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Uniqueness results on CR manifolds

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Abstract. In this paper we show the uniqueness of the constant solution to a general Yamabe-type equation on a compact pseudohermitian manifold. The Riemannian version was first proved by Bidaut-Véron and Véron. In another direction, we make use of the vanishing property of two eigenfunctions of the sub-Laplacian to conclude the nonvanishing of the first cohomology group of a compact pseudohermitian manifold. The corresponding version for the Riemannian case was first proved by Gichev.

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1. Introduction

Bidaut-Véron and Véron proved the following:

Theorem 1.1 ([1, Theorem 6.1]). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$ without boundary. Suppose u is a positive solution on M of*

$$\Delta_g u - \lambda u + u^q = 0$$

for some $\lambda > 0$ and $1 < q \leq (n+2)/(n-2)$. If the Ricci tensor of g satisfies $\mathrm{Ric} \geq \frac{n-1}{n}(q-1)\lambda g$, then u is constant with value $\lambda^{1/(q-1)}$ unless $q = (n+2)/(n-2)$ and (M, g) is isometric to $\left(\mathbb{S}^n, \frac{4\lambda}{n(n-2)} g_0\right)$. In the latter case u is given on $\mathbb{S}^n$ by the following formula

$$u = \frac{1}{(a + x \cdot \xi)^{(n-2)/2}}$$

for some $\xi \in \mathbb{R}^{n+1}$ and some constant $a > |\xi|$.

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Ilias showed in [15] that the method of Bidaut-Véron and Véron still works for the same equation on a compact manifold with convex boundary, provided that u satisfies the boundary condition $\frac{\partial u}{\partial \nu} = 0$. This corresponds to the first type of Yamabe problem on manifolds with boundary, namely finding a conformal metric with constant scalar curvature and zero mean curvature on the boundary. Similar results for the other type of Yamabe problem, which is about finding a conformal metric with zero scalar curvature and constant mean curvature on the boundary, were obtained recently by Wang [24] in dimension two.

In view of the similarity between the Yamabe problem and the CR Yamabe problem (*cf.* [16]), it is natural to ask if similar results hold for CR manifolds. In this paper we prove the corresponding version of Theorem 1.1 for CR manifolds.

Theorem 1.2. *Let (M, θ) be an orientable compact strictly pseudoconvex CR manifold of real dimension $2n + 1$ with contact form θ. Suppose that u is a positive solution on M to*

$$\Delta_b u - \lambda u + u^q = 0 \tag{1.1}$$

for some $\lambda > 0$ and $q > 1$. When $n = 1$, we assume further that the CR Paneitz operator P_0 is nonnegative:

$$\int_M f P_0 f \geq 0 \tag{1.2}$$

for any real-valued function f. Assume that

$$\left(\frac{2}{n+1}\operatorname{Ric} - Tor\right)(Z, Z) \geq \frac{\lambda}{n}(q-1)\langle Z, Z\rangle \text{ for any } Z \in T^{1,0}, \tag{1.3}$$

and

$$q \leq \frac{4n^2 + 4n}{4n^2 + 2n - 3}. \tag{1.4}$$

Assume also that one of the two inequalities in (1.3) *and* (1.4) *is strict. Then u is constant with value $\lambda^{\frac{1}{q-1}}$.*

As an application of Theorem 1.2, we will give some estimates on the value of the minimum of some functionals defined in the Folland-Stein space $S_1^2(M)$. See Theorem 3.2. It is not surprising that the assumptions in Theorem 1.2 involve the nonnegativity of the CR Paneitz operator. Note that the nonnegativity of the CR Paneitz operator has been used in the solution of the CR Yamabe problem and in the proof of the CR positive mass theorem. See [8] and [9]. We also remark that if $\lambda \leq 0$, it follows from integration by parts that (1.1) has no positive solution. In view of the CR Yamabe equation, it would be desirable to improve (1.4) showing that one can take $q \leq 1 + \frac{2}{n}$. However, our method does not seem to be able to get this better estimate. We hope to address this problem in the future.

In another direction, we consider the zero of the eigenfunctions of the sub-Laplacian. To put things into perspective, we first recall a result of Gichev. Let

(M, g) be a compact Riemannian manifold without boundary. Suppose λ is a nonzero eigenvalue of the Laplacian. Let E_λ be the eigenspace of λ, *i.e.*,

$$E_\lambda = \{u \in C^2(M) : -\Delta_g u + \lambda u = 0\}.$$

The following theorem was proved in [11] by Gichev.

Theorem 1.3. *Suppose the first de Rham cohomology group of M vanishes,* i.e., *$H^1(M) = 0$. Then for any $\lambda \neq 0$ and each pair $u, v \in E_\lambda$, there exists a point $p \in M$ such that $u(p) = v(p) = 0$.*

See also [10] and [21] for related results.

Now suppose that (M, θ) is a compact strictly pseudoconvex CR manifold of real dimension $2n + 1$ with a contact form θ. Suppose λ is a nonzero eigenvalue of the sub-Laplacian Δ_b. Let E_λ be the eigenspace of λ, *i.e.*,

$$E_\lambda = \{u \in C^2(M) : -\Delta_b u + \lambda u = 0\}.$$

We prove the corresponding version of Theorem 1.3 for CR manifolds.

Theorem 1.4. *Suppose the first de Rham cohomology group of M vanishes,* i.e., *$H^1(M) = 0$. Then for any $\lambda \neq 0$ and each pair $u, v \in E_\lambda$, there exists a point $p \in M$ such that $u(p) = v(p) = 0$.*

An immediate consequence of Theorem 1.4 is the following:

Corollary 1.5. *If there exits $\lambda \neq 0$ and a pair $u, v \in E_\lambda$ with no common zeros, then $H^1(M) \neq 0$.*

2. Preliminaries and notation

In this section we recall some basic concepts of CR geometry. Most of them can be found in [18]. See also the appendix in [5]. Let M be an orientable smooth manifold of real dimension $2n+1$. A CR structure on M is given by a complex n-dimensional subbundle $T^{1,0}$ of the complexified tangent bundle $\mathbb{C}TM$ of M, satisfying $T^{1,0} \cap T^{0,1} = \{0\}$, where $T^{0,1} = \overline{T^{1,0}}$. We assume the CR structure is integrable, that is, $T^{1,0}$ satisfies the formal Frobenius condition $[T^{1,0}, T^{1,0}] \subset T^{1,0}$. We set $G = Re(T^{1,0} \oplus T^{0,1})$, so that G is a real $2n$-dimensional subbundle of TM. Then G carries a natural complex structure $J : G \to G$, given by $J(V + \overline{V}) = i(V - \overline{V})$ for $V \in T^{1,0}$.

Let $E \subset T^*M$ denote the real line bundle $G^\perp$. Because we assumed that M is orientable and the complex structure J induces an orientation on G, E has a global nonvanishing section. A choice of such a 1-form θ defines a pseudohermitian structure on M. We associate to θ the real symmetric bilinear form $\langle\ ,\ \rangle$ on G:

$$\langle V, W\rangle = d\theta(V, JW), \quad V, W \in G,$$

called the Levi-form of θ. $\langle\, ,\, \rangle$ can be extended by complex linearity to $\mathbb{C}G$, and it induces a Hermitian form on $T^{1,0}$, which we write

$$\langle V, \overline{W}\rangle = -id\theta(V, \overline{W}), \quad V, W \in T^{1,0}.$$

We will assume that M is strictly pseudoconvex, that is, for a suitable choice of θ, $\langle\, ,\, \rangle$ is positive definite. In this case, θ defines a contact structure on M, and we call θ a contact form. Then the volume form on M is defined as $dV_\theta = \theta \wedge d\theta^n$. The volume of M is denoted by $\mathrm{Vol}(M, \theta)$, *i.e.*, $\mathrm{Vol}(M, \theta) = \int_M dV_\theta$. For simplicity, henceforth we will write $\int_M f$ instead of $\int_M f dV_\theta$.

We can choose a unique T, which is called the characteristic direction, such that $\theta(T) = 1$, $d\theta(T, \cdot) = 0$, and $TM = G \oplus \mathbb{R}T$. We can define a coframe $\{\theta, \theta^1, \theta^2, \cdots, \theta^n\}$ satisfying $\theta^\alpha(T) = 0$, which is called an admissible coframe. Its dual frame $\{T, Z_1, Z_2, \cdots, Z_n\}$ is called admissible frame. In this coframe, we have $d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$, where $h_{\alpha\bar{\beta}}$ are the coefficients of a Hermitian matrix.

Tanaka [22] and Webster [23] showed that there is a natural connection on the bundle $T^{1,0}$ adapted to a pseudohermitian structure, which is called the Tanaka-Webster connection. To define this connection, we choose an admissible coframe $\{\theta^\alpha\}$ and dual frame $\{Z_\alpha\}$ for $T^{1,0}$. Then there are uniquely determined 1-forms $\omega_{\alpha\overline{\beta}}, \tau_\alpha$ on M, satisfying

$$\begin{aligned} d\theta^\alpha &= \theta^\beta \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\alpha, \\ dh_{\alpha\bar{\beta}} &= \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha}, \\ \tau_\alpha \wedge \theta^\alpha &= 0. \end{aligned}$$

From the third equation, we can find $A_{\alpha\gamma}$ such that $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ and $A_{\alpha\gamma} = A_{\gamma\alpha}$. Here $A_{\alpha\gamma}$ is called the pseudohermitian torsion. Then Tor in (1.3) is given by

$$Tor(Z, Z) = i\sum_{\alpha,\beta=1}^{n}(A_{\bar{\alpha}\bar{\beta}}x^\alpha x^\beta - A_{\alpha\beta}x^{\bar{\alpha}}x^{\bar{\beta}}),$$

where $Z = \sum_{\alpha=1}^n x^\alpha Z_\alpha \in T^{1,0}$. With this connection, the covariant differentiation is defined by

$$DZ_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \quad DZ_{\bar{\alpha}} = {\omega_{\bar{\alpha}}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad DT = 0.$$

$\{{\omega_\alpha}^\beta\}$ are called connection 1-forms. For a smooth function f on M, we write $f_\alpha = Z_\alpha f$, $f_{\bar{\alpha}} = Z_{\bar{\alpha}} f$, $f_0 = Tf$, so that $df = f_\alpha\theta_\alpha + f_{\bar{\alpha}}\theta_{\bar{\alpha}} + f_0\theta$. The second covariant differential $D^2 f$ is the 2-tensor with components

$$\begin{aligned} &f_{\alpha\beta} = \overline{\overline{f}_{\bar{\alpha}\bar{\beta}}} = Z_\beta Z_\alpha f - {\omega_\alpha}^\gamma(Z_\beta)Z_\gamma f, \quad f_{\alpha\bar{\beta}} = \overline{\overline{f}_{\bar{\alpha}\beta}} = Z_{\bar{\beta}} Z_\alpha f - {\omega_\alpha}^\gamma(Z_{\bar{\beta}})Z_\gamma f \\ &f_{0\alpha} = \overline{\overline{f}_{0\bar{\alpha}}} = Z_\alpha T f, \quad f_{\alpha 0} = \overline{\overline{f}_{\bar{\alpha}0}} = TZ_\alpha f - {\omega_\alpha}^\gamma(T)Z_\gamma f, \quad f_{00} = T^2 f. \end{aligned}$$

Here $h_{\alpha\bar\beta}$ and $h^{\alpha\bar\beta}$ are used to lower and raise indices. In fact, when we do the computation at one point, we may assume that $h_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$. The sub-Laplacian of θ and the sub-gradient of θ are respectively denoted by Δ_b and ∇_b. Explicitly, we have

$$\Delta_b f = \sum_{\alpha=1}^{n} (f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha}).$$

For any smooth function f, we define

$$Pf = \sum_{\alpha=1}^{n} (f_{\bar\alpha}{}^{\bar\alpha}{}_{\beta} + inA_{\beta\alpha} f^{\alpha})\theta^{\beta} = (P_\beta f)\theta^\beta, \quad \beta = 1, 2, ..., n, \tag{2.1}$$

which is an operator that characterizes CR-pluriharmonic functions (see [17] for $n = 1$ and [12] for $n \geq 2$). Here

$$P_\beta f = \sum_{\alpha=1}^{n} (f_{\bar\alpha}{}^{\bar\alpha}{}_{\beta} + inA_{\beta\alpha} f^{\alpha})$$

and $\overline{P} f = (\overline{P}_\beta f)\theta^{\bar\beta}$, the conjugate of P. Moreover, the CR Paneitz operator P_0 is defined as

$$P_0 f = 4\big(\delta_b(Pf) + \overline{\delta}_b(\overline{P} f)\big). \tag{2.2}$$

Here δ_b is the divergence operator which takes $(1, 0)$-form to function by $\delta_b(\sigma_\alpha\theta^\alpha) = \sigma_\alpha,^\alpha$ and $\overline{\delta}_b(\sigma_{\bar\alpha}\theta^{\bar\alpha}) = \sigma_{\bar\alpha},^{\bar\alpha}$. For more about the CR Paneitz operator, we refer the readers to [2–4, 7].

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that u is a positive solution to (1.1). Let $u = v^{-\beta}$ where $\beta \neq 0$. Then v satisfies

$$\Delta_b v = (\beta + 1)\frac{|\nabla_b v|^2}{v} + \frac{1}{\beta}(v^{1+\beta-\beta q} - \lambda v). \tag{3.1}$$

From (2.2) in [5], we have the following Bochner-type formula which was first proved by Greenleaf in [13]:

$$\begin{aligned}
&\frac{1}{2}\Delta_b |\nabla_b f|^2 \\
&= |(\nabla^H) f|^2 + \langle \nabla_b f, \nabla_b \Delta_b f\rangle + \big(2\,\mathrm{Ric} - nTor\big)((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}) \\
&\quad - 2i \sum_{\alpha=1}^{n} (f_\alpha f_{\bar\alpha 0} - f_{\bar\alpha} f_{\alpha 0})
\end{aligned} \tag{3.2}$$

for any real-valued function f. Here $|(\nabla^H)f|^2$ is the square norm of the horizontal Hessian of f and is given by (see the line after (3.8) in [6])

$$|(\nabla^H)f|^2 = 2\sum_{\alpha,\beta=1}^{n} f_{\alpha\beta}f_{\bar\alpha\bar\beta} + 2\sum_{\alpha,\beta=1}^{n} f_{\alpha\bar\beta}f_{\bar\alpha\beta}, \tag{3.3}$$

and

$$(\nabla_b f)_{\mathbb{C}} = \sum_{\alpha=1}^{n} f_{\bar\alpha}Z_\alpha. \tag{3.4}$$

Taking $f = v$ in (3.2), multiplying it by v^γ for some γ to be chosen later, and integrating it over M, we obtain

$$A + B + C + D + E = 0. \tag{3.5}$$

Here

$$A = \int_M v^\gamma |(\nabla^H)v|^2 = J + \frac{n}{2}\int_M v^\gamma v_0^2 + \frac{1}{2n}\int_M v^\gamma(\Delta_b v)^2, \tag{3.6}$$

where

$$J = \int_M v^\gamma \left(|(\nabla^H)v|^2 - \frac{n}{2}v_0^2 - \frac{1}{2n}(\Delta_b v)^2 \right) \geq 0, \tag{3.7}$$

since

$$|(\nabla^H)v|^2 \geq 2\sum_{\alpha,\beta=1}^{n} v_{\alpha\bar\beta}v_{\bar\alpha\beta} \geq \frac{2}{n}\Big|\sum_{\alpha=1}^{n} v_{\alpha\bar\alpha}\Big|^2 = \frac{n}{2}v_0^2 + \frac{1}{2n}(\Delta_b v)^2$$

by (3.3) and the Cauchy-Schwarz inequality, and

$$B = \int_M v^\gamma \langle \nabla_b v, \nabla_b \Delta_b v\rangle, \tag{3.8}$$

$$C = -\frac{1}{2}\int_M v^\gamma \Delta_b|\nabla_b v|^2, \tag{3.9}$$

$$D = \int_M v^\gamma \big(2\,\mathrm{Ric} - nTor\big)((\nabla_b v)_{\mathbb{C}}, (\nabla_b v)_{\mathbb{C}}), \tag{3.10}$$

$$E = -2i\int_M v^\gamma \sum_{\alpha=1}^{n}(v_\alpha v_{\bar\alpha 0} - v_{\bar\alpha}v_{\alpha 0}). \tag{3.11}$$

Note that

$$\begin{aligned}
&\sum_{\alpha=1}^{n}\Big((v^x)_\alpha (v^x)_{\bar\alpha 0} - (v^x)_{\bar\alpha}(v^x)_{\alpha 0}\Big)\\
&= \sum_{\alpha=1}^{n}\Big(x v^{x-1} v_\alpha\big(x v^{x-1} v_{\bar\alpha 0} + x(x-1)v^{x-2} v_{\bar\alpha} v_0\big)\\
&\qquad - x v^{x-1} v_{\bar\alpha}\big(x v^{x-1} v_{\alpha 0} + x(x-1) v^{x-2} v_\alpha v_0\big)\Big)\\
&= x^2 v^{2x-2}\sum_{\alpha=1}^{n}(v_\alpha v_{\bar\alpha 0} - v_{\bar\alpha} v_{\alpha 0}).
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) with $x = (\gamma+2)/2$, we obtain

$$E = -\frac{8i}{(\gamma+2)^2}\int_M \sum_{\alpha=1}^{n}\Big((v^{\frac{\gamma+2}{2}})_\alpha (v^{\frac{\gamma+2}{2}})_{\bar\alpha 0} - (v^{\frac{\gamma+2}{2}})_{\bar\alpha}(v^{\frac{\gamma+2}{2}})_{\alpha 0}\Big). \tag{3.13}$$

It was proved in [5] that (see (2.3) in [5])

$$\begin{aligned}
&-2i\sum_{\alpha=1}^{n}(f_\alpha f_{\bar\alpha 0} - f_{\bar\alpha} f_{\alpha 0})\\
&= -\frac{4}{n}\big\langle Pf + \overline{P}f, d_b f\big\rangle_{L_\theta^*} - 2Tor\big((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}\big) + \frac{2}{n}\big\langle \nabla_b f, \nabla_b \Delta_b f\big\rangle
\end{aligned} \tag{3.14}$$

for a real-valued function f. Here P is the operator defined in (2.1). Combining (3.13) and (3.14) with $f = v^{\frac{\gamma+2}{2}}$, we get

$$\begin{aligned}
E &= -\frac{16}{(\gamma+2)^2 n}\int_M \langle P(v^{\frac{\gamma+2}{2}}) + \overline{P}(v^{\frac{\gamma+2}{2}}), d_b(v^{\frac{\gamma+2}{2}})\rangle_{L_\theta^*}\\
&\quad - \frac{8}{(\gamma+2)^2}\int_M Tor((\nabla_b v^{\frac{\gamma+2}{2}})_{\mathbb{C}}, (\nabla_b v^{\frac{\gamma+2}{2}})_{\mathbb{C}})\\
&\quad + \frac{8}{(\gamma+2)^2 n}\int_M \langle \nabla_b v^{\frac{\gamma+2}{2}}, \nabla_b \Delta_b v^{\frac{\gamma+2}{2}}\rangle\\
&:= I + II + III.
\end{aligned} \tag{3.15}$$

It follows from (1.3) in [5] that

$$I = \frac{4}{(\gamma+2)^2 n}\int_M v^{\frac{\gamma+2}{2}} P_0(v^{\frac{\gamma+2}{2}}), \tag{3.16}$$

where P_0 is the CR Paneitz operator defined in (2.2). By (3.4), we have

$$\big(\nabla_b v^{\frac{\gamma+2}{2}}\big)_{\mathbb{C}} = \frac{\gamma+2}{2} v^{\frac{\gamma}{2}}\sum_{\alpha=1}^{n} v_{\bar\alpha} Z_\alpha = \frac{\gamma+2}{2} v^{\frac{\gamma}{2}}(\nabla_b v)_{\mathbb{C}}, \tag{3.17}$$

which implies that

$$II=-2\int_M v^\gamma Tor\big((\nabla_b v)_{\mathbb{C}},(\nabla_b v)_{\mathbb{C}}\big). \tag{3.18}$$

A direct computation shows that

$$\begin{aligned}
&\big\langle\nabla_b v^{\frac{\gamma+2}{2}},\nabla_b\Delta_b v^{\frac{\gamma+2}{2}}\big\rangle\\
&=\frac{\gamma+2}{2}v^{\frac{\gamma}{2}}\Big\langle\nabla_b v,\nabla_b\Big(\frac{\gamma+2}{2}v^{\frac{\gamma}{2}}\Delta_b v+\frac{(\gamma+2)\gamma}{4}v^{\frac{\gamma-2}{2}}|\nabla_b v|^2\Big)\Big\rangle\\
&=\frac{(\gamma+2)^2}{4}v^\gamma\langle\nabla_b v,\nabla_b\Delta_b v\rangle+\frac{(\gamma+2)^2\gamma}{8}v^{\gamma-1}|\nabla_b v|^2\Delta_b v\\
&\quad+\frac{(\gamma+2)^2\gamma}{8}v^{\gamma-1}\langle\nabla_b v,\nabla_b(|\nabla_b v|^2)\rangle+\frac{(\gamma+2)^2\gamma(\gamma-2)}{16}v^{\gamma-2}|\nabla_b v|^4,
\end{aligned}$$

which implies that

$$\begin{aligned}
III&=\frac{2}{n}\int_M v^\gamma\langle\nabla_b v,\nabla_b\Delta_b v\rangle+\frac{\gamma}{n}\int_M v^{\gamma-1}|\nabla_b v|^2\Delta_b v\\
&\quad+\frac{\gamma}{n}\int_M v^{\gamma-1}\langle\nabla_b v,\nabla_b(|\nabla_b v|^2)\rangle+\frac{\gamma(\gamma-2)}{2n}\int_M v^{\gamma-2}|\nabla_b v|^4\\
&=\frac{2}{n}\int_M v^\gamma\langle\nabla_b v,\nabla_b\Delta_b v\rangle+\frac{\gamma}{n}\int_M v^{\gamma-1}|\nabla_b v|^2\Delta_b v\\
&\quad-\frac{\gamma}{n}\int_M v^{\gamma-1}|\nabla_b v|^2\Delta_b v-\frac{\gamma}{n}\int_M\langle\nabla_b v^{\gamma-1},\nabla_b v\rangle|\nabla_b v|^2\\
&\quad+\frac{\gamma(\gamma-2)}{2n}\int_M v^{\gamma-2}|\nabla_b v|^4\\
&=\frac{2}{n}\int_M v^\gamma\langle\nabla_b v,\nabla_b\Delta_b v\rangle-\frac{\gamma^2}{2n}\int_M v^{\gamma-2}|\nabla_b v|^4,
\end{aligned}$$

where we have used integration by parts in the second equality. Hence, by (3.8), we find

$$III=\frac{2}{n}B-\frac{\gamma^2}{2n}\int_M v^{\gamma-2}|\nabla_b v|^4. \tag{3.19}$$

Substituting (3.16), (3.18) and (3.19) into (3.15), we obtain

$$\begin{aligned}
E&=\frac{4}{(\gamma+2)^2n}\int_M v^{\frac{\gamma+2}{2}}P_0(v^{\frac{\gamma+2}{2}})-2\int_M v^\gamma Tor((\nabla_b v)_{\mathbb{C}},(\nabla_b v)_{\mathbb{C}})\\
&\quad+\frac{2}{n}B-\frac{\gamma^2}{2n}\int_M v^{\gamma-2}|\nabla_b v|^4.
\end{aligned}\tag{3.20}$$

Next we compute A in (3.6). It follows from Lemma 2.2 in [5] that

$$i\int_M \sum_{\alpha=1}^{n}(f_\alpha f_{\bar\alpha 0} - f_{\bar\alpha} f_{\alpha 0}) = n\int_M f_0^2 - \int_M Tor\big((\nabla_b f)_{\mathbb{C}}, (\nabla_b f)_{\mathbb{C}}\big) \tag{3.21}$$

for any real-valued function f. Putting $f = v^{\frac{\gamma+2}{2}}$ in (3.21), we get

$$\begin{aligned}
&i\int_M \sum_{\alpha=1}^{n}\Big((v^{\frac{\gamma+2}{2}})_\alpha (v^{\frac{\gamma+2}{2}})_{\bar\alpha 0} - (v^{\frac{\gamma+2}{2}})_{\bar\alpha}(v^{\frac{\gamma+2}{2}})_{\alpha 0}\Big)\\
&= n\int_M (v^{\frac{\gamma+2}{2}})_0^2 - \int_M Tor((\nabla_b v^{\frac{\gamma+2}{2}})_{\mathbb{C}}, (\nabla_b v^{\frac{\gamma+2}{2}})_{\mathbb{C}})\\
&= \frac{(\gamma+2)^2}{4}\left(n\int_M v^\gamma v_0^2 - \int_M v^\gamma Tor((\nabla_b v)_{\mathbb{C}}, (\nabla_b v)_{\mathbb{C}})\right),
\end{aligned}$$

where we have used (3.17). Combining this with (3.13), we obtain

$$n\int_M v^\gamma v_0^2 = -\frac{1}{2}E + \int_M v^\gamma Tor((\nabla_b v)_{\mathbb{C}}, (\nabla_b v)_{\mathbb{C}}). \tag{3.22}$$

We then compute

$$\begin{aligned}
A &= J + \frac{n}{2}\int_M v^\gamma v_0^2\\
&\quad + \frac{1}{2n}\int_M v^\gamma \Delta_b v\left((\beta+1)\frac{|\nabla_b v|^2}{v} + \frac{1}{\beta}(v^{1+\beta-\beta q} - \lambda v)\right)\\
&= J + \frac{n}{2}\int_M v^\gamma v_0^2\\
&\quad + \frac{\beta+1}{2n}\int_M v^{\gamma-1}|\nabla_b v|^2\left((\beta+1)\frac{|\nabla_b v|^2}{v} + \frac{1}{\beta}(v^{1+\beta-\beta q} - \lambda v)\right)\\
&\quad + \frac{1}{2n\beta}\int_M \Delta_b v(v^{\gamma+1+\beta-\beta q} - \lambda v^{\gamma+1})\\
&= J - \frac{1}{4}E + \frac{1}{2}\int_M v^\gamma Tor((\nabla_b v)_{\mathbb{C}}, (\nabla_b v)_{\mathbb{C}})\\
&\quad + \frac{(\beta+1)^2}{2n}\int_M v^{\gamma-2}|\nabla_b v|^4\\
&\quad + \frac{\beta q-\gamma}{2n\beta}\int_M v^{\gamma-\beta(q-1)}|\nabla_b v|^2 + \frac{\lambda(\gamma-\beta)}{2n\beta}\int_M v^\gamma|\nabla_b v|^2,
\end{aligned} \tag{3.23}$$

where the first two equalities follow from (3.1), and the last equality follows from (3.22) and integration by parts. Similarly, we can use (3.1) and integration by parts to compute B in (3.8):

$$
\begin{aligned}
B &= \int_M v^\gamma \langle \nabla_b v, \nabla_b \left((\beta+1)\frac{|\nabla_b v|^2}{v} + \frac{1}{\beta}(v^{1+\beta-\beta q} - \lambda v) \right) \rangle \\
&= (\beta+1) \int_M \langle \nabla_b \left(\frac{|\nabla_b v|^2}{v} \right), v^\gamma \nabla_b v \rangle \\
&\quad + \frac{1-\beta(q-1)}{\beta} \int_M v^{\gamma-\beta(q-1)} |\nabla_b v|^2 - \frac{\lambda}{\beta} \int_M v^\gamma |\nabla_b v|^2 \\
&= -(\beta+1)\gamma \int_M v^{\gamma-2} |\nabla_b v|^4 - (\beta+1) \int_M v^{\gamma-1} |\nabla_b v|^2 \Delta_b v \\
&\quad + \frac{1-\beta(q-1)}{\beta} \int_M v^{\gamma-\beta(q-1)} |\nabla_b v|^2 - \frac{\lambda}{\beta} \int_M v^\gamma |\nabla_b v|^2 \\
&= -(\beta+1)(\gamma+\beta+1) \int_M v^{\gamma-2} |\nabla_b v|^4 - q \int_M v^{\gamma-\beta(q-1)} |\nabla_b v|^2 \\
&\quad + \lambda \int_M v^\gamma |\nabla_b v|^2.
\end{aligned} \tag{3.24}
$$

We can also use (3.1) and integration by parts to compute C in (3.9):

$$
\begin{aligned}
C &= -\frac{1}{2} \int_M |\nabla_b v|^2 \Delta_b (v^\gamma) \\
&= -\frac{1}{2} \int_M |\nabla_b v|^2 (\gamma v^{\gamma-1} \Delta_b v + \gamma(\gamma-1) v^{\gamma-2} |\nabla_b v|^2) \\
&= -\frac{1}{2} \int_M |\nabla_b v|^2 \gamma v^{\gamma-1} \left((\beta+1)\frac{|\nabla_b v|^2}{v} + \frac{1}{\beta}(v^{1+\beta-\beta q} - \lambda v) \right) \\
&\quad - \frac{\gamma(\gamma-1)}{2} \int_M v^{\gamma-2} |\nabla_b v|^4 \\
&= -\frac{\gamma(\gamma+\beta)}{2} \int_M v^{\gamma-2} |\nabla_b v|^4 - \frac{\gamma}{2\beta} \int_M v^{\gamma-\beta(q-1)} |\nabla_b v|^2 \\
&\quad + \frac{\lambda\gamma}{2\beta} \int_M v^\gamma |\nabla_b v|^2.
\end{aligned} \tag{3.25}
$$

Substituting (3.10) and (3.20)-(3.25) into (3.5), we obtain

$$
\begin{aligned}
0 &= J + \frac{3}{(\gamma+2)^2 n} \int_M v^{\frac{\gamma+2}{2}} P_0(v^{\frac{\gamma+2}{2}}) - a_1 \int_M v^{\gamma-2} |\nabla_b v|^4 \\
&\quad - a_2 \int_M v^{\gamma-\beta(q-1)} |\nabla_b v|^2 + a_3 \int_M v^\gamma |\nabla_b v|^2 \\
&\quad + \int_M v^\gamma \big(2\,\mathrm{Ric} - (n+1) Tor \big) ((\nabla_b v)_{\mathbb{C}}, (\nabla_b v)_{\mathbb{C}}),
\end{aligned} \tag{3.26}
$$

where

$$a_1 = \frac{1}{2n}\left(\frac{4n+3}{4}\gamma^2 + ((3n+3)\beta + 2n+3)\gamma + (2n+2)(\beta+1)^2\right),$$

$$a_2 = \frac{n+1}{2n}\left(2q + \frac{\gamma}{\beta}\right),$$

$$a_3 = \frac{n+1}{2n}\lambda\left(2 + \frac{\gamma}{\beta}\right).$$

We claim that we can find $\beta \neq 0$ and γ such that

$$a_1 \leq 0, \quad a_2 \leq 0 \tag{3.27}$$

and

$$2\,\mathrm{Ric}(Z, Z) - (n+1)Tor(Z, Z) + a_3\langle Z, Z\rangle \geq 0 \quad \text{for any } Z \in T^{1,0}, \tag{3.28}$$

such that at least one of the inequalities in (3.27) and (3.28) is strict. Then Theorem 1.2 follows from (3.26) and this claim. To see this, it follows from the assumption that the CR Paneitz operator P_0 is nonnegative when $n = 1$. On the other hand, it was proved by Graham and Lee (see [12] and also Theorem 3.2 in [5]) that the CR Paneitz operator P_0 is nonnegative when $n \geq 2$. In both cases, (1.2) holds. We can now conclude that Theorem 1.2 follows from (1.2), (3.7), (3.26) and the claim.

It remains to show the claim. We set $y = 1 + 1/\beta$, $\delta = -\gamma/\beta$ such that $y \neq 1$. Since $\lambda > 0$ by assumption, the claim is equivalent to

$$(2n+2)y^2 - (2n+3)\delta y + \frac{4n+3}{4}\delta^2 - n\delta \leq 0, \tag{3.29}$$

$$2q \leq \delta, \tag{3.30}$$

and

$$2\,\mathrm{Ric}(Z, Z) - (n+1)Tor(Z, Z) \geq \frac{n+1}{2n}\lambda(\delta - 2)\langle Z, Z\rangle \quad \text{for any } Z \in T^{1,0}, \tag{3.31}$$

such that one of the inequalities in (3.29)-(3.31) is strict. It follows from (1.3) that

$$2\,\mathrm{Ric}(Z, Z) - (n+1)Tor(Z, Z) \geq \frac{n+1}{n}\lambda(q-1)\langle Z, Z\rangle \quad \text{for any } Z \in T^{1,0},$$

which implies that there exists δ such that (3.30) and (3.31) hold. If (1.3) is strict, we can choose δ such that one of the two inequalities (3.30) and (3.31) is strict. In case of equality in (1.3), then necessarily $\delta = 2q$. We choose such a δ with $\delta > 0$. In order to find $y \neq 1$ satisfying (3.29), we must have

$$(-(2n+3)\delta)^2 - 4(2n+2)\left(\frac{4n+3}{4}\delta^2 - n\delta\right) \geq 0,$$

or equivalently,

$$\delta\left(8n^2+8n-(4n^2+2n-3)\delta\right)\geq 0. \tag{3.32}$$

Since q satisfies (1.4), we can choose $\delta>0$ satisfying both (3.30) and (3.32), which implies that there exists $y\neq 1$ satisfying (3.29).

On the other hand, if equality holds in (1.3), it follows from the assumption that we have strict inequality in (1.4), which implies $\delta=2q<\frac{8n^2+8n}{4n^2+2n-3}$. In particular, the inequality in (3.32) is strict. This proves that we can find $\delta>0$ and $y\neq 1$ such that one of the inequalities in (3.29)-(3.31) is strict. This proves the claim. □

We remark that, if $n\geq 2$ and (1.3) holds, it follows from Theorem 3 in [19] (see also [13] and [20]) the first positive eigenvalue λ_1 of the sub-Laplacian satisfies

$$\lambda_1\geq\frac{\lambda}{2}(q-1).$$

When $(M,\theta)=(\mathbb{S}^{2n+1},\theta_1)$ where

$$\mathbb{S}^{2n+1}=\{z\in\mathbb{C}^{n+1}:|z|=1\}$$

is the CR sphere and the contact form θ_1 is given by

$$\theta_1=i\sum_{k=1}^{n+1}(z_k d\overline{z}_k-\overline{z}_k dz_k),$$

we have the following:

Corollary 3.1. *Suppose that u is a positive solution of* (1.1) *in* $(\mathbb{S}^{2n+1},\theta_1)$ *for some* $\lambda>0$ *and* $q>1$. *Assume that*

$$\frac{\lambda}{n}(q-1)\leq 1\quad\textit{and}\quad q\leq\frac{4n^2+4n}{4n^2+2n-3}, \tag{3.33}$$

such that one of the inequalities in (3.33) *is strict. Then u is constant with value* $\lambda^{\frac{1}{q-1}}$.

Proof. Note that the torsion of $(\mathbb{S}^{2n+1},\theta_1)$ vanishes. Hence, it follows from the definition in (2.2) that the CR Paneitz operator of $(\mathbb{S}^{2n+1},\theta_1)$ is nonnegative. Note also that the Ricci curvature of $(\mathbb{S}^{2n+1},\theta_1)$ is given by

$$\mathrm{Ric}(Z,Z)=\frac{n+1}{2}\langle Z,Z\rangle\quad\text{for any } Z\in T^{1,0}.$$

Combining these with (3.33), we can conclude that all the assumptions in Theorem 1.2 are satisfied. Now Corollary 3.1 follows from Theorem 1.2. □

We are going to apply Theorem 1.2 to give some estimates on the value of the minimum of some functionals defined in the Folland-Stein space $S_1^2(M)$. For $\lambda \geq 0$ and $1 < q \leq 1 + \frac{2}{n}$, the following expression

$$Q_{\lambda,q}(u) = \frac{\int_M (|\nabla_b u|^2 + \lambda u^2)}{(\int_M |u|^{q+1})^{\frac{2}{q+1}}}$$

is well defined in $S_1^2(M) \setminus \{0\}$ by the Folland-Stein embedding (see [16, Proposition 5.5] for example) and we have

$$\inf\left\{Q_{\lambda,q}(u) : u \in S_1^2(M) \setminus \{0\}\right\} = S_{\lambda,q} \geq 0.$$

Theorem 3.2. *Under the same assumptions as in Theorem* 1.2*, we define*

$$\Lambda = \max\left\{\lambda > 0 : \left(\frac{2}{n+1}\operatorname{Ric} - Tor\right)(Z, Z) \geq \frac{\lambda}{n}(q-1)\langle Z, Z\rangle \textit{ for any } Z \in T^{1,0}\right\}.$$

Then the following estimates hold:

(i) *if* $0 \leq \lambda \leq \Lambda$*, then*

$$S_{\lambda,q} = \lambda\big(\mathrm{Vol}(M,\theta)\big)^{\frac{q-1}{q+1}}; \tag{3.34}$$

(ii) *if* $\lambda > \Lambda$*, then*

$$\Lambda\big(\mathrm{Vol}(M,\theta)\big)^{\frac{q-1}{q+1}} < S^{\lambda,q} \leq \lambda\big(\mathrm{Vol}(M,\theta)\big)^{\frac{q-1}{q+1}}. \tag{3.35}$$

Proof. If $\lambda = 0$, then $Q_{\lambda,q}(1) = 0$ which implies that $S_{\lambda,q} = 0$, *i.e.*, (3.34) holds when $\lambda = 0$. Therefore, we assume that $\lambda > 0$ and it follows from Folland-Stein embedding that $S_{\lambda,q} > 0$. Since $1 < q < 1 + \frac{2}{n}$, it follows from Proposition 5.6 in [16] that $S_1^2(M)$ is compact in $L^{q+1}(M)$. This implies that $S_{\lambda,q}$ is achieved by some $0 \not\equiv u \in S_1^2(M)$. By replacing u with $|u|$, we can assume that u is nonnegative. Then u satisfies

$$-\Delta_b u + \lambda u = C u^q$$

for some positive constant C. Let $v = C^{\frac{1}{q-1}} u$. Then v is nonnegative and is not identically equal to zero such that

$$-\Delta_b v + \lambda v = v^q.$$

Applying in [14, Proposition A.1], we can conclude that $v > 0$. If $\lambda \leq \Lambda$, we can deduce from Theorem 1.2 that v is constant. Then

$$S_{\lambda,q} = Q_{\lambda,q}(u) = Q_{\lambda,q}(v),$$

which implies (3.34). If $\lambda > \Lambda$, we have

$$Q_{\lambda,q}(u) \leq Q_{\Lambda,q}(u)$$

for any $u \in S_1^2(M) \setminus \{0\}$, which implies that

$$S_{\Lambda,q} \leq S_{\lambda,q} \leq Q_{\lambda,q}(1) = \lambda\big(\mathrm{Vol}(M,\theta)\big)^{\frac{q-1}{q+1}}.$$

Now (3.35) follows by combining this and (3.34). □

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. By domain we mean a connected open set in M. Let D be a domain in M, and let W_0 be the closure of $C_c^2(D)$ in the Folland-Stein space $S_1^2(D)$ which consist of functions whose first derivatives (in the sense of the distribution theory) are square integrable functions.

Lemma 4.1. *Let D be a domain in M, $v \in C^2(D) \cap W_0$. Suppose $v > 0$ and $-\Delta_b v = \lambda v$ in D. Then for all $u \in W_0$, we have*

$$\int_D \langle \nabla_b u, \nabla_b u\rangle \geq \lambda \int_D u^2. \tag{4.1}$$

Equality holds if and only if $u = cv$ in D for some $c \in \mathbb{R}$.

Proof. Since $v > 0$, each $u \in C_c^2(D)$ admits the unique factorization $u = \eta v$ where $\eta \in C_c^2(D)$. Then we have

$$\begin{aligned}\int_D \langle \nabla_b u, \nabla_b u\rangle &= \int_D \langle \nabla_b(\eta v), \nabla_b(\eta v)\rangle \\ &= \int_D v^2\langle \nabla_b \eta, \nabla_b \eta\rangle + \int_D \eta^2 \langle \nabla_b v, \nabla_b v\rangle + 2\int_D \eta v\langle \nabla_b \eta, \nabla_b v\rangle.\end{aligned}$$

Note that

$$\begin{aligned}2\int_D \eta v\langle \nabla_b \eta, \nabla_b v\rangle &= \int_D v\langle \nabla_b(\eta^2), \nabla_b v\rangle \\ &= -\int_D \eta^2\langle \nabla_b v, \nabla_b v\rangle - \int_D \eta^2 v\Delta_b v \\ &= -\int_D \eta^2\langle \nabla_b v, \nabla_b v\rangle + \lambda\int_D \eta^2 v^2.\end{aligned}$$

Combining these, we have

$$\int_D \langle \nabla_b u, \nabla_b u\rangle = \int_D v^2\langle \nabla_b \eta, \nabla_b \eta\rangle + \lambda\int_D \eta^2 v^2 \geq \lambda\int_D \eta^2 v^2. \tag{4.2}$$

This proves (4.1).

Now suppose the equality holds in (4.1) for some $u \in W_0$. Let η_n be such that $\eta_n v \to u$ in W_0 as $n \to \infty$. Then we have

$$\int_D \langle \nabla_b(\eta_n v), \nabla_b(\eta_n v)\rangle \to \int_D \langle \nabla_b u, \nabla_b u\rangle \quad \text{as } n \to \infty.$$

It follows from (4.2) that

$$\lim_{n\to\infty} \int_D v^2 \langle \nabla_b \eta, \nabla_b \eta\rangle = 0.$$

Let $D' \subseteq D$ be a domain whose closure is contained in D. Standard arguments shows that any limit point of the sequence $\{\eta_n\}$ in $S_1^2(D')$ is a constant function. Hence $u = cv$ in D for some $c \in \mathbb{R}$. The converse is obvious. □

For $u \in E_\lambda$, we let

$$N_u = \{x \in M : u(x) = 0\}.$$

Then the nodal domain of u is a connected component of $M_u := M \setminus N_u$.

Lemma 4.2. *Let $u, v \in E_\lambda$ and $u, v \not\equiv 0$. Let U and V be nodal domains of u and v respectively. If $U \subseteq V$, then $u = cv$ for some $c \in \mathbb{R}$.*

Proof. Let $D = V \supseteq U$. We may assume $v > 0$ in D and $u > 0$ in U. Let $\overline{u}$ be zero outside U and coincide with u in U. Clearly, $v, \overline{u} \in W_0$. We have

$$\int_D \langle \nabla_b \overline{u}, \nabla_b \overline{u}\rangle = \int_U \langle \nabla_b u, \nabla_b u\rangle = \lambda \int_U u^2 = \lambda \int_D \overline{u}^2.$$

By Lemma 4.1, $u = cv$ in D for some $c \in \mathbb{R}$. It follows from the unique continuation that $u = cv$ on M. □

Lemma 4.3. *If $\lambda \neq 0$, then $M_u \neq M$ for any $u \in E_\lambda$.*

Proof. Suppose on the contrary, $M_u = M$. Then N_u is empty and $u(p) \neq 0$ for all $p \in M$. Without lost of generality, we may assume that $u > 0$, which implies that $\int_M u > 0$. Hence, we have

$$0 = \int_M \Delta_b u = -\lambda \int_M u \neq 0,$$

which is a contradiction. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\mathcal{U}$ and $\mathcal{V}$ be families of nodal domains for u and v respectively. Since $\lambda \neq 0$, $M_u, M_v \neq M$ by Lemma 4.3. Obviously, u and v have no common zeroes if and only if $\mathcal{C} := \mathcal{U} \cup \mathcal{V}$ is a covering of M:

$$M = \bigcup_{W \in \mathcal{C}} W. \tag{4.3}$$

We are going to prove that, under the assumption (4.3), there exists a closed 1-form in M which is not exact. Note that the covering $\mathcal{C}$ has following properties:

(A) The sets in $\mathcal{U}$ are pairwise disjoint, and the same is true for $\mathcal{V}$;

(B) Neither $U \subseteq V$ nor $U \supseteq V$ for every $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

(A) is obvious, and (B) is a consequence of Lemma 4.2. Also, it follows from Lemma 4.2 that

$$U \cap N_v \neq \varnothing, \quad V \cap N_u \neq \varnothing \quad \text{for all } U \in \mathcal{U}, V \in \mathcal{V}. \tag{4.4}$$

Thanks to (4.4), $\mathcal{C}$ is finite, since $\mathcal{V}$ covers the compact set N_u by open disjoint sets, and the same is true for $\mathcal{U}$ and N_v. This also means that a connected component X of N_u is contained in some nodal domain for v.

Let Γ be the graph whose vertex set is $\mathcal{C}$, and two distinct sets in $\mathcal{C}$ are adjacent if they have nonempty intersection. Hence, the following conditions hold for Γ:

(a) Each edge of Γ joins U and V for some $U \in \mathcal{U}$ and $V \in \mathcal{V}$;
(b) Any vertex is incident to at least two edges. It follows that Γ contains a nontrivial cycle C.

Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ be consecutive vertices of C, $Q = \partial U \cap V$. Since $Q \cap \partial V = \varnothing$ thanks to (4.3), both Q and $\partial U \setminus Q = \partial U \setminus V$ are compact. Hence there exists a smooth function f in M such that $f = 1$ in a neighbourhood of Q and $f = 0$ near $\partial U \setminus Q$. Then $df = 0$ on ∂U and the 1-form η which is zero outside U and coincides with df in U is well defined and smooth. Obviously, η is closed, and we claim that η cannot be exact.

Suppose $\eta = dF$. Then F is constant on each connected set which does not intersect $\operatorname{supp}(\eta) \subset U$. Let $U_1 = U$, $V_1 = V$,..., U_m, V_m be the cycle C, where $m > 1$. By replacing C with a shorter cycle if necessary, we may assume that

$$V_k \cap U = \varnothing \quad \text{for } 1 < k < m. \tag{4.5}$$

If a curve in V starts outside U and comes into U, then it meets U at a point of Q. Hence there exists a curve c_1 in $V \setminus U$ with endpoints in Q and U_2. Similarly, there is a curve c_2 inside $V_m \setminus U$ with endpoints in U_m and $V_m \cap \partial U$. The set

$$X = c_1 \cup U_2 \cup V_2 \cup \cdots \cup U_m \cup c_2$$

is connected, and by (A) and (4.5), $X \cap U = \varnothing$. Therefore, F is constant in X. But this contradicts to the choice of f since the closure of X has common points with Q and $\partial U \setminus Q$ (recall that $dF = df$ on U and that f takes different values on these sets). □

The following example shows that Theorem 1.4 is not true if M is not compact: If we take $(M,\theta)=(\mathbb{H}^n,\theta_0)$ to be the Heisenberg group

$$\mathbb{H}^n=\left\{(z,t):z=(z_1,...,z_n)\in\mathbb{C}^n,\,t\in\mathbb{R}\right\}$$

equipped with the standard contact form

$$\theta_0=dt+\sqrt{-1}\sum_{j=1}^{n}(z_j d\overline{z}_j-\overline{z}_j dz_j),$$

then the sub-Laplacian $\overset{\circ}{\Delta}_b$ is given by (see section 4 in [16] for example)

$$\overset{\circ}{\Delta}_b=\frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j+\overline{Z}_jZ_j),$$

where $Z_j=\frac{\partial}{\partial z_j}+\sqrt{-1}\overline{z}_j\frac{\partial}{\partial t}$ for $j=1,...,n$. Hence, we have

$$-\overset{\circ}{\Delta}_b\sin x_1=\frac{1}{4}\sin x_1\quad\text{and}\quad-\overset{\circ}{\Delta}_b\cos x_1=\frac{1}{4}\cos x_1$$

where $z_1=x_1+\sqrt{-1}y_1$. That is to say, $\sin x_1$ and $\cos x_1$ are eigenfunctions corresponding to the eigenvalue $1/4$, and they do not vanish simultaneously because $\sin^2x_1+\cos^2x_1=1$. On the other hand, we have $H^1(\mathbb{H}^n)=0$. This shows that Theorem 1.4 is not true if M is not compact.

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