mu_\varepsilon^{-1}+V(\varepsilon,N)\big)\Big).
\end{aligned}$$

Let us construct new coordinates by the tangent vectors τ_i and the normal vector ν, for $1 \le i \le N-1$ at $\bar{x}_\varepsilon$, where $d_\varepsilon = |x_\varepsilon - \bar{x}_\varepsilon|$ and $\bar{x}_\varepsilon \in \partial\Omega$. Since

$$\frac{\partial H(x_\varepsilon, x_\varepsilon)}{\partial \tau_i} = \sum_{j=1}^{N} \frac{\partial H(x_\varepsilon, x_\varepsilon)}{\partial x_j} \frac{\partial x_j}{\partial \tau_i}, \quad \text{and} \quad \frac{\partial H(x_\varepsilon, x_\varepsilon)}{\partial \nu} = \sum_{j=1}^{N} \frac{\partial H(x_\varepsilon, x_\varepsilon)}{\partial x_j} \frac{\partial x_j}{\partial \nu}.$$

Thus, (2) can be transformed into

$$\begin{aligned} & A^2 |x_\varepsilon|^{-\frac{2\alpha}{2^*-2}} \mu_\varepsilon^{2-N} \frac{\partial H(x, x_\varepsilon)}{\partial \tau_i}\Big|_{x=x_\varepsilon} \\ &= -\frac{1}{2^*} |x_\varepsilon|^{-\frac{2^*\alpha}{2^*-2}} \frac{\partial(|x|^\alpha)}{\partial \tau_i}\Big|_{x=x_\varepsilon} \int_{\mathbb{R}^N} U_{0,1}^{2^*}\, dz \Big(1 + O(\varepsilon \ln \mu_\varepsilon) + O\big((\mu_\varepsilon d_\varepsilon)^{2-N}\big) \\ &\qquad + O\big(\|\omega_\varepsilon\|\big)\Big) \\ &\quad + d_\varepsilon^{-1} O\Big((\mu_\varepsilon d_\varepsilon)^{-\frac{N-2}{2}} \big(\varepsilon \ln \mu_\varepsilon + \mu_\varepsilon^{-1} + V(\varepsilon, N)\big)\Big). \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} & A^2 |x_\varepsilon|^{-\frac{2\alpha}{2^*-2}} \mu_\varepsilon^{2-N} \frac{\partial H(x, x_\varepsilon)}{\partial \nu}\Big|_{x=x_\varepsilon} \\ &= -\frac{1}{2^*} |x_\varepsilon|^{-\frac{2^*\alpha}{2^*-2}} \frac{\partial(|x|^\alpha)}{\partial \nu}\Big|_{x=x_\varepsilon} \int_{\mathbb{R}^N} U_{0,1}^{2^*}\, dz \Big(1 + O(\varepsilon \ln \mu_\varepsilon) \\ &\qquad + O\big((\mu_\varepsilon d_\varepsilon)^{2-N}\big) + O\big(\|\omega_\varepsilon\|\big)\Big) \\ &\quad + d_\varepsilon^{-1} O\Big((\mu_\varepsilon d_\varepsilon)^{-\frac{N-2}{2}} \big(\varepsilon \ln \mu_\varepsilon + \mu_\varepsilon^{-1} + V(\varepsilon, N)\big)\Big). \end{aligned} \tag{2.8}$$

As we know that the following estimates are true:

$$\begin{aligned} H(x_\varepsilon, x_\varepsilon) &= \frac{1}{(N-2)\omega_{N-1}(2d_\varepsilon)^{N-2}} \Big(1 + O(d_\varepsilon)\Big), \\ \frac{\partial H(x_\varepsilon, x_\varepsilon)}{\partial \tau_i} &= O(d_\varepsilon^{2-N}), \\ \frac{\partial H(x_\varepsilon, x_\varepsilon)}{\partial \nu} &= -\frac{1}{\omega_{N-1}(2d_\varepsilon)^{N-1}} \Big(1 + O(d_\varepsilon)\Big). \end{aligned}$$

then, through (2.6), (2.7) and (2.8), we can solve

$$d_\varepsilon = \gamma_1 \varepsilon + \begin{cases} O(\varepsilon^{\frac{3}{2}} \ln \varepsilon) & N = 4, 5, 6 \\ O(\varepsilon^{1+\frac{2}{N-2}}) & N > 6 \end{cases} \tag{2.9}$$

and

$$\mu_\varepsilon = \gamma_2 \varepsilon^{-\frac{N-1}{N-2}} + \begin{cases} O(\varepsilon^{-\frac{N-1}{N-2}+\frac{1}{2}} \ln \varepsilon) & N = 4, 5, 6 \\ O(\varepsilon^{-\frac{N-1}{N-2}+\frac{2}{N-2}}) & N > 6. \end{cases} \tag{2.10}$$

Moreover,

$$\frac{\partial(|x|^{\alpha})}{\partial\tau_i}|_{x=x_\varepsilon} = O(\varepsilon) \text{ and } \frac{\partial(|x|^{\alpha})}{\partial\nu}|_{x=x_\varepsilon} = \alpha_0 + o(1),$$

for some $\alpha_0 \neq 0$. □

3. Existence of multiple-peak solutions

In this section, we will consider the existence result for the Hénon equation. The Lyapunov-Schmit reduction procedure will be applied to construct peak solutions.

3.1. Finite-dimensional reduction

The functional associated to problem (1.2) is

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^* - \varepsilon}\int_\Omega |x|^\alpha u^{2^*-\varepsilon}.$$

For $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_k) \in \Omega \times \Omega \times \cdots \times \Omega$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_k) \in \mathbb{R}^k_+$, we define

$$D^k_{\boldsymbol{\xi},\boldsymbol{\mu}} = \Big\{(\boldsymbol{\xi}, \boldsymbol{\mu}) \in (\Omega)^k \times \mathbb{R}^k_+ : d_i \in [L_0\varepsilon, L_1\varepsilon], \\ \mu_i \in [L_0\varepsilon^{-\frac{N-1}{N-2}}, L_1\varepsilon^{-\frac{N-1}{N-2}}], \xi_i \in B_\theta(\xi_{i,0})\Big\},$$

where $\theta > 0$ is a small constant, $d_i = \text{dist}(\xi_i, \partial\Omega)$, L_0 and L_1 are constants.

Define

$$E^k_\omega = \Big\{\omega \in H^1_0(\Omega) : \Big\langle \omega, \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}}\Big\rangle = \Big\langle \omega, \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i}\Big\rangle = 0, \\ \text{for } i = 1, \cdots, k, \ j = 1, \cdots, N\Big\}$$

and

$$(E^k_\omega)^\perp = \text{span}\left\{\frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}}, \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i}, i = 1, \cdots, k, \ j = 1, \cdots, N\right\},$$

where $\langle u, v\rangle = \int_\Omega \nabla u \nabla v$. Let

$$S^k_{\boldsymbol{\xi},\boldsymbol{\mu},\omega} = \Big\{(\boldsymbol{\xi}, \boldsymbol{\mu}, \omega) : (\boldsymbol{\xi}, \boldsymbol{\mu}) \in D^k_{\boldsymbol{\xi},\boldsymbol{\mu}}, \omega \in E^k_\omega, \text{ and } \|\omega\| \le C\Big\}$$

and

$$J(\boldsymbol{\xi}, \boldsymbol{\mu}, \omega) = I\left(\sum_{i=1}^k PW_{\xi_i,\mu_i} + \omega\right).$$

Expanding the functional J at $\omega = 0$,

$$J(\boldsymbol{\xi}, \boldsymbol{\mu}, \omega) = J(\boldsymbol{\xi}, \boldsymbol{\mu}, 0) + \langle L, \omega\rangle + \frac{1}{2}\langle Q\omega, \omega\rangle + R(\omega),$$

where

$$\langle L_\varepsilon, \omega\rangle = \sum_{i=1}^{k}\int_\Omega \nabla PW_{\xi_i,\mu_i}\nabla\omega - \int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon}\omega$$

$$\langle Q\omega, \omega\rangle = \int_\Omega |\nabla\omega|^2 - (2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega^2;$$

and

$$\begin{aligned} R(\omega) =& -\left\{\frac{1}{2^*-\varepsilon}\int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}+\omega\right)^{2^*-\varepsilon} - \int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon}\omega\right. \\ &\left. -\frac{2^*-1-\varepsilon}{2}\int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega^2\right\}. \end{aligned}$$

Lemma 3.1. *The operator Q is invertible in E_ω^k. Moreover, there exists a positive constant $\rho > 0$, such that*

$$\|Q\omega\| \geq \rho\|\omega\|, \ \forall\omega \in E_\omega^k.$$

Proof. This proof is similar to [17, Appendix A], so we only outline it briefly. We will argue by contradiction. Assume that there exists a sequence $\{\omega_n\} \in E_\omega^k$, such that

$$\|Q\omega_n\| = o(1)\|\omega_n\|,$$

where $o(1) \to 0$ as $\varepsilon_n \to 0$. For simplicity, we denote ε_n, $\xi_{i,n}$ and $\mu_{i,n}$ by ε , ξ_i and μ_i, respectiviely. Without loss of generality, we assume that $\|\omega_n\| = 1$.

For $\forall\varphi \in E_\omega^k$, we have

$$\int_\Omega \nabla\omega_n\varphi - (2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_n\varphi = o(\|\varphi\|).$$

In particular,

$$\int_\Omega |\nabla\omega_n|^2 - (2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_n^2 = o(1).$$

Set $\tilde{\omega}_n(x) = (\mu_1)^{-\frac{N-2}{2}}\omega_n(\xi_1 + \frac{x}{\mu_1})$.

Then

$$\tilde{\omega}_n \rightharpoonup \omega, \text{ weakly in } D^{1,2}(\mathbb{R}^N),$$

and

$$\tilde{\omega}_n \to \omega, \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^N).$$

Therefore,

$$-\Delta\omega - (2^* - 1)U_{0,1}^{2^*-2}\omega = 0, \text{ in } \mathbb{R}^N,$$

which implies that

$$\omega = b_0 \frac{\partial U_{0,\mu}}{\partial \mu}\Big|_{\mu=1} + \sum_{j=1}^{N} b_j \frac{\partial U_{0,1}}{\partial x_j}.$$

Since $\omega_n \in E^k_\omega$, it is easy to prove that

$$\int_{\mathbb{R}^N} \nabla\omega\nabla\frac{\partial U_{0,\mu}}{\partial \mu}|_{\mu=1} = \int_{\mathbb{R}^N} \nabla\omega\nabla\frac{\partial U_{0,1}}{\partial x_j} = 0, \quad j = 1, \cdots, N.$$

Thus, $\omega = 0$, which contradicts with $\|\omega_n\| = 1$. □

Lemma 3.2. *There exists a positive constant C, such that*

$$\|L_\varepsilon\| = O\left(\sum_{i=1}^{k}\left(\frac{1}{\mu_i} + \varepsilon \ln \mu_i\right)\right) + \sum_{i=1}^{k} V(\varepsilon, d_i, \mu_i) + O\left(\sum_{i=1, l<i}^{k} \varepsilon_{il}^{\frac{1}{2}+\theta}\right),$$

where

$$V(\varepsilon, d_i, \mu_i) = \begin{cases} O\big(\frac{1}{(\mu_i d_i)^{N-2}}\big) & N = 4, 5 \\ O\big(\frac{(\ln \mu_i d_i)^{\frac{2}{3}}}{(\mu_i d_i)^4}\big) & N = 6 \\ O\big(\frac{1}{(\mu_i d_i)^{\frac{N+2}{2}}}\big) & N > 6. \end{cases}$$

Proof. Since

$$\begin{aligned} &\int_\Omega |x|^\alpha \left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon} \omega \\ &= \sum_{i=1}^{k}\int_\Omega |x|^\alpha PW_{\xi_i,\mu_i}^{2^*-1-\varepsilon}\omega \\ &+ \begin{cases} O\Big(\sum_{i=1}^{k}\int_\Omega |x|^\alpha PW_{\xi_i,\mu_i}^{2^*-2-\varepsilon}\sum_{l\neq i} PW_{\xi_l,\mu_l}\omega\Big) & 2^*-1-\varepsilon \geq 2 \\ O\Big(\sum_{i=1}^{k}\int_\Omega |x|^\alpha PW_{\xi_i,\mu_i}^{\frac{2^*-2-\varepsilon}{2}}\sum_{l<i} PW_{\xi_l,\mu_l}^{\frac{2^*-1-\varepsilon}{2}}\omega\Big) & 2^*-1-\varepsilon < 2. \end{cases} \end{aligned}$$

Then,

$$
\begin{aligned}
\langle L_\varepsilon, \omega\rangle = {} & O\left(\sum_{i=1}^{k}\int_\Omega (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-1}\omega\right) + O\left(\varepsilon \sum_{i=1}^{k}\int_\Omega W_{\xi_i,\mu_i}^{2^*-1}\ln W_{\xi_i,\mu_i}\omega\right) \\
& + O\left(\sum_{i=1}^{k}\int_\Omega W_{\xi_i,\mu_i}^{2^*-1}\psi_{\xi_i,\mu_i}\omega\right) \\
& + \begin{cases} O\Big(\sum_{i=1}^{k}\int_\Omega W_{\xi_i,\mu_i}^{2^*-2-\varepsilon}\sum_{l\neq i} W_{\xi_l,\mu_l}\omega\Big) & 2^*-1-\varepsilon \ge 2 \\ O\Big(\sum_{i=1}^{k}\int_\Omega W_{\xi_i,\mu_i}^{\frac{2^*-2-\varepsilon}{2}}\sum_{l<i} W_{\xi_l,\mu_l}^{\frac{2^*-1-\varepsilon}{2}}\omega\Big) & 2^*-1-\varepsilon < 2. \end{cases} \\
= {} & O\Big(\sum_{i=1}^{k}\Big(\frac{1}{\mu_i} + \varepsilon\ln\mu_i\Big)\Big)\|\omega\| + \sum_{i=1}^{k} V(\varepsilon, d_i, \mu_i)\|\omega\| \\
& + O\Big(\sum_{i=1,l<i}^{k}\varepsilon_{il}^{\frac{1}{2}+\theta}\Big)\|\omega\|.
\end{aligned}
$$

□

Proposition 3.3. *There exists only one $\omega_{\boldsymbol{\xi},\boldsymbol{\mu}} \in E_\omega^k$, such that*

$$
-\Delta\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i} + \omega\right) - |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i} + \omega\right)^{2^*-1-\varepsilon} \in (E_\omega^k)^\perp. \tag{3.1}
$$

Moreover,

$$
\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\| = O\left(\sum_{i=1}^{k}\left(\frac{1}{\mu_i} + \varepsilon\ln\mu_i\right)\right) + \sum_{i=1}^{k} V(\varepsilon, d_i, \mu_i) + O\left(\sum_{i=1,l<i}^{k}\varepsilon_{il}^{\frac{1}{2}+\theta}\right).
$$

Proof. As we know, the problem is equivalent to finding a solution of

$$
L_\varepsilon + Q\omega + R'(\omega) = 0.
$$

The fixed point theorem is applied to this problem. Set

$$
S = \left\{\omega \in E_\omega^k,\ \|\omega\| \le C\left(\sum_{i=1}^{k}\left(\frac{1}{\mu_i} + \varepsilon\ln\mu_i\right) + \sum_{i=1}^{k} V(\varepsilon, d_i, \mu_i) + \sum_{i=1,l<i}^{k}\varepsilon_{il}^{\frac{1}{2}+\theta}\right)^{1-\sigma}\right\}.
$$

and define the operator

$$
A\omega := -Q^{-1}\big(L_\varepsilon + R'(\omega)\big).
$$

Then, by Lemmas 3.1 and 3.2,

$$\begin{aligned}\|A\omega\| &\leq \rho^{-1}\big(\|L_\varepsilon\| + \|\omega\|^{\min\{2^*-1-\varepsilon,2\}}\big)\\ &\leq C\left(\sum_{i=1}^{k}\left(\frac{1}{\mu_i} + \varepsilon\ln\mu_i\right) + \sum_{i=1}^{k} V(\varepsilon, d_i, \mu_i) + \sum_{i=1,l<i}^{k} \varepsilon_{il}^{\frac{1}{2}+\theta}\right)^{1-\sigma}.\end{aligned}$$

Moreover, for $\omega_1, \omega_2 \in S$, we have

$$\begin{aligned}\|A\omega_1 - A\omega_2\| &= \|Q^{-1}(R'(\omega_1) - R'(\omega_2))\|\\ &\leq \rho^{-1}(\|\omega_1\| + \|\omega_2\|)^{\min\{1,2^*-2-\varepsilon\}}\|\omega_1 - \omega_2\|\\ &\leq \frac{1}{2}\|\omega_1 - \omega_2\|.\end{aligned}$$

Therefore, A is a contraction map from S to S. By fixed point theorem, there exists a unique $\omega_{\boldsymbol{\xi},\boldsymbol{\mu}} \in S$, such that

$$\omega_{\boldsymbol{\xi},\boldsymbol{\mu}} = -Q^{-1}L_\varepsilon - Q^{-1}R'(\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}),$$

which implies

$$\begin{aligned}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\| \leq C\|L_\varepsilon\| &= O\left(\sum_{i=1}^{k}\left(\frac{1}{\mu_i} + \varepsilon\ln\mu_i\right)\right) + \sum_{i=1}^{k} V(\varepsilon, d_i, \mu_i)\\ &\quad + O\left(\sum_{i=1,l<i}^{k} \varepsilon_{il}^{\frac{1}{2}+\theta}\right).\end{aligned}$$

□

3.2. Main result

We will give the following estimates before the existence result is proved.

Lemma 3.4. *We have the estimates*

$$\begin{aligned}&\int_\Omega \nabla\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i} + \omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\\ &\quad - \int_\Omega |x|^\alpha\left(\sum_{i=1}^{k} PW_{\xi_i,\mu_i} + \omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\\ =&\ -\partial_j(|\xi_l|^\alpha)|\xi_l|^{-\frac{2^*\alpha}{2^*-2}}\beta_\varepsilon^l\int_{\mathbb{R}^N} U_{0,1}^{2^*-1}\frac{\partial U_{0,1}}{\partial z_j}z_j dz\\ &\quad + (2^*-1-\varepsilon)|\xi_l|^{-\frac{2\alpha}{2^*-2}}\beta_\varepsilon^l\frac{1}{(\mu_l)^{N-2}}\frac{\partial H(\xi_l,\xi_l)}{\partial \xi_{l,j}}\int_{\mathbb{R}^N} U_{0,1}^{2^*-2}z_j\frac{\partial U_{0,1}}{\partial z_j}\\ &\quad + O\left(\frac{\ln(\mu_l d_l)}{(\mu_l)^N(d_l)^{N+1}}\right) + O\big(\varepsilon\partial_j(|\xi_l|^\alpha)\big) + O\left(\frac{1}{(\mu_l)^2}\right) + O\left(\frac{\varepsilon}{(\mu_l)^{N-2}(d_l)^{N-1}}\right)\end{aligned}$$

and

$$
\begin{aligned}
&\int_\Omega \nabla\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&-\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
=&-|\xi_l|^{-\frac{2\alpha}{2^*-2}}\frac{N-2}{2}\frac{\varepsilon}{\mu_l}\int_{\mathbb{R}^N}\ln(1+|z|^2)U_{0,1}^{2^*-1}\frac{\partial U_{0,\lambda}}{\partial\lambda}|_{\lambda=1}dz\\
&+(2^*-1)|\xi_l|^{-\frac{2\alpha}{2^*-2}}\beta_\varepsilon^l\frac{H(\xi_l,\xi_l)}{(\mu_l)^{N-1}}\int_{\mathbb{R}^N}U_{0,1}^{2^*-2}\frac{\partial U_{0,\lambda}}{\partial\lambda}|_{\lambda=1}dz\\
&+O\left(\frac{1}{(\mu_l)^{N+1}(d_l)^N}\right)+O\left(\frac{1}{(\mu_l)^3}\right)+O\left(\frac{1}{(\mu_l)^N(d_l)^{N-2}}\right)\\
&+O\left(\frac{\varepsilon}{(\mu_l)^{N-1}(d_l)^{N-2}}\right)+\begin{cases}O\big(\frac{\ln(\mu_l d_l)}{(\mu_l)^5(d_l)^4}\big) & N=4\\ O\big(\frac{1}{(\mu_l)^{N+1}(d_l)^N}\big) & N\geq 5.\end{cases}
\end{aligned}
$$

Proof. Since $\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\in E_\omega^k$, we have

$$
\begin{aligned}
&\int_\Omega \nabla\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}\\
&-\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}\\
=&\int_\Omega \nabla\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}-\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}\\
&-(2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}+O\big(\mu_l\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|^2\big)\\
=&\int_\Omega \nabla PW_{\xi_l,\mu_l}\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}-\int_\Omega |x|^\alpha(PW_{\xi_l,\mu_l})^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}\\
&+\int_\Omega \nabla\left(\sum_{i\neq l} PW_{\xi_i,\mu_i}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}\\
&-\int_\Omega |x|^\alpha\left[\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon}-(PW_{\xi_l,\mu_l})^{2^*-1-\varepsilon}\right]\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}\\
&-(2^*-1-\varepsilon)\int_\Omega |x|^\alpha(\sum_{i=1}^k PW_{\xi_i,\mu_i})^{2^*-2-\varepsilon}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}+O\big(\mu_l\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega \nabla\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad-\int_\Omega |x|^\alpha(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}})^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&=\int_\Omega \nabla\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad-\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad-(2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad+O\big((\mu_l)^{-1}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|^2\big)\\
&=\int_\Omega \nabla PW_{\xi_l,\mu_l}\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}-\int_\Omega |x|^\alpha(PW_{\xi_l,\mu_l})^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad+\int_\Omega \nabla(\sum_{i\neq l} PW_{\xi_i,\mu_i})\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad-\int_\Omega |x|^\alpha\left[\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon}-(PW_{\xi_l,\mu_l})^{2^*-1-\varepsilon}\right]\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad-(2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}\\
&\quad+O\big((\mu_l)^{-1}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|^2\big).
\end{aligned}
$$

It is easy to compute that

$$
\begin{aligned}
\int_\Omega \nabla\left(\sum_{i\neq l} PW_{\xi_i,\mu_i}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}&=\sum_{i\neq l}\int_\Omega |\xi_i|^\alpha W_{\xi_i,\mu_i}^{2^*-1}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\xi_{l,j}}\\
&=O\left(\mu_l\sum_{i\neq l}\varepsilon_{il}\right)
\end{aligned}
$$

and

$$
\int_\Omega \nabla\left(\sum_{i\neq l} PW_{\xi_i,\mu_i}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}=O\left((\mu_l)^{-1}\sum_{i\neq l}\varepsilon_{il}\right).
$$

Since $|(a+b)^\alpha - a^\alpha - \alpha a^{\alpha-1}b| \le C(|b|^\alpha + |a|^{\alpha-2}\inf\{a^2, b^2\})$, we have

$$\begin{aligned}
&\int_\Omega |x|^\alpha \left[\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon} - (PW_{\xi_l,\mu_l})^{2^*-1-\varepsilon}\right] \frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}} \\
&= O\left(\int_\Omega \left(\sum_{i\neq l} PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\right) \\
&\quad + O\left(\int_\Omega PW_{\xi_l,\mu_l}^{2^*-2-\varepsilon} \left(\sum_{i\neq l} PW_{\xi_i,\mu_i}\right) \frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\right) \\
&\quad + O\left(\int_\Omega PW_{\xi_l,\mu_l}^{2^*-3-\varepsilon} \inf\left\{\left(\sum_{i\neq l} PW_{\xi_i,\mu_i}\right)^2, PW_{\xi_l,\mu_l}^2\right\} \frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\right) \\
&= O\left(\mu_l \sum_{i\neq l} \varepsilon_{il}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega \nabla \left(\sum_{i\neq l} PW_{\xi_i,\mu_i}\right) \nabla \frac{\partial PW_{\xi_l,\mu_l}}{\partial \mu_l} \\
&- \int_\Omega |x|^\alpha \left[\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-1-\varepsilon} - (PW_{\xi_l,\mu_l})^{2^*-1-\varepsilon}\right] \frac{\partial PW_{\xi_l,\mu_l}}{\partial \mu_l} \\
&= O\big((\mu_l)^{-1} \sum_{i\neq l} \varepsilon_{il}\big).
\end{aligned}$$

According to Lemmas A.1–A.3 and Proposition 3.3, our results follow. □

Proof of Theorem 1.4. To obtain

$$u_\varepsilon = \sum_{i=1}^k PW_{\xi_{i,\varepsilon},\mu_{i,\varepsilon}} + \omega_\varepsilon$$

is a solution of equation (1.2), we need to choose that $(\boldsymbol{\xi}, \boldsymbol{\mu}) \in D^k_{\boldsymbol{\xi},\boldsymbol{\mu}}$, such that, for any $l = 1, 2, \cdots, k$,

$$\begin{aligned}
&\int_\Omega \nabla \left(\sum_{i=1}^k PW_{\xi_i,\mu_i} + \omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right) \nabla \frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}} \\
&- \int_\Omega |x|^\alpha \left(\sum_{i=1}^k PW_{\xi_i,\mu_i} + \omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}} = 0
\end{aligned} \tag{3.2}$$

and

$$\int_\Omega \nabla\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)\nabla\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l} - \int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}+\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\right)^{2^*-1-\varepsilon}\frac{\partial PW_{\xi_l,\mu_l}}{\partial\mu_l}=0. \tag{3.3}$$

By Lemma 3.4, (3.2) and (3.3) are equivalent to, respectively,

$$\partial_j(|\xi_l|^\alpha)A-\frac{1}{(\mu_l)^{N-2}}\frac{\partial H(\xi_l,\xi_l)}{\partial\xi_{l,j}}B=O\left(\frac{\ln(\mu_l d_l)}{(\mu_l)^N(d_l)^{N+1}}\right) \tag{3.4}$$

and

$$E\frac{\varepsilon}{\mu_l}-F\frac{H(\xi_l,\xi_l)}{(\mu_l)^{N-1}}=O\left(\frac{1}{(\mu_l)^{N+1}(d_l)^N}\right), \tag{3.5}$$

where A, B, E, F are positive constants. Since

$$\begin{aligned} H(y,y)&=\frac{1}{(2d)^{N-2}}\Big(1+O(d)\Big),\\ \frac{\partial H(y,y)}{\partial\tau_i}&=O(d^{2-N}),\\ \frac{\partial H(y,y)}{\partial\nu}&=-\frac{N-2}{(2d)^{N-1}}\Big(1+O(d)\Big), \end{aligned}$$

where $d=\operatorname{dist}(y,\partial\Omega)$ sufficiently small. Let $\mu_l=t_l\varepsilon^{-\frac{N-1}{N-2}}$ and $d_l=s_l\varepsilon$, where $t_l, s_l\in[L_0,L_1]$. Then (3.4) and (3.5) become

$$F(t_l,s_l):=\alpha|\xi_l|^{\alpha-1}\langle\nabla|x|,\nu(\xi_l)\rangle|_{x=\xi_l}A-\frac{N-2}{2^{N-2}}\frac{B}{t_l^{N-2}s_l^{N-1}}=o(1), \tag{3.6}$$

$$\langle\nabla|x|,\tau_j(\xi_l)\rangle|_{x=\xi_l}=o(1),\ \text{ for } j=1,\cdots,N-1, \tag{3.7}$$

and

$$G(t_l,s_l):=\frac{E}{t_l}-\frac{F}{2^{N-2}t_l^{N-1}s_l^{N-2}}=o(1). \tag{3.8}$$

Let

$$(\tilde t_l,\tilde s_l) =\left(\frac{2\alpha|\xi^l|^{\alpha-1}A}{(N-2)B}\big(\langle\nabla|x|,\nu(\xi_l)\rangle|_{x=\xi_l}\big)\left(\frac{F}{E}\right)^{\frac{N-1}{N-2}},\ \frac{(N-2)BE}{\alpha|\xi^l|^{\alpha-1}AF}\big(\langle\nabla|x|,\nu(\xi_l)\rangle|_{x=\xi_l}\big)^{-1}\right).$$

Choose $\gamma_1=\frac12\tilde t_l$, $\gamma_2=2\tilde t_l$, $\gamma_3=2^{-\theta}\tilde s_l$, and $\gamma_4=2^\theta\tilde s_l$, where $\frac{N-2}{N-1}<\theta<1$. Then, we have, for any $s_l\in(\gamma_3,\gamma_4)$,

$$G(\gamma_1,s_l)\le\big(2-2^{N-1-(N-2)\theta}\big)\frac{E}{\tilde t_l}<0$$

and

$$G(\gamma_2, s_l) \geq \left(\frac{1}{2} - \frac{1}{2^{N-1-\theta(N-2)}}\right)\frac{E}{\tilde{t}_l} > 0.$$

Similarly, for $t_l \in (\gamma_1, \gamma_2)$,

$$F(t_l, \gamma_3) > 0 > F(t_l, \gamma_4).$$

Therefore, by the assumption that $\xi_{l,0}$ is a non-degenerate critical point of $|x|$ restricted on $\partial\Omega$, (3.6)-(3.8) has a solution in $D^k_{\boldsymbol{\xi},\boldsymbol{\mu}}$.

Appendix

A. Estimates for existence

Lemma A.1. *Assume that* $(\boldsymbol{\xi}, \boldsymbol{\mu}) \in D^k_{\boldsymbol{\xi},\boldsymbol{\mu}}$, *it holds*

$$\begin{aligned}
&\int_\Omega \nabla PW_{\xi_i,\mu_i}\nabla\frac{\partial PW_{\xi_i,\mu_i}}{\partial\xi_{i,j}} - \int_\Omega |x|^\alpha PW^{2^*-1-\varepsilon}_{\xi_i,\mu_i}\frac{\partial PW_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\\
={}& -\partial_j(|\xi_i|^\alpha)|\xi_i|^{-\frac{2^*\alpha}{2^*-2}}\beta^i_\varepsilon\int_{\mathbb{R}^N} U^{2^*-1}_{0,1}\frac{\partial U_{0,1}}{\partial z_j}z_j dz\\
&+(2^*-1-\varepsilon)|\xi_i|^{-\frac{2\alpha}{2^*-2}}\beta^i_\varepsilon\frac{1}{(\mu_i)^{N-2}}\frac{\partial H(\xi_i,\xi_i)}{\partial\xi_{i,j}}\\
&\cdot\int_{\mathbb{R}^N} U^{2^*-2}_{0,1}z_j\frac{\partial U_{0,1}}{\partial z_j} + O\left(\frac{\ln(\mu_i d_i)}{(\mu_i)^N(d_i)^{N+1}}\right) + O\big(\varepsilon\partial_j(|\xi_i|^\alpha)\big)\\
&+O\left(\frac{1}{(\mu_i)^2}\right) + O\left(\frac{\varepsilon}{(\mu_i)^{N-2}(d_i)^{N-1}}\right),
\end{aligned}$$

where $\beta^i_\varepsilon = (\mu_i)^{-\frac{N-2}{2}\varepsilon}$.

Proof. At first, we have

$$\begin{aligned}
&\int_\Omega \nabla PW_{\xi_i,\mu_i}\nabla\frac{\partial PW_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\\
={}&|\xi_i|^\alpha\int_\Omega W^{2^*-1}_{\xi_i,\mu_i}\frac{\partial PW_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\\
={}&|\xi_i|^\alpha\int_{B_{d_i}(\xi_i)} W^{2^*-1}_{\xi_i,\mu_i}\frac{\partial PW_{\xi_i,\mu_i}}{\partial\xi_{i,j}} + O\left(\frac{1}{(\mu_i)^N(d_i)^{N+1}}\right)\\
={}&-|\xi_i|^{-\frac{2\alpha}{2^*-2}}\int_{B_{d_i}(\xi_i)} U^{2^*-1}_{\xi_i,\mu_i}\frac{\partial\psi_{\xi_i,\mu_i}}{\partial\xi_{i,j}} + O\left(\frac{1}{(\mu_i)^N(d_i)^{N+1}}\right)\\
={}&-|\xi_i|^{-\frac{2\alpha}{2^*-2}}\frac{\partial H(\xi_i,\xi_i)}{\partial\xi_{i,j}}(\mu_i)^{2-N}\int_{\mathbb{R}^N} U^{2^*-1}_{0,1}dz + O\left(\frac{1}{(\mu_i)^N(d_i)^{N+1}}\right).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\int_{\Omega} |x|^{\alpha} PW_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&= \int_{B_{d_i}(\xi_i)} PW_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} + O\left(\frac{1}{(\mu_i)^N (d_i)^{N+1}}\right) \\
&= \int_{B_{d_i}(\xi_i)} |x|^{\alpha} (W_{\xi_i,\mu_i} - \psi_{\xi_i,\mu_i})^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} + O\left(\frac{1}{(\mu_i)^N (d_i)^{N+1}}\right) \\
&= \int_{B_{d_i}(\xi_i)} |x|^{\alpha} (W_{\xi_i,\mu_i} - \psi_{\xi_i,\mu_i})^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} + O\left(\frac{1}{(\mu_i)^N (d_i)^{N+1}}\right) \\
&= \int_{B_{d_i}(\xi_i)} |x|^{\alpha} W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} - (2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} |x|^{\alpha} W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&\quad + O\left(\int_{B_{d_i}(\xi_i)} |x|^{\alpha} W_{\xi_i,\mu_i}^{2^*-3-\varepsilon} (\psi_{\xi_i,\mu_i})^2 \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}}\right) + O\left(\frac{1}{(\mu_i)^N (d_i)^{N+1}}\right).
\end{aligned}
$$

Let us estimate the above terms.

$$
\begin{aligned}
&\int_{B_{d_i}(\xi_i)} |x|^{\alpha} W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&= \int_{B_{d_i}(\xi_i)} (|x|^{\alpha} - |\xi_i|^{\alpha}) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} + |\xi_i|^{\alpha} \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&= \int_{B_{d_i}(\xi_i)} (|x|^{\alpha} - |\xi_i|^{\alpha}) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \left(\frac{\partial W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} - \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \xi_{i,j}}\right) \\
&\quad + |\xi_i|^{\alpha} \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \left(\frac{\partial W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} - \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \xi_{i,j}}\right).
\end{aligned}
$$

Here, we will give more details.

$$
\begin{aligned}
&\int_{B_{d_i}(\xi_i)} (|x|^{\alpha} - |\xi_i|^{\alpha}) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&= \int_{B_{d_i}(\xi_i)} \langle \nabla(|\xi_i|^{\alpha}), x - \xi_i \rangle W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} + O\left(\frac{1}{(\mu_i)^2}\right) \\
&= \partial_j(|\xi_i|^{\alpha}) |\xi_i|^{-\frac{2^*\alpha}{2^*-2}} \beta_{\varepsilon}^i \int_{\mathbb{R}^N} U_{0,1}^{2^*-1} \frac{\partial U_{0,1}}{\partial z_j} z_j dz \\
&\quad + O\left(\partial_j(|\xi_i|^{\alpha}) \frac{1}{(\mu_i d_i)^N}\right) + O(\varepsilon \partial_j(|\xi_i|^{\alpha})) + O\left(\frac{1}{(\mu_i)^2}\right).
\end{aligned}
$$

Moreover,

$$\begin{aligned}
&\int_{B_{d_i}(\xi_i)} (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&= \int_{B_{d_i}(\xi_i)} \langle \nabla(|\xi_i|^\alpha), x-\xi_i \rangle W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{1}{(\mu_i)^{\frac{N-2}{2}}} \frac{\partial H(x,\xi_i)}{\partial \xi_{i,j}} \\
&\quad + O\left(\frac{\ln(\mu_i d_i)}{(\mu_i)^N (d_i)^{N-1}}\right) \\
&= O\left(\int_{B_{d_i}(\xi_i)} |x-\xi_i|^2 W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{1}{(\mu_i)^{\frac{N-2}{2}} (d_i)^N}\right) + O\Big(\frac{\ln(\mu_i d_i)}{(\mu_i)^N (d_i)^{N-1}}\Big) \\
&= O\left(\frac{\ln(\mu_i d_i)}{(\mu_i d_i)^N}\right).
\end{aligned}$$

The last one is

$$\begin{aligned}
&|\xi_i|^\alpha \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \left(\frac{\partial W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} - \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \xi_{i,j}}\right) \\
&= -|\xi_i|^{-\frac{2\alpha}{2^*-2}} \frac{1}{(\mu_i)^{\frac{N-2}{2}}} \int_{B_{d_i}(\xi_i)} U_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial H(x,\xi_i)}{\partial \xi_{i,j}} \\
&\quad + O\left(\frac{1}{(\mu_i)^N (d_i)^{N+1}}\right) \\
&= -|\xi_i|^{-\frac{2\alpha}{2^*-2}} \beta_\varepsilon^i \frac{1}{(\mu_i)^{N-2}} \frac{\partial H(\xi_i,\xi_i)}{\partial \xi_{i,j}} \int_{\mathbb{R}^N} U_{0,1}^{2^*-1} \\
&\quad + O\left(\frac{\varepsilon}{(\mu_i)^{N-2} (d_i)^{N-1}}\right) + O\left(\frac{1}{(\mu_i)^N (d_i)^{N+1}}\right).
\end{aligned}$$

Go back to the expansion,

$$\begin{aligned}
&(2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} |x|^\alpha W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial P W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&= (2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial P W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} \\
&\quad + (2^*-1-\varepsilon) |\xi_i|^\alpha \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \left(\frac{\partial W_{\xi_i,\mu_i}}{\partial \xi_{i,j}} - \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \xi_{i,j}}\right).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&(2^*-1-\varepsilon)\int_{B_{d_i}(\xi_i)}(|x|^\alpha-|\xi_i|^\alpha)W_{\xi_i,\mu_i}^{2^*-2-\varepsilon}\psi_{\xi_i,\mu_i}\frac{\partial PW_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\\
&=O\left(\int_{B_{d_i}(\xi_i)}|x-\xi_i|W_{\xi_i,\mu_i}^{2^*-2-\varepsilon}\psi_{\xi_i,\mu_i}\frac{\partial W_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\right)\\
&=O\left(\frac{\ln(\mu_i d_i)}{(\mu_i d_i)^{N-2}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&(2^*-1-\varepsilon)|\xi_i|^\alpha\int_{B_{d_i}(\xi_i)}W_{\xi_i,\mu_i}^{2^*-2-\varepsilon}\psi_{\xi_i,\mu_i}\left(\frac{\partial W_{\xi_i,\mu_i}}{\partial\xi_{i,j}}-\frac{\partial\psi_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\right)\\
&=(2^*-1-\varepsilon)|\xi_i|^\alpha\frac{1}{(\mu_i)^{\frac{N-2}{2}}}\frac{\partial H(\xi_i,\xi_i)}{\partial\xi_{i,j}}\int_{B_{d_i}(\xi_i)}W_{\xi_i,\mu_i}^{2^*-2-\varepsilon}(x_j-\xi_{i,j})\frac{\partial W_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\\
&\quad+O\left(\frac{\ln(\mu_i d_i)}{(\mu_i)^N(d_i)^{N+1}}\right)\\
&=(2^*-1-\varepsilon)|\xi_i|^{-\frac{2\alpha}{2^*-2}}\beta_\varepsilon^i\frac{1}{(\mu_i)^{\frac{N-2}{2}}}\frac{\partial H(\xi_i,\xi_i)}{\partial\xi_{i,j}}\int_{B_{d_i}(\xi_i)}U_{\xi_i,\mu_i}^{2^*-2}\\
&\quad\times\Big[1+\varepsilon\ln(1+(\mu_i)^2|x-\xi_i|^2)+O(\varepsilon^2(\ln(\mu_i))^2)\Big](x_j-\xi_{i,j})\frac{\partial U_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\\
&\quad+O\left(\frac{\ln(\mu_i d_i)}{(\mu_i)^N(d_i)^{N+1}}\right)\\
&=(2^*-1-\varepsilon)|\xi_i|^{-\frac{2\alpha}{2^*-2}}\beta_\varepsilon^i\frac{1}{(\mu_i)^{\frac{N-2}{2}}}\frac{\partial H(\xi_i,\xi_i)}{\partial\xi_{i,j}}\int_{B_{d_i}(\xi_i)}U_{\xi_i,\mu_i}^{2^*-2}(x_j-\xi_{i,j})\frac{\partial U_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\\
&\quad+O\left(\frac{\varepsilon^2(\ln\mu_i)^2}{(\mu_i)^{N-2}(d_i)^{N-1}}\right)+O\left(\frac{\ln(\mu_i d_i)}{(\mu_i)^N(d_i)^{N+1}}\right)\\
&=(2^*-1-\varepsilon)|\xi_i|^{-\frac{2\alpha}{2^*-2}}\beta_\varepsilon^i\frac{1}{(\mu_i)^{N-2}}\frac{\partial H(\xi_i,\xi_i)}{\partial\xi_{i,j}}\int_{\mathbb{R}^N}U_{0,1}^{2^*-2}z_j\frac{\partial U_{0,1}}{\partial z_j}\\
&\quad+O\left(\frac{\varepsilon^2(\ln\mu_i)^2}{(\mu_i)^{N-2}(d_i)^{N-1}}\right)+O\left(\frac{\ln(\mu_i d_i)}{(\mu_i)^N(d_i)^{N+1}}\right).
\end{aligned}
$$

Finally,

$$
O\left(\int_{B_{d_i}(\xi_i)}|x|^\alpha W_{\xi_i,\mu_i}^{2^*-3-\varepsilon}(\psi_{\xi_i,\mu_i})^2\frac{\partial PW_{\xi_i,\mu_i}}{\partial\xi_{i,j}}\right)=O\left(\frac{1}{(\mu_i)^N(d_i)^{N+1}}\right). \qquad \square
$$

Similarly, we have the following result:

Lemma A.2. *Assume that* $(\boldsymbol{\xi}, \boldsymbol{\mu}) \in D^k_{\boldsymbol{\xi},\boldsymbol{\mu}}$. *Then*

$$\begin{aligned}
&\int_\Omega \nabla PW_{\xi_i,\mu_i} \nabla \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} - \int_\Omega |x|^\alpha PW^{2^*-1-\varepsilon}_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
= & -|\xi_i|^{-\frac{2\alpha}{2^*-2}} \frac{N-2}{2} \frac{\varepsilon}{\mu_i} \int_{\mathbb{R}^N} \ln(1+|z|^2) U^{2^*-1}_{0,1} \frac{\partial U_{0,\lambda}}{\partial \lambda}|_{\lambda=1} dz \\
&+ (2^*-1)|\xi_i|^{-\frac{2\alpha}{2^*-2}} \beta^i_\varepsilon \frac{H(\xi_i,\xi_i)}{(\mu_i)^{N-1}} \int_{\mathbb{R}^N} U^{2^*-2}_{0,1} \frac{\partial U_{0,\lambda}}{\partial \lambda}|_{\lambda=1} dz \\
&+ O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right) + O\left(\frac{1}{(\mu_i)^3}\right) + O\left(\frac{1}{(\mu_i)^N (d_i)^{N-2}}\right) \\
&+ O\left(\frac{\varepsilon}{(\mu_i)^{N-1}(d_i)^{N-2}}\right) + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right) + \begin{cases} O\big(\frac{\ln(\mu_i d_i)}{(\mu_i)^5 (d_i)^4}\big) & N=4 \\ O\big(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\big) & N \geq 5. \end{cases}
\end{aligned}$$

Proof. Let $\beta^i_\varepsilon = (\mu_i)^{-\frac{N-2}{2}\varepsilon} = 1 - \frac{N-2}{2}\varepsilon \ln \mu_i + O((\varepsilon \ln \mu_i)^2)$. By direct computation, we have

$$\begin{aligned}
&\int_\Omega \nabla PW_{\xi_i,\mu_i} \nabla \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= \int_\Omega |\xi_i|^\alpha W^{2^*-1}_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= -|\xi_i|^\alpha \int_{B_{d_i}(\xi_i)} W^{2^*-1}_{\xi_i,\mu_i} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \mu_i} + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right) \\
&= \frac{N-2}{2(\mu_i)^{N-1}} H(\xi_i,\xi_i)|\xi_i|^{-\frac{2\alpha}{2^*-2}} \int_{\mathbb{R}^N} U^{2^*-1}_{0,1} dz + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |x|^\alpha PW^{2^*-1-\varepsilon}_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= \int_{B_{d_i}(\xi_i)} |x|^\alpha PW^{2^*-1-\varepsilon}_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right) \\
&= \int_{B_{d_i}(\xi_i)} |x|^\alpha (W_{\xi_i,\mu_i} - \psi_{\xi_i,\mu_i})^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right) \\
&= \int_{B_{d_i}(\xi_i)} |x|^\alpha W^{2^*-1-\varepsilon}_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
&\quad - (2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} |x|^\alpha W^{2^*-2-\varepsilon}_{\xi_i,\mu_i} \psi_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
&\quad + O\left(\int_{B_{d_i}(\xi_i)} W^{2^*-3-\varepsilon}_{\xi_i,\mu_i} (\psi_{\xi_i,\mu_i})^2 \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i}\right) + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right).
\end{aligned}$$

Next, will estimate for each term in the above relation.

$$\begin{aligned}
&\int_{B_{d_i}(\xi_i)} |x|^\alpha W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= \int_{B_{d_i}(\xi_i)} |x|^\alpha W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} - \int_{B_{d_i}(\xi_i)} |x|^\alpha W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= \int_{B_{d_i}(\xi_i)} (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} + \int_{B_{d_i}(\xi_i)} |\xi_i|^\alpha W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} \\
&\quad - \int_{B_{d_i}(\xi_i)} (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \mu_i} - \int_{B_{d_i}(\xi_i)} |\xi_i|^\alpha W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \mu_i}.
\end{aligned}$$

Since $|x|^\alpha - |\xi_i|^\alpha = \langle \nabla(|\xi_i|^\alpha), x-\xi_i\rangle + \langle D^2(|\xi_i|^\alpha)(x-\xi_i), x-\xi_i\rangle + O(|x-\xi_i|^3)$ in $B_{d_i}(\xi_i)$, we find

$$\begin{aligned}
&\int_{B_{d_i}(\xi_i)} (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= \int_{B_{d_i}(\xi_i)} (\langle \nabla(|\xi_i|^\alpha), x-\xi_i\rangle + \langle D^2(|\xi_i|^\alpha)(x-\xi_i), x-\xi_i\rangle) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} \\
&\quad + O\left(\int_{B_{d_i}(\xi_i)} |x-\xi_i|^3 W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i}\right) \\
&= (\Delta|\xi_i|^\alpha)\beta_\varepsilon^i \int_{B_{d_i}(\xi_i)} |x-\xi_i|^2 W_{\xi_i,\mu_i}^{2^*-1} \left(1+\frac{N-2}{2}\varepsilon \ln(1+(\mu_i)^2|x-\xi_i|^2)\right) \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} \\
&\quad + O\left(\frac{\varepsilon^2}{(\mu_i)^3}\right) + O\left(\frac{1}{(\mu_i)^4}\right) = O\left(\frac{1}{(\mu_i)^3}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_{d_i}(\xi_i)} |\xi_i|^\alpha W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= |\xi_i|^\alpha \beta_\varepsilon^i \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-1} \left(1+\frac{N-2}{2}\varepsilon \ln(1+(\mu_i)^2|x-\xi_i|^2)\right) \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} + O\left(\frac{\varepsilon^2}{\mu_i}\right) \\
&= |\xi_i|^{-\frac{2\alpha}{2^*-2}} \frac{N-2}{2} \frac{\varepsilon}{\mu_i} \int_{\mathbb{R}^N} \ln(1+|z|^2) U_{0,1}^{2^*-1} \frac{\partial U_{0,\lambda}}{\partial \lambda}|_{\lambda=1} dz \\
&\quad + O\left(\frac{\varepsilon \ln(\mu_i d_i)}{(\mu_i)^{N+1}(d_i)^N}\right) + O\left(\frac{\varepsilon^2}{\mu_i}\right).
\end{aligned}$$

Moreover, since $\psi_{\xi_i,\mu_i}(x) = \frac{H(x,\xi_i)}{(\mu_i)^{\frac{N-2}{2}}} + O\Big(\frac{1}{(\mu_i)^{\frac{N+2}{2}}(d_i)^N}\Big)$, then

$$\begin{aligned}
&\int_{B_{d_i}(\xi_i)} (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= O\left(\frac{1}{(\mu_i)^{\frac{N}{2}}(d_i)^{N-2}} \int_{B_{d_i}(\xi_i)} |x-\xi_i| W_{\xi_i,\mu_i}^{2^*-1-\varepsilon}\right) = O\left(\frac{1}{(\mu_i)^N (d_i)^{N-2}}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{B_{d_i}(\xi_i)} |\xi_i|^\alpha W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= -\frac{N-2}{2(\mu_i)^{\frac{N}{2}}} |\xi_i|^\alpha \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} H(x,\xi_i) + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right) \\
&= -\frac{N-2}{2(\mu_i)^{\frac{N}{2}}} |\xi_i|^\alpha \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-1-\varepsilon} H(x,\xi_i) + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right) \\
&= -\frac{N-2}{2(\mu_i)^{N-1}} |\xi_i|^{-\frac{2\alpha}{2^*-2}} H(\xi_i,\xi_i) \int_{\mathbb{R}^N} U_{0,1}^{2^*-1} dz \\
&\quad + O\left(\frac{\varepsilon}{(\mu_i)^{N-1}(d_i)^{N-2}}\right) + O\left(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} |x|^\alpha W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= (2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} |x|^\alpha W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} \\
&\quad - (2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} |x|^\alpha W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial \psi_{\xi_i,\mu_i}}{\partial \mu_i} \\
&= (2^*-1-\varepsilon) \int_{B_{d_i}(\xi_i)} (|x|^\alpha - |\xi_i|^\alpha) W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} \\
&\quad - (2^*-1-\varepsilon)|\xi_i|^\alpha \int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-2-\varepsilon} \psi_{\xi_i,\mu_i} \frac{\partial W_{\xi_i,\mu_i}}{\partial \mu_i} + \begin{cases} O\big(\frac{\ln(\mu_i d_i)}{(\mu_i)^5 (d_i)^4}\big) & N=4 \\ O\big(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\big) & N \geq 5 \end{cases} \\
&= (2^*-1)|\xi_i|^{-\frac{2\alpha}{2^*-2}} \beta_\varepsilon^i \frac{H(\xi_i,\xi_i)}{(\mu_i)^{N-1}} \int_{\mathbb{R}^N} U_{0,1}^{2^*-2} \frac{\partial U_{0,\lambda}}{\partial \lambda}\Big|_{\lambda=1} dz \\
&\quad + O\left(\frac{1}{(\mu_i)^N (d_i)^{N-2}}\right) + O\left(\frac{\varepsilon}{(\mu_i)^{N-1}(d_i)^{N-2}}\right) \\
&\quad + O\Big(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\Big) + \begin{cases} O\big(\frac{\ln(\mu_i d_i)}{(\mu_i)^5 (d_i)^4}\big) & N=4 \\ O\big(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\big) & N \geq 5. \end{cases}
\end{aligned}
$$

Finally,

$$
O\left(\int_{B_{d_i}(\xi_i)} W_{\xi_i,\mu_i}^{2^*-3-\varepsilon} (\psi_{\xi_i,\mu_i})^2 \frac{\partial PW_{\xi_i,\mu_i}}{\partial \mu_i}\right) = \begin{cases} O\big(\frac{\ln(\mu_i d_i)}{(\mu_i)^5 (d_i)^4}\big) & N=4 \\ O\big(\frac{1}{(\mu_i)^{N+1}(d_i)^N}\big) & N \geq 5. \end{cases} \qquad \square
$$

Lemma A.3. *Assume that* $(\boldsymbol{\xi},\boldsymbol{\mu})\in D^k_{\boldsymbol{\xi},\boldsymbol{\mu}}$. *Then we have*

$$
\begin{aligned}
&(2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\\
&= O\big(\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)+O\left(\frac{\mu_l\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|}{(\mu_l d_l)^{\frac{N-2}{2}}}\right)+O\big(\mu_l\sum_{i\neq j}\varepsilon_{ij}^{\frac{N+2}{2N}}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)+O\big(\mu_l\varepsilon\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&(2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \mu_l}\\
&= O\left(\frac{\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|}{(\mu_l)^2}\right)+O\left(\frac{\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|}{\mu_l(\mu_l d_l)^{\frac{N-2}{2}}}\right)+O\left((\mu_l)^{-1}\sum_{i\neq j}\varepsilon_{ij}^{\frac{N+2}{2N}}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\right)\\
&\quad+O\big((\mu_l)^{-1}\varepsilon\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big).
\end{aligned}
$$

Proof. Let us only consider the first one, since the second one is similar. By straight computations,

$$
\begin{aligned}
&(2^*-1-\varepsilon)\int_\Omega |x|^\alpha\left(\sum_{i=1}^k PW_{\xi_i,\mu_i}\right)^{2^*-2-\varepsilon}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\\
&=(2^*-1)\int_\Omega |x|^\alpha\sum_{i=1}^k PW^{2^*-2-\varepsilon}_{\xi_i,\mu_i}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\\
&\quad+O\left(\int_\Omega\sum_{i=1}^k PW^{2^*-3-\varepsilon}_{\xi_i,\mu_i}\inf\{PW_{\xi_i,\mu_i},PW_{\xi_j,\mu_j}\}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\right)+O\big(\mu_l\varepsilon\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)\\
&=(2^*-1)\sum_{i=1}^k\int_\Omega(|x|^\alpha-|\xi_i|^\alpha)PW^{2^*-2-\varepsilon}_{\xi_i,\mu_i}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\\
&\quad+(2^*-1)\sum_{i=1}^k|\xi_i|^\alpha\int_\Omega PW^{2^*-2-\varepsilon}_{\xi_i,\mu_i}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\\
&\quad+O\left(\mu_l\sum_{i\neq j}\varepsilon_{ij}^{\frac{N+2}{2N}}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\right)+O\big(\mu_l\varepsilon\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)\\
&=O\big(\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)+O\left(\sum_{i=1}^k\int_\Omega W^{2^*-3-\varepsilon}_{\xi_i,\mu_i}\psi_{\xi_i,\mu_i}\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\frac{\partial PW_{\xi_l,\mu_l}}{\partial \xi_{l,j}}\right)\\
&\quad+O\big(\mu_l\sum_{i\neq j}\varepsilon_{ij}^{\frac{N+2}{2N}}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)+O\big(\mu_l\varepsilon\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)\\
&=O\big(\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big)+O\left(\frac{\mu_l\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|}{(\mu_l d_l)^{\frac{N-2}{2}}}\right)+O\left(\mu_l\sum_{i\neq j}\varepsilon_{ij}^{\frac{N+2}{2N}}\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\right)+O\big(\mu_l\varepsilon\|\omega_{\boldsymbol{\xi},\boldsymbol{\mu}}\|\big).
\end{aligned}
$$

Thus, the result follows. □

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